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Integrity conditions for elastic-plastic damaged solids subjected to cyclic loading

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INTEGRITY CONDITIONS for elastic-plastic, isotropically damaged solids with isotropic and kinematic strain hardening as subjected to cyclic loading, are in question. It is assumed that the damage process is coupled with the process of plastic deformation. The shakedown conditions are assumed to be satisfied. A new sufficient condition for shakedown accounting for a mixed isotropic and kinematic hardening is developed. The problem of evaluating limit yield-condition arguments is reduced to a min-max problem. In the case of plain strain, the problem is equivalent to a hyperbolic equation in partial derivatives of the second order. A method for computing the purely elastic damaged response of the solid to the prescribed loading program is proposed. The limit yield condition with specified arguments makes it possible to obtain upper and lower estimates for the local actual limit values of the damage parameter admitted by the yield condition for the given loading program. The estimates lead to necessary and sufficient conditions of integrity. The proposed method is illustrated by an example.

1. Introduction

UNDER CERTAIN CONDITIONS, irreversible changes in the material of elastic-plastic damaged solids subjected to cyclic loading vanish after a period of adaptation, and the solid experiences only elastic deformation starting from some time on. One says in this case that the solid adapted (shook down) to the prescribed loading program; in other words, the deformation process reached a stationary (post-adaptation) stage. If the damage and plastic deformation processes are coupled, then the values of damage and internal parameters at this stage do not change. These values and the corresponding yield surface will be referred to as the limiting ones.

The shakedown theory provides us with the means to predict directly, i.e. without a detailed computation of the deformation path, whether the solid will adapt itself to the prescribed cyclic loading program or not. This gives us a chance to establish the estimates of the damage parameters limiting value. Reviews of the modern achievements in the shakedown theory are available in [1–4]. Presently the question of extension of the shakedown theory to damaged elastic-plastic solids is topical.

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HACHEMI and WEICHERT extended the Melan theorem to elastic-plastic isotropically damaged solids with unlimited [5] and limited [6] linear kinematic strain-hardening. SIEMASZKO [7] developed a method of step-by-step inadaptation analysis for elastic-plastic structures subjected to repeated loading, which accounts for nonlinear isotropic strain hardening, developing of damage, and nonlinear geometrical effects. POLIZZOTTO, BORINO and FUSCHI [8] included the damage variable into a set of internal variables, and developed an elastic-plastic damaged material model with associated constitutive relations and non-linear elasticity. Employing the D-stability principle introduced by them, they extended the static Melan shakedown theorem to this model. The theorem was also extended to the elastic damaged material model as well.

DRUYANOV and ROMAN [9] extended the Melan theorem to the model of damaged elastic-plastic solids with isotropic and isotropic-like strain hardening.

All known extensions of the Melan theorem to elastic plastic solids with isotropic damage can be formulated as follows: if there exists a time-independent field of effective residual stress tensor $\hat{\bar{\boldsymbol{\rho}}}(\mathbf{x})$, which satisfies the yield inequality $\Phi(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}(\mathbf{x},\mathbf{t}), +\hat{\bar{\boldsymbol{\rho}}}(\mathbf{x}), \chi(\mathbf{x},\mathbf{t})) < \mathbf{0}$ from some time on, then the total plastic dissipation and damage parameter are bounded. Here $\Phi(\bar{\boldsymbol{\sigma}}, \chi)$ is the yield function, $\bar{\boldsymbol{\sigma}}$ is the effective stress tensor, χ denotes the strain hardening parameter, and $\bar{\boldsymbol{\sigma}}^{\mathrm{E}}$ is the effective stress tensor representing the current, purely elastic response of the solid to the prescribed loading program.

Obviously, even if the conditions of the extended Melan theorem are satisfied, a damaged elastic plastic solid can fail due to accumulation of damage because the conditions of local integrity may be violated. For example, in the case of isotropic damage, the limit value of the damage parameter may exceed its critical value, and the solid will collapse before the plastic deformation process ceases.

To establish the conditions of integrity, FENG and YU [10, 11] supposed that the state of damage is described by a scalar quantity connected with the plastic strain tensor. They introduced a damage factor as a local average of this quantity, and assumed that the safety of structures subjected to cyclic loading is guaranteed, if the damage factor is less than its critical value. Assuming a piecewise linear yield condition, they reduced the computation of an upper bound for the damage factor to a problem of mathematical programming. Besides, a method of obtaining a lower bound was developed.

HACHEMI and WEICHERT [6, 12], WEICHERT and HACHEMI [13] employed the model of elastic plastic damaged material by Lemaitre [14] and the Generalized Standard Material Model approach (HALPHEN and NGUYEN [15]) to derive upper bounds for the accumulated plastic strain and damage parameter. As a result, they reduced the problem of determining of the safety factor to a problem of mathematical programming. A numerical method capable of controlling the current value of the damage parameter was also developed.

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A method to find lower estimates of the limit value of isotropic damage parameter was developed in DRUYANOV and ROMAN [16].

Thus, the fulfillment of the shakedown conditions is only a necessary condition for the safety of solids subjected to cyclic loading. To assure their safety, the condition of local integrity at every point of the solids has to be satisfied.

Hence, a problem appears: for the prescribed loading program, to derive the conditions of shakedown and upper and lower estimates to the limit value of damage parameter at every point of the solid, and based on them to set a priori conditions of integrity. Below, a method to solve this problem is proposed.

In the main body of the paper, the method is developed for elastic plastic solids with isotropic damage and isotropic strain hardening. Then the method is extended to solids with both kinematic and isotropic strain hardening.

The method is based on the assumption that the shakedown conditions are fulfilled. Therefore, for the sake of completeness, a novel sufficient shakedown condition accounting for kinematic strain hardening additionally to the isotropic one is proposed in Appendix 1.

A shakedown condition for structures of elastic-perfectly plastic materials with linear kinematic hardening was formulated by MELAN, as early as in 1938 [17]. A mixed linear kinematic-isotropic hardening was considered by MAIER and NOVATI [18]. Other early extensions are available in the book by KÖNIG [19]. STEIN *et al.* [20] showed how the shakedown theorems can be extended to material models with nonlinear kinematic hardening. POLIZZOTTO *et al.* extended the shakedown theory to a material model with dual internal variables and a thermodynamic potential [21]. Static and kinematic approaches to shakedown conditions for the generalized standard material model with limited kinematic/isotropic hardening was considered by NGUYEN and PHAM [22].

The quality of the obtained estimation depends on the degree of strain hardening. The less is the strain hardening, the better is the quality.

An example of application of the developed method is given.

2. Constitutive relations. Extended Melan theorem

The elastic-plastic isotropic damage material models by LEMAITRE [14] and JU [23] are employed as a basis for the formulation of constitutive relations. The formulation is incomplete: only the relations which are employed in the subsequent argumentation are formulated.

The damage and strain hardening are taken as isotropic. Thermal fluxes and inertia forces are neglected. The components of the total strain tensor ($\boldsymbol{\varepsilon}$) are assumed small, so that $\boldsymbol{\varepsilon}$ can be decomposed into plastic ($\boldsymbol{\varepsilon}^e$) and elastic ($\boldsymbol{\varepsilon}^p$) parts: $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p$.

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The Hooke law is expressed by the equation $\varepsilon^e = \mathbf{C}_0^{-1} : \bar{\sigma}$, $\bar{\sigma} = \sigma/(1-\Delta)$ where Δ is the damage parameter, $\mathbf{C}_0$ is the initial fourth-rank elastic stiffness tensor,$\bar{\sigma}$and σ are the tensors of effective and nominal stresses respectively, and $(\mathbf{C}\colon \varepsilon)_{ij} = \mathbf{C}_{ijkl}\varepsilon_{kl}$. The current (damaged) value of the elastic stiffness tensor is $\mathbf{C}=\mathbf{C}_0\ (1-\Delta)$.

It is assumed that $0\leq \Delta \leq \Delta_c<1$ where Δ_c is the critical value of the damage parameter, i. e. the material preserves its local integrity until $\Delta<\Delta_c$.

The plastic strain rate tensor is supposed to obey to the associated flow rule: $\dot{\varepsilon}^p = \dot{\lambda}\partial\Phi/\partial\bar{\sigma}$, $\dot{\lambda}\Phi(\bar{\sigma},\chi)=0$, $\dot{\lambda} \geq 0$, $\Phi(\bar{\sigma},\chi) \leq 0$ where $\Phi(\bar{\sigma},\chi) \equiv \zeta$ is the yield function, $\dot{\lambda}$ is the plastic multiplier, and χ is the strain-hardening parameter.

The yield function is assumed to be strictly convex in the argument $\bar{\sigma}$, and the inequality $\Phi(\bar{\sigma},\chi)<0$ corresponds to the interior of the yield surface $\Phi(\bar{\sigma},\chi)=0$ in the effective stress space $\bar{\sigma}$. Consequently, if $\Phi(\bar{\sigma},\chi)=0$ and $\Phi(\hat{\bar{\sigma}},\chi)<0$ where $\hat{\bar{\sigma}}$ is an effective virtual stress, then $(\bar{\sigma}-\hat{\bar{\sigma}}) : \dot{\varepsilon}^{\mathrm{p}} > 0$.

Let $\hat{\bar{\sigma}} = \dfrac{\hat{\sigma}}{1-\Delta(\mathbf{x},t)}$ where $\hat{\sigma}$ is the nominal virtual stress, and $\Delta(\mathbf{x},t)$ is the actual value of the damage parameter. Then the last inequality becomes

$$(\sigma - \hat{\sigma}) : \dot{\varepsilon}^{\mathrm{p}} \geq 0. \tag{2.1}$$

The unloading process is assumed purely elastic with the current damage value of the elastic stiffness tensor $\mathbf{C}=\mathbf{C}_0(1-\Delta)$.

The damage process is assumed to be coupled with the process of plastic deformation, i.e. the damage can develop only if the plastic deformation process is in progress.

It is supposed that the hardening is limited, i.e. there exists a material constant χ^* such that $0\leq \chi \leq \chi*$. The constant χ^* corresponds to the state of hardening saturation. See also [6].

There are two concurring growing parameters in elastic-plastic damaged solids: the parameter of isotropic strain hardening χ, and the parameter of softening Δughe first increases the yield surface, the second diminishes it. It is assumed that in the interval $0\leq \chi \leq \chi^*$ the material is stable, i.e. all the subsequent yield surfaces corresponding to increasing values of χ and Δ comprise the previous ones. In other words, this means that in the interval $0\leq \chi \leq \chi^*$the rate of strain hardening surpasses the rate of damage growth.

Thus, if there exists a field of virtual stress $\hat{\sigma}(\mathbf{x},t)$ that satisfies the inequality

$$\Phi\left(\frac{\hat{\sigma}(\mathbf{x},t)}{1-\Delta_0}, \chi_0\right) < 0 \tag{2.2}$$

for any $t \geq t_0$ where $\Delta_0 = \chi_0 = 0$ are the initial values of Δ and χ, then the inequality

$$\Phi\left(\frac{\hat{\sigma}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)}, \chi(\mathbf{x},t)\right) < 0 \tag{2.3}$$

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is valid for any $t_0 \geq 0$ until $\chi \leq \chi^*$, where $\Delta(\mathbf{x},t)$ and $\chi(\mathbf{x},t)$ are the actual values of Δ and χ. This inequality shows that the stress $\hat{\bar{\sigma}}(t) = \dfrac{\hat{\sigma}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)}$ is in the interior of the current yield surface for $t \geq 0$. Consequently, due to the assumption of material stability, inequality (2.1) preserves its format for $t \geq 0$ and $\chi_0 \leq \chi \leq \chi^*$.

Now with inequality (2.1) in hand, it is possible to extend the Melan theorem to the chosen material model [24]. The extended theorem can be formulated as follows.

If there exists a virtual stationary stress field $\hat{\mathbf{r}}(\mathbf{x})$ such that the fictitious virtual decomposition $\hat{\sigma} = \mathbf{s}^E(\mathbf{x},t) + \hat{\mathbf{r}}(\mathbf{x})$ satisfies inequality (2.2) at a time t_o, then the structure under consideration shakes down, i.e. the plastic strain rate tensor tends to zero: $\dot{\varepsilon}^p \to 0$, and the total plastic dissipation is bounded: $W < w^* < \infty$, where W is the total plastic dissipation, and w^* is a number.

This condition is not only sufficient, but also necessary for shakedown.

Here $\mathbf{s}^E(\mathbf{x},t)$ represents the fictitious actual purely elastic response of the structure to the actual value of the load and for the time-independent value of the damage parameter $\Delta_0(\mathbf{x})$ equal to its initial value (the value corresponding to the time t_0), and $\hat{\mathbf{r}}(\mathbf{x})$ is a fictitious virtual time-independent stress.

More specifically, $\mathbf{s}^E(\mathbf{x},t)$ is computed for the initial (given) value of the elastic stiffness tensor $\mathbf{C}=\mathbf{C}_0(1-\Delta_0(\mathbf{x}))$, whereas, in contrast to it, the actual elastic response $\sigma^E(\mathbf{x},t)$ is computed for the current (damaged) value of the elastic stiffness tensor $\mathbf{C}=\mathbf{C}_0(1-\Delta)$ where $\Delta(\mathbf{x},t)$ is the current value of the damage parameter.

As the initial values of the damage and hardening parameters (Δ_0, χ_0) should be given, the computation of $\mathbf{s}^E$ does not cause any principal difficulties.

3. Features of the post-adaptation stage under cyclic loading

In conditions of cyclic loading, the features of the post-adaptation stage provide us with the possibility to obtain directly, i.e. without detailed investigation of the loading path, a relation between limit values of the residual stress tensor and the damage parameter.

If the condition of the extended Melan theorem is satisfied, i.e. if there exists a stationary residual stress tensor $\hat{\rho}(\mathbf{x})$ such that the stress $\hat{\sigma} = \sigma^E(\mathbf{x},t) + \hat{\rho}(\mathbf{x})$ satisfies inequality (2.2) for any $t \geq 0$, then the structure shakes down, i.e. eventually the deformation process reaches the post-adaptation stage.

Time-independent values of the residual stress tensor ρ_s, the damage parameter Δ_s and the hardening parameter χ_s are characteristic for the post-adaptation stage of the deformation process, if it exists. These values and the corresponding yield surface will be called hereafter the limit ones.

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At the post-adaptation stage, the representative actual stress point in the effective stress space $\bar{\sigma}$ reaches the limit yield surface repeatedly, but the stress does not cause plastic deformation and damage, and the limit yield surface does not change. This is possible, if either some parts of the stress path $\bar{\sigma}$ are placed on the yield surface (neutral loading), or the stress path touches it at some isolated points. In particular, this is valid at the time instants t^* corresponding to the beginning of unloading. These time points will be named the departure instants.

At the departure instants the effective stress satisfies the equation of the yield surface, therefore the following relation is valid:

$$\zeta = \zeta^* = \Phi\Big(\frac{\sigma(t^*)}{1-\Delta_s}, \chi_s\Big)\Big) = 0. \tag{3.1}$$

The stress point cannot escape from the yield surface. According to the assumption, $\zeta<0$ for the stress points situated in the interior of the yield surface, and $\zeta=0$ for the points of it. Hence, the departure points are the points of absolute maximum of the yield function $\zeta = \Phi\,(\bar{\sigma}(t),\chi)$ with respect to t [25].

However, due to cyclic nature of loading, the yield function can have several points of local maxima in the elastic region of the deformation process. These points are situated in the interior of the yield surface.

This statement is valid for the departure instants during the whole deformation process. At the post-adaptation stage, the quantities $\boldsymbol{\rho}$, $\boldsymbol{\chi}$ and Δ do not change in time. To account for this property, it is necessary to return to the nominal stress tensor, and to employ the known decomposition $\boldsymbol{\sigma}(\mathbf{x},t)=\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t)+\boldsymbol{\rho}$.

Consequently, at the departure instants equation (3. 1) can be represented as

$$\zeta(\mathbf{x}) = \zeta^*(\mathbf{x}) = \Phi\,\Big(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)}{1-\Delta_{\mathrm{s}}(\mathbf{x})} - \boldsymbol{\eta}_{\mathrm{s}}(\mathbf{x},\boldsymbol{\eta}_{\mathrm{s}}(\mathbf{x}))\Big) = 0. \tag{3.2}$$

$$\boldsymbol{\eta}_s(\mathbf{x}) = -\frac{\boldsymbol{\rho}_{\mathrm{s}}(\mathbf{x})}{1-\Delta_{\mathrm{s}}(\mathbf{x})}\,. \tag{3.3}$$

The function

$$\bar{\boldsymbol{\sigma}}(\mathbf{x}) = \frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t)}{1-\Delta_{\mathrm{s}}(\mathbf{x})} - \boldsymbol{\eta}_{\mathrm{s}}(\mathbf{x}). \tag{3.4}$$

determines the the stress path at the point $\mathbf{x}$ of the solid, in the stress space $\bar{\boldsymbol{\sigma}}$.

Due to cyclic nature of the loading, stress path (3.4) has a number of apexes, which are specified by the loading program. It is supposed, as it usually is, that ζ is a non-decreasing function of effective stress tensor. Therefore the local and

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absolute extrema of the yield function correspond to the apexes of the stress path (3.4).

Hereafter only the post-adaptation stage is considered, so the subscript "s" is omitted.

At the post-adaptation stage, the stress path reaches the yield surface at one, two or more points. The event, when the stress path reaches the yield surface at a single point corresponds to one-sided loading and a deformation of the same sign. This is the event of ratcheting. This paper is, however, devoted to Low Cycle Fatigue. In this case the stress path reaches the yield surface at least at two points. At these points the yield function reaches its absolute maximum value equal to zero, i.e. these values of the yield function are equal to each other. To model this situation, it is necessary to require that the absolute maximums of the yield function should be minimal. As a result, the following specification of equation (3.2) is arrived at:

$$\zeta_m(\mathbf{x}) \equiv \min_{\boldsymbol{\eta}} \max_{t} \zeta(\mathbf{x}) \equiv \min_{\boldsymbol{\eta}} \max_{t} \Phi\left(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x}, t(\mathbf{x}))}{1-\Delta} - \boldsymbol{\eta}(\mathbf{x}), \chi\right) = 0. \tag{3.5}$$

The above min-max problem should be solved for fixed values of $\mathbf{x}, \Delta$ and χ.

The variables $\boldsymbol{\eta}$, Δ and $\boldsymbol{\rho}$ are connected by the relation $\boldsymbol{\rho} = (1-\Delta)\boldsymbol{\eta}$. Since Δ is fixed, the minimum of the function $\max \zeta$ should be found with respect to $\boldsymbol{\rho}$. The residual stress tensor $\boldsymbol{\rho}$ should satisfy the equilibrium equations $\nabla \cdot \boldsymbol{\rho}=0$, and the boundary conditions $\boldsymbol{\rho}\cdot\boldsymbol{\nu}=0$ at the part of the solid surface S_p, where tractions are prescribed. Here, ∇ is the vector with components $\partial/\partial x_i$, $\boldsymbol{\nu}$ is the unit vector of the external normal to S_p, and $\mathbf{a}\cdot\mathbf{b} = a_i b_i$.

REMARK 1. The equilibrium equations can be satisfied by means of introducing the Airy functions, which are defined by the min-max problem at the left-hand side of (3.5). The application of these functions to the problem of plane strain/stress of linear elasticity is widely known [26]. In the case of plane strain and the von Mises yield condition, the application of the Airy functions reduces the min-max problem (3.5) to a boundary-value problem for a hyperbolic equation in partial derivatives of the second order (Appendix 2).

The solution of the min-max problem provides us with the values $\zeta_m(\mathbf{x})$, $\boldsymbol{\eta}_m(\mathbf{x})=\boldsymbol{\rho}_m/(1-\Delta)$ and $t^*(\mathbf{x})$ at every point of the solid under consideration as a function of Δ and χ. It exists if the yield function $\zeta = \Phi(\bar{\boldsymbol{\sigma}}, \chi)$ is convex in $\bar{\boldsymbol{\sigma}}$ for any admissible values of Δ and χ.

There could be more than one solution of equation (3.5) with different values of $\boldsymbol{\eta}_m$ corresponding to different shifts of the stress path in the case, when the stress path has more than two apexes. This situation is shown in Fig. 1 where a triangular stress path ABC is considered, as an example. Three solutions of the

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min-max problem can be obtained in this case by such shifts of the triangle ABC which result in adjoining one of the sides AB, BC or CA to the yield surface. Consequently, in the case under consideration three solutions to Eq. (3.5) are possible.

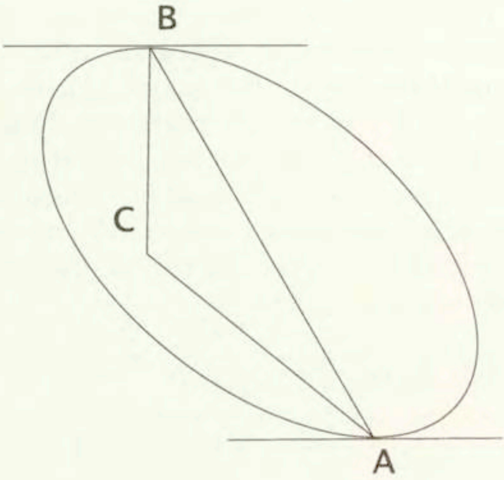

FIG. 1.

Equation (3.5) establishes a dependence between χ and Δ. This dependence defines the diameter of yield surface because changes in both χ and Δ transform the yield surfaces similarly. Notice that different solutions of (3.5) lead to different dependences, i.e. to the yield surfaces of different diameters.

Consider the von Mises yield condition for example. Its equation can have the form as $\Phi = f(\sigma) - (1-\Delta)\,\kappa(\chi) = 0$ where $\kappa(\chi)$ is the yield stress. The quantity $(1-\Delta)\kappa(\chi)$ defines the radius of the von Mises cylinder. If $f(\sigma)$ is fixed, then the radius is fixed as well.

Let us fix a pair (Δ,χ) satisfying (3.5). The quantity η_m specifies a certain position of the stress path (3.4) with respect to the yield surface defined by the pair (Δ,χ). Simultaneously η_m provides a minimal value to the function $\max\zeta$. As $\min\max\zeta$ is equal to zero, a change in η_m provides a positive value to $\max\zeta$, i.e. it shifts the stress path in such a way that at least one of its apexes falls outside the limit yield surface.

In the case when the stress path has only two apexes placed at the limit yield surface, Eq. (3.5) leads to such a position of the stress path that these apexes coincide with the opposite ends of the chord of the maximal length, whose direction coincides with the direction of the corresponding chord of the stress path (Fig. 1).

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Hence, in the classical case, when the applied load ranges between two values, the solution to equation (3.5) is unique.

If there are more than one solution to equation (3.5), the solution resulting in the best estimates, i.e. in the maximal lower estimate, and in the minimal upper estimate, should be chosen.

4. Estimating the limit value of the damage parameter and conditions of integrity

It is assumed in the subsequent argumentation that the conditions of shakedown are satisfied.

Equation (3. 5) depends on the function $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)$ which represents the current damaged, purely elastic response of the solid to the prescribed loading program. This function satisfies the system of the linear elasticity equations with Hook's law $\boldsymbol{\varepsilon}^{\mathrm{E}} = \mathbf{C}^{-1} : \boldsymbol{\sigma}^{\mathrm{E}}$ where $\mathbf{C} = \mathbf{C}_0\,(1-\Delta)$, for the boundary conditions corresponding to the departure instants t^*. Obviously, the value of Δ at $t = t^*$ is unknown in advance.

To overcome this deficiency, the following method is proposed. At the departure instant t^* the function $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*) + \boldsymbol{\rho}_m\mathbf{x}$ satisfies the yield condition. Here $\boldsymbol{\rho}_m = \boldsymbol{\eta}_m(1-\Delta)$. This property gives us the possibility of computing $\boldsymbol{\sigma}^E$ at $t = t^*$ directly, i.e. without a detailed investigation of the deformation process, by means of resolving the boundary-value problem for the system of the elasticity equations supplemented with the equations $\mathbf{C} = \mathbf{C}_0\,(1-\Delta)$ and (3. 5) for the corresponding boundary conditions. The solution of this system for fixed values of Δ and $\boldsymbol{\chi}$ provides us with the values of $t^*(\mathbf{x})$ and $\boldsymbol{\eta}_m\,(\mathbf{x},t^*)$, aside from $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)$.

For known $t^*(\mathbf{x})$, $\boldsymbol{\eta}_m\,(\mathbf{x},t^*)$ and $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x},t^*)$, equation (3.5) defines Δ as a function of χ at every point of the solid at the time instant t^*. Because Δ is a non-decreasing function of t, its values at $t = t^*$ are the limit values of Δ admitted by the yield function under the prescribed loading program.

Suppose for definiteness that ζ_m is a monotonic function of Δ. Then the extremal values of χ (χ_0 and χt^*) define the extremal limit values (bounds) of Δ admitted by the yield function and possible under the prescribed loading program: $\Delta_{\min}$ and $\Delta_{\max}$.

Hence, to determine the bounds for Δ, it is necessary to set subsequently $\chi = \chi_0$, and $\chi = \chi^*$. Then the above-mentioned system of equations becomes definite, and defines the unknown variables $t^*(\mathbf{x})$, $\boldsymbol{\eta}_m\,(\mathbf{x},t^*)$ and $\sigma t^*\mathrm{E}(\mathbf{x},t^*)$ and the bounds $\Delta_{\min}$ and $\Delta_{\max}$..

As a result, the following estimate for the limit values of Δ at every point of the solid is obtained.

$$\Delta_{\min}(\mathbf{x}) \leq \Delta\mathbf{x} \leq \Delta_{\max}(\mathbf{x}). \tag{4.1}$$

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This estimate provides us with the necessary and sufficient conditions of local integrity for the given loading program: $\Delta_{\min}<\Delta_c$ and $\Delta_{\max}<\Delta_c$, respectively. If $\Delta_{\max}<\Delta_c$, the local integrity is not violated. On the other hand, if $\Delta_{\min} \geq \Delta_c$, then the solid looses its local integrity.

The quality of the estimation depends on the difference $\delta = \Delta_{\max} - \Delta_{\min} \geq 0$. The less is δ the better is the quality; otherwise, the lower is the degree of strain hardening, the better is the quality.

The condition of overall integrity could be formulated as follows: the solid saves its overall integrity, if the maximal value of the upper estimate over the solid is less than the critical value of the damage parameter. It is supposed that the necessary condition of overall integrity coincides with that of local integrity: if the necessary condition of local integrity is violated, then the condition of overall integrity is violated as well.

5. Accounting for kinematic strain hardening.

In this section, the developed method is extended to material models with kinematic strain hardening, and additionally to the isotropic one.

Let $\boldsymbol{\beta}\boldsymbol{\varepsilon}^{\mathrm{p}}$ denote the back-stress tensor. It is assumed that the state of saturation exists for the kinematic strain hardening, i.e. the values of back-stress components are bounded by a material constant β^*: $|\beta_{ij}| \leq \beta^*$.

The yield surface equation is written as $\Phi(\bar{\sigma} - \boldsymbol{\beta}\chi) = 0$. The variables $\boldsymbol{\beta}(\boldsymbol{\varepsilon}^{\mathrm{p}})$ and $\chi(\boldsymbol{\varepsilon}^{\mathrm{p}})$ define the position and size of the yield surface.

Suppose that the solid under consideration shakes down to the prescribed loading program. All the arguments given in Secs. 3, 4 remain valid. The only difference is in the definition of $\boldsymbol{\eta}$ which becomes (see (3. 3) for comparison)

$$\boldsymbol{\eta} = -\frac{\boldsymbol{\rho}}{1-\Delta} + (1-\Delta)\boldsymbol{\beta}. \tag{5.1}$$

The method developed above proceeds from the assumption that the conditions for shakedown are fulfilled. To extend it to kinematic strain hardening, a new sufficient shakedown condition accounting for both the isotropic and kinematic strain hardening is proposed. The condition is based on the notion of the depository surface.

The yield surface $\Phi(\bar{\sigma}, \chi)=0$ is assumed regular with the principal radii of curvature R_{i} greater than β^*. Under this condition, the surface $\breve{\Phi}(\bar{\sigma}, \chi)=0$ "parallel" to $\Phi(\bar{\sigma}, \chi) = 0$ exists with the principal radii of curvature equal to $R_{\mathrm{i}} - \beta^*$. This surface will be named the depository one. The depository surface $\breve{\Phi}(\bar{\sigma},\chi) = 0$ is in the interior of the yield surface $\Phi(\bar{\sigma}, \chi) = 0$, and separated from it by a layer of constant thickness equal to β^*.

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In certain respect, the depository is similar to the "reduced elastic domain" [27], and the "sanctuary" [28].

Consider for example the von Mises yield surface. In the principal stress space it is a circular cylinder of the radius $\sigma_s\sqrt{2/3}$ where σ_s is the yield stress of the material in tension. The corresponding depository surface is also a cylinder of the radius $\sigma_s\sqrt{2/3} - \beta^*$.

Assume that the initial (at $t=0$) values of the damage and strain hardening parameters are $\Delta_0 = \chi_0 = 0$. Suppose that there exists a stationary residual stress field $\hat{\rho}$ such that the following inequality is valid for $t \geq 0$:

$$\breve{\Phi}\left(\frac{\sigma^{\mathrm{E}}(\mathbf{x},t) + \hat{\rho}(\mathbf{x})}{1-\Delta_0}\right) < 0 \tag{5.2}$$

at every point of the solid under consideration.

Under the above assumptions, the proposed theorem can be formulated as follows: the total plastic dissipation is bounded from above, if there exists a stationary field of residual stress $\hat{\rho}(\mathbf{x})$, such that the stress $\hat{\sigma} = \sigma^{\mathrm{E}}(\mathbf{x},t) + \hat{\rho}(\mathbf{x})$ satisfies inequality (5.2) for any $t \geq t_0$.

The detailed proof of this theorem is presented in the Appendix. An example of application of the theorem is given in the next section.

6. Example

Let us consider the structure shown in Fig. 2. The structure consists of three rods of the same cross-sectional area S, and the same material. The rods 2, 3 are twice as long as that of rod 1: $l_3 = l_2 = 2l_1$. The structure is loaded by a variable force $P(t)$ ranging in the interval $-P_1 \leq P \leq P_2$, $P_1 \leq P_2$, where $P(t)$ is a given function of time. The rods can bear only uniaxial tensile/compressive deformation.

Due to the symmetry, the strains and stresses in rods 2 and 3 are identical: $\varepsilon_2 = \varepsilon_3$, $\sigma_2 = \sigma_3$. The strains of rods 1 and 2 are connected by the relation: $\varepsilon_1 = 2\varepsilon_2$. The stresses in the rods satisfy the equilibrium equation: $\sigma_1 + 2\sigma_2 = \mathrm{p}$, where $p_1 \leq p(t) \leq p_2$, $p = P(t)/S$, $p_1 = P_1/S$, $p_2 = P_2/S$.

It is assumed that the damage process is coupled with the process of plastic deformation, i.e. the damage can develop only if the plastic deformation process is in progress. It is assumed also that the damage process starts simultaneously with the process of plastic deformation, i.e. the damage threshold is small enough.

In the elastic undamaged state $\sigma_1^E = 2\sigma_2^E = p/2$. Assume that rods 2, 3 remain elastic, whereas rod 1 experiences plastic deformation accompanied by damage. The yield condition of the rod 1 material is taken in the form: $\Phi = |\bar{\sigma} - \beta(\varepsilon^p)| - \kappa(\chi) = 0$, or

$$\Phi = |\sigma - (1-\Delta)\beta(\varepsilon^p)| - (1-\Delta)\kappa(\chi) = 0 \tag{6.1}$$

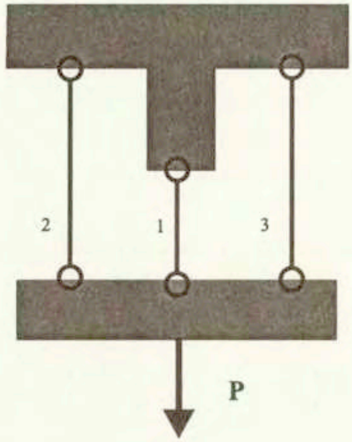

FIG. 2.

where Δ is the current value of damage parameter in rod 1, σ and ε^p are the nominal stress and plastic deformation, $\beta(\varepsilon^p)$ is the back stress, χ is the hardening parameter, and $\kappa(\chi)$ is the yield stress of undamaged material. The functions $\kappa(\chi)$ is assumed to be limited: $\kappa(\chi)$ is a known increasing function of χ for $\chi<\chi^*$, and $\kappa = \kappa(\chi^*)$=const for $\chi \geq \chi^*$. It is supposed that for $-p_1 \leq p \leq p_2$ the material is stable, i.e. the effective yield stress $\bar{\kappa} = (1-\Delta)\kappa(\chi)$ is an increasing function of time up to $\chi = \chi^*$.

The equation of the depository surface is: $\breve{\Phi} = |\boldsymbol{\sigma}| - (1-\Delta)(\kappa(\chi) - \beta^*) = 0$. Assume that $\varepsilon^p = \beta = \chi = \Delta = 0$ at t= 0. The purely elastic response of rod 1 is equal to $\sigma_1^{\mathrm{E}} = p(t)/2$. Inequality (5.2) takes the form: $\breve{\Phi} = \left|\dfrac{p}{2} + \hat{\rho}\right| < \kappa(0) - \beta^*$. Consequently, the shakedown occurs, if $\kappa(0) - \beta^* - 0.5p_2 > \hat{\rho} - \kappa(0) + \beta^* + 0.5p_1$. This inequality leads to the requirement $(p_1 + p_2) < 4(\kappa(0) - \beta^*)$. If the last inequality is valid, then it is possible to find such $\hat{\rho}$ that (5.2) would be satisfied. Consequently, the last inequality defines the constraints to values of p_1 and p_2 for which shakedown occurs.

During the damage process, the current value of the unloading Young's modulus of rod 1 is: $E = E_0(1-\Delta)$, where E_0 is its undamaged value. At the same time, according to the assumption, Young's moduli of rods 2, 3 preserve their initial value E_0. Thus the purely elastic response of the structure to the current value of the load $p(t)$, after the plastic-damage process in rod 1 has started, is $\sigma_1^{\mathrm{E}} = p\left(1 - \dfrac{1}{2-\Delta}\right)$, $\sigma_2^{\mathrm{E}} = \sigma_3^{\mathrm{E}} = \dfrac{p}{2}\dfrac{1}{2-\Delta}$. Thus, the current values of nominal stresses can be represented as: $\sigma_1 = p\left(1 - \dfrac{1}{2-\Delta}\right) + \rho_1$, $\sigma_2 = \sigma_3 = \dfrac{p}{2}\dfrac{1}{2-\Delta}\rho_2$ where ρ_1 and ρ_2 are the actual residual stresses in rods 1 and 2, 3, respectively.

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Since the residual stresses should be self-equilibrated, $2\rho_2 = \rho_1 = \rho$ where ρ is a new notation for ρ_1.

Now the yield function of rod 1 can be rephrased as $\Phi = |p\left(1 - \frac{1}{2-\Delta}\right) + \theta| - (1-\Delta)\kappa(\chi)$ with $\theta = \rho - (1-\Delta)\beta$, where $\rho\Delta\beta$ are actual values, and with θ instead for η.

At the post-adaptation stage ε^p, β, χ, Δ do not vary. Under fixed values of θ and χ, the function $\zeta = \Phi$ reaches its absolute maximum value under $p = p_2$, if $p\left(1 - \frac{1}{2-\Delta}\right) + \theta \geq 0$:$\max\Phi = \Phi_2 = p_2\left(1 - \frac{1}{2-\Delta}\right) + \theta - (1-\Delta)\kappa(\chi)$. However, if $p\left(1 - \frac{1}{2-\Delta}\right) + \theta \leq 0$, the yield function reaches its absolute maximum value under $p = -p_1$: $\max\Phi = \Phi_1 = p_1\left(1 - \frac{1}{2-\Delta}\right) - \theta - (1-\Delta)\kappa(\chi)$. The function $\max\Phi$ is minimum, if $\Phi_1 = \Phi_2$. This equation yields $\theta = -\frac{1}{2}(p_2 - p_1)\left(1 - \frac{1}{2-\Delta}\right)$. The corresponding value of the yield function is equaled to $\zeta_m \equiv \min\max\zeta = \frac{p_1 + p_2}{2}\left(1 - \frac{1}{2-\Delta}\right) - (1-\Delta)\kappa(\chi)$.

The equation $\zeta_m = 0$ determines Δ as a function of χ: $\Delta = 2 - \frac{p_1 + p_2}{2\kappa(\chi)}$.

Because χ is not greater than χ^*, and $\kappa(\chi)$ is an increasing function of χ, then

$$\Delta_{\max} = 2 - \frac{p_1 + p_2}{2\kappa(\chi^*)}. \tag{6.2}$$

This equality establishes the upper bound for Δ-variation admitted by the yield condition (6.1) under the given amplitude of the force P. One can see from (6.1) that the length of the yield segment at the σ-axis is in the inverse relation to the value of Δ. That is why the value of Δ is also in the inverse relation to the quantity $p_1 + p_2$ equaled to the amplitude of p: the larger is the amplitude, the smaller is the range of admissible values of Δ.

On the other hand, the lower estimate for Δ is obtained for χ=0

$$\Delta_{\min} = 2 - \frac{p_1 + p_2}{2\kappa(0)}. \tag{6.3}$$

If $\Delta_{\max} \leq \Delta_c$, then rod 1 preserves its integrity. If $\Delta_{\min} \geq \Delta_c$, the rod fails.

7. Concluding remarks

7.1. According to the developed method, the general algorithm of computing the damage parameter estimates (bounds) can be sketched as follows. First of all,

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the shakedown conditions for the solid under consideration have to be verified. The boundary-value problem for the system of elasticity equations supplemented with equations (3.5) and $\mathbf{C}=\mathbf{C}_0(1-\Delta)$ should be resolved for the extremal values of the strain hardening parameter and for the given boundary conditions. The solution of this system provides the bounds to the limit value of the damage parameter at every point of the solid under consideration. With the bounds in hand, it is possible to examine fulfilling of the local conditions of integrity.

In order to derive the conditions for overall integrity, the maximal value of the local upper bounds over the solid should be found and compared with the damage parameter critical value.

7.2. Under certain conditions, the proposed method can be extended to events, when the loading program is unknown as a function of time, and only the apexes of the load trajectory are given. For example, let us consider a solid subjected to a few repeated loads such that at any time point only one of the loads is active, i.e. the loads are applied in turn. The frequency of the application of loads, and the laws of their changing in time are unknown. It is possible that the loads are applied accidentally. The developed method is applicable to such situations, if only the maximal values of the loads are known.

Actually, at the time instant corresponding to the maximal value of a load, the stress tensor components reach their extremal values because they are proportional to the value of the load. Hence, for every separately taken load, the yield function reaches its maximal value at the instances corresponding to the load maximal values. Therefore the maximal values of the load can be taken as the boundary conditions for the boundary-value problem outlined in Sec. 5.

8. Summary

The method for estimating the local limit value of damage parameter was developed. The method is based on the relation between the damage and strain hardening parameters resulting from the limit yield condition, in which the rest of arguments is specified.

It was assumed that the material is linear-elastic during unloading, and the damage process was coupled with the process of plastic deformation: the first can be in progress, only if the second develops. The current stress tensor was decomposed into purely elastic and residual parts. The dependence of the current values of the elastic moduli on damage was taken into account.

The shakedown conditions were assumed to be fulfilled so that the post-adaptation (limit) stage of deformation existed. Although the limit values of damage and strain hardening parameters depend on the deformation path, and thus they are unknown in advance; nevertheless it is known that they satisfy the

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equation expressing the limit yield condition at the time points when the stress path reaches it. This equation was utilized to obtain upper and lower estimates of the limit value of damage parameter admitted by the given yield condition under the prescribed loading program. The problem was in a proper evaluating the residual stress that is connected with a parameter η.

The consideration was restricted by the requirement that the stress path reached the limit yield surface at least at two apexes. This situation is characteristic for the phenomenon of Low Cycle Fatigue. To model this situation, the parameter h is defined by the solution of min-max problem (3.5).

A system of equations was set which enables direct evaluation of the purely elastic, damaged response (σ^E) of the solid under consideration to the prescribed loading program at the departure instants.

Once the parameter η and stress σ^E have been specified, the limit yield condition issues in a relation between the admissible limit values of the damage and the strain hardening parameters. Because the strain hardening parameter is assumed bounded, this relation makes it possible to obtain the minimal and maximal limit values of the damage parameter. These values provide a priori bounds for the local limit value of the damage parameter admitted by the yield condition for the prescribed loading program.

The quality of the obtained estimate depends on strain hardening. The lower is the strain hardening, the better is the quality.

If the solid under investigation shakes down, and the upper bound at a solid point is less than the critical value of damage parameter (which is a material parameter), then the solid preserves its integrity at the point under consideration. This is a sufficient condition of local integrity. On the other hand, if the lower bound is greater than the damage critical value, then the local integrity is violated. This is a sufficient condition for local failure. It can be rephrased as a necessary condition of local integrity.

If a solid has to preserve its overall integrity, the condition of the local integrity has to be fulfilled at every point of the solid. An alternative formulation is as follows: a solid preserves its overall integrity, if the maximal value of the upper bound for damage parameter over the solid is smaller than the critical value of the damage parameter.

Appendix A. Sufficient shakedown condition accounting for kinematic strain hardening

The proposed condition is valid for classical material models with the only restriction: the kinematic strain hardening is bounded. Let $\beta(\varepsilon^p)$ denote the back-stress tensor. It is assumed that the state of saturation exists for the kine-

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matic strain hardening, i.e. the values of back-stress components are bounded by a material constant $\beta^* : |\beta_{ij}| \leq \beta^*$.

The yield surface equation is written as $\Phi(\bar{\sigma} - \beta, \chi) = 0$ where χ is the parameter of isotropic strain hardening. The parameters $\beta\varepsilon^P$ and $\chi\varepsilon^P$ define position and size of the yield surface.

It is assumed that the yield function is strictly convex in the first argument $s = \bar{\sigma} - \beta$, and the inequality $\Phi(s,\chi) < 0$ corresponds to the interior of the yield surface $\Phi(s,\chi) = 0$. Consequently, if $\Phi(s,\chi) = 0$, and $\Phi(\hat{s},\chi) < 0$ where $\hat{s} = \hat{\sigma} - \beta$, then $(s - \hat{s}) : \dot{\varepsilon}^P \geq 0$. This inequality results in inequality (2.1).

As previously, it is assumed that in the interval $0 \leq \chi \leq \chi^*$ the material is stable, i.e. inequalities (2.2), (2.3) are valid. However, if the kinematic strain hardening is taken into account, inequality (2.3) can be violated during the deformation process because the stress $\hat{\sigma}$ can fall out of the current yield surface due to its shift caused by the back-stress.

In order to make inequality (2.1) valid in the case where the back-stress is taken into account, it is necessary to modify condition (2.2). To that end, a "depository" surface $\breve{\Phi}(\bar{\sigma}, \chi) = 0$ is introduced. Its interior will be named the depository. This surface possesses the property that the requirement $\Phi(\bar{\sigma} - \beta\chi) < 0$ is satisfied, if the inequality $\breve{\Phi})(\bar{\sigma}, \chi) < 0$ is valid.

According to the assumption, the values of the back-stress components are bounded by a constant β^*. Let use consider the case when the yield surface $\Phi(\bar{\sigma},\chi) = 0$ is regular with the principal radii of curvature R_i greater than β^*. Under these conditions, the surface $\breve{\Phi}(\bar{\sigma}, \chi) = 0$ "parallel" to $\Phi(\bar{\sigma}, \chi) = 0$ can be constructed with the principal radii of curvature equal to $R_i - \beta^*$. This surface is the depository one.

The surface $\Phi(\bar{\sigma} - \beta\chi) = 0$ results from $\Phi(\bar{\sigma}\chi) = 0$ by the shift equal to β. As $|\beta_{ij}| \leq \beta^*$, the depositary surface $\Phi(\bar{\sigma}, \chi) = 0$ is in the interior of the surface $\Phi(\bar{\sigma} - \beta\chi) = 0$, or touches it. Therefore, if $\Phi(\hat{\bar{\sigma}}, \chi) < 0$, then $\Phi(\hat{\bar{\sigma}} - \beta\chi) < 0$ as well.

Assume that the initial (at $t = 0$) values of the damage and hardening parameters are zero: $\Delta_0 = \chi_0 = 0$. Suppose that there exists a stationary residual stress field $\hat{\rho}$ such that the following inequality is valid for $t \geq 0$:

$$\breve{\Phi}\left(\frac{\sigma^E(\mathbf{x}, t) + \hat{\rho}(\mathbf{x})}{1 - \Delta_0}, \chi_0\right) < 0, \tag{A.1}$$

at every point of the volume V of the solid under consideration.

According to the assumed material stability, a current yield surface comprises the previous ones. Analogously, a current depository surface also comprises the previous ones. Consequently, the following inequality holds at the every point of the volume V for any $t \geq t_0$

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$$\breve{\Phi}\left(\frac{\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)},\chi(\mathbf{x},t)\right)<0, \tag{A.2}$$

Since the depository surfaces are in the interior of the corresponding yield surfaces, inequality (A.2) results in the inequality

$$\breve{\Phi}\left(\frac{\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x},t)}{1-\Delta(\mathbf{x},t)},\beta(\mathbf{x},t),\chi(\mathbf{x},t)\right)<0, \tag{A.3}$$

Combining the above arguments it is possible to conclude that, if the stress $\hat{\sigma}(\mathbf{x},t)=\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x})$ satisfies inequality (A.1) everywhere in V for $\Delta_0=\chi_0=0$, and for any $t\geq 0$, then inequality (A. 3) is valid for any $t\geq 0$ where $\Delta(\mathbf{x},t)$, $\chi(\mathbf{x},t)$ and $\beta(\mathbf{x},t)$ are actual values, and $\chi_0\leq\chi(t)\leq\chi^{*}$.

Hence, if condition (A.1) is valid, then inequality (2. 3) holds in the case where the back-stress is taken into account.

Inequalities (A.1), (A.3) are the extensions of inequalities (2. 2), (2. 3) accounting for kinematic strain hardening.

Notice that although the virtual stress path $\sigma=\hat{\sigma}(\mathbf{x},t)$ is in the interior of the depository surface, the actual stress path $\sigma=\sigma(t,\mathbf{x})$ exits out of it.

Thus, if there exists a stationary field of residual stress $\hat{\rho}(\mathbf{x})$ such that the virtual stress $\hat{\sigma}=\sigma^{E}(\mathbf{x},t)+\hat{\rho}(\mathbf{x})$ satisfies inequality (A.1), then inequality (2. 3) is valid.

Now repeating again the arguments developed in [22], it is possible to show that in the case under consideration, the rate of plastic strain tends to zero, as well as that the total plastic dissipation is bounded.

Under the above assumptions the proposed theorem can be formulated as follows: the total plastic dissipation is bounded from above and the plastic rate strain tensor tends to zero, if there exists a stationary field of residual stress $\hat{\rho}(\mathbf{x})$, such that the stress $\hat{\sigma}=\sigma^{E}(\mathbf{x},t)+\hat{\rho}$ satisfies inequality (5. 2) for any $t\geq t_0$ where $\chi_0(\mathbf{x})$ is the initial (at $t=t_0$) value of the hardening parameter.

Obviously this condition for shakedown is not necessary, it is only sufficient.

Appendix B. Reduction of min-max problem (3.5) in the case of plane strain.

Assume that Δ and χ are fixed. Min-max problem (2.5) posses an additional condition on the residual stress field at every point of the solid. It is shown below that in the case of plane strain, min-max problem (3. 5) is equivalent to a boundary-value problem for a hyperbolic equation in partial derivatives.

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The tensor $\boldsymbol{\eta}$ is in proportion to $\boldsymbol{\rho}: \boldsymbol{\eta}(\mathbf{x})=\dfrac{\boldsymbol{\rho}(\mathbf{x})}{1-\Delta}$. As Δ is fixed, the minimum of function max ζ should be found with respect of $\boldsymbol{\rho}$. Consequently, (2.5) can be reshaped in the form

$$\zeta_m \equiv \min_{\boldsymbol{\rho}} \max_{t} \zeta = \min_{\boldsymbol{\rho}} \max_{t} \hat{\Phi}\Big(\frac{\boldsymbol{\sigma}^{\mathrm{E}}(\mathrm{x},t)+\boldsymbol{\rho}(\mathrm{x})}{1-\Delta}, \chi(\mathrm{x})\Big) = 0. \tag{B.1}$$

The minimum of the function $\max\zeta$ with respect to $\boldsymbol{\rho}$ should be found under the constraints: $\nabla\cdot\boldsymbol{\rho}=0$ and $\boldsymbol{\rho}\cdot\boldsymbol{\nu}=0$ at the part of the structure surface S_p where tractions are prescribed.

In the case of plane strain, the components of the stress tensor can be represented as: $\sigma_{11}=-\mathrm{Y}_{,22}$, $\sigma_{22}=\mathrm{Y}_{,11}$, $\sigma_{12}=\mathrm{Y}_{,12}$ where $\mathrm{Y}(\mathrm{x}_1,\mathrm{x}_2)$ is the stress function, and comma denotes partial derivative [24].

Suppose for simplicity that the surface tractions are prescribed as the product of a periodic function of time $\boldsymbol{\varphi}(t)$ and a function of boundary coordinate $\boldsymbol{\theta}(\mathbf{x})$. Then the tensor $\boldsymbol{\sigma}^{\mathrm{E}}$ is also represented as the product: $\boldsymbol{\sigma}^{\mathrm{E}}=\boldsymbol{\varphi}(t)\mathbf{r}(\mathrm{x}_1,\mathrm{x}_2)$. The components of tensor $\mathbf{r}(\mathbf{x})$ are determined by the solution of the elastic boundary-value problem under the surface traction $\boldsymbol{\theta}(\mathbf{x})$. Below r is considered as known.

Take, for example, the Mises yield function: $\Phi(\boldsymbol{\sigma}\chi)=f(\boldsymbol{\sigma}\chi)-4\mathrm{k}(\chi)^2$ where $\mathrm{k}(\chi)$ is the yield stress, and $f(\boldsymbol{\sigma})=(\sigma_{11}-\sigma_{22})^2+4\sigma_{12}^2\geq 0$. Then

$$f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})(1-\Delta)=\boldsymbol{\varphi}^2(t)f(\mathbf{r})+f(\boldsymbol{\rho})-2\boldsymbol{\varphi}(t)[(r_{11}-r_{22})(\rho_{11}-\rho_{22}+4r_{12}\boldsymbol{\rho}_{12})].$$

Suppose that the function $\boldsymbol{\varphi}(t)$ ranges between $\varphi_1=\varphi(t_1)\leq 0$ and $\varphi_2=\varphi(t_2)>0$. The function $f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})$ considered as a function of $\boldsymbol{\varphi}$ is assumed to be convex below. Therefore it reaches its absolute maximal value either at $\boldsymbol{\varphi}=\varphi_1$, or $\boldsymbol{\varphi}=\varphi_2$, depending on the relation between the values of quantities $f(\mathbf{r})$, $f(\boldsymbol{\rho})$, $2\,[(r_{11}-r_{22})(\sigma_{11}-\sigma_{22})+4r_{12}\sigma_{12}]$, which in turn depend on the values of residual stress tensor components $(\rho_{11}-\rho_{22})$ and ρ_{12}. These values of $f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})$ are denoted by f_1 and f_2 correspondingly. The absolute maximum value of $f(\bar{\boldsymbol{\sigma}}^{\mathrm{E}}+\bar{\boldsymbol{\rho}})$ is minimal, if $f_1=f_2$. This condition leads to the equation $(r_{11}-r_{22})(\rho_{11}-\rho_{22})+2r_{12}\rho_{12}=(\boldsymbol{\varphi}(t_1)+\boldsymbol{\varphi}(t_2))f(\mathbf{r})$. Expressing the components of the residual stress tensor through the stress function, we arrive at the following equation in partial derivatives with respect of the stress function Y:

$$(r_{11}-r_{22})(Y_{,11}-Y_{,22})+2r_{12}Y_{,12}=(\boldsymbol{\varphi}(t_1)+\boldsymbol{\varphi}(t_2))f\,(\mathbf{r}).$$

This equation is of hyperbolic type. It has two orthogonal families of characteristic lines, which coincide with the trajectories of shear stress of the tensor $\mathbf{r}$. The characteristic relations are:

$$p_{,\xi}\sin\alpha-q_{,\xi}\cos\alpha=-\frac{\phi_1+\phi_2}{4}\sin 2\alpha, \quad p_{,\eta}\sin\alpha=\frac{\phi_1+\phi_2}{4}\sin 2\alpha$$

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where ξ, η are the coordinates along the characteristic lines, α is the angle between the abscissa axis and the ξ-lines, and $p = Y_{,1}$, $q = Y_{,2}$.

Along the solid border S_p where the traction is prescribed $F_x = -dq/ds$, $F_y = dp/ds$ where F_x and F_y are the components of the traction corresponding to the x, y axes, and s is a coordinate along the border. Because $F_x = F_y = 0$ along S_p then at S_p of boundary p, $q =$ const.

The solution of this boundary-value problem determines the $\boldsymbol{\rho}$-field in the region of influence of the boundary conditions that is bordered by the S_p - segment of the solid boundary, and the characteristic lines originating from the ends of the S_p - segment.

The $\boldsymbol{\rho}$-field in the rest of the solid is defined by the solution of the elastic boundary-value problem under the condition of the contact stress continuity at the interfacial boundary between the elastic and plastic parts, and the conditions at the part of solid boundary where the displacement prescribed.

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