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End effects in the dynamical problem of magneto-elasticity

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In this paper we derive spatial decay bounds for the solutions of the linear dynamical problem of magneto-elasticity in a semi-infinite cylindrical region. For the forward-in-time problem we prove that an energy expression is bounded from above by a decaying exponential of a quadratic polynomial of the distance. We derive a spatial decay estimate for the backward-in-time problem as well. The proof works only if the cross-section is a finite union of rectangles with axes parallels to Ox_2 and Ox_3. As a conclusion we consider the extension of the preceding bound to the heat conduction case.

Key words: magneto-thermo-elasticity, spatial decay, comparison arguments.

1. Introduction

In recent years much attention has been directed to the study of end effects damping in several thermomechanical situations. The history and development of this question is explained in the work of Horgan and Knowles [9] and has been periodically updated by Horgan [7, 8]. We may also recall the book of Ames and Straughan [1] where the energy method is extensively used. Here, we are interested in the coupling of elastic effects and magnetic effects. As far as the author knows, there are no contributions made to the study of spatial decay in the dynamical problem of magneto-elasticity. In this paper we derive spatial decay bounds for the solutions of the linear dynamical problem of magneto-elasticity in a semi-infinite cylindrical region. For the forward-in-time problem we prove that an energy expression is bounded above by a decaying exponential of a quadratic polynomial of the distance. We derive a spatial decay estimate for the backward in time problem as well. The proof works only if the cross-section is a finite union of rectangles with axes parallels to Ox_2 and Ox_3. The class of cylinders satisfying this condition is wide. We recall that a condition of this kind was imposed in the recent work of Muñoz-Rivera and Racke [14].

A derivation of the equations and recent papers on magneto-thermo-elasticity and isothermal magneto-elasticity can be found in [2, 4, 13, 14, 16, 22]

This paper can be considered as an extension of the method proposed in the study of several linear thermoelastic problems such as those in [17-20] and/or nonlinear viscoelasticity problems as those in [21].

This paper is of interest from the mechanical and mathematical viewpoints. Spatial estimates for the forward and backward-in-time problems of the magneto-elasticity have not been studied at present. The boundary conditions of the magnetic field (see (2.3)) put forward a new mathematical difficulty to work with. In this paper we see how to overcome it when the cross-section is a finite union of rectangles with axes parallels to Ox_2 and Ox_3.

The plan of the paper is the following: In Sec. 2 we obtain our spatial decay estimate for the forward-in-time problem. To this end we prove that a certain energy measure of the solutions satisfies a one-dimensional partial differential inequality. Comparison arguments applied to this one-dimensional partial differential equation allow us to obtain our estimate. Section 3 is devoted to the study of the backward-in-time problem. A spatial decay estimate is obtained along time-spatial lines. The extension of these arguments to the heat conduction case is sketched in the last section.

In this article we use the summation an differentiation conventions. Summation over repeated indices is assumed and the differentiation with respect to the direction x_k is denoted by $,k$. Letters in boldface stand for vectors.

2. Forward-in-time problem

We consider an initial boundary value problem for the linear magneto-elasticity. The linear partial differential equations that govern the magneto-elasticity in the case of isotropic and homogeneous material are (see [5]):

$$\rho \mathbf{u}_{,tt} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\mathrm{div}\mathbf{u} - \alpha[\nabla\times\mathbf{h}]\times\mathbf{H} = \mathbf{0}, \tag{2.1}$$

$$\beta\mathbf{h}_{,t} - \Delta\mathbf{h} - \beta\nabla\times[\mathbf{v}\times\mathbf{H}] = \mathbf{0}. \tag{2.2}$$

Here $\mathbf{u}$ is the displacement, $\mathbf{v} = \mathbf{u}_t$ is the velocity and $\mathbf{h}$ the magnetic field; $\mathbf{H} = (H, 0, 0)$ is a (known) constant magnetic field and $\lambda, \mu, \rho, \alpha$, β are positive constants.

We study the system (2.1)-(2.2) in the semi-infinite cylinder $R = (0, \infty) \times D$, where D is a union of finite number of rectangles parallel to the x_2 and x_3 axes (see Fig. 1).

The boundary conditions are

$$\mathbf{u} = \mathbf{0}, \mathbf{h}.\mathbf{n} = 0,\ [\nabla\times\mathbf{h}]\times\mathbf{n} = \mathbf{0},\ \text{on } (0,\infty)\times\partial D\ t > 0. \tag{2.3}$$

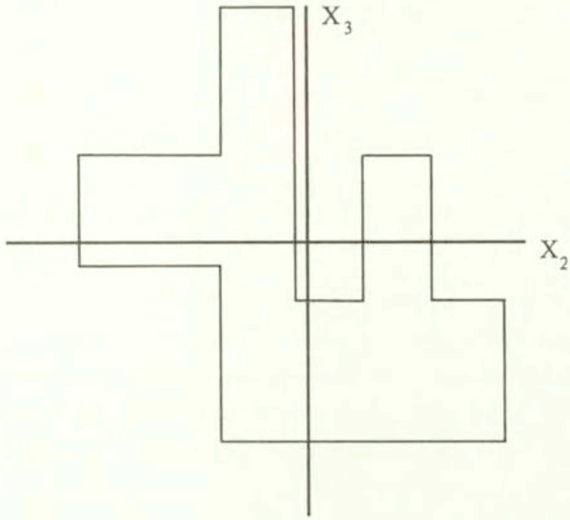

FIG. 1.

Here and from now on, let $\mathbf{n}$ be the exterior normal vector to the boundary at regular points. The initial conditions are

$$\mathbf{u}(\mathbf{x},0) = \mathbf{v}(\mathbf{x},0) = \mathbf{h}(\mathbf{x},0) = \mathbf{0} \text{ in } R. \tag{2.4}$$

Though we could impose wider asymptotic conditions, in order to make calculations easier we assume that

$$\mathbf{u}, \mathbf{v}, \mathbf{h}, \nabla\mathbf{u}, \nabla\mathbf{h} \to \mathbf{0} \text{ uniformly as } x_1 \to \infty, \text{ as } x_1^{-3}. \tag{2.5}$$

To complete the problem we should impose boundary conditions on the part of the boundary $\{0\} \times D$ for all t. But our analysis does not require the explicit knowledge of these conditions. Existence and uniqueness results for this problem could be obtained by means of semigroups methods in a similar way as in [16]. We do not consider this question here.

Now, we obtain a spatial decay estimate of the solutions of the linear problem (2.1)-(2.5). From $(2.3)_2$, we see that $h_2n_2+h_3n_3 = 0$ and then $h_{2,1}n_2+h_{3,1}n_3 = 0$. On the other hand, $(2.3)_3$ implies that

$$h_{1,2}n_2 + h_{1,3}n_3 = h_{2,1}n_2 + h_{3,1}n_3, h_{3,2}n_2 = h_{2,3}n_2, h_{3,2}n_3 = h_{2,3}n_3, \quad \text{on}(0,\infty) \times \partial D \; t > 0. \tag{2.6}$$

If we calculate $h_{i,2}h_in_2 + h_{i,3}h_in_3$ we have

$$\begin{aligned}(2.7)\quad & h_{i,2}h_in_2 + h_{i,3}h_in_3 \\ &= h_1(h_{1,2}n_2 + h_{1,3}n_3) + h_2(h_{2,2}n_2 + h_{2,3}n_3) + h_3(h_{3,2}n_2 + h_{3,3}n_3) \\ &= h_1(h_{2,1}n_2 + h_{3,1}n_3) + h_2(h_{2,2}n_2 + h_{2,3}n_3) + h_3(h_{3,2}n_2 + h_{3,3}n_3), \\ &\qquad \text{on } (0,\infty)\times\partial D,\end{aligned}$$

for all $t > 0$. But we know that $h_{2,1}n_2 + h_{3,1}n_3 = 0$. As we are assuming that the boundary of D consists of segments parallel to the axes, we obtain that either $n_2 = 0$ or $n_3 = 0$. If we assume (for instance) that $n_3 = 0$, then $h_2 = 0$ and $h_{2,3} = 0$ on the segment and then $h_{3,2} = 0$ on the segment. When $n_2 = 0$, we may repeat the arguments and obtain

$$(2.8)\qquad h_{i,2}h_in_2 + h_{i,3}h_in_3 = 0, \qquad \text{on } (0,\infty)\times\partial D,\ t > 0.$$

It will be useful to define the matrix (σ_{ij}) by:

$$(2.9)\qquad \sigma_{11} = \mu u_{1,1} + (\lambda+\mu)u_{j,j},\quad \sigma_{12} = \mu u_{1,2},\quad \sigma_{13} = \mu u_{1,3},$$

$$(2.10)\quad \sigma_{21} = \mu u_{2,1} + \alpha H h_2,\ \ \sigma_{22} = \mu u_{2,2} + (\lambda+\mu)u_{j,j} - \alpha H h_1,\ \ \sigma_{23} = \mu u_{2,3},$$

$$(2.11)\quad \sigma_{31} = \mu u_{3,1} + \alpha H h_3,\ \ \sigma_{32} = \mu u_{3,2},\ \ \sigma_{33} = \mu u_{3,3} + (\lambda+\mu)u_{j,j} - \alpha H h_1.$$

Then equation (2.1) can be written as

$$\sigma_{ij,j} = \rho \ddot{u}_i.$$

We now define a function that plays a fundamental role in this section.

$$(2.12)\qquad K(z,t) = -\int_0^t \int_{D(z)} \left[\sigma_{i1}v_i + \frac{\alpha}{\beta}h_{i,1}h_i\right] da\,ds.$$

Here $D(z)$ denotes the cross-section at a distance z from the origin; it has the same form as the domain D. The function $K(z,t)$ results from the vector field

$$\left(\sigma_{i1}v_i + \frac{\alpha}{\beta}h_{i,1}h_i,\ \sigma_{i2}v_i + \frac{\alpha}{\beta}h_{i,2}h_i,\ \sigma_{i3}v_i + \frac{\alpha}{\beta}h_{i,3}h_i\right).$$

Thus, in view of conditions (2.8), the divergence theorem implies that

$$K(z+h,t) - K(z,t) = -\frac{1}{2} \int\limits_{R(z+h,z)} (\rho v_i v_i + \mu u_{i,j} u_{i,j} + (\lambda+\mu) u_{i,i} u_{j.j} + \alpha h_i h_i)\, dv - \frac{\alpha}{\beta} \int\limits_0^t \int\limits_{R(z+h,z)} h_{i,j} h_{i,j}\, dvds,$$

where

$$R(z+h,z) = \{\mathbf{x} \in R, z < x_1 < z+h\}.$$

The asymptotic condition (2.5) implies that for finite time

$$\lim_{z\to\infty} K(z,t) = 0. \tag{2.13}$$

Using the divergence theorem we obtain that

$$K(z,t) = \frac{1}{2} \int\limits_{R(z)} (\rho v_i v_i + \mu u_{i,j} u_{i,j} + (\lambda+\mu) u_{i,i} u_{j.j} + \alpha h_i h_i) dv + \frac{\alpha}{\beta} \int\limits_0^t \int\limits_{R(z)} h_{i,j} h_{i,j}\, dvds. \tag{2.14}$$

Here $R(z)$ denotes the sub-region of R of the points that are at a distance greater than z from the plane $x_1 = 0$.

Now, we define

$$E(z,t) = \int\limits_z^\infty K(p,t) dp. \tag{2.15}$$

It follows that

$$\frac{\partial E}{\partial z} = -K(z,t), \tag{2.16}$$

and that

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{2} \int\limits_{D(z)} (\rho v_i v_i + \mu u_{i,j} u_{i,j} + (\lambda+\mu) u_{i,i} u_{j.j} + \alpha h_i h_i) da + \frac{\alpha}{\beta} \int\limits_0^t \int\limits_{D(z)} h_{i,j} h_{i,j}\, dads. \tag{2.17}$$

From the definition of the functions K and E we also obtain that

$$\frac{\partial E}{\partial t} = -\int\limits_{R(z)} \Big[\sigma_{i1}v_i + \frac{\alpha}{\beta}h_{i,1}h_i\Big]dv. \tag{2.18}$$

It is worth remarking that

$$\int\limits_{R(z)} h_i h_{i,1} dv = -\frac{1}{2}\int\limits_{D(z)} h_i h_i da. \tag{2.19}$$

Our next step is to estimate the time derivative of E in terms of the first two spatial derivatives of E. Using repeatedly the arithmetic-geometric mean inequality we can compute two positive constants A_1, A_2 such that

$$\begin{aligned} (2.20)\quad |\int\limits_{R(z)} \sigma_{i1}v_i dv| &\leq \frac{1}{2}\Big(\int\limits_{R(z)} \sigma_{i1}\sigma_{i1}dv + \int\limits_{R(z)} v_i v_i dv\Big) \\ &\leq A_1(\lambda,\mu,\alpha,H,\rho)\int\limits_{R(z)} (\rho v_i v_i + \mu u_{i,j}u_{i,j} + (\lambda+\mu)u_{i,i}u_{j.j} + \alpha h_i h_i)\,dv \\ &\leq -A_1(\lambda,\mu,\alpha,H,\rho)\frac{\partial E}{\partial z}, \end{aligned}$$

and

$$\frac{1}{2}\int\limits_{D(z)} \frac{\alpha}{\beta}h_i h_i da \leq A_2(\beta)\frac{\partial^2 E}{\partial z^2}. \quad \Big(A_2 = \beta^{-1}\Big) \tag{2.21}$$

From (2. 15)–(2. 21) we obtain the inequality:

$$\frac{\partial E}{\partial t} \leq -A_1\frac{\partial E}{\partial z} + A_2\frac{\partial^2 E}{\partial z^2}. \tag{2.22}$$

From inequality (2. 22) and after the change of variable

$$w(z,t) = \exp(b_2 t - b_1 z)E(z,t), \tag{2.23}$$

where

$$b_1 = \frac{A_1}{2A_2},\ \ b_2 = b_1^2 A_2, \tag{2.24}$$

we obtain the inequality

$$\frac{\partial w}{\partial t} \leq A_2\frac{\partial^2 w}{\partial z^2}. \tag{2.25}$$

Taking into account the asymptotic behaviour of the solutions, and using comparison arguments similar to those used in the work of HORGAN *et al.* [10, 11], we obtain the estimate (see also [17, 18])

$$(2.26)\quad E(z,t) \le \exp(b_1 z - b_2 t) \sup_{0\le s\le t} \Big[\exp(b_2 s) E(0,s)\Big] \frac{z}{(4\pi A_2)^{1/2}} \int_0^t (t-s)^{-3/2} \exp - \frac{z^2}{4A_2(t-s)}\, ds.$$

Some algebraic manipulations on the right-hand side of the estimate (2.26) (see [10, 11, 17, 18]), allow us to obtain the following estimate:

$$(2.27)\qquad E(z,t) \le \frac{B(t)}{z} \exp(b_1 z - \frac{z^2}{4tA_2}),$$

where

$$(2.28)\qquad B(t) = (4tA_2)^{1/2} \exp(-b_2 t) \sup_{0\le s\le t} \Big(\exp(b_2 s) E(s,0)\Big).$$

Thus, we have proved:

THEOREM 1. *Let (u_i, h_i) be a solution of the problem determined by the system of equations (2.1), (2.2), boundary conditions (2.3) and initial conditions (2.4) such that the asymptotic condition (2.5) is satisfied. Then, the function E defined in (2.15) satisfies the estimate (2.28).*

3. Backward-in-time problem

In this section, we obtain spatial decay estimates for the backward-in-time problem. This kind of questions are relevant from a mechanical point of view when we want to have some information about what happened in the past by means of the information that we have at this moment. Thus, we study the system determined by (2.1) (which is independent of the direction of time) and the backward-in-time version of Eq. (2.2). That is

$$(3.1)\qquad \beta \mathbf{h}_{,t} + \triangle \mathbf{h} - \beta \nabla \times [\mathbf{v} \times \mathbf{H}] = \mathbf{0}.$$

In this section, we deal with non-homogeneous initial conditions

$$(3.2)\qquad \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}),\quad \mathbf{v}(\mathbf{x},0) = \mathbf{v}^0(\mathbf{x}),\quad \mathbf{h}(\mathbf{x},0) = \mathbf{h}^0(\mathbf{x}),$$

and the boundary conditions (2.3). We assume that

$$(3.3)\qquad \int\limits_{R}\Big(\rho v_i v_i+\mu u_{i,j}u_{i,j}+(\lambda+\mu)u_{i,i}u_{j.j}+\alpha h_i h_i+\frac{\alpha}{\beta}h_{i,j}h_{i,j}\Big)\,dv<\infty,$$

for all $t\geq 0$,

and that

$$(3.4)\qquad \lim_{z\to\infty}\int\limits_{D(z)}\Big(\rho v_i v_i+\mu u_{i,j}u_{i,j}+(\lambda+\mu)u_{i,i}u_{j.j}+\alpha h_i h_i+\frac{\alpha}{\beta}h_{i,j}h_{i,j}\Big)\,da=0,$$

for all $t\geq 0$.

We consider the function

$$(3.5)\qquad F_\omega(z,t)=\int\limits_0^t\int\limits_{R(z)}\exp(\omega s)\Big(\frac{\omega}{2}(\rho v_i v_i+\mu u_{i,j}u_{i,j} + (\lambda+\mu)u_{i,i}u_{j.j}+\alpha h_i h_i)+\frac{\alpha}{\beta}h_{i,j}h_{i,j}\Big)dvds.$$

After a use of the divergence theorem, it follows that

$$(3.6)\qquad F_\omega(z,t)=\int\limits_0^t\int\limits_{D(z)}\exp(\omega s)(\sigma_{i1}v_i-\frac{\alpha}{\beta}h_{i,1}h_i)\,dads -\frac{1}{2}\int\limits_{R(z)}(\rho v_i^0 v_i^0+\mu u_{i,j}^0 u_{i,j}^0+(\lambda+\mu)u_{i,i}^0 u_{j,j}^0+\alpha h_i^0 h_i^0)\,dv +\frac{1}{2}\int\limits_{R(z)}\exp(\omega t)(\rho v_i v_i+\mu u_{i,j}u_{i,j}+(\lambda+\mu)u_{i,i}u_{j,j}+\alpha h_i h_i)\,dv.$$

Several uses of the arithmetic-geometric mean inequality allow us to obtain two positive constants B_2, B_3 (that depend on $\lambda,\mu,\alpha,\beta,H,\rho,\omega$) such that

$$(3.7)\qquad \int\limits_0^t\int\limits_{D(z)}\exp(\omega s)(\sigma_{i1}v_i-\frac{\alpha}{\beta}h_{i,1}h_i)\,dads\leq -B_2\frac{\partial F_\omega}{\partial z},$$

and

$$(3.8)\quad \frac{1}{2}\int\limits_{R(z)} \exp(\omega t)(\rho v_i v_i + \mu u_{i,j}u_{i,j} + (\lambda+\mu)u_{i,i}u_{j,j} + \alpha h_i h_i)\,dv \le B_3\frac{\partial F_\omega}{\partial t}.$$

It follows that

$$(3.9)\qquad F_\omega \le -B_2\frac{\partial F_\omega}{\partial z} + B_3\frac{\partial F_\omega}{\partial t} - S(z),$$

where

$$(3.10)\quad S(z) = \frac{1}{2}\int\limits_{R(z)} (\rho\mathbf{v}^0.\mathbf{v}^0 + \mu\nabla\mathbf{u}^0.\nabla\mathbf{u}^0 + (\lambda+\mu)(\mathrm{div}\mathbf{u}^0)^2 + \alpha\mathbf{h}^0.\mathbf{h}^0)\,dv.$$

This inequality has been studied previously (see [3, 6, 12]). Thus, we may conclude the estimate

$$(3.11)\quad F_\omega(z,t) + S(z)\Big(1-\exp\Big(-\frac{z-z_0}{B_2}\Big)\Big) \le F_\omega(z_0,t_0)\exp\Big(-\frac{z-z_0}{B_2}\Big),$$

that is satisfied along the line

$$(3.12)\qquad z + \frac{B_2}{B_3}t = z_0 + \frac{B_2}{B_3}t_0,\quad z \ge z_0.$$

We note that (3. 11) implies

$$(3.13)\quad J_\omega(z,t) + S(z)\Big(1-\exp\Big(-\frac{z-z_0}{B_2}\Big)\Big) \le \exp(\omega t_0)J_\omega(z_0,t_0)\exp\Big(-\frac{z-z_0}{B_2}\Big),$$

where

$$(3.14)\quad J_\omega(z,t) = \int\limits_0^t\int\limits_{R(z)} \Big(\frac{\omega}{2}(\rho v_i v_i + \mu u_{i,j}u_{i,j} + (\lambda+\mu)u_{i,i}u_{j.j} + \alpha h_i h_i) + \frac{\alpha}{\beta}h_{i,j}h_{i,j}\Big)\,dvds,$$

that is satisfied along the line (3. 12).

4. Magneto-thermo-elasticity

In this section, we sketch how to extend the previous results to the case of magneto-thermo-elasticity. In this section the system of equations is:

$$\rho\mathbf{u}_{,tt} - \mu\triangle\mathbf{u} - (\lambda+\mu)\nabla\mathrm{div}\mathbf{u} - \alpha[\nabla\times\mathbf{h}]\times\mathbf{H} + \gamma\nabla\theta = \mathbf{0}, \tag{4.1}$$

$$\theta_{,t} - k\triangle\theta + \gamma\mathrm{div}\mathbf{v} = 0, \tag{4.2}$$

and Eq. (2.2) as well. Here θ is the temperature measured with respect to a uniform reference temperature in the reference configuration, and γ is an arbitrary real number.

For the forward-in-time problem we supplement the boundary and the initial conditions (2.3), (2.4) with homogeneous boundary and initial conditions on the temperature

$$\theta(\mathbf{x},0) = 0, \quad \mathbf{x}\in R \quad \text{and} \quad \theta(\mathbf{x},t) = 0, \quad \mathbf{x}\in(0,\infty)\times\partial D, \tag{4.3}$$

and supplement the asymptotic condition (2.5) with

$$\theta, \nabla\theta \to \mathbf{0} \text{ uniformly as } x_1\to\infty \text{ as } x_1^{-3}. \tag{4.4}$$

If we define

$$K_\theta(z,t) = K(z,t) - \int_0^t \int_{D(z)} k\theta_{,1}\theta\, da ds. \tag{4.5}$$

It follows that

$$K_\theta(z,t) = \frac{1}{2}\int_{R(z)} (\rho v_i v_i + \mu u_{i,j}u_{i,j} + (\lambda+\mu)u_{i,i}u_{j.j} + \theta^2 + \alpha h_i h_i)\, dv \tag{4.6}$$
$$+ \int_0^t \int_{R(z)} \left(k\theta^2 + \frac{\alpha}{\beta}h_{i,j}h_{i,j}\right) dv ds.$$

If we define

$$E_\theta(z,t) = \int_z^\infty K_\theta(p,t)\, dp, \tag{4.7}$$

we may obtain an inequality of type (2.22), but now the parameters A_1 and A_2 (and consequently b_1 and b_2) may also depend on k and γ. Estimates of type (2.26), (2.28) can also be obtained in this case.

For the backward-in-time problem we assume the initial conditions (3.2) and we can supplement them by non-homogeneous initial conditions on the temperature θ. The analysis starts by assuming conditions similar to (3.3) and (3.4) and considering the function

$$G_\omega(z,t) = \int_0^t \int_{R(z)} \exp(\omega s)\Big(\frac{\omega}{2}(\rho v_i v_i + \mu u_{i,j}u_{i,j} + (\lambda+\mu)u_{i,i}u_{j.j} + \alpha h_i h_i + \theta^2) + \frac{\alpha}{\beta}h_{i,j}h_{i,j} + k\theta_{,i}\theta_{,i}\Big)\, dvds. \tag{4.8}$$

Two estimates similar to (3.11) and (3.13) can be obtained. The parameters B_2, B_3 also depend on the new constitutive constants γ and k.

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