Immersed boundary approach to stability equations for a spatially periodic viscous flow

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AN EFFICIENT NUMERICAL method for the linear stability equations of a spatially periodic channel flow is presented. The method is based on global Fourier-Chebyshev approximation of a disturbance velocity field. The physical flow domain is embedded in a larger computational domain and the boundary conditions are re-formulated as internal conditions imposed at immersed boundaries. The advantage of this approach is an avoidance of domain mapping, leading to tremendous complication of governing equations and to excessive computational cost. The results of numerical tests are presented. Favorable convergence properties with respect to the length of the Fourier expansions are demonstrated.

1. Introduction

DURING THE LAST two decades, a significant progress has been achieved in understanding the dynamics of instability and transition in many kinds of shear flows. A great deal of work in this area has been devoted to the analysis of parallel flows, like the Cuette flow, plane Poiseuille flow or flow in a pipe (Hagen-Poiseuille flow). The common feature of such flows is a very simple geometry of boundaries of the flow domain.

In the Ref. [1], FLORYAN investigated the problem of linear stability of shear flows in the presence of surface roughness. Actually, the boundary irregularities were "simulated" by a distributed surface transpiration. Thus, the flow boundaries were still straight lines and the author was able to apply directly a standard Fourier-Chebyshev method. In later works [2] and [3], the flow in a channel with corrugated walls has been investigated. The stability equations have been solved in a transformed computational space by a Chebyshev-collocation method. The transformation from the physical space (corrugated walls) to the computational space (straight walls) has generated, however, a tremendously complicated form of the governing equations. Most importantly, the stability equations in the transformed space could not be cast in the reduced form, i.e. the form involving only amplitude functions of the wall-normal velocity and vorticity components. Therefore, the algebraic eigenvalue problems solved in [2] or [3] were typically two or

more times larger than those for "simulated" roughness in [1], though using the same number of Fourier modes to the represent the disturbance fields. Consequently, the numerical difficulties in accurate determination of unstable modes have been encountered.

The objective of this paper is to propose an alternative approach. Working directly in a physical space gives an obvious advantage of dealing with a much simpler and smaller set of equations. However, the use of a spectral method based on global basic functions implies that the computational domain is a Cartesian product of two intervals, i.e. the ranges of variation of the streamwise and the wall-normal coordinate. Consequently, the computational domain of such a method is by default rectangular. On the other hand, the shape of the actual flow domain is not a rectangle – the lower and upper boundaries are wavy lines (although the magnitude of the wall corrugation is assumed to be rather small). Nevertheless, the formulation of the spectral method working directly in the physical space is indeed possible. In the Ref. [4], SZUMBARSKI and FLORYAN proposed a method based on the idea of immersed boundaries. The curvilinear physical domain is embedded in a rectangular computational domain. The physical boundaries of the flow (walls of a channel) are located in the interior of the computational domain and the conditions imposed on the velocity field at the channel walls are enforced as internal conditions. The objective of this paper is to demonstrate that the same approach can be used to obtain an effective and accurate solution method in the linear stability analysis.

The paper is organized as follows. In the Sec. 2, the problem of the basic flow determination is formulated mathematically and the numerical method is outlined. The presentation in that section follows closely the Ref. [4]. In the Sec. 3, the equations of linear stability are formulated. Section 4 is devoted to detailed description of the discretization of the stability equations using Chebyshev polynomials. Section 5 presents the major idea of the paper, i.e. the implementation of the original boundary conditions as the internal conditions in the extended computational domain. In the Sec. 6, the structure of the generalized eigenvalue problem obtained as a result of the spectral discretization is described. Finally, the results of the convergence tests are provided in the Sec. 7. Some additional technical issues are summarized in the Appendix.

2. Flow in a wavy channel

Consider the viscous flow in a plane channel with wavy boundaries. The walls can assume arbitrary shapes described by $y_L(x)$ and $y_U(x)$, where the subscript L refers to the bottom wall and the subscript U refers to the upper wall of the channel, respectively. The shape of the walls is assumed to be spatially periodic and thus the functions $y_L(x)$ and $y_U(x)$ can be expressed in the form of the

following Fourier expansions:

(2.1)
$$y_L(x) = (A_0)_L + \frac{1}{2} \sum_{k=1}^{\infty} (A_k)_L \exp(ik\alpha x) + \frac{1}{2} \sum_{k=1}^{\infty} (A_k)_L^* \exp(-ik\alpha x,)$$

$$y_U(x) = (A_0)_U + \frac{1}{2} \sum_{k=1}^{\infty} (A_k)_U \exp(ik\alpha x) + \frac{1}{2} \sum_{k=1}^{\infty} (A_k)_U^* \exp(-ik\alpha x)$$

where the wave number α characterizes the spatial period λ_x of the wall geometry, i.e. $\lambda_x = 2\pi/\alpha$. The coefficients $(A_0)_L = -1$ and $(A_0)_U = 1$ define y-coordinate of the middle lines of the wavy walls. The star superscript refers to the complex conjugation.

One can define the vertical extent of the flow domain as the interval $[Y_L, Y_U]$, where $Y_L = \min_{x \in [0,\lambda_x]} y_L(x)$ and $Y_U = \max_{x \in [0,\lambda_x]} y_U(x)$.

Since we are mostly interested in small or moderate magnitudes of wall waviness, it is reasonable to represent the velocity and pressure fields as the sums of the reference values, i.e. the values corresponding to the flow in a channel with straight boundaries (the Poiseuille flow)

(2.2)
$$\mathbf{V}_0(\mathbf{x}) = [u_0(x,y), v_0(x,y)] = [u_0(y), 0] = [1 - y^2, 0],$$

$$p_0(\mathbf{x}) = -2x/\mathrm{Re} ,$$

and the modifications caused by the change of shapes of the boundaries, namely

$$\mathbf{V}_0(\mathbf{x}) = [u(x,y), v(x,y)] = \mathbf{V}_0(\mathbf{x}) + \mathbf{V}_1(\mathbf{x})$$

(2.3)
$$= [u_0(y), 0] + [u_1(x,y), v_1(x,y)],$$

$$p(\mathbf{x}) = p_0(\mathbf{x}) + p_1(\mathbf{x}) .$$

It should be clear that for the wavy walls we have $Y_L < -1$ and $Y_U > 1$. Thus, the formula (2.3) involves evaluation of the reference velocity parabolic profile also at points located outside the original range $[-1,1]$.

Substitution of the representation (2.3) of the flow quantities into the Navier-Stokes and continuity equations results in the following form of the governing equations:

(2.4)
$$u_0 \, \partial_x u_1 + u_1 \, \partial_x u_1 + v_1 \, Du_0 + v_1 \, \partial_y u_1 = -\partial_x p_1 + \frac{1}{\mathrm{Re}} (\partial_{xx} u_1 + \partial_{yy} u_1) ,$$

$$u_0 \, \partial_x v_1 + u_1 \, \partial_x v_1 + v_1 \, \partial_y v_1 = -\partial_y p_1 + \frac{1}{\mathrm{Re}} (\partial_{xx} v_1 + \partial_{yy} v_1) ,$$

$$\partial_x u_1 + \partial_y v_1 = 0,$$

where D stands for the differential operator d/dy, i.e. $Du_0 \equiv du_0/dy$. It is convenient to formulate the problem in terms of the stream function of the velocity modification defined as $u_1 = \partial_y \Psi$, $v_1 = -\partial_x \Psi$. The following equation can be derived:

$$(2.5) \qquad (u_0 \partial_x + \partial_y \Psi \, \partial_x - \partial_x \Psi \, \partial_y) \, \Delta \Psi - D^2 u_0 \, \partial_x \Psi = \frac{1}{\mathrm{Re}} \Delta^2 \Psi,$$

where Δ denotes the Laplace operator. Since u_1 and v_1 are periodic in x with the period $\lambda_x = 2\pi/\alpha$, the stream function can be represented as

$$(2.6) \qquad \Psi(x, y) = \sum_{n=-\infty}^{n=+\infty} \Phi_n(y) \, e^{in\alpha x}$$

where $\Phi_n = \Phi_{-n}^*$ in order for Φ_n to be real. Inserting the expansion (2.6) into the Eq. (2.5), and separating the Fourier harmonics, the following system of the ordinary differential equations is obtained:

$$(2.7) \quad \left[D_n^2 - in\alpha \, \mathrm{Re} \left(u_0 D_n - D^2 u_0 \right) \right] \Phi_n$$

$$- i\alpha \, \mathrm{Re} \sum_{k=-\infty}^{k=+\infty} \left[k \, D\Phi_{n-k} D_k \Phi_k - (n-k) \, \Phi_{n-k} D_k D\Phi_k \right] = 0,$$

where $D_n = D^2 - n^2\alpha^2$. The boundary conditions at the channel walls are expressed in the following form:

$$(2.8) \qquad \begin{aligned} u_0 \left(y_L(x) \right) + u_1 \left(x, y_L(x) \right) &= 0 \quad \text{and} \quad v_1 \left(x, y_L(x) \right) = 0, \\ u_0 \left(y_U(x) \right) + u_1 \left(x, y_U(x) \right) &= 0 \quad \text{and} \quad v_1 \left(x, y_U(x) \right) = 0. \end{aligned}$$

To obtain a complete formulation two additional conditions have to be imposed. The first one is simply the selection of the (constant) value of the stream function at one of the walls. The choice of the second is, in essence, arbitrary, but two possibilities are particularly important. We can prescribe the modification of the volume flux or the modification of the average pressure gradient along the channel. In this study, the first option is chosen and the additional conditions have the following form:

$$(2.9) \qquad \begin{aligned} \Psi_0 \left(y_L(x) \right) + \Psi \left(x, y_L(x) \right) &= 0, \\ \Psi_0 \left(y_U(x) \right) + \Psi \left(x, y_U(x) \right) &= Q + q, \end{aligned}$$

where Ψ_0 denotes the stream function which corresponds to the reference flow, i.e. it is defined as $\Psi_0(y) = -\frac{2}{3} + y - \frac{1}{3}y^3$. Symbol Q denotes the volume flux

of the reference flow, namely

$$Q = \int\limits_{-1}^{1} (1 - y^2)\, dy = \frac{4}{3},$$

while symbol q denotes the assumed modification of the volume flux. In the remaining part of the paper we assume that the flow in a wavy channel preserves the original volume flux, i.e. $q = 0$.

The numerical implementation of the conditions (2.8) and (2.9) is the key problem in the numerical solution of the system (2.7). In the remaining part of this section, we shall outline briefly the immersed boundary approach. Detailed description and testing of the method can be found in [4].

First, we define the computational domain to be a rectangular region spanning in y-coordinate from Y_L to Y_U. The extent of this region in x-coordinate is simply one spatial period λ_x. Thus, the physical domain is embedded in the computational one.

The next step is to represent the stream function in a form of truncated to N leading Fourier modes

$$(2.10) \qquad \Psi\,(x, y) \approx \sum_{n=-N}^{n=N} \Phi_n\,(y)\; e^{in\alpha x}.$$

The formula (2.10) defines the stream function in the whole computational domain. Since the physical domain forms a proper subset of the computational one, the complement of the former is nonempty. In this area, the formula (2.10) defines a certain "fictitious" flow field.

The corresponding finite-dimensional system of the ordinary differential equations for the amplitude functions Φ_n, $n=0,1,..,N$ is discretized by introducing Chebyshev representations of the unknown function Φ_n as follows:

$$(2.11) \qquad \Phi_n(y) = \sum_{j=0}^{\infty} G_j^n T_j(y) \;\approx \sum_{j=0}^{J} G_j^n T_j(y)\,.$$

In the above formula, symbol T_j denotes the polynomial obtained as a results of composition of the j-th Chebyshev polynomial with the linear mapping transforming the interval $[Y_L, Y_U]$ into the interval $[-1,1]$ (see Appendix A), and G_j^n stands for the unknown expansion coefficient. The Chebyshev representations of the required derivatives $D^l\Phi$ (with l up to $l=4$) can be determined using a differentiation algorithm (see [5], also the Appendix A).

The n-th equation of the system (2.7) can be written in a general form as

$$(2.12) \qquad \Xi_n\,(\Phi_0, \Phi_1, .., \Phi_M) = 0 \quad \text{for} \quad n = 0, .., N.$$

The substitution of the Chebyshev expansions (2.11) and their derivatives into (2.12) gives the residual function

$$(2.13) \qquad R_n = \Xi_n \left(\sum_{j=0}^{J} G_j^0 T_j, \sum_{j=0}^{J} G_j^1 T_j, .., \sum_{j=0}^{J} G_j^M T_j \right), \quad n = 0, .., N.$$

The problem is converted to an algebraic, nonlinear system by imposing the orthogonality conditions

$$(2.14) \qquad \langle R_n, T_j \rangle_\omega = 0, \quad j = 0, .., J - 4, \quad n = 0, .., N.$$

The inner product used in (2.14) is defined as follows:

$$\langle f, g \rangle_\omega := \int_{Y_L}^{Y_U} f(x)\, g(x)\, \omega(x)\, dx,$$

where the Chebyshev weight function ω, along with other useful formulas, is given in the Appendix A.

The discretization method described above can be viewed as a variant of the Chebyshev-tau technique. The reader should note that the projection (2.14) is carried out onto the linear subspace spanned by the Chebyshev polynomials with the order of up to $J - 4$. The additional equations required to close the system are due to the flow boundary conditions (2.8) and the volume flux modification (2.9).

We shall describe briefly the treatment of the boundary conditions (a detailed exposition can be found in [4]). The idea is to derive explicitly the forms of the Fourier expansions of the velocity distribution along the wavy walls. Consider an arbitrary line $l := \{(x, y) : y = f(x)\}$ located entirely in the computational domain ($Y_L \leq f(x) \leq Y_U$), where the function f is periodic and it is expressed as

$$(2.15) \qquad f(x) = f_0 + \tfrac{1}{2} \sum_{k=1}^{\infty} f_k \exp(ik\alpha x) + \tfrac{1}{2} \sum_{k=1}^{\infty} f_k^* \exp(-ik\alpha x.)$$

The velocity components, computed along the line l, are the x-periodic functions and thus they can be expressed in terms of Fourier series as

$$u_l(x) \equiv u(x, f(x)) = \sum_{n=-\infty}^{+\infty} U_n\, e^{in\alpha x},$$

$$(2.16)$$

$$v_l(x) \equiv v(x, f(x)) = \sum_{n=-\infty}^{+\infty} V_n\, e^{in\alpha x}.$$

On the other hand, the same distributions can be obtained with the use of the stream function, namely

$$u_l(x) \equiv u(x, f(x)) \cong u_0(f(x)) + \sum_{n=-N}^{N} D\Phi_n(f(x))e^{in\alpha x}$$

$$= u_0(f(x)) + \sum_{n=-N}^{N} \sum_{j=0}^{J} G_j^n \, DT_j(f(x))e^{in\alpha x},$$

(2.17)

$$v_l(x) \equiv v(x, f(x)) \cong -i\alpha \sum_{n=-N}^{N} n \, \Phi_n(f(x)) \, e^{in\alpha x}$$

$$= -i\alpha \sum_{n=-N}^{N} \sum_{j=0}^{J} n \, G_j^n T_j(f(x)) \, e^{in\alpha x}.$$

The need for the Fourier expansions of the Chebyshev polynomials and their first derivative is apparent. We can write

(2.18) $$\qquad T_j(f(x)) = \sum_{k=-\infty}^{\infty} w_k^j e^{ik\alpha x}, \qquad DT_j(f(x)) = \sum_{k=-\infty}^{\infty} d_k^j e^{ik\alpha x}.$$

The expansion coefficients in (2.18) can be calculated, see Appendix A. Inserting (2.18) into (2.17), and comparing the obtained Fourier coefficients with the coefficients in (2.16), we obtain

(2.19) $$\quad U_n = F_n + \sum_{m=-N}^{N} \sum_{j=0}^{J} d_{n-m}^j G_j^m, \qquad V_n = -i\alpha \sum_{m=-N}^{N} \sum_{j=0}^{J} m \, w_{n-m}^j G_j^m.$$

In the above formula, F_n denotes the Fourier coefficient of the function $u_0(f(x))$, i.e. the velocity of the reference flow computed along the line l.

The expressions (2.19) are the bases for implementation of the boundary conditions (2.8). It is necessary to apply them separately for each wall of the channel, i.e. to set $f(x) = y_L(x)$ and $f(x) = y_U(x)$. A minor complication arises, however. It has been shown in [4] that the value of the Fourier coefficient V_0 cannot be assumed independently. Instead, we have two equations corresponding to the volume flux conditions (2.9). The final form of the algebraic equation corresponding to (2.8) and (2.9) is the following:

$$(F_n)_{L,U} + \sum_{m=-N}^{N} \sum_{j=0}^{J} \left(d_{n-m}^j \right)_{L,U} G_j^m = 0, \qquad N \geq n \geq 0,$$

$$\sum_{m=-N}^{N} \sum_{j=0}^{J} m \left(w_{n-m}^{j}\right)_{L,U} G_j^m = 0, \quad N \geq n \geq 1,$$

(2.20)

$$(H_0)_L + \sum_{m=-N}^{N} \sum_{j=0}^{J} G_j^m (w_m^j)_L^* = 0,$$

$$(H_0)_U + \sum_{m=-N}^{N} \sum_{j=0}^{J} G_j^m (w_m^j)_U^* = Q + q,$$

where the subscripts L and U mean that the formulas are to be applied to the lower and the upper wall separately, $(H_0)_L$ and $(H_0)_U$ denote the coefficients of the zero Fourier mode of the functions $\Psi_0(y_L(x))$ and $\Psi_0(y_U(x))$. The precise meaning of the above equations is the following. The first group of the equations expresses the fact that $N+1$ leading Fourier modes of the boundary distribution of the x-components of the flow velocity must vanish. The second group of the equations plays the same role with respect to the y-component of the velocity field. As we have already mentioned, the equation for the zero Fourier mode is excluded, as it is actually satisfied automatically. The role of the two last equations is to set the average values of the complete (the reference flow and the modification) stream function at the channel boundaries.

The complete nonlinear system consists of the Eqs. (3.4) and (3.11). It can be written in a form including the complex Chebyshev coefficients of the amplitude function with only non-negative indices. Thus, its dimension is $(N+1) \cdot (K+1)$. It can be solved efficiently using an algorithm, which takes advantage of the particular structure of the equations. Such algorithm has been described in details in [4].

3. The equations of linear stability

The complete velocity field of the flow in the channel with wavy walls is x-periodic. For the sake of further convenience, the components of the velocity will be expressed as the following Fourier expansions

(3.1) $\quad U(x,y) = \sum_{n=-\infty}^{\infty} F_U^n(y) \exp(in\alpha x), \quad V(x,y) = \sum_{n=-\infty}^{\infty} F_V^n(y) \exp(in\alpha x).$

Clearly, we have $F_U^{-n} = (F_U^n)^*$ and $F_V^{-n} = (F_V^n)^*$, since both functions in (3.1) are real.

The amplitude functions in (3.1) can be expressed in terms of the amplitude functions of the stream function, namely

$$\text{(3.2)} \quad \begin{aligned} F_U^0(y) &= u_0(y) + D\Phi_0(y), \quad F_U^n(y) \equiv \left(F_U^{-n}(y)\right)^* = D\Phi_D(y), \quad n > 0, \\ F_V^n(y) &\equiv \left(F_V^{-n}(y)\right)^* = -in\alpha\Phi_n(y), \quad n \geq 0. \end{aligned}$$

The linear analysis of stability of the flow defined by the formula (3.1) and (3.2) consists in investigation of the response of the flow to a small disturbance. Such analysis consists of four essential steps. In the first step, the class of admissible disturbances should be defined. Secondly, the equations governing the dynamics of admissible disturbances should be derived. As a result of this step, the eigenvalue differential problem is obtained. Thirdly, using appropriate discretization procedure, the differential problem has to be transformed to a tractable algebraic eigenvalue problem. Finally, the algebraic eigenvalue problem is to be solved in order to determine the most "dangerous" forms of disturbances, their amplification rates, critical Reynolds number and so on.

Since the main flow is x-periodic, the admissible form of a disturbance velocity field in the linear stability theory is

$$\text{(3.3)} \quad \upsilon(t, x, y, z) = \sum_{m=-\infty}^{+\infty} \left[g_u^m(y), g_v^m(y), g_w^m(y)\right] \exp\left[i\left(t_m x + \beta z - \sigma t\right)\right]$$

where $t_m = \delta + m\alpha$ and δ is a (real) Floquet exponent. Thus, the admissible disturbance is, in general, quasi-periodic in variable x.

Once the representation (3.3) is inserted into the linearized Navier-Stokes equations (in the velocity-vorticity formulation) and into the continuity equation then, after a rather lengthy algebra, the following equations are obtained:

$$S^m g_v^m = -\mathrm{Re} \sum_{n=1}^{\infty} \left(L_u^{m,n} g_u^{m-n} + L_v^{m,n} g_v^{m-n} + L_w^{m,n} g_w^{m-n}\right.$$

$$\left. + M_u^{m,n} g_u^{m+n} + M_v^{m,n} g_v^{m+n} + M_w^{m,n} g_w^{m+n}\right),$$

$$\text{(3.4)} \quad Q^m\left(t_m g_w^m - \beta g_u^m\right) + \mathrm{Re}\beta \cdot DF_u^0 g_v^m = i\mathrm{Re} \sum_{n=1}^{\infty} \left(N_u^{m,n} g_u^{m-n} + N_v^{m,n} g_v^{m-n}\right.$$

$$\left. + N_w^{m,n} g_w^{m-n} + K_u^{m,n} g_u^{m+n} + K_v^{m,n} g_v^{m+n} + K_w^{m,n} g_w^{m+n}\right),$$

$$it_m g_u^m + D g_v^m + i\beta g_w^m = 0, \quad m = \ldots -2, -1, 0, 1, 2, \ldots$$

The left-hand side operators appearing in the Eq. (3.4) are defined as

$$\text{(3.5)} \quad \begin{aligned} S^m &= \left(D^2 - k_m^2\right)^2 - i\mathrm{Re}\left[\left(t_m F_u^0 - \sigma\right)\left(D^2 - k_m^2\right) - t_m D^2 F_u^0\right], \\ Q^m &= D^2 - k_m^2 - i\mathrm{Re}\left(t_m F_u^0 - \sigma\right), \end{aligned}$$

where $k_m^2 = t_m^2 + \beta^2$. The operators (3.5) can be regarded as generalizations of Orr-Sommerfeld and Squire operators. The operators in the right-hand side of the equations are rather complicated and we skip their explicit form. For the case $n = 1$, these operators have been presented by Floryan [1]. Besides, we will further transform the Eq. (3.4) to obtain a more tractable form.

It is well known that the equations of linear stability of a parallel flow can be converted into the equations for wall-normal components of velocity and vorticity fields of the disturbance flow. Similar procedure can be applied to the equations (3.4). The main advantage is the reduction of the number of unknowns and lowering the dimension of the algebraic eigenvalue problem obtained as a result of discretization of the system (3.4).

As a first step, the following functions are introduced

$$(3.6) \qquad \theta^m(y) = t_m g_w^m(y) - \beta g_u^m(y), \quad m = .. - 2, -1, 0, 1, 2, .. .$$

Then, using the definition (3.6) and the spectral form of the continuity equation

$$i t_m g_u^m(y) + D g_v^m + i \beta g_w^m = 0, \quad m = .. - 2, -1, 0, 1, 2, .. ,$$

the inverse relations can be derived

$$(3.7) \qquad g_u^m = (i t_m D g_v^m - \beta \theta^m)/k_m^2, \quad g_w^m = (i \beta D g_v^m + t_m \theta^m)/k_m^2 .$$

The reader should notice that θ^m is closely related to the Fourier representation of the y-components of the disturbance vorticity field $\boldsymbol{\upsilon}$. Indeed, one has

$$(3.8) \quad \omega_y = \partial_z v_x - \partial_x v_z = \sum_{n=-\infty}^{\infty} i(\beta g_u^n - t_m g_w^n) \exp[i(t_n x + \beta z - \sigma t)]$$

$$= -i \sum_{n=-\infty}^{\infty} \theta^n \exp[i(t_n x + \beta z - \sigma t)].$$

The stability equations expressed in terms of the amplitude functions $\{g_v^n, \theta^n\}$ assume the following form:

$$S^m g_v^m + \sum_{n=1}^{\infty} (\bar{G}_v^{m,n} g_v^{m+n} + G_v^{m,n} g_v^{m-n} + \bar{G}_\theta^{m,n} \theta^{m+n} + G_\theta^{m,n} \theta^{m-n}) = 0$$

$$m = .., -2, -1, 0, 1, 2, ..$$

$$(3.9)$$

$$Q^m \theta^m + \mathrm{Re}\beta \, D F_u^0 g_v^m$$

$$+ \sum_{n=1}^{\infty} (\overline{E}_v^{m,n} g_v^{m+n} + E_v^{m,n} g_v^{m-n} + \overline{E}_\theta^{m,n} \theta^{m+n} + E_\theta^{m,n} \theta^{m-n}) = 0.$$

In the above, the operators S^m and Q^m are defined as previously by (3.5). The expressions of the other operators are rather long and therefore they are presented the Appendix B.

The disturbance velocity field should vanish at the boundaries of the flow domain. Thus, the Eq. (3.9) are supplemented by homogeneous boundary conditions imposed at the wavy walls of the channel. It will be explained in Sec. 5 how the treatment of the boundary conditions applied for the main flow in Sec. 2 can be adopted in the stability analysis.

4. The numerical method for the stability equations

4.1. Spectral discretization of the stability equations

The structure of the set of the stability Eq. (3.9) is simple – for each integral number we have a pair of two differential equations, the first of the fourth order and the other of the second order. Clearly, all these pairs of the equations are coupled. Supplemented by appropriate homogeneous boundary conditions, the Eq. (3.9) define the eigenvalue problem. Nontrivial solutions can exist only for certain combinations of the following parameters: the Reynolds number Re, wave numbers α and β, the Floquet exponent δ and the complex amplification rate σ. In the analysis of temporal stability, we are interested in determination of the amplification rate σ as a function of the remaining parameters. Mathematically, a linear differential eigenvalue problem must be solved. After suitable discretization and truncation, a finite-dimensional algebraic problem is obtained. It can be solved numerically with the use of standard tools, for instance those from the LAPACK library.

The numerical method used to solve the stability equations can be regarded as a variant of the spectral Chebyshev-Galerkin method. The unknown amplitude functions are sought in the form of (truncated) Chebyshev expansions

$$(4.1) \qquad g_v^n(y) = \sum_{k=0}^{K_V} \Gamma_k^n T_k(y), \quad \theta^n(y) = \sum_{k=0}^{K_\theta} \Theta_k^n T_k(y).$$

As in the Sec. 2, the Chebyshev polynomials are defined on the interval $[Y_L, Y_U]$. The further procedure consists in three steps:

1. Insertion of the above expansions into the stability equations.
2. Projection of the equations obtained after the first step onto the subspace spanned by a finite number of the Chebyshev polynomials. For each fourth-order equation, the projection is made by computing the weighted inner products with the functions $T_0, T_1, .., T_{K_V-4}$. For each second-order equation the projection is made by computing the weighted inner products with the polynomials $T_0, T_1, .., T_{K_\theta-2}$. The additional six equations (for

each fourth/second-order pair) come from the direct enforcement of the homogeneous boundary conditions.

3. Truncation of the originally infinite set of the stability equations to its finite-dimensional approximation. It means that in the computations, the range of the summation index "m" in the expansion (3.3) will be $-M, .., M$.

We first consider the discretization of the m-th fourth-order equation in (3.9) After the Chebyshev representations are inserted and the projections are made, one has

$$(4.2) \quad \sum_{k=0}^{K_V} \langle S^m T_k, T_j \rangle_\omega \, \Gamma_k^m$$

$$+ \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{K_V} \langle \bar{G}_V^{m,n} T_k, T_j \rangle_\omega \, \Gamma_k^{m+n} + \sum_{k=0}^{K_V} \langle G_V^{m,n} T_k, T_j \rangle_\omega \, \Gamma_k^{m-n} \right\}$$

$$+ \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{K_\theta} \langle \bar{G}_\theta^{m,n} T_k, T_j \rangle_\omega \, \Theta_k^{m+n} + \sum_{k=0}^{K_\theta} \langle G_\theta^{m,n} T_k, T_j \rangle_\omega \, \Theta_k^{m-n} \right\} = 0$$

$$j = 0, .., K_V - 4$$

where the inner product $\langle \cdot, \cdot \rangle_\omega$ is defined as described in the Appendix A.

Analogous procedure applied to the m-th second-order Eq. (3.9) yields

$$(4.3) \quad \sum_{k=0}^{K_\theta} \langle Q^m T_k, T_j \rangle_\omega \, \Theta_k^m + \beta \mathrm{Re} \cdot \sum_{k=0}^{K_V} \langle D F_u^0 T_k, T_j \rangle_\omega \, \Gamma_k^m$$

$$+ \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{K_V} \langle E_V^{m,n} T_k, T_j \rangle_\omega \, \Gamma_k^{m-n} + \sum_{k=0}^{K_V} \langle \bar{E}_V^{m,n} T_k, T_j \rangle_\omega \, \Gamma_k^{m+n} \right\}$$

$$+ \sum_{n=1}^{\infty} \left\{ \sum_{k=0}^{K_\theta} \langle E_\theta^{m,n} T_k, T_j \rangle_\omega \, \Theta_k^{m-n} + \sum_{k=0}^{K_\theta} \langle \bar{E}_\theta^{m,n} T_k, T_j \rangle_\omega \, \Theta_k^{m+n} \right\} = 0,$$

$$j = 0, .., K_\theta - 2.$$

Insertion of the explicit forms of all operators (see Appendix B) into the formulas (4.2) and (4.3) leads to rather complicated expressions. However, all of them can be computed in terms of seventeen elemental structures with two or three indices. Each entry of a double-index structure is defined by the Chebyshev integral of the products of a Chebyshev polynomial and an even-order derivative (up to 4th order) of another one. Each entry of a triple-index structure is defined

by the Chebyshev integral of the product of a Chebyshev polynomial, a derivative (up to 3rd order) of another one, and one of the amplitude functions defined in (3.2) or one of its derivatives (up to 2nd order).

4.2. Numerical implementation of the boundary conditions

Main feature of the numerical method applied for the determination of the main flow is that the computational domain is extended to overlap the interior of the channel with wavy walls. Consequently, the basic functions for spectral expansions are the Chebyshev polynomials defined in the non-standard interval $[Y_L, Y_U]$. The boundary conditions at the channel walls are enforced as the internal conditions imposed along the wavy lines immersed in the interior of the extended computational domain.

The same technique can be applied to the homogeneous boundary conditions formulated for the disturbance velocity field in the stability analysis.

The derivation of the boundary conditions for the disturbance velocity field consists in four steps:

1. Derivation of the Fourier representation of the disturbance velocity distributions along the wavy walls.
2. Extracting the Fourier coefficients and setting them to zero.
3. Re-writing the obtained expressions in terms of the expansion coefficients of the amplitude functions g_V^m and θ^m.
4. Truncating to a finite number of Fourier modes in order to get the final computable form.

The derivation procedure will be demonstrated on the example of the y-component v_y computed for the bottom wall $y = y_L(x)$. The distribution along the bottom wall can be expressed as

$$(4.4) \quad v_y(x, y_L(x), z) = \sum_{n=-\infty}^{\infty} g_V^n(y_b(x)) \exp\left[i\left(t_n x + \beta z - \sigma t\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \left(w_k^j\right)_L \Gamma_j^n \right) \exp\left(ik\alpha x\right) \right] \exp\left[i\left(t_n x + \beta z - \sigma t\right)\right]$$

$$= \sum_{m=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \left(w_{m-n}^j\right)_L \Gamma_j^n \right) \right] \exp\left[i\left(t_m x + \beta z - \sigma t\right)\right].$$

Let us rewrite (4.4) in the following form

(4.5) $v_y(x, y_L(x), z) \exp\left[-i\left(\delta x + \beta z\right)\right] \exp\left(i\sigma t\right)$

$$= \sum_{m=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \left(w_{m-n}^j\right)_L \Gamma_j^n \right) \right] \exp\left(i m\alpha x\right).$$

If we set all Fourier coefficients in right-hand side of (4.5) to zero then, at any time instant, the y-component of the disturbance velocity field will satisfy the homogeneous boundary condition at the bottom wall. Indeed, both the Floquet exponent δ and the spanwise wave number β are real, so the factor $\exp[-i(\delta x + \beta z)]$ is bounded in space. Thus, the second step of the derivation procedure yields

(4.6) $$\sum_{n=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \left(w_{m-n}^j\right)_L \Gamma_j^n \right) = 0, \quad m = ..-2.-1, 0, 1, 2, ...$$

Similar two equations are obtained for the remaining velocity components. Then, replacing the subscript "L" with "U" gives an analogous triple of equations for the upper wall.

In the third step, the conditions for the components v_x and v_z have to be expressed in terms of the functions g_V^m and θ^m. Using the relation (3.7), we end up with following formulas:

$$\sum_{n=-\infty}^{\infty} \left(\frac{it_n}{k_n^2} \sum_{k=1}^{\infty} \left(d_{m-n}^k\right)_L \Gamma_k^n - \frac{\beta}{k_n^2} \sum_{k=0}^{\infty} \left(w_{m-n}^k\right)_L \Theta_k^n \right) = 0,$$

$$m = .., -2, -1, 0, 1, 2, ..$$

(4.7)

$$\sum_{n=-\infty}^{\infty} \left(\frac{i\beta}{k_n^2} \sum_{k=1}^{\infty} \left(d_{m-n}^k\right)_L \Gamma_k^n + \frac{t_n}{k_n^2} \sum_{k=0}^{\infty} \left(w_{m-n}^k\right)_L \Theta_k^n \right) = 0,$$

$$m = .., -2, -1, 0, 1, 2, ..$$

for the bottom wall. Again, a pair of analogous conditions for the upper wall is obtained by replacing the subscript "L" with "U".

This way, we obtain three additional homogeneous equations for each wall and each number (Fourier mode number) "m". To make finite computations possible, all sums in above expressions have to be truncated. The final form of

the equations expressing the boundary conditions at the channel walls is

$$\sum_{n=-M}^{M} \left(\sum_{k=0}^{K_V} \left(w_{m-n}^k \right)_{L,U} \Gamma_k^n \right) = 0, \qquad m = -M, .., 0, .., M,$$

(4.8)
$$\sum_{n=-M}^{M} \left(\frac{it_n}{k_n^2} \sum_{k=1}^{K_V} \left(d_{m-n}^k \right)_{L,U} \Gamma_k^n - \frac{\beta}{k_n^2} \sum_{k=0}^{K_\theta} \left(w_{m-n}^k \right)_{L,U} \Theta_k^n \right) = 0,$$
$$m = -M, .., 0, .., M,$$

$$\sum_{n=-M}^{M} \left(\frac{i\beta}{k_n^2} \sum_{k=1}^{K_V} \left(d_{m-n}^k \right)_{L,U} \Gamma_k^n + \frac{t_n}{k_n^2} \sum_{k=0}^{K_\theta} \left(w_{m-n}^k \right)_{L,U} \Theta_k^n \right) = 0$$
$$m = -M, .., 0, .., M.$$

In the above formula, the subscripts "L" and "U" refer to the bottom and upper walls, respectively.

4.3. Structure of the algebraic eigenvalue problem

The unknowns can be collected in the following block vector

(4.9) $\quad \mathbf{z} = \left[\left\{ \Gamma_0^n, \Gamma_1^n, .., \Gamma_{K_V-1}^n, \Gamma_{K_V}^n ; \Theta_0^n, \Theta_1^n, .., \Theta_{K_\theta-1}^n, \Theta_{K_\theta}^n \right\}, \right.$

$$\left. n = -M, .., 0, ..M \right]^T.$$

Then, Eqs. (4.2), (4.3) and (4.8) can be written in the form of the homogeneous linear system

(4.10) $$\mathbf{P} \cdot \mathbf{z} = 0.$$

The matrix $\mathbf{P}$ has a block structure. The block dimension of $\mathbf{P}$ is $2*M+1$. All blocks are matrices with dimension $K_V + K_\theta + 2$. The position of a block inside the matrix $\mathbf{P}$ can be described with the use of two block indices: row index m and the column index n. The range of these indices is $[-M, .., M]$. Each block with the row index m consists of $K_V - 3$ rows corresponding to the m-th Eq. (4.2), $K_\theta - 1$ rows corresponding to the m-th Eq. (4.3) and six rows corresponding to the m-th Eq. (4.8) enforcing the homogeneous boundary conditions.

The eigenvalue problem is defined by the characteristic equations for the homogeneous system, i.e. det $\mathbf{P} = 0$. In the case of temporal stability, one seeks the amplification exponent σ as a function of other parameters of the problem. In

such a case the eigenvalue problem is linear and can be expressed in the standard form as

(4.11) $$\mathbf{P}_0 z = \sigma \mathbf{P}_1 z.$$

The matrices defining the generalized eigenvalue problem (4.11) can be obtained directly by extracting those parts of the stability equations, which do not contain σ. Alternatively, the matrix $\mathbf{P}$ can be considered as a function of the attenuation exponent σ and then the matrices in (4.11) can be determined as follows:

(4.12) $$\mathbf{P}_0 = \mathbf{P}(0), \quad \mathbf{P}_1 = \mathbf{P}_0 - \mathbf{P}(1).$$

The generalized eigenvalue problem (4.11) can be solved using standard routines from the LAPACK library. However, the investigation of variations of selected eigenvalues/eigenvectors with a change of various parameters (Reynolds number, roughness geometry, wave numbers etc.) necessitates an efficient tracing algorithm. It seems that the **Inverse Iterations Method** provides an appropriate tool for this task. The IIM can also be used to improve the accuracy of particular eigensolutions obtained by other methods. Although an exposition of this method can be found in many textbooks on the numerical algebra, for the sake of completeness it is presented in the Appendix C.

5. The convergence tests

In this section, we present results of the convergence tests. The numerical computations have been carried out for the Reynolds number Re=5000 and the wave numbers α=3 and β=2. The shape of the channel walls is assumed in the following form:

$$y_L(x) = -1 - S\cos(\alpha x), \quad y_U(x) = 1 + S\cos(\alpha x) ,$$

where S=0.02. The main flow is computed using fifteen Fourier modes (N=14) and sixty (J=59) Chebyshev polynomials per each amplitude function Φ_n. The assumed length of the Fourier representation of the main flow ensures that the error in enforcement of the boundary conditions (2.8) is less than 10^{-12}. The convergence test has been designed in the following way. For the flow parameters chosen for the test, two unstable modes exist: the Squire mode (Sq) and the Orr-Sommerfeld mode (OS). This situation is analogous to the case of distributed wall suction, discussed by the author in [6]. The corresponding eigenvalues are evaluated using different number of modes M in the disturbance Fourier expansion (3.3). With increasing number of modes, the representation of the disturbance

is getting more accurate. We also assume that the number of Fourier modes of the main flow accounted for in the stability computations is the largest possible, i.e. it is equal to M.

When the number of modes M increases, the convergence is expected. The computations are repeated with a different number of the Chebyshev polynomials. This way the convergence with respect to K_v and K_θ can be established as well. The latter issue is, however, less important because the efficiency of immersed boundary approach to the boundary conditions depends mostly on the convergence properties with respect to the Fourier expansions.

The results of the test computations are presented in Table 1. Rapid convergence of the evaluated eigenvalues is observed. Surprisingly enough, the computed values are quite accurate even for M=3. The values for M=7 and M=8 match to within nine significant digits. In the numerical stability analysis, such accuracy is more than satisfactory.

Table 1. The unstable eigenvalues computed for different values of the Fourier modes M and Chebyshev polynomials K_v and K_θ.

M	$\sigma \cdot 10^3$	$K_v = K_\theta = 49$	$K_v = 59, K_\theta = 49$	$K_v = K_\theta = 59$
3	Sq	1.923180264	1.923152536	1.923125632
	OS	1.382477688	1.382428066	1.382420663
4	Sq	1.922667419	1.922629066	1.922598856
	OS	1.381969904	1.381912186	1.381896911
5	Sq	1.922661537	1.922622306	1.922592749
	OS	1.381965322	1.381907517	1.381891484
6	Sq	1.922662554	1.922623455	1.922593912
	OS	1.381966101	1.381908421	1.381892490
7	Sq	1.922662452	1.922623367	1.922593787
	OS	1.381966016	1.381908339	1.381892384
8	Sq	1.922662449	1.922623364	1.922593784
	OS	1.381966013	1.381908337	1.381892378

Practically, $M = 4$ or $M = 5$ is sufficient to obtain very good approximations of the leading eigenvalues. If one is interested in the corresponding eigensolutions, the accuracy of the boundary conditions should be also considered. It is instructive to compute the error in the boundary conditions for the components of the disturbance velocity field. Obviously, the value of the error is meaningful providing that an eigensolution is normalized. Table 2 shows the maximum norm of the boundary error computed at the bottom wall for the x-component of the disturbance velocity field. The latter is normalized so that its maximal value in the flow domain is the unity. The accuracy of the boundary conditions is im-

proving with increasing number of the Fourier modes. In the number of modes is sufficiently large, addition of one mode reduces the norm of the error by a certain factor, i.e. the error diminishes at exponential or spectral rate. The observed rate of convergence can be roughly characterized as reduction by two orders of the magnitude per each three more modes.

Table 2. The maximal boundary error of the velocity component v_x of the eigensolutions computed using different number of the Fourier modes. The number of the Chebyshev polynomials is $K_v = K_\theta = 59$.

M	3	4	5	6	7	8
Sq	$1.4767 \cdot 10^{-3}$	$3.2391 \cdot 10^{-4}$	$1.1662 \cdot 10^{-4}$	$3.1286 \cdot 10^{-5}$	$6.8561 \cdot 10^{-6}$	$1.5373 \cdot 10^{-6}$
OS	$1.4590 \cdot 10^{-3}$	$3.5159 \cdot 10^{-4}$	$1.2026 \cdot 10^{-4}$	$3.2257 \cdot 10^{-5}$	$7.0622 \cdot 10^{-6}$	$1.5943 \cdot 10^{-6}$

6. Concluding remarks

The numerical method described above offers an efficient alternative for the traditional approach based of the domain transformation. The technique of immersed boundaries seems to be particularly useful when the magnitude of boundary corrugation is sufficiently small, i.e. it is not larger than several percent of the channel width. This range of values it exactly what one would reasonably call the "wall roughness". The methods based on a domain transformation are not much competitive here, because of the "overhead" due to extreme complication of the stability equations in the computational domain. Another possibility is to resort to domain perturbation methods (DPM). The idea is to transfer the boundary conditions from a corrugated boundary to a straight centerline, and get an approximate form of boundary conditions with the use of a Taylor expansion of a certain order. First-order procedure is well described in [7]. The applicability of this approach to flows in a corrugated channel has been recently investigated by Cabal et al. [8] and compared with the domain transformation and immersed boundary techniques. The essential conclusion is that the DPM (even of higher orders) cannot provide a sufficient accuracy in enforcement of the boundary conditions, particularly when the magnitude of wall corrugation becomes sufficiently large for flow destabilization. Thus, it is rather unlikely that DPM applied to the stability problem considered here would be of any use.

There are, certainly, some limitations of efficiency or even applicability of the immersed boundary method. The magnitude of a corrugation has been already mentioned. The other problem is posed by large values of the wave numbers. It has been shown in [4] that computations with large values of the wave number α require much larger numbers of the Fourier modes and Chebyshev polynomials.

At the same time, the rate of convergence, although exponential, becomes much slower. Consequently, the overall efficiency of the methods significantly drops and the computational cost becomes prohibitive. For such extreme cases, the method based on the domain transformation, although much more complicated in derivation and implementation, would be a better option (see [7]). On the other hand, the wave numbers characterizing unstable forms of disturbances turn out to fall within the range of small or moderate values ($\alpha,\beta{<}10$), where the immersed boundary approach is preferable.

Appendix A. Basic functions

Assume that the computational domain is the interval $[Y_L, Y_U]$. The transformation from the computational domain to the standard domain, i.e. the interval $[-1,1]$, is given as

$$(A.1) \qquad [Y_L, Y_U] \ni y \ \longrightarrow \ \eta(y) = p \cdot y + q \in [-1, 1]$$

where
$p = 2/(Y_U - Y_L)$ and $q = -(Y_L + Y_U)/(Y_U - Y_L)$.

The following set of functions (polynomials) can be defined

$$(A.2) \qquad T_j(y) = t_j(p \cdot y + q), \quad j = 0, 1, 2.., $$

where t_j denotes the standard Chebyshev polynomial.

If the following weight function is introduced

$$(A.3) \qquad \omega(y) = \Omega(p \cdot y + q), \quad \Omega(\eta) = \left(1 - \eta^2\right)^{-\frac{1}{2}}$$

then the functions defined by (A.2) satisfy an orthogonality condition, namely

$$(A.4) \qquad \int_{Y_L}^{Y_U} T_j(y)\, T_k(y)\, \omega(y)\, dy$$

$$= \frac{1}{p} \int_{-1}^{1} t_j(\eta)\, t_k(\eta)\, \Omega(\eta)\, d\eta = \frac{1}{p} \begin{cases} 0 & \text{if } j \neq k, \\ \pi & \text{if } j = k = 0, \\ \pi/2 & \text{if } j = k > 0. \end{cases}$$

The derivative of the basic function T_j can be expressed in terms of the functions $T_0, T_1, .., T_{j-1}$ as follows:

$$(A.5) \qquad DT_j(y) = \sum_{k=0}^{j-1} c_j^k \cdot T_k(y).$$

The expansion coefficients in (A.5) are related to the coefficients in Chebyshev differentiation formulae

(A.6)
$$Dt_j(\eta) = \sum_{k=0}^{j-1} C_j^k \cdot t_k(\eta),$$

namely, we have $c_j^k = p\,C_j^k$. The coefficients $\left\{ C_j^k \right\}$ are the following:

$$C_j^k = \begin{cases} 2j, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases} \qquad \text{– when } j \text{ is an even number,}$$

$$C_j^k = \begin{cases} 0, & k \text{ is odd,} \\ 2j, & k > 0 \text{ and even} \\ j, & k = 0. \end{cases} \qquad \text{– when } j \text{ is an odd number,}$$

The multiplication rule for the basic functions $\{T_j\}$ is the same as for the standard Chebyshev polynomials, i.e.

(A.7)
$$T_j \cdot T_k = \frac{1}{2}T_{j+k} + \frac{1}{2}T_{|j-k|}.$$

Another important operation is an inner product of a pair of the expansions in $\{T_j\}$. Consider two functions given as follows:

$$g(y) = \sum_{j=0}^{M} g_j T_j(y), \quad h(y) = \sum_{k=0}^{N} h_k T_k(y).$$

Then the inner product is defined as follows:

(A.8)
$$\int_{Y_L}^{Y_U} g(y)h(y)\omega(y)\,dy = \frac{\pi}{p}\left(g_0 h_0 + \frac{1}{2} \sum_{k=1}^{\min(M,N)} g_k h_k \right).$$

Implementation of the boundary conditions in the channel with wavy walls requires computations of the Fourier expansions of the following composite functions:

$$T_j(f(x)) = \sum_{k=-\infty}^{k=+\infty} w_k^j \exp(ik\alpha x), \quad DT_j(f(x)) = \sum_{k=-\infty}^{k=+\infty} d_k^j \exp(ik\alpha x).$$

Since the basic functions are real, the coefficients satisfy obvious relations

$$w^j_{-k} = \left(w^j_k\right)^*, \quad d^j_{-k} = \left(d^j_k\right)^*, \quad k = 0, 1, .., \quad j = 0, 1, ...$$

The real function f is assumed in the form of the following Fourier series:

$$f(x) = f_0 + \frac{1}{2} \sum_{k=1}^{\infty} f_k \exp(ik\alpha x) + \frac{1}{2} \sum_{k=1}^{\infty} f_k^* \exp(-ik\alpha x).$$

Using the recursive definition of the Chebyshev polynomials, one arrives at the following formulas:

(A.9)

$$w^{j+1}_k = p \left(\sum_{n=0}^{\infty} f_n w^j_{k-n} + \sum_{n=0}^{\infty} f_n^* w^j_{k+n} \right) + 2qw^j_k - w^{j-1}_k,$$

$$d^{j+1}_k = p \left(2w^j_k + \sum_{n=0}^{\infty} f_n d^j_{k-n} + \sum_{n=0}^{\infty} f_n^* d^j_{k+n} \right) + 2qd^j_k - d^{j-1}_k.$$

In order to use (A.9), the coefficients for $j = 0$ and $j = 1$ have to be defined explicitly. Since

$$B_0(f(x)) \equiv 1, \quad B_0'(f(x)) \equiv 0,$$

$$B_1(f(x)) = p \cdot f(x) + q = pf_0 + q + \sum_{n=1}^{\infty} \frac{1}{2}pf_n \exp(in\alpha x) + \sum_{n=1}^{\infty} \frac{1}{2}pf_n^* \exp(-in\alpha x),$$

$$DB_1(f(x)) \equiv p,$$

one obtains the following starting formulas:

(A.10)

$$w^0_0 = 1, \quad w^0_k = 0 \quad \text{for} \quad k > 0,$$

$$w^1_0 = pf_0 + q, \quad w^1_k = \frac{1}{2}pf_k \quad \text{for} \quad k > 0,$$

$$d^0_k = 0 \quad \text{for} \quad k \geq 0,$$

$$d^1_0 = p, \quad d^1_k = 0 \quad \text{for} \quad k > 0.$$

Appendix B. The operators in the stability equations (3.9)

The differential operators appearing in the stability equations (3.9) are defined as follows

$$G_v^{m,n} = \text{Re} \cdot \left[\frac{i\,n\alpha}{k_{m-n}^2} \left(\beta^2 - t_m t_{m-n} \right) DF_u^n D + \frac{k_m^2}{k_{m-n}^2} \left(\beta^2 + t_{m-n} t_{m-2n} \right) F_v^n D \right.$$

$$+ \frac{i}{k_{m-n}^2} \left(2n\alpha\beta^2 - t_m k_{m-n}^2 \right) F_u^n D^2 + \frac{i}{k_{m-n}^2} \left(n\alpha t_m - k_m^2 \right) F_v^n D^3$$

$$\left. + i\,k_m^2 t_{m-2n} F_u^n + i\,t_m D^2 F_u^n \right],$$

$$\overline{G}_v^{m,n} = \text{Re} \cdot \left[\frac{i\,n}{\alpha} k_{m+n}^2 \left(t_m t_{m+n} - \beta^2 \right) (DF_u^n)^* D \right.$$

$$+ \frac{k_m^2}{k_{m+n}^2} \left(\beta^2 + t_{m+n} t_{m+2n} \right) (F_v^n)^* D + \frac{i}{k_{m+n}^2} \left(-2n\alpha\beta^2 - t_m k_{m+n}^2 \right) (F_u^n)^* D^2$$

$$\left. + \frac{i}{k_{m+n}^2} \left(-n\alpha t_m - k_m^2 \right) (F_v^n)^* D^3 + i\,k_m^2 t_{m+2n} (F_u^n)^* + i\,t_m \left(D^2 F_u^n \right)^* \right],$$

$$G_\theta^{m,n} = \text{Re} \cdot \left[\frac{1}{k_{m-n}^2} 2n\alpha\,\beta t_{m-n} F_u^n D + \frac{n\alpha\beta}{k_{m-n}^2} \left(t_m + t_{m-n} \right) DF_u^n \right.$$

$$\left. - \frac{i\,n\alpha\beta k_m^2}{k_{m-n}^2} F_v^n - \frac{i\,n\alpha\beta}{k_{m-n}^2} F_v^n D^2 \right],$$

$$\overline{G}_\theta^{m,n} = \text{Re} \cdot \left[-\frac{1}{k_{m+n}^2} 2n\alpha\,\beta t_{m+n} (F_u^n)^* D - \frac{n\alpha\beta}{k_{m+n}^2} \left(t_m + t_{m+n} \right) (DF_u^n)^* + \right.$$

$$\left. + \frac{i\,n\alpha\beta k_m^2}{k_{m+n}^2} (F_v^n)^* + \frac{i\,n\alpha\beta}{k_{m+n}^2} (F_v^n)^* D^2 \right],$$

$$E_v^{m,n} = \text{Re} \cdot \left[\beta DF_u^n - \frac{i\,n\alpha\beta}{k_{m-n}^2} F_v^n D^2 \right],$$

$$\overline{E}_v^{m,n} = \text{Re} \cdot \left[\beta \left(DF_u^n \right)^* + \frac{i\,n\alpha\beta}{k_{m+n}^2} (F_v^n)^* D^2 \right],$$

$$E_\theta^{m,n} = \text{Re} \cdot \left[-i\,t_m F_u^m - \frac{1}{k_{m-n}^2} \left(\beta^2 + t_m t_{m-n} \right) F_v^n D \right],$$

$$\overline{E}_\theta^{m,n} = \text{Re} \cdot \left[-i\,t_m \left(F_u^m \right)^* - \frac{1}{k_{m+n}^2} \left(\beta^2 + t_m t_{m+n} \right) (F_v^n)^* D \right].$$

Appendix C. Inverse iterations method

In this work, the following form of IIM has been used:

START : λ_0 − inititial approximation of an eigenvalue,

$\qquad$ $\mathbf{z}_0$ − inititial approximation of an eigenvector. $\|\mathbf{z}_0\| = 1$

$\qquad$ $p_0 = 0$

REPEAT for k $= 0, 1, ..$:

1) Solve $(\mathbf{A} - \lambda_0 \mathbf{B})\, \mathbf{w}_{k+1} = \mathbf{B}\mathbf{z}_k$

2) Compute $p_{k+1} = \langle \mathbf{w}_{k+1}, \mathbf{z}_k \rangle^{-1}$

3) If $|p_{k+1} - p_k| > \varepsilon$ then

$\qquad$ normalize $\mathbf{z}_{k+1} = \mathbf{w}_{k+1} / \|\mathbf{w}_{k+1}\|_2$

$\qquad$ go to step 1

Else

$\qquad$ compute the eigenvalue $\lambda = \lambda_0 + p_{k+1}$

$\qquad$ compute the normalized eigenvector $\mathbf{z} = \mathbf{w}_{k+1} / \|\mathbf{w}_{k+1}\|_2$

$\qquad$ STOP

End If

END.

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Received January 31, revised version April 26, 2002.