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Mean value and bounding formulae for heat conduction problems

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SOME MEAN VALUE formulae and bounds on the thermal energy for the steady-state heat conduction problems are proven. The formulation is based on the analogy which exists between the linear elasticity and heat conduction. The formalism of the applied analogy follows the Wojnar's approach. Some examples illustrate the applications of the theorems derived.

1. Governing equations

CONSIDER A BODY that occupies a closed and limited region $\bar{B}$ of volume V in R^3. The set of inner points of $\bar{B}$ is denoted by B and the set of points on the boundary of $\bar{B}$ is denoted by ∂B, $\bar{B} = B \cup \partial B$. Point P of B is indicated by the vector $\mathbf{OP} = \mathbf{p} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ in a given orthogonal Cartesian coordinate system $0x_1x_2x_3$ with the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The temperature difference field in the body $\bar{B}$ is given by $\Theta = \Theta(x_1, x_2, x_3)$ [1, 2]. The heat flux vector is denoted by $\mathbf{q} = \mathbf{q}(x_1, x_2, x_3) = q_1(x_1, x_2, x_3)\,\mathbf{e}_1 + q_2(x_1, x_2, x_3)\,\mathbf{e}_2 + q_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\,\mathbf{e}_3$ in $\bar{B}$.

Following WOJNAR [1], we introduce the thermal intensity vector field $\mathbf{t}$ by the definition

$$\mathbf{t} = -\nabla\Theta. \tag{1.1}$$

In Eq. (1.1) ∇ is the gradient (del) operator.

The field equation of the steady-state heat conduction problem are the heat balance equation [6, 7, 8]

$$-\nabla \cdot \mathbf{q} + r = 0 \qquad \text{in } B, \tag{1.2}$$

the Fourier law of heat conduction [6, 7, 8]

$$\mathbf{q} = \mathbf{K} \cdot \mathbf{t}, \tag{1.3}$$

and the thermal intensity-temperature field relation (1.1).

In Eqs. (1.2), (1.3) the dot denotes the scalar product according to [3, 4, 5], $\mathbf{K} = \mathbf{K}(x_1, x_2, x_3)$ is the heat conductivity tensor field which is symmetric and positive definite, and the distributed heat source in B is indicated by $r = r(x_1, x_2, x_3)$.

On the boundary surface ∂B the heat source q_s is defined at every regular point of ∂B as

$$q_s = \mathbf{q}(x_1, x_2, x_3) \cdot \mathbf{n}, \qquad (x_1, x_2, x_3) \in \partial B, \tag{1.4}$$

where $\mathbf{n}$ is the outward unit normal vector to ∂B at point (x_1, x_2, x_3).

2. Mean value formulae

2.1. Mean thermal intensity vector

Throughout this paper, the mean value of a continous function $f = f(x_1, x_2, x_3)$ on $\bar{B}$ is denoted by

$$< f >= \frac{1}{V} \int_B f dV .$$

THEOREM 1. *Let Θ be a temperature field, let $\mathbf{t}$ be the corresponding thermal intensity vector field, and suppose that Θ and $\mathbf{t}$ are continuous on $\bar{B}$. Then the mean value of $\mathbf{t}$ depends only on the boundary values of Θ and is given by*

$$< \mathbf{t} >= -\frac{1}{V} \int_{\partial B} \Theta \mathbf{n} \, dA . \tag{2.1}$$

P r o o f. The validity of the relation (2.1) follows from Eq. (1.1) and the divergence theorem [3, 4, 5]. In Eq. (2.1) dA is the surface element.

2.2. Mean heat flux vector

At first, we formulate two integral relations which will be used to derive the governing relationships.

Let $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3)$ and $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3)$ be regular vector fields in B. From the divergence theorem we get the first integral relation

$$\int_B (\mathbf{u} \circ \nabla) \cdot \mathbf{v} \, dB = \int_{\partial B} \mathbf{u} (\mathbf{n} \cdot \mathbf{v}) \, dA - \int_B \mathbf{u} (\nabla \cdot \mathbf{v}) \, dV . \tag{2.2}$$

Here, the tensor product of two vectors is denoted by a small circle and its definition is given in [3, 4, 5].

The second integral relation is

$$\int_B \mathbf{v} \cdot \nabla U \, dV = \int_{\partial B} U \mathbf{n} \cdot \mathbf{v} \, dA - \int_B U \nabla \cdot \mathbf{v} \, dV, \tag{2.3}$$

where $U = U(x_1, x_2, x_3)$ is an arbitrary continuously differentiable scalar-valued function in $\bar{B}$. The relation (2.3) is a direct consequence of Eq. (2.2), from it we obtain by the substitution $\mathbf{u} = \mathbf{c}U$, where $\mathbf{c}$ is an arbitrary constant, vector different from zero vector.

THEOREM 2. *The mean heat flux corresponding to the Eqs. (1.2), (1.4) depends only on the associated boundary heat flux and the distributed heat source and is given by*

$$< \mathbf{q} >= \frac{1}{V} \left[\int_{\partial B} q_s \mathbf{p} \, dA - \int_B r \mathbf{p} \, dV \right]. \tag{2.4}$$

P r o o f. The proof of the mean heat flux theorem is based on the vector identity (2.2), from it we obtain by the substitutions $\mathbf{u} = \mathbf{p}, \mathbf{v} = \mathbf{q}$ and the application of the identity $\mathbf{p} \circ \nabla = \mathbf{1}$, where $\mathbf{1}$ is the unit tensor.

THEOREM 3. *For a homogeneous material we have*

$$< \mathbf{q} >= \mathbf{K} < \mathbf{t} > . \tag{2.5}$$

P r o o f. The validity of the relation (2.5) follows from the Eq. (1.3) and the definition of the mean value of $\mathbf{t}$ and $\mathbf{q}$.

Next theorem refers to a nonhomoegenous body and it can be derived from the Fourier (1.3), the divergence theorem and the mean heat flux theorem (2.4).

THEOREM 4. *For a nonhomogeneous body the solution of Eqs. (1.1), (1.2) and (1.3) satisfies the relation*

$$\int_{\partial B} \Theta \mathbf{K} \cdot \mathbf{n} \, dA = \int_B r \mathbf{p} \, dV - \int_{\partial B} q_s \mathbf{p} \, dA, \tag{2.6}$$

where q_s is given by Eq. (1.4).

3. Upper and lower bounds on the thermal energy

Two types of the boundary value problems of heat conduction are considered.

The first one is formulated by the field equations (1.1), (1.2), (1.3) with vanishing heat source in B and the following boundary conditions:

$$\Theta = \Theta_1 \quad \text{on } \partial B_1 \quad \text{and} \quad \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } \partial B_2 , \tag{3.1}$$

where $\Theta_1 = \Theta_1(x_1, x_2, x_3)$ $(x_1, x_2, x_3) \in \partial B_1$ is a given function and $\partial B = \partial B_1 U \partial B_2$ $\partial B_1 \cap \partial B_2 = \{0\}$. It may be $\partial B_1 = \partial B$ and $\partial B_2 = \{0\}$.

The second type of the heat conduction problem is described by the field Eqs. (1.1), (1.2), (1.3) and the boundary conditions

$$\Theta = 0 \quad \text{on } \partial B_1 \quad \text{and} \quad \mathbf{q} \cdot \mathbf{n} = q_2 \quad \text{on } \partial B_2 \,. \tag{3.2}$$

It may be $\partial B_2 = \partial B$ and $\partial B_1 = \{0\}$. The prescribed heat flux $q_2 = q_2(x_1, x_2, x_3)$ $(x_1, x_2, x_3) \in \partial B_2$ and the given internal heat source $r = r(x_1, x_2, x_3)$ $(x_1, x_2, x_3) \in B$ must satisfy the global balance equation

$$\int_B r \, dV - \int_{\partial B_2} q_2 \, dA = 0, \qquad \text{if} \quad \partial B_2 = \partial B \,. \tag{3.3}$$

The definition of the thermal energy corresponding to the thermal intensity field $\mathbf{t}$ on $\bar{B}$ is as follows [1]:

$$E_K\{\mathbf{t}\} = \frac{1}{2} \int_B \mathbf{t} \cdot \mathbf{K} \cdot \mathbf{t} \, dV \,. \tag{3.4}$$

The heat flux energy corresponding to the heat flux vector $\mathbf{q}$ on $\bar{B}$ is defined as [1]:

$$E_R\{\mathbf{q}\} = \frac{1}{2} \int_B \mathbf{q} \cdot \mathbf{R} \cdot \mathbf{q} \, dV \,, \tag{3.5}$$

where $\mathbf{R} = \mathbf{R}(x_1, x_2, x_3)$ is the inverse matrix of $\mathbf{K} = \mathbf{K}(x_1, x_2, x_3)$ (the thermal resistivity tensor [1, 2, 7]). Of course, $\mathbf{R} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{R} = \mathbf{1}$ and $\mathbf{R}$ is symmetric and positive definite second-order tensor field on $\bar{B}$. If $\mathbf{t}$ and $\mathbf{q}$ satisfy the Fourier law (1.3) then we have $E_K\{\mathbf{t}\} = E_R\{\mathbf{q}\}$.

The aim of this section is to establish the upper and lower bounds for the thermal energy computed from the solutions of the above mentioned types of heat conduction problems.

THEOREM 5. *Let U_1 be the thermal energy computed from the solution of the boundary value problem whose boundary conditions are specified in Eq. (3.1), and for it the internal heat sources vanish. In this case, we have*

$$\frac{\left(\int_{\partial B_1} \mathbf{c} \cdot \mathbf{n} \, \Theta_1 dA \right)^2}{\int_B \mathbf{c} \cdot \mathbf{R} \cdot \mathbf{c} \, dV} \leq U_1 \leq \int_B \nabla F \cdot \mathbf{K} \cdot \nabla F \, dV \,, \tag{3.6}$$

where $F = F(x_1, x_2, x_3)$ *is a sufficiently smooth scalar field defined on* $\bar{B}$ *satisfying the "temperature" boundary condition*

$$F(x_1, x_2, x_3) = \Theta_1(x_1, x_2, x_3) \quad (x_1, x_2, x_3) \in \partial B_1; \tag{3.7}$$

futhermore, $\mathbf{c} = \mathbf{c}(x_1, x_2, x_3)$ *is a sufficiently smooth vector field on* $\bar{B}$ *satisfying the conditions*

$$\nabla \cdot \mathbf{c} = 0 \text{ in } B \quad \text{and} \quad \mathbf{c} \cdot \mathbf{n} = 0 \text{ on } \partial B_2 \quad \text{and} \quad \int_B \mathbf{c}^2 dB \neq 0. \tag{3.8}$$

Equality in (3.6) is reached if

$$F = \Theta \text{ in } \bar{B} \quad \text{and} \quad \mathbf{c} = \lambda \mathbf{q} \text{ in } \bar{B}, \tag{3.9}$$

where λ *is an arbitrary constant different from zero.*

P r o o f. The proof of the upper bound relation is based on the equation

$$\int_B \mathbf{q} \cdot \nabla \Theta \, dV = \int_B \mathbf{q} \cdot \nabla F \, dV. \tag{3.10}$$

Putting in Eq. (2.3) $\mathbf{v} = \mathbf{q}$ and $U = F$, we get

$$\begin{aligned} \int_B \nabla F \cdot \mathbf{q} \, dB &= \int_{\partial B} F \mathbf{q} \cdot \mathbf{n} \, dA - \int_B F \nabla \cdot \mathbf{q} \, dV = \int_{\partial B_1} F \mathbf{q} \cdot \mathbf{n} \, dA \\ &= \int_{\partial B_1} \Theta \mathbf{q} \cdot \mathbf{n} \, dA = \int_{\partial B} \Theta \mathbf{q} \cdot \mathbf{n} \, dA \\ &= \int_B \nabla \Theta \cdot \mathbf{q} \, dV + \int_B \Theta \nabla \cdot \mathbf{q} \, dV = \int_B \nabla \Theta \cdot \mathbf{q} \, dV, \end{aligned}$$

thus the relation (3.1) is valid. The application of the Fourier law (1.3) and the definition of $\mathbf{t}$ gives

$$\int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla \Theta \, dV = \int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla F \, dV. \tag{3.11}$$

The combination of the Eq. (3.11) with the Schwarz inequality

$$\left(\int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla F \, dV \right)^2 \leq \int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla \Theta \, dV \int_B \nabla F \cdot \mathbf{K} \cdot \nabla F \, dV. \tag{3.12}$$

yields the upper bound relation formulated in (3.6).

In order to prove the lower bound relation formulated in (3.6) we substitute into Eq. (2.3) $\mathbf{v} = \mathbf{c}$ and $U = \Theta$. This substitution yields

$$\int_B \mathbf{c} \cdot \nabla\Theta \, dV = \int_{\partial B_1} \Theta_1 \mathbf{c} \cdot \mathbf{n} \, dA . \tag{3.13}$$

By application of the Schwarz inequality we can write

$$\left(\int_B \mathbf{c} \cdot \nabla\Theta \, dV \right)^2 = \left(\int_B \mathbf{c} \cdot \mathbf{R} \cdot \mathbf{q} \, dV \right)^2 \leq \int_B \mathbf{c} \cdot \mathbf{R} \cdot \mathbf{c} \, dV \int_B \mathbf{q} \cdot \mathbf{R} \cdot \mathbf{q} \, dV . \tag{3.14}$$

From Eq. (3.13) and the inequality relation (3.14) we obtain the lower bound formula of the relation (3.6).

THEOREM 6. *Let U_2 be the thermal energy computed from the solution of the boundary value problem whose boundary conditions are specified in Eq. (3.2). In this case, we have*

$$\frac{\left(\int_{\partial B_2} f q_2 \, dA - \int_B r f \, dV \right)^2}{\int_B \nabla f \cdot \mathbf{K} \cdot \nabla f \, dV} \leq U_2 \leq \int_B \mathbf{C} \cdot \mathbf{R} \cdot \mathbf{C} \, dV , \tag{3.15}$$

where $f = f(x_1, x_2, x_3)$ is a sufficiently smooth scalar field defined on $\bar{B}$ and it satisfies the conditions

$$f = 0 \quad \text{on } \partial B_1 \quad \text{and} \quad \int |\nabla f|^2 \, dV \neq 0; \tag{3.16}$$

futhermore $\mathbf{C} = \mathbf{C}(x_1, x_2, x_3)$ is a sufficiently smooth vector field defined on $\bar{B}$ satisfying the conditions

$$-\nabla \cdot \mathbf{C} + r = 0 \quad \text{in } B \quad \text{and} \quad \mathbf{C} \cdot \mathbf{n} = q_2 \quad \text{on } \partial B_2. \tag{3.17}$$

Equality in (3.15) can be reached only if

$$f = \lambda\Theta \quad \text{in } \bar{B} \quad \text{and} \quad \mathbf{C} = \mathbf{q} \quad \text{in } \bar{B} , \tag{3.18}$$

where λ is an arbitrary constant different from zero.

P r o o f. Putting in Eq. (2.3) $\mathbf{v} = \mathbf{q}$ and $U = f$, we obtain

$$\int_B \mathbf{q} \cdot \nabla f \, dV = \int_{\partial B} f \mathbf{q} \cdot \mathbf{n} \, dA - \int_B f \nabla \cdot \mathbf{q} \, dV = \int_{\partial B_2} f q_2 \, dA - \int_B r f \, dV. \tag{3.19}$$

We note here that

$$\int_B \mathbf{q} \cdot \nabla f dV = \int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla f dV. \tag{3.20}$$

From the Schwarz inequality we get

$$\left(\int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla f dV \right)^2 \leq \int_B \nabla \Theta \cdot \mathbf{K} \cdot \nabla \Theta \, dV \int_B \nabla f \cdot \mathbf{K} \cdot \nabla f dV. \tag{3.21}$$

The combination of Eqs. (3.19), (3.20) with inequality (3.21) gives the lower bound relation formulated in (3.15).

To prove the upper bound formulated in the relation (3.15), we start from Eq. (2.3). Putting in (2.3) $\mathbf{v} = \mathbf{C}$, $U = \Theta$, we obtain

$$\begin{aligned} \int_B \mathbf{C} \cdot \nabla \Theta \, dV &= \int_{\partial B} \Theta \mathbf{n} \cdot \mathbf{C} \, dA - \int_B \Theta \nabla \cdot \mathbf{C} \, dV = \int_{\partial B} \Theta \mathbf{n} \cdot \mathbf{q} \, dA - \int_B \Theta \nabla \cdot \mathbf{q} \, dV \\ &= \int_B \nabla \Theta \cdot \mathbf{q} \, dV + \int_B \Theta \nabla \cdot \mathbf{q} \, dV - \int_B \Theta \nabla \cdot \mathbf{q} \, dV, \end{aligned}$$

thus, we have

$$\int_B \mathbf{C} \cdot \nabla \Theta \, dV = \int_B \mathbf{q} \cdot \nabla \Theta \, dV. \tag{3.22}$$

Equation (3.22) can be written in the form

$$\int_B \mathbf{C} \cdot \mathbf{R} \cdot \mathbf{q} \, dV = \int_B \mathbf{q} \cdot \mathbf{R} \cdot \mathbf{q} \, dB. \tag{3.23}$$

The combination of Eq. (3.23) with the Schwarz inequality

$$\left(\int_B \mathbf{C} \cdot \mathbf{R} \cdot \mathbf{q} \, dV \right)^2 \leq \int_B \mathbf{C} \cdot \mathbf{R} \cdot \mathbf{C} \, dV \int_B \mathbf{q} \cdot \mathbf{R} \cdot \mathbf{q} \, dV \tag{3.24}$$

leads to the upper bound formulated in (3.15).

We know that, in the Schwarz inequality used to prove the relations (3.6) and (3.15), the sign of equality is valid when the functions appearing in them are *not* linearly independent. This fact and the structure of the boundary value problems considered determine these cases when the equality holds in (3.6) and in (3.15).

The upper bounds in (3.6) for the case $\partial B_1 = \partial B$ and in (3.15) for the case $\partial B_2 = \partial B$, were derived by the use of principles of "minimum potential thermal energy" and "minimum complementary heat flux energy" in [1].

4. Examples

4.1. Example 1. Concentrated heat sources.

The solid body under the action of two concentrated "inner" heat sources located at points P_1 and P_2 is considered. The boundary surface of the body is free from the heat flux, $q_s = 0$ on ∂B, and we have

$$r = Q\left[\delta\left(\mathbf{p} - \mathbf{p}_2\right) - \delta\left(\mathbf{p} - \mathbf{p}_1\right)\right] \qquad \mathbf{p}_i = \mathbf{0P}_i \quad (i = 1,\ 2)\ , \tag{4.1}$$

where the symbol $\delta(...)$ denotes the Dirac function.

The application of the formula (2.4) gives the result

$$\langle \mathbf{q} \rangle = -\frac{Q}{V}\left(\mathbf{p}_2 - \mathbf{p}_1\right) \tag{4.2}$$

which shows that the mean heat flux vector is parallel to the vector $\mathbf{P_1P_2} = \mathbf{p_2} - \mathbf{p_1}$.

4.2. Example 2. Hollow body subjected to surface heat flux.

The body considered is bounded by the closed surfaces A_1 and A_2. The closed surface A_1 is "inner" and the closed surface A_2 is the "outer" boundary surface of body B. It is assumed that the inner heat source r is given, and on the whole boundary of B which is $\partial B = A_1 U A_2$ the surface heat flux is known, that is

$$q_s = q_1 \quad \text{on} \quad A_1 \quad \text{and} \quad q_s = q_2 \quad \text{on} \quad A_2\,. \tag{4.3}$$

The relation between r, q_1 and q_2 is as follows (global heat balance equation):

$$\int_{A_1} q_1\, dA + \int_{A_2} q_2\, dA = \int_B r\, dV. \tag{4.4}$$

The mean heat flux vector in this problem is

$$\langle \mathbf{q} \rangle = \frac{1}{V}\left(\int_{A_1} \mathbf{p} q_1\, dA + \int_{A_2} \mathbf{p}\, q_2 - \int_V r\mathbf{p}\, dA\right). \tag{4.5}$$

Let us consider the case $r = 0$ in B and assume that q_1, q_2 are constants. For this case from Eqs. (4.4), (4.5) we obtain

$$\langle \mathbf{q} \rangle = \frac{A_2}{V} q_2 (\mathbf{p}_2 - \mathbf{p}_1) = \frac{A_2}{V} \mathbf{G}_1 \mathbf{G}_2, \tag{4.6}$$

where

$$\mathbf{0G}_1 = \frac{1}{A_i} \int_{A_i} \mathbf{p}\, dA\,, \qquad (i = 1, 2)\,. \tag{4.7}$$

4.3. Example 3. Heat conduction in cylindrical body.

This example illustrates the application of the relation (2.6). The body considered is cylindrical bounded by a cylindrical, surface which is A_3 and two planes normal to A_3 at $x_3 = 0$ and $x_3 = L$. The end cross-sections of the cylindrical bar-like body are A_1 at $x_3 = 0$ and A_2 at $x_3 = L$. The equation of A_3 is $c(x_1, x_2) = 0$ and $0 \leq x_3 \leq L$. The following problem of steady-state heat conduction is analysed:

$$\Theta(x_1, x_2, 0) = T_1(x_1, x_2) \qquad \text{on } A_1, \tag{4.8}$$

$$\Theta(x_1, x_2, L) = T_2(x_1, x_2) \qquad \text{on } A_2, \tag{4.9}$$

$$\mathbf{q} \cdot \mathbf{n} = Q(x_1, x_2, x_3) \quad \text{on } A_3. \tag{4.10}$$

In Eqs. (4.8), (4.9), (4.10) T_1, T_2 and Q are given and it is assumed that the "internal" heat source is known. The form of the thermal conductivity tensor K is as follows:

$$\begin{aligned} \mathbf{K}(x_1, x_2, x_3) &= k_{11}(x_1, x_2, x_3)\, \mathbf{e}_1 \circ \mathbf{e}_1 + k_{22}(x_1, x_2, x_3)\, \mathbf{e}_2 \circ \mathbf{e}_2 \\ &+ k_{12}(x_1, x_2, x_3)\, (\mathbf{e}_1 \circ \mathbf{e}_2 + \mathbf{e}_2 \circ \mathbf{e}_1) \\ &+ k_{33}(x_1, x_2, x_3)\, \mathbf{e}_3 \circ \mathbf{e}_3. \end{aligned} \tag{4.11}$$

We introduce the heat flux resultants at the end cross-sections A_1, A_2 by the definition

$$Q_i = \int_{A_i} \mathbf{q} \cdot \mathbf{n}_i\, dA \qquad (i = 1, 2)\,. \tag{4.12}$$

We note $\mathbf{n}_1 = -\mathbf{e}_3$ and $\mathbf{n}_2 = \mathbf{e}_3$. The outer normal vector on the boundary surface segment A_3 is

$$\mathbf{n} = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2, \qquad \left(n_1^2 + n_2^2 = 1\right)\,. \tag{4.13}$$

Let Q_3 be defined as

$$Q_3 = \int\limits_{\partial A_2} \left(\int\limits_0^L Q\left(x_1, x_2, x_3\right) dx_3 \right) ds, \tag{4.14}$$

where ∂A_2 is the boundary curve of the cross-section A_2 and s is an arc coordinate defined on ∂A_2.

The global balance equation for the body B bounded by the surface $\partial B = A_1 U A_2 U A_3$ is

$$Q_1 + Q_2 + Q_3 - R = 0, \tag{4.15}$$

where

$$R = \int\limits_B r\left(x_1, x_2, x_3\right) dV. \tag{4.16}$$

In Eq. (4.15) Q_1 and Q_2 are unknown. Their values can be computed by the use of relation (2.6). From Eq. (2.6) we get

$$\int\limits_{\partial B} \Theta \mathbf{e}_3 \cdot \mathbf{K} \cdot \mathbf{n}\, dA = \int\limits_B x_3 r dV - \int\limits_{\partial B} x_3 q_s\, dA\,. \tag{4.17}$$

We define the following quantities:

$$R_1 = \int\limits_B x_3 r\, dV, \tag{4.18}$$

$$Q_4 = \int\limits_{\partial A_2} \left(\int\limits_0^L x_3 Q\, dx_3 \right) ds, \tag{4.19}$$

$$\bar{k}_1 = \frac{1}{A} \int\limits_{A_1} k_{33}(x_1, x_2, 0) dA, \tag{4.20}$$

$$\bar{k}_2 = \frac{1}{A} \int\limits_{A_2} k_{33}(x_1, x_2, L) dA, \tag{4.21}$$

$$\overline{T}_1 = \frac{1}{A\bar{k}_1} \int_{A_1} k_{33}(x_1, x_2, 0) T_1(x_1, x_2) dA, \tag{4.22}$$

$$\overline{T}_2 = \frac{1}{A\bar{k}_2} \int_{A2} k_{33}(x_1, x_2, L) T_2(x_1, x_2) dA, \tag{4.23}$$

$$A = \int_{A_1} dA = \int_{A_2} dA. \tag{4.24}$$

Introducing the quantities defined above into the Eq. (4.13) we obtain

$$Q_2 = -\frac{\bar{k}_2 \overline{T}_2 - \bar{k}_1 \overline{T}_1}{L} A - \frac{Q_4}{L} + \frac{R_1}{L}. \tag{4.25}$$

For the case $r = 0$ in B, and $\mathbf{n} \cdot \mathbf{q} = 0$ on A_3, Eq. (4.25) gives

$$Q_2 = -Q_1 = -k \frac{\overline{T}_2 - \overline{T}_1}{L} A, \qquad \text{if } \bar{k}_1 = \bar{k}_2 = k. \tag{4.26}$$

We mention that the heat conduction problem determined by the boundary conditions (4.8), (4.9) and (4.10) is a three-dimensional boundary value problem which, in general, does not have the solution in a closed form.

4.4. Example 4. Bounds for the thermal energy.

Homogeneous isotropic body is bounded by two similar ellipsoids of revolution described by the surfaces $\nu = \nu_1$ and $\nu = \nu_2$ $(0 < \nu_1 < \nu_2)$, where

$$\nu^2 = x_1^2 + x_2^2 + \left(\frac{x_3}{\alpha}\right)^2, \qquad \alpha \geq 1. \tag{4.27}$$

The "inner" boundary surface of body B is A_1 and on it $\nu = \nu_1$, the "outer" boundary surface of body B is A_2 and on it $\nu = \nu_2$. The inner heat sources vanish, $r = 0$ in B, and we have the next "temperature" boundary condition

$$\Theta_1 = T = \text{const on } A_1, \tag{4.28}$$

$$\Theta_1 = 0 \text{ on } A_2. \tag{4.29}$$

The function

$$F(x_1, x_2, x_3) = T \frac{\nu_1}{\nu_2 - \nu_1} \left(\frac{\nu_2}{\nu} - 1\right) \tag{4.30}$$

satisfies all conditions which are prescribed in (3.6) for $F = F(x_1, x_2, x_3)$. Since the hollow body considered is isotropic and homogeneous, the thermal conductivity tensor has the form $\mathbf{K} = k\mathbf{1}$ ($k = \text{const}$) and we have

$$\int_B \nabla F \cdot \mathbf{K} \cdot \nabla F dV = k \int_B |\nabla F|^2 \, dV.$$

A simple calculation gives

$$U_1 \leq T^2 \frac{4\pi k}{3\left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right)} \left(2\alpha + \frac{1}{\alpha}\right). \tag{4.31}$$

To obtain the lower bound for U_1, we use in (3.6) the divergence-free vector field $\mathbf{c} = \mathbf{c}(x_1, x_2, x_3)$

$$\mathbf{c} = \frac{\mathbf{p}}{p^3} \qquad \left(p = \sqrt{x_1^2 + x_2^2 + x_3^2}\right). \tag{4.32}$$

It is very easy to show that

$$\int_{\partial B_1} \mathbf{c} \cdot \mathbf{n} \Theta_1 dA = T \int_{A_1} \frac{\mathbf{p} \cdot \mathbf{n}}{p^3} dA = 4\pi T, \tag{4.33}$$

$$\int_B \mathbf{c} \cdot \mathbf{R} \cdot \mathbf{c} \, dV = \frac{1}{k} \int_B \frac{dV}{p^4} = 4\pi h(\alpha) \left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right) \frac{1}{k}, \tag{4.34}$$

where

$$h(\alpha) = \frac{1}{2}\left(\alpha \operatorname{arc\,tg} \alpha + \frac{\sqrt{\alpha^2 - 1}}{\alpha}\right) \text{ for } \alpha > 1 \text{ and } h(\alpha) = 1 \text{ for } \alpha = 1. \tag{4.35}$$

The combination of (3.6) with Eqs. (4.33), (4.34) leads to the lower bound

$$U_1 \geq \frac{4\pi k}{\left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right) h(\alpha)} T^2. \tag{4.36}$$

The bounds (4.31) and (4.36) for $\alpha = 1$ give the same result; in this case the boundary surfaces A_1 and A_2 are concentric spheres with their centers at the origin of the coordinate system and ν denotes the distance from the origin.

5. Conclusions

The main purpose of the present paper was to show how one can use the analogy which exists between the linear elasticity and heat conduction. The formulation of the mean value theorems of steady-state heat conduction problems follows the formulation of mean displacement, and the mean stress theorem done by GURTIN [9] in linear elasticity.

The upper and lower bounds for the heat flux are derived by the application of Schwarz inequality, avoiding the application of the minimum principles of potential thermal energy and complementary heat-flux energy which were developed by WOJNAR [1].

Examples 1 and 2 present the computation for two problems of the mean heat flux vectors.

Example 3 discusses the heat conduction in a bar-like body by the application of equation based on the concept of mean thermal intensity and the mean heat flux vector.

Example 4 gives two side bounds for the thermal energy of a hollow body bounded by two similar ellipsoids of revolution. In this example, the test functions applied can be used for the case of nonhomogeneous anisotropic bodies which have shape as the body in Example 4.

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