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Partial material replacement without stress redistribution

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A PARTIAL MATERIAL replacement causes stress redistribution in comparison to the original structure made of a homogeneous material. The article presents a possibility to design a geometry of the replacement which keeps the state of stress unchanged. It is shown that for a class of two-dimensional configurations, it is possible to find a solution of this problem. Conditions for the existence of an appropriate geometry are given. A method to obtain the shape of the replaced part is proposed.

1. Introduction

WE CONSIDER AN ELASTIC body initially made of one material and subjected to a single load system. Then some part of the body is replaced by a part of the same shape but made of another material. This partial material replacement usually causes essential stress redistribution in comparison to the same structure made of one material. The choice of the shape of a bimaterial interface significantly influences the redistribution. Defining any norm (measure) of the stress change, we can consider a function which maps the set of all possible interfaces into the value of the stress redistribution norm. Considering the lowest possible value of this norm it is clear that since the norm is nonnegative, the lowest (theoretical) value is zero. The physical interpretation of this value requires the existence of an interface which does not cause the stress redistribution in the whole domain. We are going to analyze the possibility of existence of the replacement shape which preserves the stress state unchanged.

Such a problem seems to lead to a contradiction. The solution of a Boundary Value Problem (BVP) of elasticity is dependent on the distribution of material properties. The change of mechanical properties in any part of the domain should lead to significant changes in the BVP solution. Considering the problem of the body with a replaced part from the viewpoint of mechanics, it is obvious that the stress redistribution depends on the mechanical properties of materials, the geometry of the body, the shape of a material interface, the applied loads and kinematic boundary conditions.

We are going to show in the paper that, even if the properties of the original and new material are given and the geometry of the domain is determined, there may exist a shape of the interface which does not lead to any change of stress distribution. We will show cases when such an interface exists. Then we will try to establish a simple method of finding such an interface for a limited class of problems.

The idea of this research was introduced by the analysis of biomechanical problems related to the stability of an implant–bone system. An inserted implant (the replacement) causes stress redistribution which initiates the process of bone material adaptation known as *remodeling*. This process causes essential structural changes in the bone material and these changes belong to important factors limiting the implant service time [3]. The majority of known remodeling theories assume that the process is caused by stress/strain redistribution [7]. The most effective way to avoid an unwanted effect is to prevent its cause. A design which would cause no redistribution of stress would prevent any mechanically induced remodeling and eliminate one of the reasons of implant loosening. However, at this stage of development, the presented theory is of a purely theoretical significance for biomechanics, since assumptions adopted in this paper are too restrictive for their practical application.

Apart from biomechanics, the presented results can be interesting for any problem where a bimaterial interface occurs and a single load system dominates. At the bimaterial interface high stress concentrations occur [6]. The existence of such a special shape of the interface which does not cause the stress redistribution allows one to obtain a structure made of two materials but maintaining the same stress state as the structure made of one material, and therefore free of any stress concentrations caused by the bimaterial interface.

Finally, what is most important, this solution presents some unexpected properties of elasticity which contradict our belief and is interesting *per se*.

2. Problem formulation and trivial solutions

2.1. Primary problem

We define a primary problem considering a body occupying domain $\mathcal{D}$, with prescribed boundary conditions: kinematic on $\partial\mathcal{D}_u$ of boundary $\partial\mathcal{D}$ and static on part $\partial\mathcal{D}_\sigma$, as depicted in Fig. 1. The linear elastic material properties are described by the tensor E_{ijkl}.

This is the classical BVP of the theory of elasticity. The problem is well-posed and can be solved for the unknown fields of displacements u_i^I, stresses σ_{ij}^I and strains ε_{ij}^I.

In what follows we assume that the solution of the primary problem i.e. fields u_i^I, σ_{ij}^I and ε_{ij}^I are known.

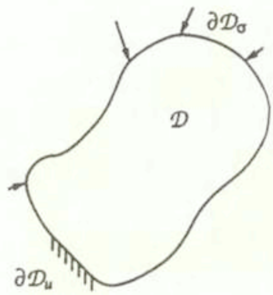

FIG. 1. Primary problem

2.2. Secondary problem

Let us modify the primary problem assuming the existence of an interface Γ which forms two subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$ of domain $\mathcal{D}$, as shown in Fig. 2.

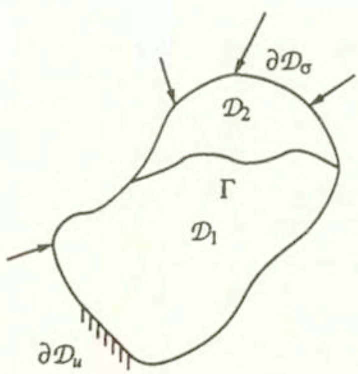

FIG. 2. Secondary problem

The material properties in subdomain $\mathcal{D}_1$ remain the same as in the primary problem, while in subdomain $\mathcal{D}_2$ the material properties are described by another elasticity tensor $\tilde{E}_{ijkl}$.

Kinematic boundary conditions are prescribed on boundary part $\partial\mathcal{D}_{1u}$ of subdomain $\mathcal{D}_1$ and static boundary conditions are prescribed on $\partial\mathcal{D}_{1\sigma}$. Only static boundary conditions are prescribed on boundary $\partial\mathcal{D}_2$ of subdomain $\mathcal{D}_2$.

This new BVP will be called a secondary problem. The secondary problem is still well-posed and can also be solved for the fields: displacements u_i^{II}, stresses σ_{ij}^{II} and strains ε_{ij}^{II}. One primary problem can generate a series of secondary problems with different locations of Γ. Each of the secondary problems has a solution u_i^{II}, σ_{ij}^{II}, ε_{ij}^{II}.

We expect that the solution of a secondary problem is highly dependent on the location of Γ. It is also expected that the introduction of another material causes redistribution of stresses, i.e. everywhere in $\mathcal{D}$

$$\Delta\sigma_{ij} = \sigma_{ij}^{II} - \sigma_{ij}^{I} \neq 0. \tag{2.1}$$

2.3. Redistribution problem

Our objective is to find a special case of the secondary problem with such a shape of interface Γ which gives no stress redistribution

$$\Delta\sigma_{ij} = 0 \tag{2.2}$$

in the whole domain $\mathcal{D}$ provided that such a line Γ exists. This is equivalent to the condition:

$$\sigma_{ij}^{II} = \sigma_{ij}^{I}. \tag{2.3}$$

In what follows we will refer to this problem as the problem of stress redistribution or shortly, as the problem of redistribution.

In order to distinguish the solution of the secondary problem from its special case when condition Eq. (2.3) is met, another notation is introduced. For the solution of the stress redistribution problem we denote the solution of the secondary problem in subdomain $\mathcal{D}_1$ (where material properties are E_{ijkl}) by u_i, σ_{ij}, ε_{ij} and in subdomain $\mathcal{D}_2$ (where material properties are $\tilde{E}_{ijkl}$) by $\tilde{u}_i, \tilde{\sigma}_{ij}, \tilde{\varepsilon}_{ij}$.

Due to the properties of Eq. (2.3) we have: $\sigma_{ij} \equiv \sigma_{ij}^{I}$ in $\mathcal{D}_1$, $\tilde{\sigma}_{ij} \equiv \sigma_{ij}^{I}$ in $\mathcal{D}_2$ and therefore we could use σ_{ij}^{I} as the distribution of stresses in the solution of the redistribution problem, but we will use the notation σ_{ij} and $\tilde{\sigma}_{ij}$ in order to emphasize which subdomain is considered.

It should be noted that for a known shape of Γ, the verification of Eq. (2.3) can be performed by a direct solution of the primary and secondary problems.

2.4. Trivial solutions

It is unbelievable that the redistribution problem may possess a solution. The solution of the problem which satisfies condition Eq. (2.3) contradicts our experience. In order to prove that the considered problem is solvable, a few simple solutions should be presented prior to the formulation of a consistent theory.

Let us consider a rectangular panel subjected to pure tension in plane stress conditions. The panel is made of two linear, elastic, isotropic and homogenous materials with a material interface perpendicular to the axis of tension, as shown in Fig. 3. In order to preserve the pure tension condition, we assume that the kinematic boundary conditions at the left-hand edge of the panel should allow for free expansion in direction y.

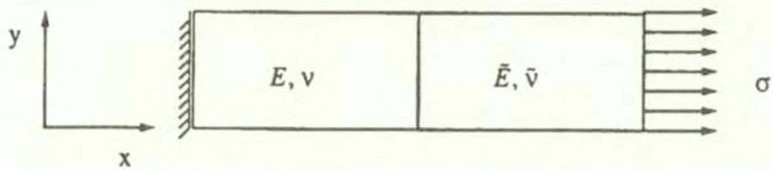

FIG. 3. Panel with straight interface

Let us virtually separate the panel along the material interface and assume pure tension in both parts. Then both parts of the panel deform independently. The elongation of each part in the direction of tension is governed by Young's modulus. In the direction perpendicular to the axis of tension, the value of Poisson's ratio governs the width change of each part. The strain value in the direction perpendicular to the axis of tension (in the system of coordinates shown in Fig. 3) is given by:

$$\varepsilon_{yy} = -\frac{\nu}{E}\sigma \tag{2.4}$$

where the values of E and ν are taken for the relevant material.

If both sides of the interface shrank in the same way, the displacements along both sides of the interface would be compatible and both parts would deform in the same way, either separately or as a whole. Therefore we can join them back because both the displacement continuity and equilibrium are satisfied and the complete solution of BVP is obtained.

The condition of equal transverse shrinking of both parts leads to the following relation between the material constants:

$$\frac{\tilde{E}}{E} = \frac{\tilde{\nu}}{\nu}. \tag{2.5}$$

The above solution can be viewed as a solution of the redistribution problem while the primary problem would concern the same panel made of one material. This gives us an example of a material replacement without stress redistribution, but in fact this problem is different from that originally specified, since it imposes conditions on the material properties instead of conditions on the shape of the interface.

In order to find an interface shape which gives no redistribution after material replacement, it is sufficient to consider a straight but inclined interface in the previously considered example, see Fig. 4. It could be shown that it is sufficient to consider the condition of equal elongations of both sides of the interface. After rewriting the strain tensor in a local set of coordinates chosen in such a way that the first local axis ξ coincides with the direction of the interface, the elongation of the interface is given by:

$$\varepsilon_{\xi\xi} = \varepsilon_{xx}\cos^2\alpha + \varepsilon_{yy}\sin^2\alpha. \tag{2.6}$$

where α is the angle between the global axis x and the local axis ξ.

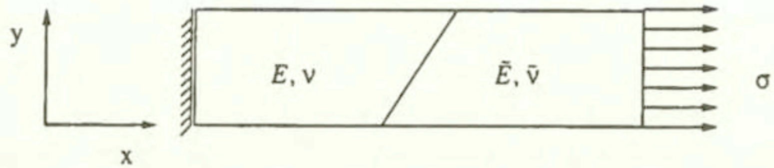

FIG. 4. Panel with inclined interface

The condition of equal elongation of both sides of the interface leads to the following formula for the inclination angle:

$$\tan^2\alpha = \frac{\tilde{E} - E}{\tilde{E}\nu - E\tilde{\nu}}. \tag{2.7}$$

It can be verified that the solution of the primary and the secondary problem with the inclination angle given by Eq. (2.7) yields the same stress field. This means that for a given primary problem such as the panel made of one material subjected to pure tension, one can replace a part of this panel with another material and, as long as the interface is a straight line with an inclination given by Eq. (2.7), there will be no stress redistribution. This is a simple example of a solution of the main problem considered in this paper. It is also a proof that the problem has a solution at least in one case.

It is worth noting that, while in the secondary problem the stress field is homogeneous, the field of strain is not. There is a jump of strains along the material interface. As a consequence of this, the deformation is non-symmetric, as depicted in Fig. 5.

It has been shown that a solution of the problem exists in the case of the pure tension. Another simple example could also be demonstrated, the case of

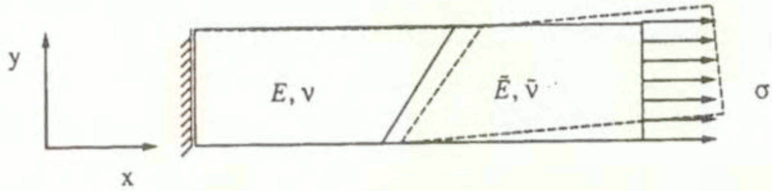

FIG. 5. Undeformed (solid line) and deformed (dashed line) shape of panel with inclined interface

uniform tension (compresion) state of stress where a solution does not exist (the simple proof will be given later in this paper).

At this point it is clear that the problem of stress redistribution can possess a solution but it is also possible that even in a simple case, such a solution may not exists. Therefore, it is difficult to find a condition which gives us information about the existence of the solution for a given case.

Instead of searching for the universal condition which asserts existence, in the following part we focus our attention on a more practical problem, how to find the interface location. We assume that this interface exists and try to establish the conditions which allow us to find its location.

3. Interface location

The problem of finding the shape of such a replacement in the most general case is far too difficult to consider. In this paper we simplify the problem essentially (and therefore restrict the validity of our solution) by adopting the following assumptions:

(a) displacements and strains are small,

(b) plane stress conditions are adopted,

(c) materials are homogeneous, isotropic and linear elastic, therefore tensor E_{ijkl} is completely defined by Young's modulus E and Poisson's ratio ν, and the tensor $\tilde{E}_{ijkl}$ by values $\tilde{E}$ and $\tilde{\nu}$,

(d) body forces are neglected,

(e) a single load is applied to the structure,

(f) the interface of materials is perfectly bonded,

(g) the replacement is external, which means that a part of the structure made from the new material includes a boundary of the domain $\partial\mathcal{D}_2 \cap \partial\mathcal{D} \neq \emptyset$; in other words, the replacement is not completely surrounded by the body,

(h) there are no kinematic boundary conditions placed on the part of boundary which belongs to the replacement $\partial\mathcal{D}_{2\sigma} = \partial\mathcal{D}_2$.

We restrict our aim to establishing the theoretical formulation. The numerical application of the presented idea is not straightforward and has been described separately [5].

The stress redistribution problem leads to searching for the unknown interface line Γ. The formulation of the problem provides us a simple condition for verification if a particular line is an actual solution of the redistribition problem (by means of condition (2.3)) but it does not give us any practical method of searching for the interface location.

In the following part of the paper we will try to anlyse the consequences of the existence of the stress redistribution problem in order to find an auxiliary condition which helps us to determine the interface location.

It is useful to consider the problem as given in two subdomains. The solution of the whole redistribution problem can be split into two smaller problems given in two subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$ defined by line Γ. Then we have two BVPs and in addition, we have to specify the interface conditions which couple the solutions of the subproblems to obtain a proper solution of BVP in whole domain domain $\mathcal{D}$.

Interface conditions [1] require:
continuity of displacements along interface Γ:

$$u_i = \tilde{u}_i; \tag{3.1}$$

equilibrium of interface

$$\sigma_{ij}n_j = \tilde{\sigma}_{ij}n_j. \tag{3.2}$$

Standard interface conditions of type Eq. (3.1) and Eq. (3.2) introduce $\tilde{u}_i$, u_i, $\sigma_{ij}n_j$ and $\tilde{\sigma}_{ij}n_j$ as unknowns of the problem at each point of Γ. None of these variables is prescribed along Γ and there are six independent components of the stress tensors, while formula Eq. (3.2) gives only two equations which relate them.

For the redistribution problem the stresses along the interface are not unknown, since stresses should be the same as in the primary problem everywhere in $\mathcal{D}$ and therefore also along Γ. Then, in addition to Eq. (3.2), because of condition Eq. (2.3) we also have

$$\sigma_{ij}n_j = \sigma^I_{ij}n_j. \tag{3.3}$$

Therefore, the interface equilibrium Eq. (3.2) is satisfied as an identity, but because of condition Eq. (3.3), the value of $\sigma_{ij}n_j$ and $\tilde{\sigma}_{ij}n_j$ along Γ becomes prescribed. We obtain prescribed static boundary conditions at the boundary Γ of both subdomains.

Now the problem can be seen as two, almost uncoupled, BVPs with prescribed tractions along Γ (as shown in Fig. 6) and with an additional condition given by Eq. (3.1).

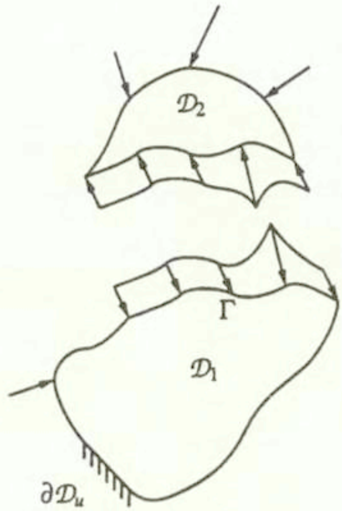

FIG. 6. Problem separated into two problems in subdomains

In subdomain $\mathcal{D}_1$ a separate BVP can be formulated, with prescribed tractions Eq. (3.3) along Γ. The material is the same as in the primary problem, therefore the whole solution of BVP is identical to the solution of the primary problem in subdomain $\mathcal{D}_1$. The field of stress σ_{ij} is the same as σ^I_{ij}, the field of strain ε_{ij} is the same as ε^I_{ij} and the field of displacement u_i is the same as u^I_i. The last statement leads to the conclusion that displacements along Γ are also known, so we can write in addition to Eq. (3.1) that along Γ:

$$u_i = u^I_i. \tag{3.4}$$

Next, we can consider the second subdomain. The boundary of $\mathcal{D}_2$ is formed by part $\partial\mathcal{D}_{2\sigma}$ and by interface Γ. At boundary $\partial\mathcal{D}_{2\sigma}$, the tractions are prescribed. At boundary Γ the static boundary conditions are also prescribed, since substituting Eq. (3.3) into Eq. (3.2) we obtain:

$$p_i = \tilde{\sigma}_{ij} n_j = \sigma^I_{ij} n_j. \tag{3.5}$$

Moreover, at Γ also the kinematic boundary conditions are prescribed. Substituting Eq. (3.4) into Eq. (3.1) we obtain:

$$\tilde{u}_i = u^I_i. \tag{3.6}$$

Therefore, in the second subdomain we have a BVP with both kinematic and static boundary conditions prescribed along Γ. For a problem of linear elasticity, exactly one type of boundary conditions should be prescribed. When both types are specified, it can (and usually will) lead to a contradiction and can form an ill-posed problem.

The problem formulated in subdomain $\mathcal{D}_1$ is an example of a BVP problem which can formally be specified with an excessive number of boundary conditions and which does not lead to a contradiction. Formally, in this BVP problem in $\mathcal{D}_1$ we can have along Γ given tractions Eq. (3.3) and displacements Eq. (3.4), but a solution satysfying both conditions exists.

This is the unique possibility. For prescribed tractions there is exactly one function of displacements which gives the proper solution. Conversely, for prescribed displacements there is exactly one function of tractions which does not lead to contradiction.

The application of the same reasoning to $\mathcal{D}_2$ leads to an alternative method of verification if a given Γ is a solution of the redistribution problem. We can specify one of boundary conditions Eq. (3.5) or Eq. (3.6) along Γ, solve the BVP in $\mathcal{D}_2$ and check if the second condition is satisfied. Then with both equilibrium and continuity along Γ satisfied, we can join both subdomains to form one body and obtain a complete solution which satisfies Eq. (2.3).

When the interface line Γ is unknown, we can assume that one of these boundary conditions is specified and use the other one to determine the shape of Γ if it exists. The choice which boundary condition is prescribed is arbitrary. In the following considerations tractions along Γ are specified.

Now, we focus our attention on the virtually separated subdomain $\mathcal{D}_2$. Taking assumption (h) into account, the BVP in $\mathcal{D}_2$ is given in stresses. Because of assumptions (a)–(d), the stress distribution in such a problem is independent of material properties of the body. It is then clear that the state of stress $\tilde{\sigma}_{ij}$ is the same as the state of stress σ^I_{ij} in the solution of the primary problem in subdomain $\mathcal{D}_2$, while the same stress $\tilde{\sigma}_{ij}$ with new material of properties $\tilde{E}$, $\tilde{\nu}$ gives a different strain $\tilde{\varepsilon}_{ij}$. It is also clear that the field of displacements will change, however displacements for the secondary problem in $\mathcal{D}_2$ can be obtained with an accuracy corresponding to unknown rigid body motions since the problem is given in stresses.

This leads us to another formulation of the redistribution problem:

> Find such a line Γ defining subdomain $\mathcal{D}_2$ that for given stresses σ^I_{ij} and material properties $\tilde{E}$, $\tilde{\nu}$ we can satisfy prescribed displacements Eq. (3.6) along Γ.

4. Displacements along Γ

Now, we focus our attention on the separated BVP given in $\mathcal{D}_2$ with an additional condition of prescribed displacement Eq. (3.6) which has to be satisfied along part Γ of the boundary.

We analyse the consequences of this problem in order to find the unknown shape of Γ in the redistribution problem.

Equation (3.6) is formulated in terms of displacements and is not convenient for further analysis. A better possibility is given by rewriting this equation in terms of displacement differentials. The displacement along any curve C can be expressed as

$$u_i(x_C) = u_i(x_{C_0}) + \int\limits_{x_{C_0}}^{x_C} du_i. \tag{4.1}$$

Applying Eq. (4.1) to Γ it can be stated that, assuming the existence of one point $x_0 \in \Gamma$ where equation Eq. (3.6) is satisfied, it is *equivalent* to compare the displacement differentials along Γ:

$$d\tilde{u}_i = du_i^I. \tag{4.2}$$

For any field, the differentials of displacements can be expressed in terms of derivatives using the chain rule, and can be expressed in terms of small strain and small rotation tensors. In an extended form it gives (all superscripts are dropped):

$$\begin{aligned} du_1 &= \varepsilon_{11}dx_1 + \varepsilon_{12}dx_2 + \omega_{12}dx_2, \\ du_2 &= \varepsilon_{12}dx_1 + \varepsilon_{22}dx_2 - \omega_{12}dx_1. \end{aligned} \tag{4.3}$$

For the plane stress conditions the tensor of small rotation ω_{ij} has only two nonzero components, each of the same absolute value but with an opposite sign, so it can be described by a single scalar value ω:

$$\omega = \omega_{12} = -\omega_{21}. \tag{4.4}$$

Rearranging equation Eq. (4.2) and taking into account Eqs. (4.3) and Eq. (4.4) we obtain:

$$\begin{aligned} (\varepsilon_{11} - \tilde{\varepsilon}_{11})\, dx_1 + (\varepsilon_{12} - \tilde{\varepsilon}_{12} + \omega - \tilde{\omega})\, dx_2 &= 0, \\ (\varepsilon_{12} - \tilde{\varepsilon}_{12} - \omega + \tilde{\omega})\, dx_1 + (\varepsilon_{22} - \tilde{\varepsilon}_{22})\, dx_2 &= 0. \end{aligned} \tag{4.5}$$

Now, we introduce an additional notation in order to simplify the equations. We define the difference of strains as:

$$e_{ij} = \varepsilon_{ij} - \varepsilon_{ij}. \tag{4.6}$$

It should be noted that e_{ij} depends on the properties of both materials. In a similar manner we define the difference of rotations:

$$\Omega = \omega - \tilde{\omega}. \tag{4.7}$$

Rewriting the set of Eqs. (4.5) with the new notation we obtain:

$$\begin{aligned} &e_{11}dx_1 + (e_{12} + \Omega)\, dx_2 = 0. \\ &(e_{12} - \Omega)\, dx_1 + e_{22}dx_2 = 0. \end{aligned} \tag{4.8}$$

The set of Eqs. (4.8) is a system of linear homogeneous equations with unknowns dx_1 and dx_2. Such a system always possesses the trivial solution with the unknowns equal to zero. It has a nontrivial solution if the determinant of the system is equal to zero. This happens only if value of Ω satisfies the condition:

$$\Omega^2 = e_{12}^2 - e_{11}e_{22}. \tag{4.9}$$

Since Eq. (3.6) and its equivalent form Eq. (4.8) are well–defined only along Γ, the unknowns dx_1 and dx_2 define increments along Γ. Then, the local slope of Γ at a given point is:

$$\frac{dx_2}{dx_1} = -\frac{e_{11}}{e_{12} + \Omega} = \frac{\Omega - e_{12}}{e_{22}}. \tag{4.10}$$

The rightmost term of Eq. (4.10) is valid except for the case $e_{22} = 0$. In order to handle properly any case of e_{ij}, we have to examine special cases:

1. If $e_{22} = 0$ and $e_{12} \neq 0$, then we obtain inclination

$$\frac{dx_2}{dx_1} = -\frac{e_{11}}{2e_{12}}. \tag{4.11}$$

2. If $e_{22} = 0$ and $e_{12} = 0$, then we obtain a vertical line

$$dx_1 = 0. \tag{4.12}$$

It should be noted that Eq. (4.10) (and its special cases) is a first order differential equation which can be used for finding Γ if proper initial conditions can be specified. This requires that at least a single point belonging to Γ is known.

4.1. Remarks

Equation (4.9) gives us a value of Ω squared. This makes it necessary to determine when the RHS of Eq. (4.9) is nonnegative. The sign of the RHS of Eq. (4.9) depends on the state of stress and material constants of both materials. This makes it dependent on seven variables: three components of stress tensor $\tilde{\sigma}_{ij}$ and four material constants. Such a function is far too complex to be analysed. However, the properties of this function with respect to the stress state can be examined. In order to simplify the analysis we consider the stress tensor in its principal directions. This does not restrict the generality of our considerations since the existence of a solution does not depend on the choice of a coordinate system.

After rewriting equation Eq. (4.9) in terms of principal stresses, the condition for the nonnegative value of RHS of Eq. (4.9) leads to the following inequality:

$$0 \geq [k\sigma_1 - \sigma_2]\,[k\sigma_2 - \sigma_1] \tag{4.13}$$

where

$$k = \frac{\tilde{E} - E}{\tilde{E}\nu - E\tilde{\nu}}. \tag{4.14}$$

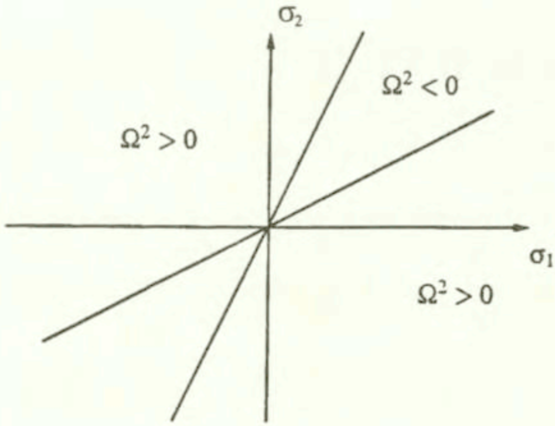

FIG. 7. Existence of solution on the plane of principal stresses

Inequality Eq. (4.13) splits the plane σ_1, σ_2 into four parts as depicted in Fig. 7. In the part where condition Eq. (4.13) is not satisfied, the solution does not exist. This means that such a point of domain can not belong to Γ. When at a given point the value of Ω can be computed, the existence of Γ is not guaranteed yet and must be investigated using further conditions.

It is clear from Fig. 7 that in the case of uniform tension or compression ($\sigma_1 = \sigma_2$) the solution does not exist.

Equation (4. 6) defines a tensor of strain differences e_{ij}. The RHS of Eq. (4. 9) is the value of the second invariant of e_{ij} taken with a negative sign. This makes it clear that condition Eq. (4. 9) is independent of the choice of a coordinate system.

4.2. Interpretations

Now, we review some interpretations of the obtained formulae Eq. (4. 9) and Eq. (4. 10). By definition Eq. (4. 7) Ω is a difference between the value of rotation in the primary problem and the value of rotation in the secondary problem. The rotation tensor component ω can be interpreted as the angle of rotation of an infinitesimally small segment between the undeformed and deformed state. Therefore, in case of the secondary problem, any curve crossing interface which is smooth before deformation, exhibits a slope discontinuity at Γ after deformation.

The physical interpretation of the equations describing the continuity of displacements along the material interface is also worth examining. It becomes quite clear after assuming that Eqs. (4. 8) are written in a local set of coordinates with the first axis coinciding with the tangent to the curve Γ ($dx_2 = 0$). The first Eq. $(4.8)_1$ then becomes a condition of equal elongation of both sides of the interface. The second one becomes a condition of equal displacements in the direction perpendicular to the interface. This condition is dependent on Ω, therefore the value of Ω is responsible for a gap or overlap.

4.3. Final condition

We now extend the meaning of Eq. (4. 9). This equation must be satisfied along the material interface Γ. It has been pointed out that the RHS of Eq. (4. 9) can be computed everywhere in domain $\mathcal{D}$. Then, it is useful to introduce a new field $\bar{\Omega}$ such that:

$$\bar{\Omega}^2 = e_{12}^2 - e_{11}e_{22} \tag{4.15}$$

in $\mathcal{D}$. The value of $\bar{\Omega}^2$ can be computed from the solution of the primary problem without knowing Γ. Then, condition Eq. (4. 9) can be rewritten as:

$$\Omega^2 = \bar{\Omega}^2 \tag{4.16}$$

and must be satisfied along Γ.

We note that the sign of $\bar{\Omega}$ computed from Eq. (4. 15) is in fact undetermined. Since along Γ the sign of Ω should remain the same, setting the sign at any arbitrary point determines the whole solution. This is the reason why both possible signs of $\bar{\Omega}$ should be examined as separate solutions. Then we will rewrite Eq. (4.16) in the form:

$$\Omega = \bar{\Omega} \tag{4.17}$$

to be satisfied along Γ, keeping in mind that it actually shows two possibilities

$$\bar{\Omega} = \pm\sqrt{e_{12}^2 - e_{11}e_{22}}. \tag{4.18}$$

Let us look at the definition of Ω given by equation Eq. (4.7). The value of the rotation, either ω or $\tilde{\omega}$, is not independent of the strain field. It is given by the following relation [2]:

$$\omega = \omega_0 + \int (\varepsilon_{11,2} - \varepsilon_{12,1})dx_1 + (\varepsilon_{12,2} - \varepsilon_{22,1})dx_2. \tag{4.19}$$

This means that for a given strain field the whole field of rotation is determined over the domain (or subdomain) except for the unknown value of ω_0. The field of ω is determined by the solution of the primary problem since the value of ω_0 can be computed from boundary conditions on $\partial\mathcal{D}_u$. The field of $\tilde{\omega}$ is not completely defined, since the value of $\tilde{\omega}_0$ is unknown. This value can easily be computed from the condition of displacement continuity if at least one point of curve Γ is known. In the case of unknown Γ and the secondary problem given in stresses, the value of $\tilde{\omega}$ is determined with a parameter. This means that at any point that is supposed to belong to Γ, the value of $\tilde{\omega}_0$ can be chosen to satisfy condition Eq. (4.17). Then, at any other point of Γ the value of $\tilde{\omega}$ is fixed.

Thus, the field of Ω over the whole domain is determined with a single unknown scalar parameter. This is the reason why a directional derivative of Ω is used. Since condition Eq. (4.17) has to be satisfied only along Γ, the increments of both sides of Eq. (4.17) should be equal along Γ:

$$\frac{d\Omega}{ds} = \frac{d\bar{\Omega}}{ds}. \tag{4.20}$$

After expanding both sides of equation Eq. (4.20) according to the chain rule we obtain:

$$\frac{\partial\Omega}{\partial x_1}dx_1 + \frac{\partial\Omega}{\partial x_2}dx_2 = \frac{\partial\bar{\Omega}}{\partial x_1}dx_1 + \frac{\partial\bar{\Omega}}{\partial x_2}dx_2. \tag{4.21}$$

After substituting the definition of $\bar{\Omega}$ Eq. (4.15) and its derivatives, taking into account Eqs. (4.7), (4.19) and rearranging Eq. (4.21), we obtain the following condition to be satisfied along Γ:

$$\left(\bar{\Omega}_{,1} - e_{11,2} + e_{12,1}\right)dx_1 + \left(\bar{\Omega}_{,2} - e_{12,2} + e_{22,1}\right)dx_2 = 0. \tag{4.22}$$

The relation between dx_1 and dx_2 is determined from the local slope of Γ and is given by Eq. (4.10)

$$e_{22}\left(\bar{\Omega}_{,1} - e_{11,2} + e_{12,1}\right) + \left(\bar{\Omega}_{,2} - e_{12,2} + e_{22,1}\right)\left(\bar{\Omega} - e_{12}\right) = 0. \tag{4.23}$$

In special cases not covered by Eq. (4.10) equation (4.22) leads to other expressions:

1. If $e_{22} = 0$ and $e_{12} \neq 0$, then we use Eq. (4.11)

$$2e_{12}\left(\bar{\Omega}_{,1} - e_{11,2} + e_{12,1}\right) - \left(\bar{\Omega}_{,2} - e_{12,2} + e_{22,1}\right)e_{11} = 0. \tag{4.24}$$

2. If $e_{22} = 0$ and $e_{12} = 0$, then we use Eq. (4.12)

$$\bar{\Omega}_{,2} - e_{12,2} + e_{22,1} = 0. \tag{4.25}$$

It should be noted that condition (4.23) depends on the derivatives of the strain field. This is the reason why for a homogeneous state of stress this condition is satisfied as an identity and then the solution is given by a straight line which can be obtained from condition Eq. (4.10).

In order to simplify the notation we define a function F as:

$$F = \begin{cases} e_{22}\left(\bar{\Omega}_{,1} - e_{11,2} + e_{12,1}\right) + \left(\bar{\Omega}_{,2} - e_{12,2} \right. \\ \quad \left. + e_{22,1}\right)\left(\bar{\Omega} - e_{12}\right) & \text{if } e_{22} \neq 0, \\ 2e_{12}\left(\bar{\Omega}_{,1} - e_{11,2} + e_{12,1}\right) + \\ \quad -\left(\bar{\Omega}_{,2} - e_{12,2} + e_{22,1}\right)e_{11} & \text{if } e_{22} = 0 \text{ and } e_{12} \neq 0, \\ \bar{\Omega}_{,2} - e_{12,2} + e_{22,1} & \text{if } e_{22} = 0 \text{ and } e_{12} = 0. \end{cases} \tag{4.26}$$

With this definition of function F we can simply write conditions Eq. (4.23), Eq. (4.24), and Eq. (4.25) as:

$$F = 0. \tag{4.27}$$

We have shown that if some line Γ is the sought interface then along this line, condition (4.27) has to be satisfied. However we have not shown that the inverse is true. In fact, it can be shown by just one example that it is not. Therefore, the condition (4.27) is only the necessary condition, but lines along which condition (4.27) is satisfied are the only possible locations of the sought interface. As a consequence of this, we have reduced the problem of stress redistribution from searching among the infinite number of possible locations of Γ to a finite number of its locations.

4.4. Practical use of derived condition

The solution of the primary problem gives us stresses $\tilde{\sigma}_{ij}$ as functions of coordinates. The values of e_{ij} can be obtained from stresses $\tilde{\sigma}_{ij}$ which determine ε_{ij} and $\tilde{\varepsilon}_{ij}$. The value of $\tilde{\varepsilon}_{ij}$ was originally defined in $\mathcal{D}_2$ only. However, one can compute $\tilde{\varepsilon}_{ij}$ from σ^I_{ij} everywhere in $\mathcal{D}$, although it can be meaningless in the neighbourhood of the kinematic boundary conditions. This makes it possible to obtain e_{ij} without the prior knowledge of Γ. Then the field(s) of $\bar{\Omega}$ and its derivatives can be computed. Substituting all these functions into Eq. (4.26), a parametric equation for F can be obtained:

$$F\left(e_{ij}(x,y), e_{ij,k}(x,y), \bar{\Omega}(x,y), \bar{\Omega}_{,i}(x,y)\right) = F(x,y). \tag{4.28}$$

It is quite difficult and apparently not necessary to solve this parametric equation. The most practical way to obtain the location of lines where condition Eq. (4.27) is satisfied is to draw contour lines of the surface given by F over the entire domain.

When drawn, the contour lines of zero level show us all possible locations of Γ.

The absence of the zero level contour informs us that a solution does not exist. When one or more zero level contours appear, it is neccessary to verify each one separately. In fact, we should consider two such surfaces F^+ and F^-, according to the choice of the sign in Eq. (4.18)

This approach is very efficient. In fact we need to solve a simple linear static problem (primary), then compute the values and draw the contours of F^+ and F^- and finally verify by direct computation every line satisfying Eq. (4.27). Prior to this verification we can exclude some lines when they violate assumptions (g) or (h).

5. Example

The formula (4.27) gives a solution for the problem of finding the shape of Γ. We will now show an example of a solution predicted by Eq. (4.27). This example is a case of an inhomogeneous state of stress, in order to actually verify condition Eq. (4.27) rather than Eq. (4.10). We show a simple example obtained in a rather artificial way. We find an example which is a solution of the redistribution problem and then compute the functions given by Eq. (4.27) in order to check if we obtain the correct solution.

We assume the shape of Γ and determine the stress which satisfies Eq. (4.27). For the sake of simplicity, the shape of Γ is chosen as a straight line described by the function:

$$y = x - 1. \tag{5.1}$$

We will seek the solution in a class of linearly varying fields of stress. After lengthy but simple derivations, one of the possible stress fields is found as:

$$\begin{aligned} \sigma_{11} &= -2137x, \\ \sigma_{22} &= 1645x + 3362y, \\ \sigma_{12} &= -3572 - 3362x + 2137y. \end{aligned} \tag{5.2}$$

The following material constants are adopted: $E = 20$ GPa, $\nu = 0.1$, $\tilde{E} = 210$ GPa, $\tilde{\nu} = 0.3$. After computing the displacements for the primary problem and the secondary problem it is verified that the displacements along the material interface are equal, which makes us sure that it is the complete solution of the redistribution problem.

Therefore, any part of plane xy (finite or not) can be considered as a primary problem as long as we guarantee that the stresses inside are given by Eq. (5. 2). Then, the static boundary conditions at the appropriate part of the boundary can be obtained from Eq. (5. 2). The kinematic boundary conditions (if any) have to be determined from the displacements resulting from Eq. (5. 2), but again the rigid body motion is arbitrary. Then the solution of the redistribution problem is given by Eq. (5. 1) provided that the chosen domain contains any part of this line and the chosen kinematic boundary condition does not violate assumption (h).

In order to demonstrate the application of the presented approach, the domain $0 \leq x \leq 2$ and $0 \leq y \leq 4$ is chosen arbitrarily. The static boundary conditions are prescribed at all boundaries. In addition to this, one can assume appropriate pointwise supports at lower corners of the rectangle in order to prevent the rigid body motions. They do not influence the presented example, however they might be essential in case of (numerical) verification of the obtained solution.

Then, we can verify the validity of formula (4. 27), by computing F using stresses given by Eq. (5. 2). After applying the procedure described previously, the surfaces created by functions $F(x, y)$ are drawn over the domain. The values of $F^{-}(x, y)$ are negative everywhere in this domain and the solution does not exist. Figure 8 (left) shows the contour lines of function $F^{+}(x, y)$. In Fig. 8 (right) only the zero contours are left. This shows us two lines satisfying Eq. (4. 27). Then we should verify each line separately, by solving the secondary problem, if it is the actual solution of the redistribution problem. It should be noted that the known solution of the redistribution problem (5. 1) has been detected by this approach.

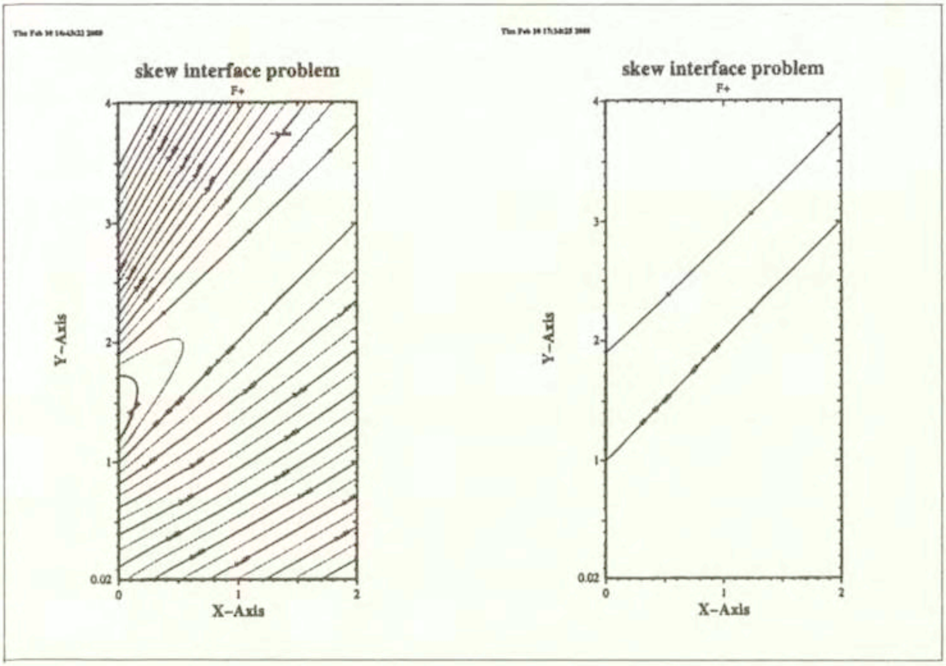

FIG. 8. Plot of contour lines of $F^+(x,y) = \text{const}$ (left) and contours of zero level (right)

6. Conclusions

In this paper we have formulated the problem of stress redistribution, i.e. the problem of finding a shape of a partial material replacement which preserves the same stress as in the body before replacement. We have shown some trivial examples which prove that this problem may have a solution. The paper demonstrates how significantly we can influence the redistribution caused by a bimaterial interface. It is also a proof that the solution of such a problem does not violate the fundamental laws of mechanics and, moreover, it contradicts the common belief that a bimaterial interface has to lead to stress redistribution, as long as the material properties of both materials are different.

Then we have discussed the question of searching for such a specific material interface. For a simplified problem we have derived the neccessary condition for the existence of a solution. This converts the problem of searching among an infinite number of interface locations to the verification of finite number of lines. We have demonstrated that the use of this condition is a practical method of finding that interface. However, the proposed method of solution has obvious limitations. The application of this approach is strongly limited by the set of adopted assumptions.

The article presents the ultimate approach to the problem of redistribution: we can obtain no redistribution or no solution at all. In practice, a partial solution of the problem focused on the reduction of redistribution could be equally worthwhile.

The redistribution problem brings some side effects, such like undesired deformations. The significance of these effects can be assessed in case of a given application, since the same effects may be irrelevant for one problem while they can disqualify the solution for other problems.

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