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On material objectivity and reduced constitutive equations

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THE PRINCIPLE OF MATERIAL frame indifference, as it is usually stated, actually consists of two distinct assumptions. Firstly, that the stresses transform like objective tensors under change of the observer, and secondly, that the constitutive equations do not depend on the observer (form-invariance). As a consequence, superimposed rigid body motions also do not effect the material response. In the present work these three statements are formulated independently. The mutual relations between them can be clearly and generally worked out by group-theoretical concepts. If only two of these principles hold for a certain class of materials, then *reduced forms* exist, i. e. forms of constitutive equations that identically fulfil these principles. A general definition of reduced forms is given, its existence is proven, and a method for their construction is formulated and applied to the case of simple elastic materials.

1. Introduction

IN HIS DISSERTATION published in 1955, NOLL stated that under a change of frame, the stresses transform in an objective manner. By this "principle of isotropy of space", or of "objectivity of material properties" as he called it later, he obtained *reduced forms* for simple materials that identically satisfy this principle. This notion of objectivity became quite popular. It was exploited in all branches of continuum physics where material equations are formulated, in order to reduce them. More recently, TRUESDELL/NOLL (1965 Sec. 19 p. 44 ff.) discussed the history of the concept of *material frame-indifference*. Here, one finds the statement. "In fact, *two* principles have been stated and studied. According to the first, which may be called the "Hooke-Poisson-Cauchy form", constitutive equations must be invariant under a superimposed rigid rotation of the *body*. According to the second, which may be called the "Zaremba-Jaumann form", an arbitrary change of *observer* is allowed." This fact could not be expressed clearer. In contrast to this, however, in the rest of the article, the only distinction between the two versions is attributed to orientation, always positive for the Hooke-Poisson-Cauchy form, and either positive or negative for the Zaremba-Jaumann form.

To our knowledge, the consequences of the distinction between these two approaches or interpretations has apparently never really been considered, or worked out, in detail. One purpose of the current work is to attempt just to do this, building on an earlier work (SVENDSEN and BERTRAM [16]). As it turns out, with the help of precise group-theoretic definitions for change of observers or frames, sometimes called *Euclidean observer transformations*, on the one hand, and for superimposed rigid body motions on the other, one can identify three distinct concepts contained in standard formulations of "objectivity" or "frame-indifference". In a sense, the weakest of the three, *Euclidean frame-indifference* (EFI) requires physical quantities associated with Euclidean observers such as stress and heat flux to transform corrotationally between them. Secondly, *form-invariance* (FI) requires (the *form* of) the constitutive relation to be independent of observer. Lastly, *indifference with respect to superimposed rigid-body motions* (IRBM) requires the material response to be independent of arbitrary rotations of the material body with respect to a single observer. As shown in this work, EFI and FI together represent the concept of *material frame indifference* as stated by Truesdell/Noll. In addition, we show that *material frame indifference* is equivalent to IRBM. More precisely, one can show that any two of the concepts EFI, FI and IRBM imply the third one. Or in other words, if one of these principles holds, the remaining two become equivalent.

As discussed in detail in earlier work (e.g., SVENDSEN and BERTRAM [16]), the concepts of FI and IRBM or *material frame indifference* are constitutive in nature, while EFI represents a general principle, apparently holding for all materials[1)] As such assumptions, IRBM or *material frame indifference* appear to be quit reasonable for most material classes subject to non-extreme (i.e., in terms of acceleration and spin) conditions. As is clear from the work of Noll and others, the exploitation of *material frame indifference* leads to a drastic simplification of or *reduction in* the form of constitutive equations. Indeed, invariance with respect to superimposed rigid body motions or, equivalently, material–frame indifference, lead to *reduced forms*. Neither more, nor less can be obtained[2)].

[1)]The subtlety of the concepts and issues inherent in the notion of "objectivity" has led to a number of disagreements between various authors in the literature. One such disagreement had to do with the material behaviour of rarified gases. Using the results from IKENBERRY/TRUESDELL [6], MÜLLER [8] showed that such gases violate IRBM, or equivalently *material frame indifference*. More recently, MURDOCH [9] attempted to refute Müller's conclusion by showing that such materials actually satisfy *material frame indifference*. Because, like Truesell/Noll before him, he tacitly assumed FI from the start in his treatment, however, he could not have shown this, despite his conclusion to the contrary. Indeed, in effect, what he showed was that such gases satisfy EFI. Note that, in contrast to his successors, NOLL [11, 12] clearly stated "In any system of reference, Galilean or not, the constitutive equations must be the same".

[2)]APPLEBY and KADIANAKIS [1] clearly demonstrate that Euclidean frame-indifference is

A second purpose of the present work is the development of a general representaion for such reduced forms and the formulation of a procedure for their construction. More precisely, with the help of an abstract group-theoretic representation for constitutive equations, we are able to (*i*) define the concept of a *reduced form* in a rather abstract way, (*ii*) show their existence, and (*iii*) give a general procedure to construct them.[3)]

2. Kinematics in the euclidean space

Let $\mathscr{V}$ be the three-dimensional vector-space associated with the Euclidean point space. For brevity, we introduce the following tensor sets:

$\mathscr{L}in$:= the set of all tensors or linear mappings on $\mathscr{V}$

$\mathscr{I}nv$:= the set of all invertible tensors

$\mathscr{S}kw$:= the set of all skew–symmetric tensors

$\mathscr{S}ym$:= the set of all symmetric tensors

$\mathscr{P}sym$:= the set of all symmetric positive-definite tensors

$\mathscr{O}rth$:= the set of all orthogonal tensors.

A subscript + indicates the subset of tensors with positive determinants. Thus, e. g., $\mathscr{O}rth^+$ is the set of *proper* orthogonal tensors. As usual, R stands for the reals. For two sets $\mathscr{A}$ and $\mathscr{B}$, let

- $\mathscr{M}ap(\mathscr{A}, \mathscr{B})$ denote the set of all mappings from $\mathscr{A}$ into $\mathscr{B}$
- $\mathscr{B}ij(\mathscr{A}, \mathscr{B})$ denote the set of all bijections from $\mathscr{A}$ onto $\mathscr{B}$

We now come to the basic Euclidean kinematics. As usual, an **observer** or a *frame of reference* can be represented via a reference point for the position

not enough to obtain reduced forms, by expressing the whole matter in a frame-independent or "intrinsic" way. "It is interesting to note that this approach does not eliminate the need for an invariance principle for equations of state equivalent in effect to the principle of frame-indifference". They use invariance with respect to superimposed rigid-body motions in the form of "invariance under rotations of space time", i. e. essentially the same as we do. This is, however, by no means *equivalent in effect* to Euclidean frame-indifference

[3)] In Wang [20, 21] and in Williams [22], the theories of invariant forms have been suggested, which fulfil the restrictions imposed by these three principles and by an assumed material symmetry at the same time. However, these presentations become rather complicated and not practical. The present suggestion is not based on either of those and takes only into account the universal restrictions and not the individual ones.

vectors and a vector triad. Let $\mathscr{B}$ represent a material body manifold, $\mathscr{T}$ an open time interval, and ξ an observer, and

$$\kappa_\xi \ : \mathscr{B} \times \mathscr{T} \to \mathscr{V}$$

$$(P,t) \mapsto \mathbf{r}_\xi$$

the *motion of the body* $\mathscr{B}$ during $\mathscr{T}$ with respect to ξ, assigning to each material point P and each instant t the position vector in the Euclidean space.

This mapping is subject to certain regularity requirements which depend on the specific context and thus, shall not be specified in general. The same holds for all time-dependent mappings in the rest of this work, without further mention.

As usual, we have the *velocity*

$$\mathbf{v}_\xi(P,t) = \frac{\partial}{\partial t}\kappa_\xi(P,t) = \kappa_\xi(P,t)^\bullet$$

and the *acceleration*

$$\mathbf{a}_\xi(P,t) = \frac{\partial^2}{\partial t^2}\kappa_\xi(P,t) = \kappa_\xi(P,t)^{\bullet\bullet}.$$

Further, the spatial differential of κ_ξ is the linear mapping from the tangent space $\mathscr{T}_P\mathscr{B}$ to the body manifold at P onto the space of the Euclidean shifters

$$\mathbf{K}_\xi(P,t) = d\kappa_\xi(P,t) : \mathscr{T}_P\mathscr{B} \to \mathscr{V}$$

at $(P,\ t)$, called the *local placement* at P and time t in the motion κ_ξ. It is customary, but not necessary (as we know from NOLL [11, 12]), to use a *reference-placement*

$$\kappa_0 \ : \mathscr{B} \to \mathscr{V}$$

$$P \ \mapsto \ \mathbf{X}$$

of the body, and to define the *motion of the body* relative to it by

$$\chi_\xi(X,t) := \kappa_\xi(\kappa_0^{-1}(\mathbf{X}),t).$$

which induces the mapping

$$\chi_\xi \ : \kappa_0(\mathscr{B}) \times \mathscr{T} \to \mathscr{V}$$

$$(\mathbf{X},t) \ \mapsto \ \mathbf{r}_\xi.$$

Note that the time derivatives of χ_ξ and κ_ξ coincide

$$\mathbf{v}_\xi(\kappa_0^{-1}(\mathbf{X}),t) = \frac{\partial}{\partial t}\chi_\xi(\mathbf{X},t),$$

$$\mathbf{a}_\xi(\kappa_0^{-1}(\mathbf{X}), t) = \frac{\partial^2}{\partial t^2}\chi_\xi(\mathbf{X}, t),$$

whereas the corresponding differential, the *deformation gradient*, is according to the chain rule

$$\mathbf{F}_\xi(\mathbf{X}, t) = d\chi_\xi(\mathbf{X}, t) = d\kappa_\xi(d\kappa_0^{-1}(\mathbf{X}), t) = \mathbf{K}_\xi\,\mathbf{K}_0^{-1} \in \mathscr{Inv}.$$

The function

$$\chi_\xi^X(\mathbf{Y}, t) := \chi_\xi(\mathbf{X}, t) + \mathbf{F}_\xi(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X}) \tag{2.1}$$

stands for the motion of an infinitesimal neighborhood of $\mathbf{X}$; in what follows, this will be referred to as an *infinitesimal motion around* $\mathbf{X}$. The determinant

$$\det\ \mathbf{F}_\xi(\mathbf{X}, t) = \frac{\rho_{\xi 0}(\mathbf{X})}{\rho_\xi(\mathbf{X}, t)}$$

relates the mass density in the reference placement to that of the current placement. For invertibility we have det $\mathbf{F}_\xi(\mathbf{X}, t) \neq 0$ at all times and points. So, its sign is either strictly positive *or* negative, but never changes sign. As the choice of the reference placement is arbitrary, one can do it in such a way, that det $\mathbf{F}_\xi > 0$ without loss of generality. The advantage is twofold. Firstly, we avoid the ambiguity in the above expression. Secondly, it is then *possible* (but not necessary) that the body occupies the reference placement at some time during its motion.

Other important kinematic quantities are the spatial *velocity gradient*

$$\mathbf{L}_\xi := \mathbf{F}_\xi^\circ\,\mathbf{F}_\xi^{-1} \in \mathscr{Lin},$$

$$\mathbf{D}_\xi := \frac{1}{2}\left(\mathbf{L}_\xi + \mathbf{L}_\xi^T\right) \in \mathscr{Sym},$$

and its skew-symmetric part, the *spin tensor*

$$\mathbf{W}_\xi := \frac{1}{2}(\mathbf{L}_\xi - \mathbf{L}_\xi^T) \in \mathscr{Skw},$$

such that

$$\mathbf{L}_\xi = \mathbf{W}_\xi + \mathbf{D}_\xi,$$

T denoting the transpose of a tensor.

The deformation gradient can be subjected to the polar decomposition

$$\mathbf{F}_\xi = \mathbf{R}_\xi\mathbf{U}_\xi \in \mathscr{Inv}^+, \quad \mathbf{U}_\xi \in \mathscr{Psym}, \quad \mathbf{R}_\xi \in \mathscr{Orth}^+$$

with the *right stretch tensor* $\mathbf{U}_\xi$ and the *local rotation tensor* $\mathbf{R}_\xi$.

3. Euclidean observers

Again, all these kinematical concepts generally depend on the *Euclidean observer* denoted by the suffix ξ. As usual, any such observer can be represented via a reference point for the position vectors and a vector triad, both appearing time-independent to the observer by definition.

Although each observer has his own reference instant $t = 0$, this plays no role in continuum mechanics, as most concepts are based on time *differences*. Consequently, this effect of change of observer will not be taken into account here. On the other hand, the spatial part of the transformation is of central importance. If we change the observer from ξ to η, the position vectors are subjected to the **Euclidean transformation**

$$\mathbf{r}_\eta = \mathbf{Q}_\eta^\xi \mathbf{r}_\xi + \mathbf{c}_\eta^\xi \tag{3.1}$$

with $\mathbf{Q}_\eta^\xi \in \mathcal{O}rth^+$ and $\mathbf{c}_\eta^\xi \in \mathcal{V}$, both time-dependent. Although it is possible that different observers use different orientations, we assume for simplicity and without loss of generality, that $\det \mathbf{F} > 0$ for all observers. Consequently, all Euclidean transformations are orientation preserving, and $\mathbf{Q}$ is proper orthogonal at all times. We emphasize that $\mathbf{Q}_\eta^\xi$ and $\mathbf{c}_\eta^\xi$ are uniquely determined (as functions of time) by the two involved observers ξ and η, being the same for all bodies and all motions. As such, the pair $E := \{\mathbf{Q}, \mathbf{c}\}$ of time-dependent $\mathbf{Q}$ in $\mathcal{O}rth^+$ and $\mathbf{c}$ in $\mathcal{V}$ completely determine such change of observer or Euclidean transformation.

Now, let η, ξ, ζ, be three observers. Analogously to (3.1), we have the transformations

$$\mathbf{r}_\zeta = \mathbf{Q}_\zeta^\eta \mathbf{r}_\eta + \mathbf{c}_\zeta^\eta$$

and

$$\mathbf{r}_\zeta = \mathbf{Q}_\zeta^\xi \mathbf{r}_\xi + \mathbf{c}_\zeta^\xi$$

between them, which yield

$$\mathbf{Q}_\zeta^\xi = \mathbf{Q}_\zeta^\eta \mathbf{Q}_\eta^\xi$$

and

$$\mathbf{c}_\zeta^\eta = \mathbf{Q}_\zeta^\xi \mathbf{c}_\xi^\eta + \mathbf{c}_\zeta^\xi.$$

For the inverse Euclidean transformation we find

$$\mathbf{Q}_\xi^\eta = (\mathbf{Q}_\eta^\xi)^{-1} = (\mathbf{Q}_\eta^\xi)^T$$

and

$$\mathbf{c}_\xi^\eta = -\mathbf{Q}_\xi^\eta \mathbf{c}_\eta^\xi,$$

or, equivalently,

$$E = \{\mathbf{Q}, \mathbf{c}\} \;\Leftrightarrow\; E^{-1} = \{\mathbf{Q}^T, -\mathbf{Q}^T\mathbf{c}\}.$$

Trivially, there is a neutral Euclidean transformation $I = \{\mathbf{I}, \mathbf{o}\}$ such that $\mathbf{Q}$ is the identity tensor $\mathbf{I}$ and $\mathbf{c}$ is the null-vector $\mathbf{o}$ at all times. Clearly,

$$E \circ E^{-1} = I = E^{-1} \circ E$$

holds for all Euclidean transformations E. Here, $\circ$ denotes the composition of mappings. In fact, by these properties the Euclidean transformations form a group under composition, the **Euclidean group** $\mathscr{G}$.

Euclidean transformations with $\mathbf{c}^{\bullet\bullet} \equiv \mathbf{o}$ and $\mathbf{Q}^{\bullet} = \mathbf{0}$ are called *Galilean transformations* which become important as the invariance groups of balance equations, what is beyond the scope of the present considerations.

Clearly, the set of all observers is equipotent to the set of all time-functions with values in $\mathscr{O}rth^+ \times \mathscr{V}$. Therefore, it is equivalent whether a certain property is (*i*) valid for *one* observer and remains valid under *all* transformations in $\mathscr{G}$, or is (*ii*) valid for *all* observers.

As almost all physical quantities depend on observers, we have to specify the actions of Euclidean transformations on these quantities. For kinematical quantities, these actions can be uniquely derived from (3.1). It is a common practice to introduce the reference placement to be the same for all observers. This is by no means necessary, but it simplifies some transformations without real loss of generality.

The actions of an $E_\xi^\eta \in \mathscr{G}$ on the following quantities are

- on the velocity

$$\mathbf{v}_\eta = \mathbf{Q}_\eta^\xi \mathbf{v}_\xi + \mathbf{Q}_\eta^{\xi\bullet}\, \mathbf{Q}_\eta^\xi (\mathbf{r}_\xi - \mathbf{c}_\eta^\xi) + \mathbf{c}_\eta^{\xi\bullet},$$

- on the acceleration

$$\mathbf{a}_\eta = \mathbf{Q}_\eta^\xi \mathbf{a}_\xi + \mathbf{c}_\eta^{\xi\bullet\bullet} + 2\mathbf{Q}_\eta^{\xi\bullet} \mathbf{v}_\xi + \mathbf{Q}_\eta^{\xi\bullet\bullet} \mathbf{r}_\xi,$$

- on the infinitesimal motion around $\mathbf{X}$

$$\chi_\eta^{\mathbf{X}}(\mathbf{Y}, t) = \mathbf{Q}_\eta^\xi [\chi_\xi(\mathbf{X}, t) + \mathbf{F}_\xi(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X})] + \mathbf{c}_\eta^\xi = \mathbf{Q}_\eta^\xi (\chi_\xi^{\mathbf{X}}(\mathbf{Y}, t)) + \mathbf{c}_\eta^\xi,$$

- on the deformation gradient

$$\mathbf{F}_\eta = \mathbf{Q}_\eta^\xi \mathbf{F}_\xi, \tag{3.2}$$

- on the velocity gradient

$$\mathbf{L}_\eta = \mathbf{Q}_\eta^\xi \mathbf{L}_\xi \mathbf{Q}_\xi^\eta + \mathbf{Q}_\eta^{\xi\bullet} \mathbf{Q}_\xi^\eta,$$

- on the deformation rate

$$\mathbf{D}_\eta = \mathbf{Q}_\eta^\xi \mathbf{D}_\xi \mathbf{Q}_\xi^\eta,$$

- on the spin tensor

$$\mathbf{W}_\eta = \mathbf{Q}_\eta^\xi \mathbf{W}_\xi \mathbf{Q}_\xi^\eta + \mathbf{Q}_\eta^{\xi\bullet} \mathbf{Q}_\xi^\eta,$$

- on the right stretch tensor

$$\mathbf{U}_\eta = \mathbf{U}_\xi,$$

- and on the local rotation tensor

$$\mathbf{R}_\eta = \mathbf{Q}_\eta^\xi \mathbf{R}_\xi.$$

If we also include the mass density, the *temperature* $\theta_\xi \in \mathscr{R}$, and the (spatial) *temperature gradient* $\mathbf{g}_\xi \in \mathscr{V}$, we assume in addition the actions of $\mathscr{G}$

- on the densities

$$\rho_\eta = \rho_\xi; \qquad \rho_{0\eta} = \rho_{0\xi},$$

- on the temperature

$$\theta_\eta = \theta_\xi,$$

- on the temperature gradient

$$\mathbf{g}_\eta = \mathbf{Q}_\eta^\xi \, \mathbf{g}_\xi.$$

Generally speaking, the action a of a group $\mathscr{G}$ on some set $\mathscr{A}$ is a group-(homo) morphism

$$a: \mathscr{G} \to \mathscr{B}ij(\mathscr{A}, \mathscr{A})$$

from $\mathscr{G}$ to the group of all automorphisms (= bijections) of $\mathscr{A}$. This means that it is compatible with the four group axioms

- $a(E_2 \circ E_1) = a(E_2) \circ a(E_1)$
- $a((E_3 \circ E_2) \circ E_1) = a(E_3 \circ (E_2 \circ E_1))$
- $a(I) = I_{\mathscr{A}}$
- $a(E_1^{-1}) = a(E_1)^{-1}$

$\forall \ E_i \in \mathscr{G}$.

By the above examples, we see that the actions of $\mathcal{G}$ on different quantities are in general different. While some of them depend on both $\mathbf{Q}$ and $\mathbf{c}$ (like velocity, acceleration), others do not depend on $\mathbf{c}$ (like all gradients), or depend neither on $\mathbf{Q}$ nor on $\mathbf{c}$ (mass density, temperature). Most, but not all of the above actions are **instantaneous**, i. e., only the momentary values of $\mathbf{Q}$ and $\mathbf{c}$ enter the transformation. To make this clearer, we write the arguments of an example for such an action, namely that of the deformation gradient

$$\mathbf{F}_\eta(\mathbf{X}, t) = \mathbf{Q}_\eta^\xi(t)\mathbf{F}_\xi(\mathbf{X}, t).$$

As a counter example, the action for the spin tensor is not instantaneous in this sense, as also $\mathbf{Q}_\eta^{\xi\bullet}$ enters.

In many cases, there are more actions than just that induced by the unique identity $I = \{\mathbf{I}, \mathbf{o}\}$ in $\mathcal{G}$ that leave some physical quantity unchanged. And for some $E \in \mathcal{G}$, there are often more inverse actions than just the one belonging to $E^{-1} \in \mathcal{G}$. It also happens that certain actions commute, whereas $\mathcal{G}$ is clearly a non-commutative group.

Two kinds of physical quantities are very important for what follows, namely the corrotational ones and the invariant ones. We call a quantity φ **corrotational** (sometimes also called *objective* or *tensorial*) if

$$\varphi_\eta = \varphi_\xi \qquad \text{for scalars,}$$

$$\varphi_\eta = \mathbf{Q}_\eta^\xi \varphi_\xi \qquad \text{for vectors,}$$

$$\varphi_\eta = \mathbf{Q}_\eta^\xi \varphi_\xi \, \mathbf{Q}_\xi^\eta \qquad \text{for tensors,}$$

and **invariant** if

$$\varphi_\eta = \varphi_\xi \qquad \text{in all cases.}$$

Clearly, the actions on corrotational and invariant quantities are instantaneous.

Note that only $\mathbf{Q}_\eta^\xi$ acts on corrotational and invariant quantities. In particular, the translational acceleration $\mathbf{c}_\eta^{\xi\bullet\bullet}$ and the angular velocity $\mathbf{Q}_\eta^{\xi\bullet}$ do not influence corrotational quantities. Examples include:

- for invariant quantities: ρ, θ, $\mathbf{U}$,
- for corrotational quantities: ρ, $\mathbf{g}$, $\mathbf{D}$,

whereas

- $\mathbf{F}$, $\mathbf{R}$ are acted on instantaneously, and
- the others $\mathbf{v}$, $\mathbf{a}$, $\mathbf{L}$, $\mathbf{W}$ are neither corrotational nor instantaneous.

4. Superimposed rigid body motions

Apart from (and independent of) observer changes, another transformation class is very useful in continuum mechanics, namely that of *superimposed rigid body motions*. Here, all quantities of this section are taken with respect to a fixed, but arbitrary observer, if not otherwise stated.

DEFINITION 1. *Let χ be the motion of a body $\mathscr{B}$. Let $\{\boldsymbol{Q}(t),\ \boldsymbol{c}(t)\}$ be functions of time with values in $\mathscr{O}rth^+ \times \mathscr{V}$. Then*

$$\chi^*(\boldsymbol{X},t) = \boldsymbol{Q}(t)\chi(\boldsymbol{X},t) + \boldsymbol{c}(t) \tag{4.1}$$

is called ***superimposed rigid body motion*** *(RBM) of χ.*

Clearly, χ^* is a motion of $\mathscr{B}$ iff χ is. Conversely, then, χ is also a superimposed rigid body motion of χ^*. Mathematically, the superimposed rigid body motions are identical to Euclidean transformations, and thus form the same group $\mathscr{G}$. Physically, however, two observers watching the same motion is something quite different from one observer watching two different motions. This distinction must be kept in mind, even if formally the same notations appear in the formulas.

As a consequence, the actions of RBMs on all kinematical quantities are the same as those of the Euclidean group in the preceding section, if we drop the observer indices ξ and η. If the temperature is considered as a material state property, then the same holds for the temperature and its gradient. For other quantities, however, the actions of Euclidean transformations can be different from those of RBM's, as we will see later.

5. Constitutive equations

In continuum mechanics, it is customary to consider the kinematical quantities as independent variables, and all dynamical ones such as stresses, couples, forces, etc. as dependent ones. If generalized to thermodynamics, motion and temperature are considered as independent, whereas heat flux, energy, entropy, stresses, etc. are taken as dependent.

For elastic materials, only the current values of the variables appear in the constitutive equations. For materials with memory, however, past values can also influence the present values of the dependent variables. In such cases, higher time derivatives, finite kinematic process segments or even the (semi-infinite) history of the motion may appear as arguments in the constitutive equations.

Let $\mathscr{X}$ be a set or space of such **independent variables** of a certain class of materials, and $\mathscr{Y}$ a set of corresponding **dependent variables**. In most cases, the identification of $\mathscr{Y}$ is clear and the same for a broad class of materials.

The set $\mathscr{X}$, however, depends on the specific framework and/or materials under consideration.

For non-polar, purely mechanical behavior, for example, $\mathscr{Y}$ is just the set of all symmetric tensors $\mathscr{Sym}$, each of them being a candidate for the Cauchy stress tensor. On the other hand, for simple materials, $\mathscr{X}$ could be chosen as

- the set of all semi-infinite deformation histories $\mathbf{F}(\tau)$, $-\infty < \tau \leqslant t$;
- the set of all finite deformation processes $\mathbf{F}(\tau)$, $0 \leqslant \tau \leqslant t$.

The easiest case is that of a simple non-polar elastic material. Here, the Cauchy stresses are assumed to depend on the current value of the deformation gradient $\mathbf{F}(t)$, and $\mathscr{X}$ equals the set of all invertible tensors with positive determinant.

For a simple viscous gas or fluid, the stresses depend on the mass density ρ and the current velocity gradient $\mathbf{L}$, and $\mathscr{X}$ is the set-product of the positive reals and second order tensors.

The identification of the spaces of variables is, in general, not a trivial task (see BERTRAM [2, 4]). But it becomes easier to solve, if it is restricted to specific material classes.

With this we can give the notion of a constitutive equation a rather general form.

Principle of Determinism: *For a given class of materials, there exist two sets, $\mathscr{X}$ and $\mathscr{Y}$, and for any observer ξ a* ***constitutive equation*** $f_\xi \in \mathscr{Map}(\mathscr{X}, \mathscr{Y})$.

We assume that a change of observer, represented by an element of $\mathscr{G}$, induces the actions

- on the independent variables

$$a : \mathscr{G} \to \mathscr{Bij}(\mathscr{X}, \mathscr{X}) \mid E \quad \mapsto \quad a_E := a(E);$$

- on the dependent variables

$$b : \mathscr{G} \to \mathscr{Bij}(\mathscr{Y}, \mathscr{Y}) \mid E \mapsto b_E := b(E).$$

If these actions are specified, then the action

- on the constitutive equations

$$c : \mathsf{G} \to \mathscr{Bij}\{\mathscr{Map}(\mathscr{X}, \mathscr{Y}), \mathscr{Map}(\mathscr{X}, \mathscr{Y})\} \mid E \mapsto c_E := c(E)$$

is determined by

$$f_\eta = c_E(f_\xi) := b_E \circ f_\xi \circ a_E^{-1} \qquad \forall\, E \in \mathscr{G}. \tag{5.1}$$

So, if a constitutive equation has once been identified by one observer, then by virtue of EFI it is determined for all other observers.

How can these actions be determined? As the members of $\mathscr{X}$ are kinematical quantities, their transformations can be deduced from (3.1) uniquely. For the dependent variables, however, the action b cannot be deduced from (3.1), but rather is the subject of another principle. In particular, Cauchy stresses, heat flux, internal or free energy, and entropy are assumed to be corrotational. Nobody has ever been able to prove this assumption generally, and it will probably never be possible to do so. Therefore, the following assumption is axiomatic in nature.

EFI: Euclidean frame-indifference (ZAREMBA, 1903; JAUMANN, 1906)[4)] *The dependent variables in $\mathscr{Y}$ are corrotational (or objective) under the action b of the Euclidean group.*

This means for the Cauchy stresses

$$\mathbf{T}_\eta = \mathbf{Q}_\eta^\xi \, \mathbf{T}\xi \, \mathbf{Q}_\eta^{\xi T}, \tag{5.2}$$

for the heat flux

$$\mathbf{q}_\eta = \mathbf{Q}_\eta^\xi \, \mathbf{q}_\xi,$$

and for certain scalar variables φ like internal or free energy, entropy, etc.

$$\varphi_\eta = \varphi_\xi.$$

Clearly, the corrotationality of the dependent variables does not hold for all choices of them. If the Cauchy stresses are corrotational, then the 2. Piola-Kirchhoff stresses are invariant under Euclidean transformations. Therefore, the above form of EFI depends on the specific choice of $\mathscr{Y}$ as spatial ones.

As an example, we consider an *elastic material*, where the independent variables consist of the current deformation gradient $\mathbf{F}$, and the dependent ones of the Cauchy stress tensor $\mathbf{T}$. Hence,

$$\mathscr{X} \equiv Inv^+, \quad Y \equiv \mathscr{S}ym.$$

Then, by (3.2), and the EFI in the form of (5.2), we obtain the transformation

$$f_\eta(\mathbf{F}_\eta) = \mathbf{Q}_\xi^\eta \, f_\xi(\mathbf{Q}_\xi^\eta \mathbf{F}_\eta) \, \mathbf{Q}_\eta^\xi$$

between the constitutive equations via the relative rotation $\mathbf{Q}_\xi^\eta$.

In general, the constitutive function also depends on the observer. Only for certain classes of materials constitutive equations themselves are invariant and the following principle is valid (see BERTRAM [3]).

4) For historical sources see TRUESDELL/NOLL [18], p. 47, App. 19 A, TRUESDELL [19], Ch. 3.

FI: **Form-invariance**[5] *The constitutive equation f is* ***invariant*** *under Euclidean transformations, i. e.*

$$f_\eta \equiv f_\xi \tag{5.3}$$

holds for all observers.

Note that *together*, EFI and FI imply that the induced action c_E (5.1) of the Euclidean group on $\mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ is the identity, i. e.

$$c_E(f_\xi) = f_\xi \qquad \forall\ E \in \mathcal{G}. \tag{5.4}$$

The consequences of this principle will be investigated later.

Let us next consider the actions of superimposed rigid body motions on the constitutive equations. The action of $\mathcal{G}$ on $\mathcal{X}$ resulting from (4.1) is formally the same as that of the Euclidean group. In contrast to EFI, and like FI, the principle to follow does not generally hold, as counterexamples show.

IRBM: **Indifference with respect to superimposed rigid body motions** (Hooke 1678, Poisson 1831, Cauchy 1829)

The dependent variables are corrotational under the action of the superimposed rigid body motions $\mathcal{G}$ (which thus coincides with b).[6]
Since the observer ξ is held fixed, here, this takes the form

$$f_\xi \circ a_E = b_E \circ f_\xi \quad \text{or} \quad c_E(f_\xi) = f_\xi, \tag{5.5}$$

formally analogous to (5.4).

For a mechanical material, this would mean that the Cauchy stresses will simply be rotated together with the body, but not modified otherwise. This condition does not hold for rarified gases under fast rotations (see MÜLLER [8]), what is discussed in another paper SVENDSEN, BERTRAM, [16].

By (5.5), we immediately see the following

PROPOSITION 1. *Let $f_\xi \in \mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ be a constitutive equation which fulfills* IRBM *or* FI . *Then the following implication holds for all Euclidean transformations E :*

$$a_E \ \text{ is the identity on } \mathcal{X} \Rightarrow\ b_E \ \text{ is the identity on } f_\xi(\mathcal{X}).$$

This will be trivially the case if E is the identity. But this is, by no means, the only case.

Now, by comparison of the different invariance principles in the forms (5.1), (5.3), (5.5), we immediately obtain the following

[5] In relativity EINSTEIN [5] called the analogous *Principle of Relativity.*

[6] LEIGH([7], as an exception, introduces this carefully seperated from EFI and FI. See also MUSCHIK [10] and SPEZIALE [15].

PROPOSITION 2. *Let $f_\xi \in \mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ be a constitutive equation. Then the following implications hold:*

- EFI $\wedge$ FI $\Rightarrow$ IRBM : $c_E(f_\xi) = f_\eta \quad \wedge \quad f_\eta = f_\xi \quad \Rightarrow \quad c_E(f_\xi) = f_\xi$,
- EFI $\wedge$ IRBM $\Rightarrow$ FI : $c_E(f_\xi) = f_\eta \quad \wedge \quad c_E(f_\xi) = f_\xi \quad \Rightarrow \quad f_\eta = f_\xi$,
- FI $\wedge$ IRBM $\Rightarrow$ EFI : $f_\eta = f_\xi \quad \wedge \quad c_E(f_\xi) = f_\xi \quad \Rightarrow \quad c_E(f_\xi) = f_\eta$.

In many papers and books, influenced by TRUESDELL/NOLL [18], form-invariance is tacitly assumed as a part of material frame indifference, which is here equivalent to EFI and FI together. As the above proposition shows, EFI and IRBM are then indistinguishable. In such a theory, materials, the response of which is affected by superimposed rigid body motions, cannot be described.

6. Reduced constitutive equations

If a material satisfies EFI, it is sufficient to identify the constitutive equation for one single observer, since by (5.1) it can immediately be transformed to any other Euclidean observer. By this procedure EFI is identically fulfilled, i. e., no further restrictions are imposed on the constitutive equation by EFI.

If in addition to EFI, IRBM or, equivalently, FI holds, the number of candidates for constitutive equations in some $\mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ is drastically reduced. In the rest of this section, this reduction will be worked out. For this purpose, we consider exclusively materials for which both EFI and IRBM are satisfied, so that FI also holds. Let us denote all such constitutive equations in $\mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ by $\mathcal{R}ed(\mathcal{X}, \mathcal{Y})$.

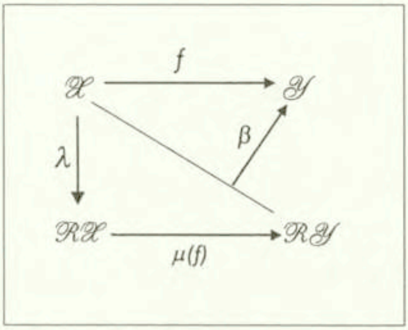
Our aim is to find representations for this set of constitutive equations. For this purpose, we will define two sets $\mathcal{RX}$ and $\mathcal{RY}$, called *reduced sets of independent* and *dependent variables*, respectively, such that any element of $\mathcal{M}ap(\mathcal{RX}, \mathcal{RY})$ uniquely corresponds to an element of $\mathcal{R}ed(\mathcal{X}, \mathcal{Y})$. This means that $\mathcal{R}ed(\mathcal{X}, \mathcal{Y})$ and $\mathcal{M}ap(\mathcal{RX}, \mathcal{RY})$ are isomorphic by means of a natural bijection.

Note that $\mathcal{RX}$ and $\mathcal{RY}$ are not necessarily subsets of $\mathcal{X}$ or $\mathcal{Y}$, respectively, but rather independent. Let us put this in precise terms by the following concept of reduced forms.

DEFINITION 2. *Let $\mathcal{R}ed(\mathcal{X}, \mathcal{Y}) \subset \mathcal{M}ap(\mathcal{X}, \mathcal{Y})$ be as before. Let $\mathcal{RX}$ and $\mathcal{RY}$ be two sets, $\lambda \in \mathcal{M}ap(\mathcal{X}, \mathcal{RX})$ a surjection, $\beta \in \mathcal{M}ap(\mathcal{X} \times \mathcal{RX}, \mathcal{Y})$, and $\mu \in \mathcal{B}ij\{\mathcal{R}ed(\mathcal{X}, \mathcal{Y}), \mathcal{M}ap(\mathcal{RX}, \mathcal{RY})\}$. Then $(\mathcal{RX}, \mathcal{RY}, \lambda, \beta, \mu)$ is called a* **reduced form**, *if*

$$f(x) = \beta[x, \mu(f) \circ \lambda(x)] \qquad \forall\, x \in \mathcal{X}, \ \forall\, f \in \mathcal{R}ed(\mathcal{X}, \mathcal{Y}). \tag{6.1}$$

The practical benefit from such a reduced form is the following. We could pick out any function of $\mathscr{Map}(\mathscr{RX}, \mathscr{RY})$, and this would correspond uniquely (by μ^{-1}) to a constitutive equation in $\mathscr{Map}(\mathscr{X}, \mathscr{Y})$, which automatically satisfies the three principles, and hence is in $\mathscr{Red}(\mathscr{X}, \mathscr{Y})$.



The following construction not only assures that the concept of reduced forms is not vacuous, but also gives a general procedure to construct the reduced forms.

7. Construction of a reduced form

The construction consists of three parts. Firstly, we will construct the reduced spaces of the independent variables, secondly, of the dependent variables and finally, the bijection μ is defined so that the condition (6.1) will be fulfilled.

1. *Construction of the reduced state of independent variables*
   We introduce an equivalence relation $\sim$ on $\mathscr{X}$ in the following way. Let x_1, $x_2 \in \mathscr{X}$, then $x_1 \sim x_2$, if there exists an $E \in \mathscr{G}$, such that $a_E(x_1) = x_2$[7].
   Let $\underline{\mathscr{X}}$ be the quotient space of $\sim$, and $\varepsilon : \mathscr{X} \to \underline{\mathscr{X}}$ be the natural surjection of $\sim$. Moreover, let η be a section of the fiber bundle $\underline{\mathscr{X}}$ (sometimes called a selection function), i.e., an $\eta : \underline{\mathscr{X}} \to \mathscr{X}$, such that $\varepsilon \circ \eta = \mathbf{I}_{\underline{\mathscr{X}}}$, the identity on $\underline{\mathscr{X}}$. In general, η is not unique.
   We define $\mathscr{RX} := (\eta \circ \varepsilon)(\mathscr{X})$, which is clearly a subset of $\mathscr{X}$. Then

$$\lambda := \eta \circ \varepsilon : \mathscr{X} \to \mathscr{RX}$$

   is surjective by definition. Consequently, λ has the following property: $\lambda(x) \sim x, \forall x \in \mathscr{X}$, therefore an $E(x) \in \mathscr{G}$ exists such that $\lambda(x) = a_{E(x)}(x)$.
2. *Construction of the reduced set of dependent variables*
   Let $\mathscr{RY} \equiv \mathscr{Y}$, simply.

[7] This means that x_1 and x_2 lie in the same orbit of a.

3. *Construction of the bijection* μ
Let $f \in \mathscr{Red}(\mathscr{X}, \mathscr{Y})$. We define

$$\mu : \mathscr{Red}(\mathscr{X}, \mathscr{Y}) \rightarrow \mathscr{Map}(\mathscr{RX}, \mathscr{RY})$$

by the restriction

$$\mu : f \mapsto f|_{\mathscr{RX}}$$

and

$$\beta : \mathscr{X} \times \mathscr{RY} \rightarrow \mathscr{Y}$$

by

$$\beta(x, y) := b_{E(x)}^{-1}(y)$$

with the specific $E(x) \in \mathscr{G}$ that gives $\lambda(x) = a_{E(x)}(x)$, and therefore generally depends on x. Then, by IRBM and this $E(x)$,

$$b_{E(x)} \circ f(x) = f \circ a_{E(x)}(x) = f \circ \lambda(x) = f|_{\mathscr{RX}} \circ \lambda(x) \qquad \forall x \in \mathscr{X}$$

and

$$\beta[x, \mu(f) \circ \lambda(x)] = \beta[x, f_{\mathscr{RX}} \circ \lambda(x)] = \beta[x, f \circ a_{E(x)}(x)]$$

$$= \beta[x, b_{E(x)} \circ f(x)] = b_{E(x)}^{-1} \circ b_{E(x)} \circ f(x) = f(x) \qquad \forall x \in \mathscr{X}.$$

The inverse of μ is given by

$$\mu^{-1} : g(x) \mapsto f(x) = \beta[x, g \circ \lambda(x)] \qquad \forall\, x \in \mathscr{X}.$$

We will next show that such $\mu^{-1}(g)$ fulfils the invariance-requirement (5.5) for all $g \in \mathscr{Map}(\mathscr{RX}, \mathscr{RY})$. We take an arbitrary transformation $Q \in \mathscr{G}$. Clearly, $a_Q(x)$ lies in the same orbit as $\lambda(x)$ and x: $\lambda(x) \sim x \sim a_Q(x)$, for all $x \in \mathscr{X}$. Thus

$$\lambda(x) = \lambda \circ a_Q(x).$$

We now evaluate the above expression for $f := \mu^{-1}(g)$ at $a_Q(x)$ and obtain

$$f(a_Q(x)) = \beta[a_Q(x), g \circ \lambda(a_Q(x))] = f \circ a_Q(x) = \beta[a_Q(x), g \circ \lambda \circ a_Q(x)]$$

$$= b_{E^{-1}} \circ g \circ \lambda \circ a_Q(x) = b_{E^{-1}} \circ g \circ \lambda(x)$$

with the specific transformation $\underline{E} := E(a_Q(x)) \in \mathscr{G}$ that gives $\lambda(a_Q(x)) = a_{\underline{E}}(a_Q(x))$. It can easily be seen that $\underline{E} = E(x)\, Q^{-1}$ as

$$\lambda(x) = a_{E(x)}(x) = \lambda \circ a_Q(x) = a_{\underline{E}} \circ a_Q(x) = a_{\underline{E}Q}(x) \qquad \forall\, x \in \mathscr{X}.$$

Therefore

$$b_{\underline{E}} = b_{E(x)Q^{-1}} = b_{E(x)} \circ b_{Q^{-1}} \;\Leftrightarrow\; b_{\underline{E}^{-1}} = b_Q \circ b^{-1}_{E(x)}.$$

We continue with this

$$f \circ a_Q(x) = b_Q \circ b^{-1}_{E(x)} \circ g \circ \lambda(x) = b_Q \circ f(x).$$

Thus, $\mu^{-1}(g) \in \mathscr{Red}(\mathscr{X}, \mathscr{Y})$.

We show that $\mu \circ \mu^{-1}$ is the identity on $\mathscr{Map}(\mathscr{RX}, \mathscr{RY})$.

$$\begin{aligned}\mu \circ \mu^{-1}(g)(x) &= \beta[x, g \circ \lambda(x)]\,|_{\mathscr{RX}} \qquad \forall\, x \in \mathscr{RX}\\ &= b^{-1}_{E(x)} \circ g \circ\; a_{E(x)}(x)\end{aligned}$$

with

$$\lambda(x) = a_{E(x)}(x).$$

In this particular case with the restriction to $\mathscr{RX}$, λ is the identity and so is $a_{E(x)}$. By Proposition 1, $b_{E(x)}$ as well as $b^{-1}_{E(x)}$ are also identities. Thus

$$\mu \circ \mu^{-1}(g)(x) = g(x) \qquad \forall\, x \in \mathscr{RX}, \qquad \forall\, g \in \mathscr{Map}(\mathscr{RX}, \mathscr{RY}).$$

On the other hand

$$\begin{aligned}\mu^{-1} \circ \mu(f)(x) =& \beta[x, \mu(f) \circ \lambda(x)]\\ =& \beta[x, f\,|_{\mathscr{RX}} \circ \lambda(x)]\\ =& \beta[x, f \circ a_{E(x)}(x)]\\ =& \beta[x, b_{E(x)} \circ f(x)]\\ =& b^{-1}_{E(x)} \circ b_{E(x)} \circ f(x)\\ =& f(x) \qquad \forall\, x \in \mathscr{X}, \;\; \forall\, f \in \mathscr{Red}(\mathscr{X}, \mathscr{Y}).\end{aligned}$$

Therefore $\mu^{-1} \circ \mu$ is the identity on $\mathscr{Red}(\mathscr{X}, \mathscr{Y})$, and μ is shown to be a bijection. Thus we have proven the following

Theorem. *Let $\mathscr{Red}(\mathscr{X}, \mathscr{Y})$ be non-empty. Then $(\mathscr{RX}, \mathscr{RY}, \lambda, \beta, \mu)$ is a reduced form.*

Remark 1. In the construction no use has been made of the corrotationality of $\mathscr{Y}$ under the action b. The theorem remains valid for any other action b.

REMARK 2. The construction of this reduced form is based on the choice of η . Apart from trivial cases, this selection function is not unique, and each choice gives rise to a different reduced form.

To illustrate these concepts, a simple and well-known example shall be given next.

EXAMPLE. Let us choose the class of *simple elastic materials*, in which the stresses at a material point are assumed to depend on the current infinitesimal motion around the point. For a fixed $\mathbf{X} \in \mathscr{B}$ and $t \in \mathscr{T}$, χ^X is uniquely determined by the vector $\mathbf{r} = \chi_\xi(\mathbf{X},t) \in \mathscr{V}$ and the tensor $\mathbf{F}_\xi(\mathbf{X}, t) \in \mathscr{I}nv^+$ according to (2.1). The following identifications specify the sets and functions for this class:

- $\mathscr{X} \equiv \mathscr{V} \times \mathscr{I}nv^+$

and

- $\mathscr{Y} \equiv \mathscr{S}ym$.

Each element $\{\mathbf{r}, \mathbf{F}\} \in \mathscr{X}$ stands for an infinitesimal motion, and each element $\mathbf{T} \in \mathscr{S}ym$ for the Cauchy stress tensor. An elastic constitutive equation is then, according to the principle of determinism,

$$f : \mathscr{V} \times \mathscr{I}nv^+ \to \mathscr{S}ym$$

$$\{\mathbf{r}, \mathbf{F}\} \mapsto \mathbf{T}.$$

The action of $\mathscr{G}$ on $\mathscr{X}$ is (see 3.1, 3.2)

$$a_E : \mathscr{I}nv^+ \to \mathscr{I}nv^+$$

$$\{\mathbf{r}, \mathbf{F}\} \mapsto \{\mathbf{Qr} + \mathbf{c}, \mathbf{QF}\} \qquad \text{with } E = \{\mathbf{Q}, \mathbf{c}\},$$

and on $\mathscr{Y}$

$$b_E : \mathscr{S}ym \to \mathscr{S}ym$$

$$\mathbf{T} \mapsto \mathbf{Q}\ \mathbf{T}\ \mathbf{Q}^T$$

with $E = \{\mathbf{Q}, \mathbf{c}\}$, i. e. corrotational. Now $\mathscr{R}ed(\mathscr{X}, \mathscr{Y})$ consists of all $f \in \mathscr{M}ap(\mathscr{X}, \mathscr{Y})$, such that

$$\mathbf{Q} f(\mathbf{r}, \mathbf{F}) \mathbf{Q}^T = f(\mathbf{Qr} + \mathbf{c}, \mathbf{QF}) \qquad \forall \mathbf{c} \in \mathscr{V}, \ \forall \mathbf{Q} \in \mathscr{O}rth^+.$$

We now exemplify the three steps of the above Proof for this class of material.

1. Two pairs $\{\mathbf{r}_1, \mathbf{F}_1\}$, $\{\mathbf{r}_2, \mathbf{F}_2\} \in \mathscr{X}$ are considered as equivalent, if there exists a $\mathbf{c} \in \mathscr{V}$ and a $\mathbf{Q} \in \mathscr{O}rth^+$, such that $\mathbf{r}_2 = \mathbf{Qr}_1 + \mathbf{c}$ and $\mathbf{F}_2 = \mathbf{QF}_1$. The first condition can always be fulfilled. By the polar decomposition $\mathbf{F}_i$

$= \mathbf{R}_i\, \mathbf{U}_i$, $i = 1,2$, $\mathbf{R}_i \in \mathcal{O}rth^+$, $\mathbf{U}_i = (\mathbf{F}_i^T\, \mathbf{F}_i)^{1/2} \in \mathcal{P}sym$, the second one is fulfilled iff $\mathbf{U}_1 = \mathbf{U}_2$, i. e.

$$\{\mathbf{r}_1, \mathbf{F}_1\} \sim \{\mathbf{r}_2, \mathbf{F}_2\} \Leftrightarrow (\mathbf{F}_1^T \mathbf{F}_1)^{1/2} = (\mathbf{F}_2^T \mathbf{F}_2)^{1/2}.$$

Thus, our choice is $E = \{\mathbf{Q} = \mathbf{R}_2\, \mathbf{R}_1^T,\ \mathbf{c} = \mathbf{r}_2 - \mathbf{Q}\mathbf{r}_1\}$. We identify $\mathcal{RX} \equiv \{\mathbf{o}\} \times \mathcal{P}sym \subset \mathcal{X} \equiv \mathcal{V} \times \mathcal{I}nv^+$

$$\lambda :\ \mathcal{V} \times \mathcal{I}nv^+ \to \{\mathbf{o}\} \times \mathcal{P}sym$$

$$\{\mathbf{r}, \mathbf{F}\} \mapsto \{\mathbf{o}, \mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}\},$$

which is clearly surjective. Obviously, $\lambda(\mathbf{r}, \mathbf{F}) = \{\mathbf{o}, \mathbf{U}\} \sim \{\mathbf{r}, \mathbf{F}\}$, so that with $\mathbf{c} \equiv -\mathbf{Q}\ \mathbf{r} \in \mathcal{V}$ and $\mathbf{Q} \equiv \mathbf{R}^T \in \mathcal{O}rth^+$ we obtain

$$\lambda(\mathbf{r}, \mathbf{F}) = \mathbf{a}_{E(x)}(\mathbf{r}, \mathbf{F}) = \{\mathbf{o}, \mathbf{U}\}$$

with

$$E(x) = \{\mathbf{R}^T = (\mathbf{F}^T\mathbf{F})^{-1/2}\mathbf{F}^T, -\mathbf{R}^T\mathbf{r}\} \in \mathcal{G}.$$

Thus $\{\mathbf{o}\} \cup \mathcal{P}sym$ stands for the quotient space $\underline{\mathcal{X}}$, and the selection function η is the inclusion of $\{\mathbf{o}\} \cup \mathcal{P}sym$ in $\mathcal{V} \times \mathcal{I}nv^+$.

2. We take $\mathcal{RY} \equiv \mathcal{S}ym$, whose elements stand for the back-rotated or *relative stress tensor*

(7.1) $$\mathbf{T}_{\rm rel} := \mathbf{R}^T\mathbf{T}\ \mathbf{R}.$$

3. Let

(7.2) $$\beta(\{\mathbf{r}, \mathbf{F}\}, \mathbf{T}_{\rm rel}) := b^{-1}_{E(x)}(\mathbf{T}_{\rm rel}) = \mathbf{Q}^T\mathbf{T}_{\rm rel}\mathbf{Q} = \mathbf{T}$$

and

$$\mu(f) := f\,|_{\{\mathbf{o}\} \times \mathcal{P}sym}.$$

Now we have by (7.1)

$$b_{E(x)} \circ f(\mathbf{r}, \mathbf{F}) = \mathbf{R}^T f(\mathbf{r}, \mathbf{F})\mathbf{R} = f(\mathbf{o}, \mathbf{U}) = f \circ \lambda(\mathbf{r}, \mathbf{F}) = f\,|_{\{\mathbf{o}\} \times \mathcal{P}sym}(\mathbf{o}, \mathbf{U}),$$

and

$$\begin{aligned} \beta[\{\mathbf{r}, \mathbf{F}\}, \mu(f) \circ \lambda(\mathbf{r}, \mathbf{F})] &= \beta[\{\mathbf{r}, \mathbf{F}\}, f\,|_{\{\mathbf{o}\} \times \mathcal{P}sym}(\mathbf{o}, \mathbf{U})] \\ &= \beta[\{\mathbf{r}, \mathbf{F}\}, f(\mathbf{o}, \mathbf{R}^T\mathbf{F})] \\ &= \beta[\{\mathbf{r}, \mathbf{F}\}, \mathbf{R}^T f(\mathbf{r}, \mathbf{F})\mathbf{R}]) = \mathbf{T} = f(\mathbf{r}, \mathbf{F}). \end{aligned}$$

As the first argument $\mathbf{o}$ of $\mu(f)$ is trivial, we can drop it. Hence, IRBM does not allow the stresses to depend on $\mathbf{r}$. Moreover, the dependence of the stresses on $\mathbf{F} = \mathbf{RU}$ is only arbitrary in the stretching part $\mathbf{U}$, but rather specific in the rotational part $\mathbf{R}$.

As mentioned before, the selection function η is not unique, and this is not the only reduced form. One could also have taken the right Cauchy-Green tensor $\mathbf{C} = \mathbf{U}^2$ or the Green tensor $\frac{1}{2}(\mathbf{C} - \mathbf{I})$ instead of $\mathbf{U}$, etc.

In the literature, one often finds the following argument. Let

$$\psi : \mathscr{X} \equiv \mathscr{I}nv^{+} \to \mathscr{Y} \equiv \mathscr{R}$$

$$\mathbf{F} \mapsto \psi(\mathbf{F})$$

be the *hyperelastic energy*. Then by IRBM

$$\psi(\mathbf{F}) = \psi(\mathbf{Q}\ \mathbf{F}) \qquad \forall \mathbf{Q} \in \mathscr{O}rth^{+}, \qquad \forall \mathbf{F} \in \mathscr{I}nv^{+} \tag{7.3}$$

and by the polar decomposition one obtains the reduced form

$$\psi(\mathbf{F}) = \psi(\mathbf{U})$$

with $\mathbf{Q} \equiv \mathbf{R}^T$, and one concludes that any function $\psi(\mathbf{U})$ identically fulfils the IRBM. Of course, this reasoning is a shorthand[8)] for saying: iff ψ fulfils (7.3), then it can be represented by

$$\psi(\mathbf{F}) = \psi|_{\mathscr{P}sym} \circ \lambda_\psi(\mathbf{F})$$

with

$$\lambda_\psi : \mathscr{I}nv^{+} \to \mathscr{P}sym$$

$$\mathbf{F} \mapsto \mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}.$$

8. Conclusions

Euclidean frame-indifference (EFI) determines the action of changes of observer or Euclidean transformations on the dependent variables such as Cauchy stresses. On the other hand, IRBM describes the effect of superimposed rigid body motions on the stresses. It is violated, if the response of the material depends on accelerations, spins, etc.

[8)] This shorthand is essentially correct, but has been misunderstood [see RIVLIN and SMITH, [14]. Unfortunately, also these critical authors overlooked the fact that the condition (7.2) has nothing to do with Euclidean frame-invariance].

Although there exist certain mathematical similarities, these two principles represent completely distinct notions. In SVENDSEN and BERTRAM [16], as well as in the current work, we have attempted to work these and other aspects of these principles out in detail. While EFI appears to be generally valid on the basis of our understanding of stresses, IRBM is clearly violated for certain materials such as kinetic gases. As such, both classes of materials, namely those that do satisfy the IRBM, and those that fail to do so, are described in the current formulation.

When investigating the reasons for the long and controversial debate on this issue, a third assumption comes into play. The dependence of the constitutive equations on the observer can be further specified as form-invariant.

It turns out, that this assumption is equivalent to IRBM, if the validity of EFI is assumed. As such, EFI and IRBM are indistinguishable whenever the observer dependence of the constitutive equations is not taken into account, i. e., whenever FI holds. And this is common practice.

We have stated FI with certain emphasis, although it is rather formal and difficult to interpret physically. This has two reasons. Firstly, it is often assumed without mentioning. And secondly, it has strong consequences for the material theory, as we have seen.

Once having understood the structure of the mutual dependences of these principles, the following approach seems to be natural and physically adequate.

1. State EFI as a fundamental principle which is generally valid in continuum physics.
2. Define a special class of materials by the condition IRBM, but not as a general principle for all materials. Under EFI, FI is necessary and sufficient for IRBM to hold.
3. For this specific class of materials that obey IRBM, obtain the reduced forms via the procedure developed in the last section.

We have shown a way to define and to construct reduced forms in a rather general context. The underlying structure is the same for many different applications in physics and other fields. One has a set of independent and dependent variables, $\mathscr{X}$ and $\mathscr{Y}$, respectively. Then one considers mappings from $\mathscr{X}$ to $\mathscr{Y}$ that are restricted by invariance-properties under certain transformation groups. The problem is to find two sets $\mathscr{R}\mathscr{X}$ and $\mathscr{R}\mathscr{Y}$, such that any mapping from $\mathscr{R}\mathscr{X}$ to $\mathscr{R}\mathscr{Y}$ corresponds to exactly one mapping from $\mathscr{X}$ to $\mathscr{Y}$ that fulfills these invariances. Clearly, the stronger these conditions are, the greater is the reduction.

The suggested procedure to construct reduced forms is a mathematical reformulation and generalization of what NOLL [12] suggested in the context of elastic materials.[9] This procedure has to be specified for the individual class of materials under consideration so that we can really benefit from this reduction. For many classes of materials, this has already been done long ago. For complicated material classes such as higher-order (non-simple) materials, non-local materials, Cosserat materials, mixtures, materials with micro-structure or internal length scales, this analysis of reduced forms can be expected to be advantageous in the future.

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[9] see also TRUESDELL [17]

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