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Variational principles of bending problems of thin plates

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IN THIS PAPER, via the semi-inverse method proposed previously by the present author, a family of variational principles of bending problems of plates is derived directly from their governing equations and boundary conditions, without using the Lagrange multiplier method. In this method, an energy-like trial functional is constructed with a certain unknown function, which can be identified step by step. A new generalized variational principle is obtained.

Key words: Thin Plate, Variational Theory, Semi-Inverse Method, Trial-Functional

1. Introduction

CHIEN [1] STUDIED THE GENERALIZED variational principles with multi-variables of bending problems of thin plates by means of the method of high-order Lagrange multipliers [2] and the involutory transformations, for the purpose of reducing the order of differentiations for the variables in the minimum potential energy principle and minimum complementary energy principle. Reissner gave several generalizations for elasticity [3] and the plate theory [4], WASHIZU [5] suggested a functional to deal with the "corner forces" which appear on the edge at the points of discontinuity of the torsiónal moments. Classification of various variational theorems was given by CHIEN in [6].

In using the Lagrange multiplier method to eliminate the constraints, however, one might always come across variational crisis [1, 2, 7–12] (some of Lagrange multipliers become zero during the process of variation, or some constraints can be eliminated even in the case the multiplier can be identified; in some special cases, wrong field equations might be obtained after substituting the identified multiplier into the augmented functional). We explained this phenomenon in our previous publications [7, 8, 9, 11] and pointed out in [12] that the so-called variational crisis is the inherent attribution of the Lagrange multiplier method. Several ways have been proposed to eliminate the crisis: a modified Lagrange multiplier method is suggested in [10], and a semi-inverse method is proposed in [13, 14]. Some applications of the semi-inverse method can be found in [15–17].

It has been shown in [1] that we cannot obtain a generalized variational principle by the Lagrange multiplier method due to the variational crisis (see Eq. (7.5) in [1]); in this paper, we apply the semi-inverse method [13, 14] to derive a family of generalized variational principles directly from the differential equations and boundary conditions, and no variational crisis occurs due to absence of Lagrange multipliers in our procedure.

2. Generalized variational principle of thin plate bending problems

The differential equation of deflection due to bending of a thin plate reads [1,18]

$$\nabla^2\nabla^2 w = \frac{\bar{f}}{D}, \tag{2.1}$$

where D is the flexural rigidity, $\bar{f}$ is the given lateral load, w is the lateral deflection of the plate. It can be seen clearly that the differential equation (2.1) requires strong local differentiability (smoothness). While the variable in its corresponding variational partner(see Eq. (2. 2)) is in second order of differentiations, it can be written in the form (the boundary conditions will be taken into consideration at the end of this section)

$$J(w) = \iint \left\{ \frac{1}{2}\left(\nabla^2 w\right)^2 - \frac{\bar{f}}{D} w \right\} dxdy. \tag{2.2}$$

The field associated with w must be continuous and must possess continuous second-order derivatives. In the context of finite elements, it is well known that satisfaction of the continuity of the second-order derivatives across the element boundaries is difficult to achieve [19]. So the high order of differentiation in the variational functional (2.2) leads to complications in the finite element calculation, and consequently, inconveniences appear in numerical computations. For the purpose of simplification in the finite element computation, we often introduce some additional canonical variables by means of involutory transformations [1] to reduce the order of differentiations. According to CHIEN [1], we have the following transformations:

$$\varphi_\alpha = w_{,\alpha}, \tag{2.3}$$

$$k_{\alpha\beta} = -w_{,\alpha\beta}, \tag{2.4}$$

$$w_{,\alpha\beta} = \frac{1}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}), \tag{2.5}$$

(2.6) $$Q_\alpha = M_{\alpha\beta,\beta},$$

(2.7) $$M_{\alpha\beta} = D_{\alpha\beta\nu\delta}k_{\gamma\delta},$$

where the Greek indices are the dummy indices taking the values 1 or 2, $w_{,\alpha\beta} = \partial^2 w/\partial x_\alpha \partial x_\beta$, $w_{,\alpha}$ is the slope of the deflection surface, Q_α and $M_{\alpha\beta}$ are the shearing force and the bending moment, respectively.

The equilibrium equation (2.1) can be rewritten in the form

(2.8) $$M_{\alpha\beta,\alpha\beta} + \bar{f} = 0.$$

As illustrated in [1], it is difficult to search for a generalized variational principle with five sets of independent variations $(w, \varphi_\alpha, k_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha)$ by Lagrange multipliers. However, it is a straightforward approach to deduce various generalized variational principles by the semi-inverse method [13, 14].

In order to establish a generalized variational principle with five sets of independent variations $(w, \varphi_\alpha, k_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha)$, whose stationary conditions satisfy the field equations (2.3), (2.5), (2.6), (2.7) and (2.8), we can construct a trial functional in the form

(2.9) $$J(w, \varphi_\alpha, k_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha) = \iint (\frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} + F) dxdy.$$

Note: Eqs. (2.4) are still considered as constraints.

There exist other ways to construct the trial functionals, details can be found in the [13, 14]. Let us illustrate the procedure of identification of the unknown function F step by step.

STEP 1

Taking variation with respect with $k_{\alpha\beta}$, i.e.

$$\delta_k J = \iint (D_{\alpha\beta\nu\delta} k_{\nu\delta} + \delta F/\delta k_{\alpha\beta}) \delta k_{\alpha\beta} dxdy = 0,$$

we have the following trial Euler equation:

$(2.10)_1$ $$D_{\alpha\beta\nu\delta} k_{\nu\delta} + \frac{\delta F}{\delta k_{\alpha\beta}} = 0,$$

where $\delta F/\delta\varphi = \partial F/\partial\varphi - (\partial F/\partial\varphi_{,\alpha})_{,\alpha}$ is called the variational derivative.

The above trial Euler equation $(2.10)_1$ should satisfy Eq. (2.7), accordingly we can set

$(2.10)_2$ $$\frac{\delta F}{\delta k_{\alpha\beta}} = -M_{\alpha\beta}.$$

The unknown F can be preliminarily identified as follows:

$$F = -k_{\alpha\beta}M_{\alpha\beta} + f. \tag{2.11}$$

The trial functional (2.9), therefore, can be rewritten as follows:

$$J = \iint \left(\frac{1}{2}D_{\alpha\beta\nu\delta}k_{\alpha\beta}k_{\nu\delta} - k_{\alpha\beta}M_{\alpha\beta} + f\right)dxdy, \tag{2.12}$$

where f is a newly introduced unknown, and should be, in general, free of the variations $k_{\alpha\beta}$ and their derivatives.

STEP 2

By the same manipulation as that used in STEP 1, we can obtain the following trial Euler equations for $\delta M_{\alpha\beta}$.

$$(2.13)_1 \qquad \delta M_{\alpha\beta}: \qquad -k_{\alpha\beta} + \frac{\delta f}{\delta M_{\alpha\beta}} = 0.$$

We assume that the above Euler equations $(2.13)_1$ satisfy the field equations (2.5), and in view of constraints (2.4), the above trial Euler equations reduce to

$$(2.13)_2 \qquad \frac{\delta f}{\delta M_{\alpha\beta}} = -w_{,\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}).$$

Thus we can preliminarily identify the unknown f as follows:

$$(2.14)_1 \qquad f = -\varphi_{\alpha,\beta}M_{\alpha\beta} + g_1,$$

or

$$(2.14)_2 \qquad f = \varphi_{\alpha}M_{\alpha\beta,\beta} + g_2,$$

or in a more general form

$$(2.14)_3 \qquad f = -m\varphi_{\alpha,\beta}M_{\alpha\beta} + n\varphi_{\alpha}M_{\alpha\beta,\beta} + g, \quad \text{with} \quad m + n = 1,$$

where g, g_1 and g_2 are unknowns to be determined later, and they should be expressed by the functions $w, \varphi_\alpha, Q_\alpha$ and/or their derivatives.

Substituting $(2.14)_3$ into (2.12) we obtain a modified trial functional

$$J = \iint \left(\frac{1}{2}D_{\alpha\beta\nu\delta}k_{\alpha\beta}k_{\nu\delta} - k_{\alpha\beta}M_{\alpha\beta} - m\varphi_{\alpha,\beta}M_{\alpha\beta} + n\varphi_{\alpha}M_{\alpha\beta,\beta} + g\right)dxdy. \tag{2.15}$$

STEP 3

Continually we have the following trial Euler equations for $\delta\varphi_\alpha$:

$$(2.16)_1 \qquad \delta\varphi_\alpha : \qquad (m+n)M_{\alpha\beta,\beta} + \tfrac{\delta g}{\delta\varphi_\alpha} = 0,$$

which should satisfy the equations (2.6); therefore we have

$$(2.16)_2 \qquad \frac{\delta g}{\delta\varphi_\alpha} = -Q_\alpha.$$

From above relation, we can express the unknown g as follows:

$$(2.17) \qquad g = -Q_\alpha\varphi_\alpha + h,$$

where h is a newly introduced unknown to be determined later, and should in general be expressed as w, Q_α and/or their derivatives.

STEP 4

Substituting (2.17) in (2.15) to modify the trial functional, making the modified trial functional stationary with respect to Q_α, we have

$$(2.18)_1 \qquad \delta Q_\alpha : \ -\varphi_\alpha + \frac{\delta h}{\delta Q_\alpha} = 0,$$

which should satisfy the equations (2.3); accordingly, we have

$$(2.18)_2 \qquad \frac{\delta h}{\delta Q_\alpha} = w_{,\alpha},$$

$$(2.19) \qquad h = Q_\alpha w_{,\alpha} + h',$$

where h' is an unknown to be determined, and should be only the function of w and/or its derivative.

STEP 5

Substituting (2.19) into the last modified trial functional, finally we can obtain the last trial Euler equations:

$$(2.20)_1 \qquad \delta w : \ -Q_{\alpha,\alpha} + \tfrac{\delta h'}{\delta w} = 0,$$

which should satisfy Eq.(2.8). In view of (2.6), we have

$$(2.20)_2 \qquad \frac{\delta h'}{\delta w} = M_{\alpha\beta,\alpha\beta} = -\bar{f},$$

$$(2.21) \qquad h' = -\bar{f}w.$$

Finally we obtain the following functional:

$$(2.22)_1 \qquad J = \iint L dx dy,$$

in which

$$(2.22)_2 \quad L = \frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - \bar{f} w - M_{\alpha\beta}(k_{\alpha\beta} + m\varphi_{\alpha,\beta}) + n\varphi_\alpha M_{\alpha\beta,\beta} - Q_\alpha(\varphi_\alpha - w_{,\alpha}),$$

and where m and n are arbitrary constants with $m + n=1$. It follows that the continuity requirements for the variables in (2.22) are less stringent. The presence of free parameters (m and n) offers an opportunity for a systematic derivation of the energy-balanced finite elements [15].

Next we will illustrate how to use the semi-inverse method to eliminate the constraints of boundary conditions. The boundary conditions usually encountered are given below,

$$(2.23) \qquad H_n = \bar{H}_n, \quad \text{on } \Gamma_\sigma$$

$$(2.24) \qquad w = \bar{w}, \quad \text{on } \Gamma_w$$

where $H_n = Q_n + M_{ns,s}$, and $\Gamma = \Gamma_\sigma + \Gamma_w$ covers the complete boundary.

To eliminate the constrains (2.23) and (2.24) , we construct a trial functional as follows:

$$(2.25)_1 \qquad J_{GVP} = \iint \tilde{L} dx dy + \int_{\Gamma_\sigma} G dS + \int_{\Gamma_w} P dS,$$

where G and P are unknowns to be determined, and $\tilde{L}$ is defined as (by setting m =1, n=0 in $(2.22)_2$)

$$(2.25)_2 \qquad \tilde{L} = \frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - \bar{f} w - M_{\alpha\beta}(k_{\alpha\beta} + \varphi_{\alpha,\beta}) - Q_\alpha(\varphi_\alpha - w_{,\alpha}).$$

With the help of Green's theorem, on the boundary Γ_σ we have following trial Euler equations:

$$(2.26) \qquad \delta w: \quad \frac{\delta G}{\delta w} = -Q_\alpha n_\alpha = -Q_n,$$

$$(2.27) \qquad \delta\varphi_s: \quad \frac{\delta G}{\delta \varphi_s} = M_{ns},$$

$$\delta M_{ns}: \quad \frac{\delta G}{\delta M_{ns}} = 0. \tag{2.28}$$

The above trial Euler equations should satisfy the boundary conditions (2.23) or the identities including the equations (2.3)-(2.8).

In view of (2.23), from (2.26) we have

$$\frac{\delta G}{\delta w} = -(\bar{H} - M_{ns,s}). \tag{2.29}$$

Accordingly, the unknown G can be written as follows:

$$G = -\bar{H}w - M_{ns}w_{,s} + G_1, \tag{2.30}$$

in which G_1 should be free of w or its derivatives.

In combination with (2.28), and in view of (2.3), we obtain

$$\frac{\delta G_1}{\delta M_{ns}} = w_{,s} = \varphi_s. \tag{2.31}$$

From (2.27) and (2.31) we can determine the unknown G_1 as follows:

$$G_1 = M_{ns}\varphi_s. \tag{2.32}$$

Therefore we have finally identified the unknown G which is written in the form

$$G = -\bar{H}w - M_{ns}(w_{,s} - \varphi_s). \tag{2.33}$$

Using the same procedure we have the following trial Euler equations on the boundary Γ_w:

$$\delta w: \quad \frac{\delta P}{\delta w} = -Q_n, \tag{2.34}$$

$$\delta \varphi_s: \quad \frac{\delta P}{\delta \varphi_s} = M_{ns}, \tag{2.35}$$

$$\delta Q_n: \quad \frac{\delta P}{\delta Q_n} = 0, \tag{2.36}$$

$$\delta M_{ns}: \quad \frac{\delta P}{\delta M_{ns}} = 0. \tag{2.37}$$

From (2.34) and (2.35), we determine the unknown P as follows:

$$P = -Q_n w + M_{ns}\varphi_s + P_1, \tag{2.38}$$

where P_1 should be free of w and φ_s or their derivatives; if not, the unknown P should be determined again.

In combination with (2.36), and in view of the boundary conditions (2.24), we have

$$\frac{\delta P_1}{\delta Q_n} = w = \bar{w}, \tag{2.39}$$

$$P_1 = Q_n \bar{w} + P_2. \tag{2.40}$$

Substituting (2.38) and (2.40) into (2.36), we obtain

$$\frac{\partial P_2}{\partial M_{ns}} = -\varphi_s = -w_{,s}. \tag{2.41}$$

We temporarily express the unknown P_2 as follows:

$$P_2 = -M_{ns} w_{,s} + P_3. \tag{2.42}$$

It should be stressed that in P_2 there exist the terms involving $w_{,s}$, therefore the unknown P should be determined again. The unknown P can be rewritten as follows:

$$P = -Q_n(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s}) + P_3, \tag{2.43}$$

where P_3 should be expressed as w or its derivative.

Substituting (2.43) into (2.33), and in view of (2.34), we obtain:

$$\frac{\delta P_3}{\delta w} = -M_{ns,s}. \tag{2.44}$$

Accordingly, the unknown P_3 can be written, in general, as follows:

$$P_3 = -M_{ns,s}(w - \bar{w}). \tag{2.45}$$

It can be proved that such unknown P_3 satisfies, in view of equations (2.3) and (2.24), the equations (2.37). Finally we obtain the following generalized variational principle

$$J_{GVP} = \iint \tilde{L} dx dy + \int_{\Gamma\sigma} \{-\bar{H}w - M_{ns}(w_{,s} - \varphi_s)\} dS \tag{2.46}$$
$$+ \int_{\Gamma w} \{-(Q_n + M_{ns,s})(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s})\} dS.$$

where $\tilde{L}$ is defined by Eq.$(2.25)_2$.

P r o o f. Making the above functional stationary, we have the following Euler's equations:

$$\delta k_{\alpha\beta} : \text{equations } (2.7);$$
$$\delta M_{\alpha\beta} : \kappa_{\alpha\beta} + \varphi_{\alpha\beta} = 0;$$
$$\delta Q_\alpha : \text{equations } (2.3);$$
$$\delta \varphi_\alpha : \text{equations } (2.6);$$
$$\delta w : \text{equations } (2.8);$$

on the Γ_σ:

$$\delta w : \text{equation } (2.23);$$
$$\delta \varphi_s : -M_{ns} + M_{ns} = 0;$$
$$\delta M_{ns} : w_{,s} - \varphi_s = 0;$$

on the Γ_w:

$$\delta w : Q_n - Q_n = 0;$$
$$\delta \varphi_s : -M_{ns} + M_{ns} = 0;$$
$$\delta Q_n : \text{equation } (2.24);$$
$$\delta M_{ns} : -(w - \bar{w})_{,s} + (\varphi_s - w_{,s}) = 0.$$

It can be clearly seen that on the boundary, some of Euler equations are identities or they satisfy the boundary conditions or they have been already derived as Euler equations in the process of variation.

Let us consider the singular points where M_{ns} is discontinuous. The singular corner might exist on the surface Γ_w or Γ_σ or at $\Gamma_w \cap \Gamma_\sigma$. In this paper, we only discuss the later case. We assume that on the boundary, the corner exists at transition points from Γ_σ to Γ_w and *vice versa*, while all the remaining parts of the boundary are assumed to be smooth.

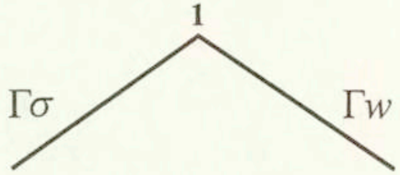

FIG. 1. Singular point

By a similar analysis, we can transform Eq. (2.45) into another functional

$$(2.47)\quad J_{GVP} = \iint \tilde{L}dxdy + \int\limits_{\Gamma\sigma} \{-\bar{H}w - M_{ns}(w_{,s} - \varphi_s)\}dS + \int\limits_{\Gamma w} \{-(Q_n + M_{ns,s})(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s})\}dS - \sum (w - \bar{w})(Q_n + M_{ns,s} - \bar{H}_n)\Big\|_{\Gamma u},$$

where $H_n = Q_n + M_{ns,s}$, and the notation $\sum(\cdot)(\cdot)\|_{\Gamma_u}$ has the same meaning as that in [6,18], which means summation over all the Γ_u. If Eq. (2.23) is replaced by

$$(2.48)\qquad M_{ns} = \bar{M}_{ns}, \quad \text{on } \Gamma_\sigma$$

then the last term at the right-hand side of Eq. (2.46) should be replaced by

$$-\sum (w - \bar{w})(M_{ns} - \bar{M}_{ns})\Big\|_{\Gamma u}.$$

From the generalized variational principle (2.46), various variational principles with smaller number of independent variables can be readily obtained by constraining the functional (2.46) by some field equations or boundary conditions. For example, enforcing the functional (2.46) by the field equation (2.3) results in a new functional, which reads

$$(2.49)_1\qquad J_1 = \iint \left\{\frac{1}{2}D_{\alpha\beta\nu\delta}k_{\alpha\beta}k_{\nu\delta} - \bar{f}w - M_{\alpha\beta}(k_{\alpha\beta} + \varphi_{\alpha,\beta})\right\}dxdy + IB,$$

where

$$(2.49)_2 \quad IB = \int_{\Gamma\sigma} \{-\bar{H}w - M_{ns}(w_{,s} - \varphi_s)\} dS + \int_{\Gamma w} \{-(Q_n + M_{ns,s})(w - \bar{w}) + M_{ns}(\varphi_s - w_{,s})\} dS.$$

The functional $(2.49)_1$ is under constraints of the Eqs. (2.3) and (2.4).

Constraining the functional (2.47) by the Eq. (2.4), we obtain

$$(2.50) \qquad J_2 = \iint \left\{\frac{1}{2} D_{\alpha\beta\nu\delta} k_{\alpha\beta} k_{\nu\delta} - \bar{f}w\right\} dxdy + IB,$$

which is in agreement with the Eq. (2.2).

3. Conclusion

In this paper, the author has systematically discussed the semi-inverse method of establishing generalized variational principles in thin plate bending problems. The variational model can be readily obtained without any variational crisis. A family of generalized variational principles can be readily obtained by specifying the parameters m and n. The Lagrange function $(2.22)_2$, as far as the author knows, has never appeared in any literature.

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