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Semi-inverse solutions in nonlinear theory of elastic shells

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THE SEMI-INVERSE METHOD is applied to the solution of static problems in the nonlinear theory of elastic shells. This method consists in construction of particular solutions in such a way that the initial system of equations is reduced to a system of a smaller number of independent variables. Two nonlinear models, one of the Love type and another of the Cosserat type, are considered. For these models, several two-parameter families of finite deformations are found; then the system of partial differential equations of equilibrium reduces to a system of nonlinear ordinary differential equations. The semi-inverse solutions found are valid for prismatic and toroidal shells, as well as for shells of revolution. These solutions are of practical significance. They describe torsion of a prismatic shell under large angles of twist, strong flexure of a thin-walled cylinder of arbitrary cross-section, spatial bending of a shell having the shape of a sector of a surface of revolution, straightening of a toroidal shell to a cylindrical surface, and some other types of large deformations.

1. Basic statements

Let σ be the reference (undeformed) surface of a shell. We refer the position vector of a particle of σ to Gaussian coordinates q^α ($\alpha = 1,2$) and $\mathbf{r}(q^1, q^2) = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$. Here, x_k ($k = 1,2,3$) are Cartesian coordinates of a point of σ, and $\mathbf{i}_k$ ($k = 1,2,3$) is a fixed orthonormal basis. The coefficients of the first and the second fundamental forms of the surface σ are

$$
\begin{gathered}
g_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta, \quad b_{\alpha\beta} = \frac{\partial \mathbf{r}_\alpha}{\partial q^\beta} \cdot \mathbf{n} = -\mathbf{r}_\alpha \cdot \frac{\partial \mathbf{n}}{\partial q^\beta}, \quad \mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial q^\alpha}, \\
\mathbf{r}^\beta \cdot \mathbf{r}_\alpha = \delta_\alpha^\beta, \quad \mathbf{r}^\beta \cdot \mathbf{n} = 0,
\end{gathered}
\tag{1.1}
$$

where $\mathbf{n}$ – is a unit normal to σ and δ_α^β is the Kronecker symbol.

The surface Σ of the shell after deformation is referred to the coordinates q^α as well, and the position of a particle of Σ is described by the position vector

$\boldsymbol{R}(q^1, q^2) = X_1\boldsymbol{i}_1 + X_2\boldsymbol{i}_2 + X_3\boldsymbol{i}_3$, X_k being the Cartesian coordinates of the point of the surface Σ whose normal vector is $\boldsymbol{N}$,

The components of the fundamental forms of Σ are

$$
(1.2)\qquad
\begin{gathered}
G_{\alpha\beta} = \boldsymbol{R}_\alpha \cdot \boldsymbol{R}_\beta, \quad B_{\alpha\beta} = \frac{\partial \boldsymbol{R}_\alpha}{\partial q^\beta} \cdot \boldsymbol{N} = -\boldsymbol{R}_\alpha \cdot \frac{\partial \boldsymbol{N}}{\partial q^\beta},\\
\boldsymbol{R}_\alpha = \frac{\partial \boldsymbol{R}}{\partial q^\alpha}, \quad \boldsymbol{R}^\beta \cdot \boldsymbol{R}_\alpha = \delta^\beta_\alpha, \quad \boldsymbol{R}^\beta \cdot \boldsymbol{N} = 0.
\end{gathered}
$$

First we consider the model of a nonlinearly elastic shell of the Love type (Koiter [1], Pietraszkiewicz [2,3], Galimov [4] and Zubov [5]). The equations of equilibrium of the shell expressed in terms of the stress and couple resultants are

$$
(1.3)\qquad
\begin{gathered}
\nabla_\alpha(\nu^{\alpha\beta} - \mu^{\alpha\delta}B^\beta_\delta) - B^\beta_\delta \nabla_\alpha \mu^{\alpha\delta} + F^\beta = 0, \quad (\beta = 1,2),\\
\nabla_\alpha \nabla_\beta \mu^{\alpha\beta} + B_{\alpha\beta}(\nu^{\alpha\beta} - B^\alpha_\delta \mu^{\delta\beta}) + F = 0,\\
F = \boldsymbol{F} \cdot \boldsymbol{N}, \quad F^\beta = \boldsymbol{F} \cdot \boldsymbol{R}^\beta, \quad B^\alpha_\delta = B_{\gamma\delta} G^{\alpha\gamma}, \quad G^{\alpha\gamma} = \boldsymbol{R}^\alpha \cdot \boldsymbol{R}^\gamma,
\end{gathered}
$$

where $\mathbf{F}$ is the vector intensity of the force load on Σ, $\nu^{\alpha\beta}$ and $\mu^{\alpha\beta}$ are the resultant stress and couple tensors, respectively, and ∇_α is the symbol of the covariant derivative in the metric $G_{\alpha\beta}$. The constitutive equations take the form

$$
(1.4)\qquad
\begin{gathered}
\chi\sqrt{\frac{G}{g}}\nu^{\alpha\beta} = \frac{\partial W}{\partial G_{\alpha\beta}}, \quad \chi\sqrt{\frac{G}{g}}\mu^{\alpha\beta} = \frac{\partial W}{\partial B_{\alpha\beta}},\\
G = G_{11}G_{22} - G_{12}^2, \quad g = g_{11}g_{22} - g_{12}^2,\\
\chi = \begin{cases} 1, & \alpha = \beta,\\ 2, & \alpha \neq \beta. \end{cases}
\end{gathered}
$$

For a homogeneous isotropic shell, its specific (per unit area of surface σ) potential energy of deformation W is a function of the following nine variables:

$$
(1.5)\qquad
\begin{gathered}
g^{\alpha\beta}b_{\alpha\beta}, \quad g^{\alpha\beta}G_{\alpha\beta}, \quad g^{\alpha\beta}B_{\alpha\beta}, \quad (b_{11}b_{22} - b_{12}^2)/g, \quad G/g,\\
(B_{11}B_{22} - B_{12}^2)/g, \quad b^{\alpha\beta}G_{\alpha\beta}, \quad b^{\alpha\beta}B_{\alpha\beta}, \quad g^{\alpha\beta}g^{\gamma\delta}G_{\alpha\gamma}B_{\beta\delta},\\
b^{\alpha\beta} = g^{\alpha æ}g^{\beta\lambda}b_{æ\lambda}, \quad g^{\beta\lambda} = \boldsymbol{r}^\beta \cdot \boldsymbol{r}^\lambda.
\end{gathered}
$$

Suppose that $\partial\Sigma$, the edge of the shell in the deformed configuration, is loaded by a distributed force of linear density $\boldsymbol{Q} = Q^\alpha \boldsymbol{R}_\alpha + Q\boldsymbol{N}$ and distributed moment of linear density $\boldsymbol{d} \times \boldsymbol{N}$. Now the boundary conditions take the form

$$\sqrt{\frac{G}{g}}m_\alpha(\nu^{\alpha\beta} - 2B^\beta_\delta \mu^{\alpha\delta}) = \varepsilon(Q^\beta - B^\beta_\alpha)d^\alpha), \quad (\beta = 1, 2),$$

(1.6)
$$\sqrt{\frac{G}{g}}m_\alpha m_\beta \mu^{\alpha\beta} = \varepsilon m_\alpha d^\alpha,$$

$$\sqrt{\frac{G}{g}}m_\beta \nabla_\alpha \mu^{\alpha\beta} + \frac{d}{ds}\left(\sqrt{\frac{G}{g}}\varepsilon^{-2}\tau^\delta m_\beta G_{\alpha\delta}\mu^{\alpha\beta}\right) = \varepsilon Q + \frac{d}{ds}\left(\varepsilon^{-1}\tau^\delta d_\delta\right),$$

$$\boldsymbol{d} = d^\alpha \boldsymbol{R}_\alpha = d_\beta \boldsymbol{R}^\beta, \quad \boldsymbol{m} = m_\alpha \boldsymbol{r}^\alpha, \quad \boldsymbol{\tau} = \tau^\delta \boldsymbol{r}_\delta, \quad \varepsilon = \sqrt{\tau^\alpha \tau_\beta G_{\alpha\beta}}.$$

Here $\boldsymbol{m}$ and $\boldsymbol{\tau}$ are the unit normal vector and the unit tangent vector to the boundary contour $\partial\sigma$ in the undeformed configuration of the shell, and s is the length parameter of $\partial\sigma$.

In what follows, we shall consider particular exact solutions of the equilibrium equations (1.3). These solutions refer to shells of certain geometries. The solutions constitute some families (classes) of finite deformation for which the initial nonlinear system of partial differential equations of two independent variables, q^1, q^2, reduces to a system of ordinary differential equations whose unknown functions are functions of only one variable.

2. Spatial bending of a cylindrical shell

Suppose that a shell in the reference configuration is a cylindrical surface with its generator parallel to the x_1-axis. The cross-section of the surface by the plane x_2x_3 is an arbitrary closed or nonclosed curve having no points of self-intersection. Its equation is given by functions $x_2(s)$ and $x_3(s)$, s being the length parameter of the curve. Let us take the Gaussian coordinates $q^1 = x_1$ and $q^2 = s$. Let a prime denote a derivative with respect to s. We have

(2.1)
$$\boldsymbol{r}_1 = \boldsymbol{i}_1, \quad \boldsymbol{r}_2 = x_2'\boldsymbol{i}_2 + x_3'\boldsymbol{i}_3.$$

It follows that the quantities $g_{\alpha\beta}$ and $b_{\alpha\beta}$ do not depend on x_1. Let us consider the following family of deformations of a cylindrical shell:

(2.2)
$$X_1 = \gamma(s)\sin(ax_1 + \lambda(s)),$$
$$X_2 = \alpha(s) + lx_1,$$
$$X_3 = \gamma(s)\cos(ax_1 + \lambda(s)),$$

where a and l are constants and α,γ, and λ are functions of one variable. When $l = \lambda = 0$, the formulae (2.2) describe bending of the shell in the plane x_1, x_3

such that each generator of the cylinder becomes a part of a circumference. If $l \neq 0$ and $\lambda \neq 0$, the generator is a helical line. Using (2.2) we find that

$$\begin{aligned} &\boldsymbol{R}_1 = a\gamma(s)\boldsymbol{e}_1 + l\boldsymbol{i}_2, \quad \boldsymbol{R}_2 = \gamma(s)\lambda'(s)\boldsymbol{e}_1 + \alpha'(s)\boldsymbol{i}_2 + \gamma'(s)\boldsymbol{e}_3, \\ &\boldsymbol{e}_1 = \boldsymbol{i}_1\cos(ax_1 + \lambda(s)) - \boldsymbol{i}_3\sin(ax_1 + \lambda(s)), \\ &\boldsymbol{e}_3 = \boldsymbol{i}_1\sin(ax_1 + \lambda(s)) + \boldsymbol{i}_3\cos(ax_1 + \lambda(s)), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\boldsymbol{N} = N_1(s)\boldsymbol{e}_1 + N_2(s)\boldsymbol{i}_2 + N_3(s)\boldsymbol{e}_3, \\ &\frac{\partial \boldsymbol{R}_1}{\partial q^1} = a^2\gamma\boldsymbol{e}_3, \quad \frac{\partial \boldsymbol{R}_1}{\partial q^2} = \frac{\partial \boldsymbol{R}_2}{\partial q^1} = a\gamma'\boldsymbol{e}_1 - a\gamma\lambda'\boldsymbol{e}_3, \\ &\frac{\partial \boldsymbol{R}_2}{\partial q^2} = (2\gamma'\lambda' + \gamma\lambda'')\boldsymbol{e}_1 + \alpha''\boldsymbol{i}_2 + (\gamma'' - \gamma\lambda'^2)\boldsymbol{e}_3.) \end{aligned} \tag{2.4}$$

The vectors $\boldsymbol{e}_1, \boldsymbol{i}_2, \boldsymbol{e}_3$ constitute an orthonormal basis. With regard to (1.2) and (2.3) we can conclude that the quantities $G_{\alpha\beta}, B_{\alpha\beta}$ do not depend on x_1. The Christoffel symbols $\Gamma^{\beta}_{\lambda\delta}$ that participate in the covariant derivatives ∇_α also depend only on the length parameter s. For an isotropic homogeneous shell, by (1.4) and (1.5), the tensors $\nu^{\alpha\beta}$ and $\mu^{\alpha\beta}$ are functions of s only. Let us suppose that the external surface load F^β, F, as well as for a nonclosed cylinder the forces Q_β, Q, d^α acting on the direct edges of the shell $s = s_1$ and $s = s_2$, do not depend on x_1. Now the equilibrium equations (1.3) supplemented with the boundary conditions (1.6) at $s = s_1$ and $s = s_2$ constitute a nonlinear boundary value problem for a system of ordinary differential equations with unknown functions $\alpha(s), \gamma(s)$ and $\lambda(s)$. When $l = \lambda(s) = 0$ we have $G_{12} = B_{12} = \nu^{12} = \mu^{12} = 0$. Under conditions $F^1 = 0$ and $Q^1 = d^1 = 0$, one of the equations of equilibrium (1.3) as well as one of the boundary conditions (1.6) is satisfied identically.

3. Twisting of a cylindrical shell

Let us change the Cartesian coordinates x_k to the cylindrical coordinates r, φ, z using the formulae $x_1 = r\cos\varphi, x_2 = r\sin\varphi, x_3 = z$, and consider a prismatic cylindrical shell whose generator lines are parallel to the z-axis. The curve in the cross-section of the surface σ by the plane $z = \text{const}$ will be defined by the function $r = r(\varphi)$. We take $q^1 = z, q^2 = \varphi$ as the Gaussian coordinates on σ. Now we have

$$\begin{aligned} &\boldsymbol{r}_1 = \boldsymbol{i}_3, \quad \boldsymbol{r}_2 = r'\boldsymbol{e}_r + r\boldsymbol{e}_\varphi, \quad r' \equiv dr/d\varphi, \\ &\boldsymbol{e}_r = \boldsymbol{i}_1\cos\varphi + \boldsymbol{i}_2\sin\varphi, \quad \boldsymbol{e}_\varphi = -\boldsymbol{i}_1\sin\varphi + \boldsymbol{i}_2\cos\varphi, \\ &g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = r'^2 + r^2. \end{aligned} \tag{3.1}$$

Let us denote by R, Φ, Z the cylindrical coordinates of a point of the deformed surface Σ so that $X_1 = R\cos\Phi, X_2 = R\sin\Phi, X_3 = Z$. Consider the following family of deformations of a cylindrical shell:

$$
\begin{gathered}
R = R(\varphi),\\
\Phi = \varphi + \psi z + v(\varphi),\\
Z = \alpha z + a\varphi + w(\varphi),\\
\psi, a, \alpha = \mathrm{const}.
\end{gathered}
\tag{3.2}
$$

These formulae describe torsion of the shell with the angle of twist ψ, longitudinal stretching, and longitudinal shift. In this, the cross-section of the shell, $z = \mathrm{const}$, has a deplanation characterized by function w; besides it is deformed in its plane what is described by the functions $R(\varphi)$ and $v(\varphi)$. When the shell is deformed according to (3.2), the generators of the initially cylindrical surface become helical lines. If the cross-section of the shell is a closed line, the functions R, v, and w must be 2π-periodic, and the quantity $2\pi a$ is equal to the modulus of the Burgers vector of a screw dislocation.

By (3.2) we have

$$
\begin{aligned}
&\boldsymbol{R}_1 = \alpha \boldsymbol{i}_3 + \psi R \boldsymbol{e}_\Phi, \quad \boldsymbol{R}_2 = (a + w')\boldsymbol{i}_3 + R'\boldsymbol{e}_R + R(1+v')\boldsymbol{e}_\Phi,\\
&\frac{\partial \boldsymbol{R}_1}{\partial q^1} = -\psi^2 R \boldsymbol{e}_R, \quad \frac{\partial \boldsymbol{R}_1}{\partial q^2} = \frac{\partial \boldsymbol{R}_2}{\partial q^1} = -\psi R(1+v')\boldsymbol{e}_R + \psi R'\boldsymbol{e}_\Phi,\\
&\frac{\partial \boldsymbol{R}_2}{\partial q^2} = [R'' - R(1+v')^2]\boldsymbol{e}_R + [Rv'' + 2R'(1+v')]\boldsymbol{e}_\Phi + w''\boldsymbol{i}_3,
\end{aligned}
\tag{3.3}
$$

$$
\boldsymbol{e}_R = \boldsymbol{i}_1\cos\Phi + \boldsymbol{i}_2\sin\Phi, \quad \boldsymbol{e}_\Phi = -\boldsymbol{i}_1\sin\Phi + \boldsymbol{i}_2\cos\Phi,
$$

$$
\begin{gathered}
G_{11} = \alpha^2 + \psi^2 R^2, \quad G_{12} = \alpha(a + w') + \psi R^2(1+v'),\\
G_{22} = (a+w')^2 + R^2(1+v')^2 + R'^2.
\end{gathered}
\tag{3.4}
$$

From (3.3),(3.4) it follows that the quantities $G_{\alpha\beta}, B_{\alpha\beta}, \Gamma^\beta_{\delta\lambda}$ do not depend on z. Thus, for an isotropic homogeneous shell, (1.3) is a system of ordinary differential equations with respect to the unknown functions $R(\varphi), v(\varphi), w(\varphi)$. Here, the components of external surface forces F^α, F and edge load Q, Q^α, d^α at $\varphi = \varphi_1$ and $\varphi = \varphi_2$ (for an open shell) should not depend on z.

If σ is a sector of a circular cylindrical shell, at $F^\alpha = 0$, the above system has the simple solution

$$
R = R_0, \quad v(\varphi) = A\varphi, \quad w(\varphi) = 0, \quad R_0, A = \mathrm{const}.
\tag{3.5}
$$

Let us consider the problem of a screw dislocation in a closed circular cylindrical shell which in the initial state, before dislocation, has the radius r_0. It is easy to verify with the use of (3.1) and (3.4) that there is an isometric deformation (it is the bending) of the cylindrical surface that is defined by the formulae

$$R(\varphi) = R_0 = \sqrt{r_0^2 - a^2}, \quad v(\varphi) = 0, \quad w(\varphi) = 0,$$
$$\psi = -\frac{a}{r_0\sqrt{r_0^2 - a^2}}, \quad \alpha = \sqrt{1 - \frac{a^2}{r_0^2}} \tag{3.6}$$

By (3.6), the cylinder is twisted and its radius decreases. Deformation (3.6) is a solution of the equilibrium equations (1.3) for a momentless elastic shell when the external forces are absent. Indeed, now, due to the momentless state of the shell, $\mu^{\alpha\beta} \equiv 0$ and $\nu^{\alpha\beta} \equiv 0$ since the deformation is isometric. For a sufficiently thin shell the assumption that the shell is momentless is quite accurate.

4. Bending, shifting, and twisting of a sector of a shell of revolution

Let the surface σ be a surface of revolution or a sector thereof. Let the equation of the meridian in cylindrical coordinates r, φ, z be $r = r(z)$. As the Gaussian coordinates we take $q^1 = z, q^2 = \varphi$. For the reference configuration we have

$$\mathbf{r}_1 = r'\mathbf{e}_r + \mathbf{i}_3, \quad \mathbf{r}_2 = r\mathbf{e}_\varphi. \tag{4.1}$$

Let us consider the following family of deformations using the circular cylindrical coordinates r, φ, z and R, Φ, Z:

$$R = R(z), \quad \Phi = K\varphi + \beta(z), \quad Z = l\varphi + \gamma(z), \quad K, l = \text{const}. \tag{4.2}$$

By (4.2) we find that

$$\begin{gathered}
\mathbf{R}_1 = R'\mathbf{e}_R + R\beta'\mathbf{e}_\Phi + \gamma'\mathbf{i}_3, \quad \mathbf{R}_2 = KR\mathbf{e}_\Phi + l\mathbf{i}_3, \\
\mathbf{N} = N_1(z)\mathbf{e}_R + N_2(z)\mathbf{e}_\Phi + N_3(z)\mathbf{i}_3, \\
\frac{\partial \mathbf{R}_1}{\partial q^1} = (R'' - R\beta'^2)\mathbf{e}_R + (2R'\beta' + R\beta'')\mathbf{e}_\Phi + \gamma''\mathbf{i}_3, \\
\frac{\partial \mathbf{R}_1}{\partial q^2} = \frac{\partial \mathbf{R}_2}{\partial q^1} = -KR\beta'\mathbf{e}_R + KR'\mathbf{e}_\Phi, \\
\frac{\partial \mathbf{R}_2}{\partial q^2} = -K^2R\mathbf{e}_R.
\end{gathered} \tag{4.3}$$

It follows that the coefficients of the quadratic forms $G_{\alpha\beta}, B_{\alpha\beta}$ and the Christoffel symbols $\Gamma^{\beta}_{\delta\lambda}$ for the surface Σ do not depend on φ. Thus for an isotropic homogeneous shell, the stress and couple resultants $\nu^{\alpha\beta}$ and $\mu^{\alpha\beta}$ are functions of one variable z and now equations (1.3) are ordinary differential equations with respect to the unknown functions $R(z), \beta(z), \gamma(z)$ if the loads F^{β}, F do not depend on φ. Expressions (4.2) include the following important particular deformations:

1. $K = 1, l = 0$ corresponds to twisting and axisymmetrical deformation of a shell of revolution. This special case of the semi-inverse solution (4.2) was found earlier by ZUBOV [5]. The results of numerical solution of the problem of twisting and inflation of a shell of revolution made of highly-elastic material are presented by ZUBOV and OVSEENKO [6].
2. $K = -1, l = 0$ corresponds to the twisting and axisymmetrical deformation of a shell of revolution that is turned inside out.
3. $K > 0, l = 0$ corresponds to the twisting and axisymmetrical deformation of a shell with a disclination. If $l = 0, \beta(z) = 0$ then twisting of the shell is absent. Now $G_{12} = B_{12} = 0, \nu^{12} = \mu^{12} = 0$ and if $F^2 = 0$, then one of the three equations of equilibrium in (1.3) is satisfied identically.

5. Straightening and twisting of a sector of a shell of revolution

Let the equations of the meridian of the surface of revolution be $r = r(s), z = z(s)$, where r, φ, z are cylindrical coordinates in space, s is the length parameter of the meridian, and $q^1 = s, q^2 = \varphi$ are taken as the Gaussian coordinates. Using the Cartesian coordinates of the deformed surface X_k we define the following deformation of the shell:

$$\begin{gathered} X_1 = u(s)\sin\eta\varphi + v(s)\cos\eta\varphi, \\ X_2 = æ\varphi + w(s), \\ X_3 = u(s)\cos\eta\varphi - v(s)\sin\eta\varphi, \\ æ, \eta = \text{const}. \end{gathered} \tag{5.1}$$

By (5.1), we obtain

$$\begin{gathered} \boldsymbol{r}_1 = r'\boldsymbol{e}_r + z'\boldsymbol{i}_3, \quad \boldsymbol{r}_2 = r\boldsymbol{e}_{\varphi}, \\ \boldsymbol{R}_1 = v'\boldsymbol{h}_1 + w'\boldsymbol{i}_2 + u\boldsymbol{h}_3, \quad \boldsymbol{R}_2 = \eta u\boldsymbol{h}_1 + æ\boldsymbol{i}_2 - \eta v\boldsymbol{h}_3, \\ \boldsymbol{h}_1 = \boldsymbol{i}_1\cos\eta\varphi - \boldsymbol{i}_3\sin\eta\varphi, \quad \boldsymbol{h}_3 = \boldsymbol{i}_1\sin\eta\varphi + \boldsymbol{i}_3\cos\eta\varphi, \\ \boldsymbol{N} = N_1(s)\boldsymbol{h}_1 + N_2(s)\boldsymbol{i}_2 + N_3(s)\boldsymbol{h}_3, \end{gathered}$$

$$\begin{aligned}\frac{\partial \boldsymbol{R}_1}{\partial q^1} &= v''\boldsymbol{h}_1 + w''\boldsymbol{i}_2 + u''\boldsymbol{h}_3,\\ \frac{\partial \boldsymbol{R}_1}{\partial q^2} = \frac{\partial \boldsymbol{R}_2}{\partial q^1} &= \eta u'\boldsymbol{h}_1 - \eta v'\boldsymbol{h}_3,\\ \frac{\partial \boldsymbol{R}_2}{\partial q^2} &= -\eta^2 v\boldsymbol{h}_1 - \eta^2 u\boldsymbol{h}_3.\end{aligned} \tag{5.2}$$

Using (5.2) we can prove that the quantities $G_{\alpha\beta}, B_{\alpha\beta}, \nu^{\alpha\beta}, \mu^{\alpha\beta}, \Gamma^{\beta}_{\lambda\delta}$ do not depend on φ. Let F^{β}, F in (1.3) be independent of φ. Then we again obtain a system of ordinary differential equations with respect to $u(s), v(s), w(s)$.

The case $\eta = 0, w(s) = 0$ in (5.1) corresponds to the deformation of unbending (straightening) without twisting of a sector of the shell of revolution into a cylindrical surface. Now $G_{12} = B_{12} = \nu^{12} = \mu^{12} = 0$, and if $F^2 = 0$, then one of the three equilibrium equations (1.3) is satisfied identically.

A careful analysis of the above semi-inverse solutions (2.2),(3.2),(4.2), and (5.1) shows that the assumption of homogeneity of the shell is not necessary. It is sufficient if the shell is homogeneous along only one coordinate. This means that, besides the arguments (1.5), the specific energy of the shell, W, can also depend explicitly on one of the coordinates q^1 or q^2.

6. Semi-inverse solutions for elastic shells of the Cosserats type

Unlike the classical model of the Love type employed above, in the Cosserats theory of a shell, a particle of the surface Σ is characterized not only by its position in space $\boldsymbol{R}(q^1, q^2)$ but by its orientation given by the rotation $\mathbf{H}(q^1, q^2)$, where $\mathbf{H}$ is a proper orthogonal tensor. The equilibrium equations of a shell of the Cosserats type are ZHILIN [7] and ZUBOV [8]

$$\operatorname{div}(\mathbf{P}\cdot\mathbf{H}) + \boldsymbol{f} = 0, \quad \operatorname{div}(\mathbf{K}\cdot\mathbf{H}) + [(\operatorname{grad}\boldsymbol{R})^{\mathrm{T}}\cdot\mathbf{P}\cdot\mathbf{H}]_{\times} + \boldsymbol{l} = 0, \tag{6.1}$$

$$\mathbf{P} = \frac{\partial W(\mathbf{U},\mathbf{L})}{\partial \mathbf{U}}, \quad \mathbf{K} = \frac{\partial W(\mathbf{U},\mathbf{L})}{\partial \mathbf{L}}, \tag{6.2}$$

$$\begin{gathered}\mathbf{U} = (\operatorname{grad}\boldsymbol{R})\cdot\mathbf{H}^{\mathrm{T}}, \quad \mathbf{L} = \frac{1}{2}\boldsymbol{r}^{\alpha}\otimes\left(\frac{\partial \mathbf{H}}{\partial q^{\alpha}}\cdot\mathbf{H}^{\mathrm{T}}\right)_{\times},\\ \operatorname{grad}\boldsymbol{\Psi} \equiv \boldsymbol{r}^{\alpha}\otimes\frac{\partial \boldsymbol{\Psi}}{\partial q^{\alpha}}, \quad \operatorname{div}\boldsymbol{\Psi} \equiv \boldsymbol{r}^{\alpha}\cdot\frac{\partial \boldsymbol{\Psi}}{\partial q^{\alpha}}.\end{gathered} \tag{6.3}$$

Here W is the specific energy of the shell, $\mathbf{U}$ the surface stretch tensor, $\mathbf{L}$ is the surface bending tensor, $\boldsymbol{f}$ is the vector intensity of the force load on σ, $\boldsymbol{l}$ is the vector intensity of the couple load on σ, and $\mathbf{P}$ and $\mathbf{K}$ are the resultant stress and couple tensors, respectively. The symbol $\mathbf{A}_\times$ denotes a vector invariant of the second order of the tensor $\mathbf{A}$.

Equations (6.1) can be transformed to a form that is more appropriate for us:

$$\operatorname{div}\mathbf{P} - (\mathbf{P}^{\mathrm{T}}\cdot\mathbf{L})_\times + \boldsymbol{f}^* = 0, \quad \operatorname{div}\mathbf{K} - (\mathbf{K}^{\mathrm{T}}\cdot\mathbf{L} + \mathbf{P}^{\mathrm{T}}\cdot\mathbf{U})_\times + \boldsymbol{l}^* = 0, \tag{6.4}$$
$$\boldsymbol{f}^* = \boldsymbol{f}\cdot\mathbf{H}^{\mathrm{T}}, \quad \boldsymbol{l}^* = \boldsymbol{l}\cdot\mathbf{H}^{\mathrm{T}}.$$

Since in the Cosserats-type shell the field of rotations $\mathbf{H}(q^1,q^2)$ is kinematically independent of the field of displacements of the surface σ, the above semi-inverse solutions (2.2), (3.2), (4.2), and (5.1) should be supplemented with the field of rotation $\mathbf{H}$. These expressions will be written out for each of the four families of semi-inverse solutions.

Spatial bending of a cylindrical shell:

$$\mathbf{H}(s,x_1) = H_{mn}(s)\boldsymbol{i}_m\otimes\boldsymbol{e}_n, \quad (m,n=1,2,3), \tag{6.5}$$
$$\boldsymbol{e}_2 = \boldsymbol{i}_2.$$

From (2.2), (6.3) and (6.5) we obtain

$$\mathbf{U} = U_{mn}(s)\boldsymbol{i}_m\otimes\boldsymbol{i}_n, \quad \mathbf{L} = L_{mn}(s)\boldsymbol{i}_m\otimes\boldsymbol{i}_n.$$

Torsion of a cylindrical shell:

$$\mathbf{H}(\varphi,z) = H_{mn}(\varphi)\boldsymbol{a}_m\otimes\boldsymbol{A}_n, \tag{6.6}$$
$$\boldsymbol{a}_1=\boldsymbol{e}_r, \quad \boldsymbol{a}_2=\boldsymbol{e}_\varphi, \quad \boldsymbol{a}_3=\boldsymbol{i}_3, \quad \boldsymbol{A}_1=\boldsymbol{e}_R, \quad \boldsymbol{A}_2=\boldsymbol{e}_\Phi, \quad \boldsymbol{A}_3=\boldsymbol{i}_3.$$

From (3.2), (6.3) and (6.6) we obtain

$$\mathbf{U} = U_{mn}(\varphi)\boldsymbol{a}_m\otimes\boldsymbol{a}_n, \quad \mathbf{L} = L_{mn}(\varphi)\boldsymbol{a}_m\otimes\boldsymbol{a}_n.$$

Bending, shifting, and twisting of a sector of a shell of revolution:

$$\mathbf{H}(z,\varphi) = H_{pt}(z)\boldsymbol{a}_p\otimes\boldsymbol{A}_t, \quad (p,t=1,2,3). \tag{6.7}$$

From (4.2), (6.3) and (6.7) we obtain

$$\mathbf{U} = U_{pt}(z)\boldsymbol{a}_p\otimes\boldsymbol{a}_t, \quad \mathbf{L} = L_{pt}(z)\boldsymbol{a}_p\otimes\boldsymbol{a}_t.$$

Straightening and twisting of a sector of a shell of revolution:

$$\mathbf{H}(s,\varphi) = H_{pt}(s)\boldsymbol{a}_p \otimes \boldsymbol{h}_t, \quad \boldsymbol{h}_2 = \boldsymbol{i}_2. \tag{6.8}$$

From (5.1), (6.3) and (6.8) we obtain

$$\mathbf{U} = U_{pt}(s)\boldsymbol{a}_p \otimes \boldsymbol{a}_t, \quad \mathbf{L} = L_{pt}(s)\boldsymbol{a}_p \otimes \boldsymbol{a}_t.$$

In (6.5)-(6.8) H_{mn} and H_{pt} are the proper orthogonal matrixes.

Notice that in a general case, the potential energy of deformation W of the shell dependes on the tensors $\mathbf{U}$ and $\mathbf{L}$, and some parameters, being fixed during the deformation process. These parameters may depend upon the coordinates q^1, q^2. Therefore, the $\operatorname{grad} W$, in general, is not coincided with the vector

$$\boldsymbol{r}^\alpha \operatorname{tr}\left(\frac{\partial W}{\partial \mathbf{U}} \cdot \frac{\partial \mathbf{U}^{\mathrm{T}}}{\partial q^\alpha} + \frac{\partial W}{\partial \mathbf{L}} \cdot \frac{\partial \mathbf{L}^{\mathrm{T}}}{\partial q^\alpha}\right).$$

If the relation

$$\boldsymbol{i}_1 \cdot \operatorname{grad} W = \frac{\partial W}{\partial U_{mn}}\frac{\partial U_{mn}}{\partial x_1} + \frac{\partial W}{\partial L_{mn}}\frac{\partial L_{mn}}{\partial x_1} \tag{6.9}$$

is satisfied, then we shall call the cylindrical shell of the Cosserat type the homogeneous along the coordinate x_1, count of along the cylinder generator.

If the relation

$$r\boldsymbol{e}_\varphi \cdot \operatorname{grad} W = \frac{\partial W}{\partial U_{pt}}\frac{\partial U_{pt}}{\partial \varphi} + \frac{\partial W}{\partial L_{pt}}\frac{\partial L_{pt}}{\partial \varphi} \tag{6.10}$$

is satisfied, then we shall call the shell of revolution of the Cosserat type homogeneous along the coordinate φ.

We see that for deformations described by (2.2), (3.2), (4.2), (5.1) and (6.5)-(6.8), the components of the tensors $\mathbf{U}, \mathbf{L}$ depend on one variable only. If the shell is homogeneous along a coordinate then, by (6.2), (6.9) and (6.10), this is also valid for quantities $P_{mn} = \boldsymbol{i}_m \cdot \mathbf{P} \cdot \boldsymbol{i}_n$, $K_{mn} = \boldsymbol{i}_m \cdot \mathbf{K} \cdot \boldsymbol{i}_n$, $P_{pt} = \boldsymbol{a}_p \cdot \mathbf{P} \cdot \boldsymbol{a}_t$ and $K_{pt} = \boldsymbol{a}_p \cdot \mathbf{K} \cdot \boldsymbol{a}_t$. Thus, under the condition that the components of the external loads $\boldsymbol{i}_m \cdot \boldsymbol{f}^*$, $\boldsymbol{i}_m \cdot \boldsymbol{l}^*$, $\boldsymbol{a}_p \cdot \boldsymbol{f}^*$ and $\boldsymbol{a}_p \cdot \boldsymbol{l}^*$ depend on only one coordinate, the equilibrium equations (6.4) are ordinary differential equations. In these systems of equations, besides the unknown functions participating in equations (2.2), (3.2), (4.2), and (5.1), the proper orthogonal matrixes H_{mn}, H_{pt} from the relations (6.5)-(6.8) appear as unknowns.

7. Conclusions

In the conclusion we would like to emphasize that a set of one-dimensional deformations of nonlinear elastic shells is analysed in monographs of ANTMAN [9] and LIBAI and SIMMONDS [10]. The solutions describing pure bending of a cylindrical shell, torsion, inflation and extension of a circular tube, eversion of a spherical shell, axisymmetric deformation of shells of revolution are contained there as well as some other kinds of deformations. Two-parameter families of solutions (2.2), (3.2), (4.2), and (5.1) represented in our paper contain a much wider class of one-dimensional deformations of shells than those of ANTMAN [9] and LIBAI and SIMMONDS [10]. The statements (2.2), (3.2), (4.2), and (5.1) in particular, contain the following new one-dimensional deformations: twisting of a cylindrical shell with a dislocation, straightening and twisting of a sector of a shell of revolution, eversion and twisting of shells of revolution, formation of a disclination in a shell of revolution.

Such statements as (6.5)-(6.8) being constructed for the rotation fields corresponding to one-dimensional deformations of the shells of the Cosserat type are also obtained for the first time.

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