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Physical reasons for abandoning plastic deformation measures in plasticity and viscoplasticity theory

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CONSTITUTIVE EQUATIONS, which characterize the response of a material to future loadings, must depend on state variables that, in principle, can be measured without any prior knowledge of the past history of deformation of the material. This notion of state is consistent with that proposed by ONAT [3] and it is consistent with GILMAN'S comment [4] on physical problems with using total strain as a state variable in plasticity theory. Within the context of this notion of state, elastic strain is a state variable, whereas the total strain and plastic strains are not state variables since they are measured with respect to an arbitrary reference configuration. Alternative constitutive equations which are formulated in terms of elastic deformation measures have been discussed in the literature for finite deformations of elastically isotropic and anisotropic elastic-plastic and elastic-viscoplastic materials. These constitutive equations have the physical properties that they are independent of the choice of the reference configuration, and they do not utilize any measures of total deformation or plastic deformation. The main objectives of this paper are to discuss physical reasons for abandoning total and plastic deformation measures in plasticity and viscoplasticity theory, and to present an alternative small deformation theory which is formulated in terms of elastic strain. Also, aspects of alternative finite deformation theories are reviewed.

1. Introduction

THE MAIN OBJECTIVES of this paper are to discuss physical reasons for abandoning total and plastic deformation measures in plasticity and viscoplasticity theory and to present an alternative small deformation theory which is formulated in terms of elastic strain. This point of view is motivated by the notion that deformation measures are, by definition, relative measures which depend explicitly on the choice of the reference from which they are defined. More specifically, total strain is a measure of the deformation of the body from a fixed, but arbitrary reference configuration. Consequently, the total strain measure depends explicitly on the choice of the reference configuration and thus inherits all of the

arbitrariness of this choice. Similarly, plastic deformation and strain measures are also defined relative to the reference configuration so they too inherit certain arbitrariness. Some ideas related to this perspective have been presented in [1,2].

To be more specific, it is emphasized that given a sample of material in its present configuration, there is no set of experiments that can be used to determine completely the past history of deformation of the material or the arbitrary choice of the reference configuration. Moreover, the physical response of a material, which is characterized by its constitutive equation, cannot depend on the arbitrary choice of the reference configuration. Consequently, such constitutive equations must depend on variables which characterize the state of the material. To avoid arbitrariness, these state variables should have the property that, in principle, they can be measured (given enough identical samples of the material in its present configuration) without any prior knowledge of the past history of deformation of the material. This notion of state is consistent with that proposed by ONAT [3] and it is consistent with GILMAN'S comment [4] on physical problems using the strain as a state variable in plasticity theory:

> "It seems very unfortunate to me that the theory of plasticity was ever cast into a mold of stress-strain relations because 'strain' in the plastic case has no physical meaning that is related to the *material* of the body in question. It is rather like trying to deduce some properties of a liquid from the shape of the container that holds it. The plastic behaviour of a body depends on its structure (crystalline and defect), and on the system of stresses that is applied to it."

Further in this regard, it is noted that some quantities like position, velocity, temperature, force and stress on the outer surface of a body are presumed to be measurable directly. Whereas, other quantities like a hardening variable in plasticity theory are presumed to be measurable indirectly by interpreting the results of experimental data using a general but specific constitutive equation. Moreover, since only constitutive equations for simple materials are considered, it is sufficient to confine attention to the response of uniform, homogeneous materials. Also, the complications of analyzing a sample of material which has been subjected to inhomogeneous deformations and which can have residual stresses, is not considered here.

An outline of this paper is as follows. Section 2 distinguishes between the notions of material states and configurations and it describes some properties of a thermoelastic solid. Section 3 presents a dissipation inequality for the purely mechanical theory and Sec. 4 discusses the main ideas of this paper within the context of the small deformation theory. Then, Sec. 5 and 6 briefly review alternative constitutive equations for finite deformations of elastically isotropic and elastically anisotropic materials, respectively, and Sec. presents conclusions.

Throughout the text, vectors and tensors are denoted by bold- faced symbols; $\mathbf{a} \cdot \mathbf{b}$ denotes the usual dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$;$\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ denotes the dot product of two second order tensors; $\mathrm{tr}(\mathbf{A})$ denotes the trace of $\mathbf{A}$; $\mathbf{B}^{\mathrm{T}}$ denotes the transpose of $\mathbf{B}$; $\mathbf{A}^{-1}$ denotes the inverse of $\mathbf{A}$; $\mathbf{a}\otimes\mathbf{b}$ denotes the tensor product of $\mathbf{a}$ and $\mathbf{b}$; and $\mathbf{I}$ is the unit second order tensor. Also, the usual summation convention is implied over all repeated indices except (e,p,m) which are used to denote elastic, plastic, and microstructural quantities, respectively.

2. Material states, configurations and a thermoelastic solid

The notion of a material state is both mathematical and physical. From the mathematical point of view, a material state is the collection of all variables that are needed to predict the response of the material to future mechanical and thermal loadings. On the other hand, from the physical point of view, these state variables are restricted only to those variables which can be measured, in principle, by experiments on identical samples of the material. For the present discussion, it is convenient to introduce the notion of the *Reference State*.

> The *Reference State* of a material is any state of the material when it is stress-free and at absolute reference temperature θ_0.

Here, a thermoelastic solid is assumed to have a unique shape in its *Reference State*. Consequently, since the stress and temperature can be measured, it is always possible to determine this reference shape by unloading the material to its *Reference State*. This also means that it is natural to introduce a measure of elastic strain from this reference shape. Thus, the elastic strain and the absolute temperature θ are the state variables which characterize a thermoelastic solid. Moreover, by definition, the elastic strain vanishes whenever the material is in its *Reference State*.

The notion of a configuration is a mathematical mapping of the position of all material points onto the physical space, as well as a mapping of all of the state variables onto an appropriately dimensioned vector space. In particular, with reference to a fixed origin, a material point Y is mapped onto the position vector $\mathbf{X}$ in the fixed reference configuration and the same material point is mapped onto the position vector $\mathbf{x}$ in the present configuration at time t.

Consequently, a configuration is a mathematical representation of the physical state of the material. Moreover, since the notion of the *Reference State* fixes only the values of the stress and the temperature, there are an infinite number of configurations of a body in its *Reference State*. These configurations include all superposed rigid body motions as well as a group of homogeneously deformed configurations for elastic-plastic materials.

3. Rate of material dissipation in the purely mechanical theory

To present the physical argument in its simplest terms, it is convenient to focus attention on the purely mechanical theory for which θ equals the reference temperature θ_0. More specifically, with reference to the present configuration, it is convenient to consider a material region P with smooth closed surface ∂P. Now, the rate of material dissipation $\mathcal{D}$ can be defined by the equation

$$\int_P \mathcal{D} dv = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}}, \tag{3.1}$$

where a superposed dot denotes material time differentiation, $\mathcal{W}$ is the rate of work due to the specific (per unit mass) external body force $\mathbf{b}$ and the surface tractions $\mathbf{t}$, $\mathcal{K}$ is the kinetic energy, and $\mathcal{U}$ is the internal strain energy due to the specific strain energy function Σ, which are defined by

$$\mathcal{W} = \int_P \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial P} \mathbf{t} \cdot \mathbf{v} da, \quad K = \int_P \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} dv, \quad U = \int_P \rho \Sigma dv. \tag{3.2}$$

In these formulas, ρ is the present value of the mass density, $\mathbf{v}$ is the absolute velocity of a material point, dv is the present element of volume and da is the present element of area.

Next, using the conservation of mass, the balances of linear and angular momentum, the fact that the traction vector is related to the Cauchy stress $\mathbf{T}$ and the outward unit normal vector $\mathbf{n}$ to ∂P by the formula $\mathbf{t}=\mathbf{T}\mathbf{n}$, and using standard continuity assumptions, it can be shown that the local form of the dissipation $\mathcal{D}$ becomes

$$\mathcal{D} = \mathbf{T} \cdot \mathbf{D} - \rho \dot{\Sigma} \geq 0. \tag{3.3}$$

In this expression, $\mathbf{D}$ is the symmetric part of the velocity gradient $\mathbf{L}=\partial \mathbf{v}/\partial \mathbf{x}$ and $\mathbf{W}$ is its skew-symmetric part, which are defined by

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T). \tag{3.4}$$

Moreover it is assumed that the rate of material dissipation $\mathcal{D}$ must be non-negative for all motions.

4. Small deformation theory of elastic-viscoplastic materials

A review of the small deformation theory of plasticity and thermoplasticity can be found in [5]. Within the context of this theory, the plastic strain $\boldsymbol{\varepsilon}_p$ can be introduced through an evolution equation for its rate of the form

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{D}_p, \quad \mathbf{D}_p = \Gamma \bar{\mathbf{D}}_p, \quad \Gamma \geq 0, \quad \bar{\mathbf{D}}_p^T = \bar{\mathbf{D}}_p, \tag{4.1}$$

where $\mathbf{D}_p$ represents the relaxation effects of plasticity on the stress. In this equation, the symmetric second order tensor $\bar{\mathbf{D}}_p$ controls the direction of plastic strain rate and the non-negative scalar Γ influences the magnitude of plastic strain rate. Also, it is common to introduce hardening variables which control both the isotropic hardening and directional hardening [6]. However, here, it suffices to consider only isotropic hardening κ which is determined by integrating the evolution equation

$$\dot{\kappa} = \Gamma K, \tag{4.2}$$

where K requires a constitutive equation. Moreover, for nonporous metals, it is usually assumed that plastic deformation is isochoric so that $\bar{\mathbf{D}}_p$ must satisfy the restriction

$$\bar{\mathbf{D}}_p \cdot \mathbf{I} = 0, \tag{4.3}$$

which causes $\boldsymbol{\varepsilon}_p$ to be a deviatoric tensor. Furthermore, it can be observed from the evolution equations (4.1) and (4.2) that when Γ vanishes, both the plastic strain rate and hardening rate vanish, so the material response becomes elastic.

If both of the evolution equations (4.1) and (4.2) are homogeneous of order one in time, then the constitutive equations characterize rate-insensitive plasticity. Otherwise, the constitutive equations characterize rate-sensitive viscoplasticity. For plasticity theory with a yield function [5], the scalar Γ in (4.1) is determined by a consistency condition in both the stress-space and strain-space [7] formulations. For the overstress formulation of viscoplasticity [8,9], the yield function is retained but the consistency condition is no longer enforced. Also, an alternative unified formulation of viscoplasticity can be proposed which does not use either a yield function or the consistency condition [10-12].

Total strain $\boldsymbol{\varepsilon}$ is a measure of the deformation of the material from a specified fixed reference configuration. Within the context of the small deformation theory, the total strain can be separated into a pure measure of dilatation ε and a deviatoric tensor $\boldsymbol{\varepsilon}'$, which is a pure measure of distortion, such that

$$\boldsymbol{\varepsilon} = \frac{\varepsilon}{3}\mathbf{I} + \boldsymbol{\varepsilon}', \quad \varepsilon = \boldsymbol{\varepsilon} \cdot \mathbf{I}, \quad \boldsymbol{\varepsilon}' \cdot \mathbf{I} = 0. \tag{4.4}$$

Moreover, these quantities can be obtained by integrating equations for their rates which can be approximated by the formulas

$$\dot{\varepsilon} = \mathbf{D} \cdot \mathbf{I}, \quad \dot{\boldsymbol{\varepsilon}} = \mathbf{D}' = \mathbf{D} - \frac{1}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{I}, \tag{4.5}$$

where $\mathbf{D}'$ is the deviatoric part of $\mathbf{D}$.

For an elastic material, it is common to express the stress as a unique function of the total strain $\boldsymbol{\varepsilon}$ that is insensitive to the history of loading. In contrast, the

response of an elastic-plastic material depends on the history of loading and the stress can have different values for the same value of total strain. One of the reasons for introducing the plastic strain $\boldsymbol{\varepsilon}_{\mathrm{p}}$ in plasticity theory is to account for this history-dependence. Also, since it is usually assumed that stress depends on elastic strain $\boldsymbol{\varepsilon}_{\mathrm{e}}$, it is necessary to introduce an additional definition of elastic strain as a function of total strain and plastic strain. In the small deformation theory this definition is simply the difference between the total and plastic strains

$$\boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p \tag{4.6}$$

Furthermore, using the fact that $\boldsymbol{\varepsilon}_{\mathrm{p}}$ is a deviatoric tensor, it follows that the elastic strain can be represented in the form

$$\boldsymbol{\varepsilon}_e = \frac{\varepsilon_e}{3}\mathbf{I} + \boldsymbol{\varepsilon}'_e, \quad \varepsilon_e = \varepsilon, \quad \boldsymbol{\varepsilon}'_e = \boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}_p, \tag{4.7}$$

where ε_e is a measure of elastic dilatation and $\boldsymbol{\varepsilon}'_e$ is a measure of elastic distortion.

Next, it is assumed that the stress $\mathbf{T}$ and the strain energy Σ are functions of the elastic strains only

$$\mathbf{T} = \mathbf{T}(\varepsilon_e, \boldsymbol{\varepsilon}'_e), \quad \Sigma = \Sigma(\varepsilon_e, \boldsymbol{\varepsilon}'_e), \tag{4.8}$$

and that the constitutive equation for stress satisfies the restriction that

$$\mathbf{T}(0,0) = 0. \tag{4.9}$$

It then follows from (4.5) and (4.7) that the dissipation inequality (3.3) reduces to

$$\mathcal{D} = \left[\mathbf{T} - \rho\frac{\partial\Sigma}{\partial\varepsilon_e}\mathbf{I} - \rho\frac{\partial\Sigma}{\partial\boldsymbol{\varepsilon}'_e}\right]\cdot\mathbf{D} + \rho\frac{\partial\Sigma}{\partial\boldsymbol{\varepsilon}'_e}\cdot\mathbf{D}_p \geq 0. \tag{4.10}$$

For plasticity theory with a yield function [5,7], it can be shown by considering elastic response or elastic unloading, that the stress must be determined by derivatives of the strain energy function such that

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}', \quad p = -\rho\frac{\partial\Sigma}{\partial\varepsilon_e}, \quad \mathbf{T}' = \rho\frac{\partial\Sigma}{\partial\boldsymbol{\varepsilon}'_e}, \tag{4.11}$$

where p is the pressure and $\mathbf{T}'$ is the deviatoric stress. Thus, the dissipation inequality reduces to

$$\mathcal{D} = \mathbf{T}'\cdot\mathbf{D}_p \geq 0, \tag{4.12}$$

This condition places a restriction on the tensor $\bar{\mathbf{D}}_p$ which ensures that plasticity is dissipative. Next, for viscoplasticity, with or without a yield function, the

stress is assumed to be given by the expressions (4.11) for all motions so that the dissipation inequality again reduces to (4.12). Moreover, it is noted that more general constitutive equations which include dependence of the energy on the hardening variables can be considered without difficulty [13,14].

Within the context of this formulation, it is necessary to specify initial conditions for the variables

$$(\varepsilon,\ \varepsilon',\ \varepsilon_p,\ \kappa), \tag{4.13}$$

in order to integrate the evolution equations (4.1), (4.2) and (4.5). The main objective of this paper is to emphasize that total strain and plastic strain are not state variables that can be measured in the present configuration, so that an alternative formulation of plasticity theory is required which depends only on measurable quantities.

To this end, it is noted that in the present loaded configuration it is possible to measure the force acting on a surface and its surface area, so that it is possible to measure the traction vector acting on an arbitrary surface of the body. Since attention is confined to homogeneous deformations with homogeneous states of stress, it follows that all components of the stress tensor $\mathbf{T}$ can be determined by measuring the traction vector on three planes whose normals are linearly independent.

Next, it is emphasized that the constitutive equation for stress (4.11) depends only on the elastic strains ε_e and ε'_e. Thus, assuming that this functional form is invertible, it is possible to determine the values of $(\varepsilon_e, \varepsilon'_e)$ as functions of the stress $\mathbf{T}$. Consequently, the current values$(\varepsilon_e, \varepsilon'_e)$ of elastic strain can be determined indirectly by measuring the current value of stress $\mathbf{T}$. In this regard, it is important to emphasize that in contrast with the stress $\mathbf{T}$, which can be determined by direct measurements, the values $(\varepsilon_e, \varepsilon'_e)$ of elastic strain are determined only indirectly because they depend on the specific choice of the constitutive equations. This fundamental difference between quantities that are determined by direct measurements and other quantities that are determined only by indirect measurements is a consequence of the essential physical fact that different materials respond differently to the same stress state. More specifically, the elastic strains for rubber and steel are different when the two materials are subjected to the same stress state.

The hardening variable κ is another example of a quantity that depends on specific constitutive assumptions and can only be determined by indirect measurements. For example, in the simplest theory with a yield function based on the von Mises stress, the value of κ determines the current value of the yield strength. Given a specific definition of yielding (either based on a specified amount of inelastic deformation or based on a specified amount of change in stiffness), experiments can be performed on a finite set of identical samples to

determine the current value of the yield strength. For a more general theory, the hardening variable κ measures the resistance to plastic flow and usually appears in the scalars Γ and K in (4.2). Again, by comparing the predictions of a specified set of constitutive equations with experimental data (on a finite set of identical samples), it is possible to determine the initial value of hardening that is consistent with the proposed set of constitutive equations.

Physically, the above discussion indicates that the values of the elastic strains ε_e and $\boldsymbol{\varepsilon}'_e$, and the value of κ can be measured, in principle, in the present configuration. In particular, the initial values of these variables

$$(\varepsilon_e, \boldsymbol{\varepsilon}'_e, \kappa), \tag{4.14}$$

can be measured without any prior knowledge of the past history of loading. Thus, these quantities are state variables in the sense described in the Introduction.

In contrast, it follows from the definition $(4.7)_3$ that only the difference between the total deviatoric strain $\boldsymbol{\varepsilon}'$and the plastic stain $\boldsymbol{\varepsilon}_p$ can be measured. More specifically, it is obvious that the elastic strain and the stress remain unaffected by the transformation that subtracts an arbitrary deviatoric tensor $\bar{\boldsymbol{\varepsilon}}'$ from both the total deviatoric strain $\boldsymbol{\varepsilon}'$ and the plastic strain $\boldsymbol{\varepsilon}_p$ such that

$$\boldsymbol{\varepsilon}'^{*} = \boldsymbol{\varepsilon}' - \bar{\boldsymbol{\varepsilon}}', \quad \boldsymbol{\varepsilon}^{*}_p = \boldsymbol{\varepsilon}_p - \bar{\boldsymbol{\varepsilon}}', \tag{4.15}$$

and the elastic strain $\boldsymbol{\varepsilon}'^{*}_e$ associated with these transformed variables becomes

$$\boldsymbol{\varepsilon}'^{*}_e = \boldsymbol{\varepsilon}'^{*} - \boldsymbol{\varepsilon}^{*}_p = \boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon}_p = \boldsymbol{\varepsilon}'_e. \tag{4.16}$$

Physically, this means that the initial values for $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\varepsilon}_p$ can never be measured without prior knowledge of the past history of deformation since they always include the arbitrariness of the deviatoric tensor $\bar{\boldsymbol{\varepsilon}}'$. Consequently, the initial values of these variables, which are required in the integrate the evolution equations $(4.5)_2$ and $(4.1)_1$, cannot be determined uniquely.

In other words, only the difference between the total deviatoric strain $\boldsymbol{\varepsilon}'$ and the plastic strain $\boldsymbol{\varepsilon}_p$ influences the response of the material. Consequently, the individual actual values of these quantities have no physical meaning, as was clearly stated by GILMAN [4]. This also means that the notions of total strain and plastic strain should be abandoned in the formulation of plasticity theories.

For the small deformation theory, it is quite easy to formulate constitutive equations for plasticity and viscoplasticity which are free from these physical inconsistencies. In particular, it is possible to differentiate equations $(4.7)_{2,3}$ with respect to time and to use the expressions $(4.1)_1$ and (4.5) to develop evolution equations for the elastic strains $\boldsymbol{\varepsilon}_e$ and $\boldsymbol{\varepsilon}'_e$ directly of the forms

$$\dot{\boldsymbol{\varepsilon}}_e = \mathbf{D} \cdot \mathbf{I}, \quad \dot{\boldsymbol{\varepsilon}}'_e = \mathbf{D}' - \mathbf{D}_p. \tag{4.17}$$

Now, since the elastic strains ε_e and $\boldsymbol{\varepsilon}'_e$ are state variables, which can be measured in the present configuration, the initial conditions which are required to integrate these evolution equations can be determined without arbitrariness. The resulting theory is characterized by the constitutive equations for the strain energy $(4.8)_2$, the stress (4.11), the rate of plastic dissipation $\mathbf{D}_p$, the expression for the hardening rate K (4.2), and the evolution equations (4.2) and (4.17). This theory has the properties that it is independent of the choice of the reference configuration, and it does not utilize any measures of total deformation or plastic deformation. The influence of plastic deformation only enters the constitutive equations through the rate of relaxation $\mathbf{D}_p$, which is a function of state variables that can be determined in the present configuration only. Also, when Γ vanishes, the plastic deformation rate vanishes and the constitutive equations characterize the usual small deformation theory of elastic materials.

A rather standard set of constitutive equations for an elastically isotropic material can be obtained by specifying

$$\rho\Sigma = \frac{1}{2}k\varepsilon_e^2 + \mu\boldsymbol{\varepsilon}'_e \cdot \boldsymbol{\varepsilon}'_e, \quad \bar{\mathbf{D}}_p = \boldsymbol{\varepsilon}'_e, \quad K = m(Z_1 - \kappa), \tag{4.18}$$

where the material constants k and μ are the bulk modulus and the shear modulus, respectively, m is a material constant that controls the rate of hardening, and Z_1 controls the saturated value of hardening. It then follows from (4.11) and $(4.1)_2$ that

$$p = -k\varepsilon_e, \quad \mathbf{T}' = 2\mu\boldsymbol{\varepsilon}'_e, \quad \mathbf{D}_p = \Gamma\boldsymbol{\varepsilon}'_e = \frac{\Gamma}{2\mu}\mathbf{T}', \tag{4.19}$$

which shows that the rate of relaxation $\mathbf{D}_p$ is consistent with the classical Prandtl-Reuss form. SWEGLE and GRADY [15,16] have developed an overstress model for modeling viscoplasticity in shock waves which is equivalent to specifying Γ in the form

$$\Gamma = \Gamma_0 \frac{<\sigma_e - \kappa>^2}{\kappa_0^2}, \quad \sigma_e^2 = \frac{3}{2}\mathbf{T}' \cdot \mathbf{T}', \tag{4.20}$$

where Γ_0 is a material constant controlling the strain-rate sensitivity, κ_0 is the annealed value of hardening, σ_e is the von Mises effective stress, and the McAuley brackets are defined by

$$<x> = \frac{x + |x|}{2}. \tag{4.21}$$

Alternatively, a modified version of the Bodner-Partom viscoplasticity model without a yield function can be obtained by taking Γ in the form [17]

$$\Gamma = \Gamma_0 \exp\left[-\frac{1}{2}\left\{\frac{\kappa}{\sigma_e}\right\}^{2n}\right]. \tag{4.22}$$

For large values of stress ($\sigma_e >> \kappa$) or very high strain rates, the rate-dependence of the material is controlled almost entirely by the material constant Γ_0. On the other hand, at lower stresses ($\sigma_e < \kappa$) or lower strain rates, the rate-dependence of the material is controlled mainly by the material constant n. Both of the functions (4.20) and (4.22) cause the evolution equation $(4.17)_2$ to be a stiff differential equation which requires special numerical methods that have been developed in [18-20].

5. Finite deformation of elastically isotropic elastic-viscoplastic materials

ECKART [21] seems to have been the first to develop a properly invariant theory of elastically isotropic inelastic solids which does not depend on the choice of the reference configuration or any measure of total or plastic deformations. LEONOV [22], independently, developed the same theory for describing the response of polymeric liquids. The main idea in this theory is to propose an evolution equation directly for an elastic deformation tensor. Specifically, the symmetric tensor $\mathbf{B}_e$ is introduced as a measure of elastic deformation which is determined by the evolution equation

$$\dot{\mathbf{B}}_e = \mathbf{L}\mathbf{B}_e + \mathbf{B}_e\mathbf{L}^T - 2\mathbf{D}_p, \quad \mathbf{D}_p = \Gamma\bar{\mathbf{D}}_p, \tag{5.1}$$

where Γ and $\bar{\mathbf{D}}_p$ have the same physical meanings as those quantities related to $(4.1)_2$. Next, a pure measure of elastic dilation J_e is defined by the formula

$$J_e = [\det\mathbf{B}_e]^{1/2}, \tag{5.2}$$

so that

$$\dot{J}_e = \frac{1}{2}J_e\left[\dot{\mathbf{B}}_e \cdot \mathbf{B}_e^{-1}\right] = J_e[\mathbf{D}\cdot\mathbf{I} - \Gamma\bar{\mathbf{D}}_p \cdot \mathbf{B}_e^{-1}]. \tag{5.3}$$

Consequently, the condition that plastic deformation rate is isochoric and does not influence the elastic dilation, requires $\bar{\mathbf{D}}_p$ to satisfy the restriction that

$$\bar{\mathbf{D}}_p \cdot \mathbf{B}_e^{-1} = 0. \tag{5.4}$$

Thus, it is possible to use the notions proposed by Flory [23] to define a pure measure of elastic distortional deformation $\mathbf{B}'_e$ as a unimodular symmetric tensor

$$\mathbf{B}'_e = J_e^{-2/3}\mathbf{B}_e, \quad \det\mathbf{B}'_e = 1. \tag{5.5}$$

Then, the evolution equations for the pure measure of elastic dilation J_e and the pure measure of elastic distortional deformation $\mathbf{B}'_e$ can be written in the forms

$$\dot{J}_e = J_e \mathbf{D} \cdot \mathbf{I}, \quad \dot{\mathbf{B}}'_e = \mathbf{L}\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}^T - \frac{2}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{B}'_e - 2\Gamma\bar{\mathbf{D}}_p. \tag{5.6}$$

Moreover, a specific form for $\bar{\mathbf{D}}_p$ and a numerical procedure for integrating the evolution equation $(5.6)_2$ can be found in [20].

Under superposed rigid body motions (SRBM) it is well known that the mass density ρ in the present configuration and the kinematic quantities $\mathbf{D}$ and $\mathbf{W}$ transform to the values ρ^+, $\mathbf{D}^+$, $\mathbf{W}^+$ in the superposed configuration such that

$$\rho^+ = \rho, \quad \mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \quad \mathbf{W}^+ = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega}, \tag{5.7}$$

where $\mathbf{Q}(t)$ is an arbitrary proper orthogonal tensor function of time only characterizing the rigid rotation, and $\boldsymbol{\Omega}(t)$ is the skew-symmetric tensor function of time related to $\dot{\mathbf{Q}}$

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}, \quad \det\mathbf{Q} = 1, \quad \dot{\mathbf{Q}} = \boldsymbol{\Omega}\mathbf{Q}, \quad \boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega}. \tag{5.8}$$

Also, the stress $\mathbf{T}$ and the strain energy Σ transform by the formulas

$$\mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \quad \Sigma^+ = \Sigma. \tag{5.9}$$

Next, with the help of the assumption that $\Gamma, \bar{\mathbf{D}}_p$ and K satisfy the transformation relations

$$\Gamma^+ = \Gamma, \quad \bar{\mathbf{D}}_p^+ = \mathbf{Q}\bar{\mathbf{D}}_p\mathbf{Q}^T, \quad K^+ = K, \tag{5.10}$$

it follows that $\mathbf{B}_e, J_e, \mathbf{B}'_e$ and κ transform under SRBM by

$$\mathbf{B}_e^+ = \mathbf{Q}\mathbf{B}_e\mathbf{Q}^{\mathbf{T}}, \quad J_e^+ = J_e, \quad \mathbf{B}_e^{'+} = \mathbf{Q}\mathbf{B}_e^{'}\mathbf{Q}^T, \quad K^+ = K. \tag{5.11}$$

Here, and throughout the text, a superposed $(^+)$ is added to any variable to denote its value in the superposed configuration.

Furthermore, the stress and the strain energy are functions of elastic deformation quantities

$$\mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e), \quad \Sigma = \Sigma, (J_e, \mathbf{B}'_e) \tag{5.12}$$

where the constitutive equation for stress satisfies the restriction

$$\mathbf{T}(1, \mathbf{I}) = 0 \tag{5.13}$$

However, since Σ must remain unaltered by SRBM, it can be a function of $\mathbf{B}'_e$ only through its two nontrivial invariants, which can be expressed in the forms

$$\alpha_1 = \mathbf{B}'_e \cdot \mathbf{I}, \quad \alpha_2 = \mathbf{B}'_e \cdot \mathbf{B}'_e, \tag{5.14}$$

so that Σ becomes

$$\Sigma = \Sigma(J_e, \alpha_1, \alpha_2). \tag{5.15}$$

Next, using the evolution equation $(5.6)_2$, it can be shown that

$$\dot{\alpha}_1 = 2\left[\mathbf{B}'_e - \frac{1}{3}(\mathbf{B}'_e \cdot \mathbf{I})\mathbf{I}\right] \cdot \mathbf{D} - 2\Gamma \bar{\mathbf{D}}_p \cdot \mathbf{I}, \tag{5.16}$$

$$\dot{\alpha}_2 = 4\left[\mathbf{B}'^2_e - \frac{1}{3}(\mathbf{B}'_e \cdot \mathbf{B}'_e)\mathbf{I}\right] \cdot \mathbf{D} - 4\Gamma \bar{\mathbf{D}}_p \cdot \mathbf{B}'_e.$$

Thus, the constitutive equations

$$p = -\rho_0 \frac{\partial \Sigma}{\partial J_e},$$

$$\mathbf{T}' = 2\rho \frac{\partial \Sigma}{\partial \alpha_1}\left[\mathbf{B}'_e - \frac{1}{3}(\mathbf{B}'_e \cdot \mathbf{I})\mathbf{I}\right] + 4\rho \frac{\partial \Sigma}{\partial \alpha_2}\left[\mathbf{B}'^2_e - \frac{1}{3}(\mathbf{B}'_e \cdot \mathbf{B}'_e)\mathbf{I}\right], \tag{5.17}$$

can be used to reduce the dissipation inequality (3.3) to the form

$$\mathcal{D} = 2\Gamma\left[\rho \frac{\partial \Sigma}{\partial \alpha_1}\mathbf{I} + 2\rho \frac{\partial \Sigma}{\partial \alpha_2}\mathbf{B}'_e\right] \cdot \bar{\mathbf{D}}_p \geq 0, \tag{5.18}$$

which restricts the tensor $\bar{\mathbf{D}}_p$. Also, in $(5.17)_1$, use has been made of the conservation law of of mass which relates the density ρ to the density ρ_0 in the *Reference State*

$$\rho J_e = \rho_0. \tag{5.19}$$

This theory for elastically isotropic elastic-viscoplastic materials is characterized by the constitutive equations for the strain energy (5.15), the stress (5.17), the rate of plastic dissipation $(5.1)_2$, the expression for the hardening rate K (4.2), and the evolution equations (4.2) and (5.6). Assuming that the stress is an invertible function of the elastic deformation quantities J_e and $\mathbf{B}'_e$, it follows that

$$\{J_e, \mathbf{B}'_e, \kappa\}, \tag{5.20}$$

are state variables since they can be measured, in principle, without any prior knowledge of the past history of deformation of the material. In particular, the initial values of these quantities, which are required to integrate the evolution equations (4.2) and (5.6), can be measured without arbitrariness. Also, when Γ vanishes, the plastic deformation rate vanishes and the constitutive equations characterize general elastically isotropic materials, with $\mathbf{B}_e$ reducing to the left Cauchy-Green deformation tensor $\mathbf{B}$.

A simple specific set of constitutive equations can be obtained by specifying the strain energy Σ and the plastic deformation rate tensor $\bar{\mathbf{D}}_p$ in the forms [13,14]

$$\rho_0\Sigma = k[1 - J_e - \ln(\mathrm{J_e})] + \frac{1}{2}\mu(\alpha_1 - 3), \quad 2\bar{\mathbf{D}}_{\mathrm{p}} = \mathbf{B'}_{\mathrm{e}} - \frac{3}{\mathbf{B'}_{\mathrm{e}}^{-1}\cdot\mathbf{I}}\mathbf{I}, \tag{5.21}$$

where again the material constants k and μ are the bulk modulus and the shear modulus, respectively. It then follows from (5.17) and (5.19) that stresses become

$$p = k\left[\frac{1}{J_e} - 1\right], \quad \mathbf{T'} = J_e^{-1}\mu\left[\mathbf{B'}_e - \frac{1}{3}(\mathbf{B'}_e\cdot\mathbf{I})\mathbf{I}\right]. \tag{5.22}$$

Also, the constitutive equation for K is given by $(4.18)_3$, Γ can be taken in the forms (4.20) or (4.21), and the numerical methods developed in [18-20] remain applicable to this finite deformation theory.

6. Finite deformation of elastically anisotropic elastic-viscoplastic materials

A critical review of finite deformation theories of elastically anisotropic elastic-plastic materials has been presented by NAGHDI [24]. In order to model anisotropic response, these theories are usually formulated in terms of deformation tensors that are related to the reference configuration and hence are trivially invariant under SRBM. However, as has been previously discussed, the specific choice of the reference configuration is a part of the history of the material that can never be determined by experiments on identical samples of the material in its present state. Consequently, it is necessary to consider an alternative formulation that is capable of describing the material anisotropy. One such formulation has been motivated by the physical discussion presented by BESSELING [25] and has been developed in terms of physically based microstructural variables [13,14].

The response of a material that is elastically anisotropic depends on the orientation of the material relative to the loading direction. This means that there are specific material directions which are related to the microstructure of the material and which can be determined by experiments. Within the context

of this alternative formulation of plasticity theory [13,14], these microstructural directions in the present configuration are represented by a triad of linearly independent vectors $\mathbf{m}_i (i = 1, 2, 3)$, which characterize the present state of the material. In this theory, it is convenient to introduce the metric

$$m_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j, \tag{6.1}$$

which measures deformation of the microstructure relative to the *Reference State* of the material. Moreover, the vectors $\mathbf{m}_i$ can be normalized so that $\mathbf{m}_i$ become an orthonormal set of vectors whenever the material is in its *Reference State*

$$m_{ij} = \delta_{ij} \quad \text{in the } \textit{Reference State}, \tag{6.2}$$

where δ_{ij} is the Kronecker delta symbol. However, for a general material state, the metric m_{ij} attains the values different from δ_{ij}.

The general theory requires evolution equations for the vectors $\mathbf{m}_i$. In order to motivate the forms of these evolution equations, it is recalled that the rate of change of a macroscopic material line element $d\mathbf{x}$ in the present configuration can be expressed in the form

$$d\dot{\mathbf{x}} = \mathbf{L}\, d\mathbf{x}. \tag{6.3}$$

An important physical characteristic of the vectors $\mathbf{m}_i$ is that, in general, they characterize the elastic deformation of the microstructure which is not directly connected with the deformation of macroscopic material line elements $d\mathbf{x}$. To model this physical distinction between the evolution of the microstructure and the macroscopic total deformation rate, the evolution equations for $\mathbf{m}_i$ are specified by modifying (6.3), such that

$$\dot{\mathbf{m}}_i = \mathbf{L}_m \mathbf{m}_i, \quad \mathbf{L}_m = \mathbf{L} - \mathbf{L}_p. \tag{6.4}$$

In these equations, $\mathbf{L}_m$ denotes the microstructural deformation rate, and $\mathbf{L}_p$ is a second order tensor that characterizes the relaxation effects of plasticity. In general, $\mathbf{L}_p$ has a symmetric part $\mathbf{D}_p$ and a skew-symmetric part $\mathbf{W}_p$

$$\mathbf{L}_p = \mathbf{D}_p + \mathbf{W}_p, \quad \mathbf{D}_p = \frac{1}{2}(\mathbf{L}_p + \mathbf{L}_p^T), \quad \mathbf{W}_p = \frac{1}{2}(\mathbf{L}_p - \mathbf{L}_p^T), \tag{6.5}$$

and $\mathbf{D}_p$ and $\mathbf{W}_p$ can be expressed in the forms

$$\mathbf{D}_p = \Gamma \bar{\mathbf{D}}_p, \quad \mathbf{W}_p = \Gamma \bar{\mathbf{W}}_p, \tag{6.6}$$

where $\Gamma, \bar{\mathbf{D}}_p$ and $\mathbf{W}_p$ require constitutive equations. In this regard it should be mentioned that the term $\mathbf{W}_p$, which is currently referred to as the plastic spin in

crystal plasticity [26], is not new and was required in the constitutive equations for elastically anisotropic material response proposed by BESSELING [25].

Since $\mathbf{m}_i$ are normalized vectors that measure the deformation relative to the *Reference State*, it is possible to introduce the dilatation J_e and a pure measure of distortional deformation m_{ij}, both measured relative to the *Reference State*, through the definitions

$$J_e = \mathbf{m}_1 \times \mathbf{m}_2 \cdot \mathbf{m}_3 > 0, \quad m'_{ij} = J_e^{-2/3} m_{ij}. \tag{6.7}$$

Moreover, by using (6.4), it can be shown that

$$\dot{J}_e = J_e[\mathbf{D} \cdot \mathbf{I} - \mathbf{D}_p \cdot \mathbf{I}]. \tag{6.8}$$

Consequently, the condition that plastic deformation rate is isochoric and does not influence the elastic dilation requires $\bar{\mathbf{D}}_p$ to satisfy the condition that

$$\bar{\mathbf{D}}_p \cdot \mathbf{I} = 0. \tag{6.9}$$

Also, it is possible to express the derivatives of J_e and m'_{ij} in the forms

$$\dot{J}_e = J_e \mathbf{D} \cdot \mathbf{I}, \quad \dot{m}'_{ij} = 2\left[\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij} \mathbf{I}\right] \cdot \mathbf{D} - 2[\mathbf{m}'_i \otimes \mathbf{m}'j] \cdot \mathbf{D}_p, \tag{6.10}$$

where the vectors $\mathbf{m}'_i$ are defined by

$$\mathbf{m}'_i = J_e^{-1/3} \mathbf{m}_i. \tag{6.11}$$

Thus, it follows from (6.4) and $(6.10)_2$ that spin tensor $\mathbf{W}_p$ affects only the orientation of the vectors $\mathbf{m}_i$, and specifically does not influence the rate of change of the elastic distortional deformation tensor m'_{ij}.

These constitutive equations are properly invariant under SRBM provided that the constitutive equation for $\mathbf{L}_p$ satisfies the transformation relations

$$\mathbf{L}_p^+ = \mathbf{Q} \mathbf{L}_p \mathbf{Q}^T. \tag{6.12}$$

Consequently, various other quantities transform under SRBM as follows:

$$\Gamma^+ = \Gamma, \quad \bar{\mathbf{D}}_p^+ = \mathbf{Q} \bar{\mathbf{D}}_p \mathbf{Q}^T, \quad \bar{\mathbf{W}}_p^+ = \mathbf{Q} \bar{\mathbf{W}}_p \mathbf{Q}^T, \tag{6.13}$$

$$\mathbf{m}_i^+ = \mathbf{Q} \mathbf{m}_i, \quad \mathbf{m}'^+_i = \mathbf{Q} \mathbf{m}'_i,$$

$$J_e^+ = J_e, \quad m_{ij}^+ = m_{ij}, \quad m'^+_{ij} = m'_{ij}.$$

Since J_e and m'_{ij} measures of elastic deformation from the *Reference State*, it follows that the stress and the strain energy are functions of the forms

$$\mathbf{T} = \mathbf{T}(J_e, m'_{ij}, \mathbf{m}'_i), \quad \Sigma = \Sigma(J_e, \mathbf{m}'_i), \tag{6.14}$$

where the constitutive equation for stress satisfies the condition

$$\mathbf{T}(1, \delta_{ij}, \mathbf{m}'_i) = 0. \tag{6.15}$$

However, since Σ must remain unaltered by SRBM, it can be a function of $\mathbf{m}'_i$ only through the metric m'_{ij} so that Σ becomes

$$\Sigma = \Sigma(J_e, m'_{ij}). \tag{6.16}$$

Thus, with the help of (6.10), it can be shown that the constitutive equations

$$p = -\rho_0 \frac{\partial \Sigma}{\partial J_e}, \quad \mathbf{T}' = 2\rho \frac{\partial \Sigma}{\partial m'_{ij}} \left[\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij} \mathbf{I} \right], \tag{6.17}$$

and the condition(6.9) can be used to reduce the dissipation inequality (3.3) to the form

$$D = \Gamma \mathbf{T}' \cdot \bar{\mathbf{D}}_p \geq 0, \tag{6.18}$$

which restricts the tensor $\bar{\mathbf{D}}_p$. Also, in $(6.17)_1$ use has been made of the law of conservation of mass (5.19).

This theory for elastically anisotropic elastic-viscoplastic materials is characterized by the constitutive equations for the strain energy (6.16), the stress (6.17), the relaxation effects of plasticity (6.5), (6.6), the definitions (6.7), the expression for the hardening rate K (4.2), and the evolution equations (4.2) and $(6.4)_1$. In this theory, the microstructural vectors $\mathbf{m}_i$ are state variables since they can be measured, in principle, without any prior knowledge of the past history of deformation of the material. When Γ vanishes, the relaxation effects of plasticity vanish and the constitutive equations characterize general elastically anisotropic materials.

For a general anisotropic response, the initial values of $\mathbf{m}_i$ which are required to integrate the evolution equations $(6.4)_1$, can be measured. Moreover, $\mathbf{m}_i$ are directly connected to identifiable directions in the microstructure of the material. Depending on the symmetry properties of the specific material under consideration, there can be some degree of arbitrariness in the determination of the initial values of $\mathbf{m}_i$. However, any such arbitrariness must be reflected in corresponding restrictions on the symmetry of the strain energy function which cause the resulting material response to be uninfluenced by this arbitrariness. For example, if the material is elastically isotropic, then there is no physical experiment that can distinguish between the directions $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$. For this case, the strain energy function must depend on m'_{ij} only through its two invariants, which are related to the elastic deformation tensor $\mathbf{B}'_e$ by the equations

$$\mathbf{B}'_e = \mathbf{m}'_i \otimes \mathbf{m}'_i, \quad \alpha_1 = \mathbf{B}'_e \cdot \mathbf{I} = m'_{ii}, \quad \alpha_2 = \mathbf{B}'_e \cdot \mathbf{B}'_e = m'_{ij} m'_{ij}. \tag{6.19}$$

Due to the summation over the repeated indices in these expressions, it is obvious that these variables cannot distinguish between the 1, 2 and 3 directions in the material's microstructure.

Since J_e and m'_{ij} are trivially invariant under SRBM, the strain energy function can be an arbitrary function of its arguments which satisfies restrictions related to (6.15). Moreover, the tensors $\bar{\mathbf{D}}_p$ and $\bar{\mathbf{W}}_p$ are properly invariant under SRBM (6.13) if they are expressed in terms of components relative to $\mathbf{m}_i$ of the forms

$$\bar{\mathbf{D}}_p = \bar{D}^p_{ij}(\mathbf{m}_i \otimes \mathbf{m}_j), \quad \bar{\mathbf{W}}_p = \bar{W}^p_{ij}(\mathbf{m}_i \otimes \mathbf{m}_j), \tag{6.20}$$

where $\bar{D}^p_{ij}$ and $\bar{W}^p_{ij}$ are arbitrary functions (which remain unaltered by SRBM) of the state variables $\{J_e, m'_{ij}\}$. Specific examples of these tensors can be developed to incorporate standard expressions used in crystal plasticity [26] without difficulty.

7. Conclusions

The discussion in this paper emphasizes the notion that neither total strain nor plastic strain are measurable quantities in the present configuration of an elastic-plastic material. Consequently, total strain and plastic strain are not state variables and therefore should be abandoned in the formulation of constitutive equations. An alternative approach to the development of constitutive equations for elastically isotropic response of inelastic materials [21, 22] has been reviewed in both the small deformation and the large deformation contexts. Within the context of this alternative approach, evolution equations are proposed directly for elastic deformation quantities and hardening (4.14) or (5.20), instead of for total deformation, plastic deformation and hardening (4.13). Moreover, since Cauchy stress is measurable in the present configuration, the elastic deformations can be obtained by inverting the constitutive equations for stress (4.19) or (5.22), so the initial conditions for the evolution equations (4.17) for (5.6) can be determined without ambiguity. In contrast, the initial conditions for the evolutions of total strain (4.5) and plastic strain (4.1) cannot be determined without arbitrariness which has no physical relevance to the prediction of subsequent material response.

An alternative formulation for elastically anisotropic inelastic materials has also been discussed which introduces evolution equations (6.4) for three vectors characterizing the absolute orientation and elastic deformation of the microstructure. Again, the initial values of these vectors are measurable so the evolution equations can be integrated without ambiguity. Also, these vectors are used to determine the stress by constitutive equations of the form (6.17). This alternative method should be contrasted with the more standard methods (discussed in

[13]), such as the one used for crystal plasticity [26], which requires integration of the evolution equations for the total deformation gradient $\mathbf{F}$ and the plastic deformation tensor $\mathbf{F}_p$

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad \dot{\mathbf{F}}_p = \mathbf{\Lambda}_p \mathbf{F}_p, \tag{7.1}$$

and which includes a definition of elastic deformation $\mathbf{F}_e$

$$\mathbf{F}_e = \mathbf{F}\,\mathbf{F}_p^{-1}, \tag{7.2}$$

where the function $\mathbf{\Lambda}_p$ requires a constitutive equation. For elastically anisotropic response the Cauchy stress depends on both $\mathbf{F}$ and $\mathbf{F}_p$, neither of which can be measured without arbitrariness in the present configuration [13].

Within the context of the formulation associated with (7.1) and (7.2), it is common to define the symmetric tensors $\mathbf{C}_e$ and $\mathbf{B}_e$ by

$$\mathbf{C}_e = \mathbf{F}_e^T \mathbf{F}_e, \quad \mathbf{B}_e = \mathbf{F}_e \mathbf{F}_e^T. \tag{7.3}$$

Then, it can be shown that these tensors are determined by the evolution equations

$$\dot{\mathbf{F}}_e = \mathbf{L}\mathbf{F}_e - \mathbf{F}_e \mathbf{\Lambda}_p, \tag{7.4a}$$

$$\dot{\mathbf{C}}_e = 2\mathbf{F}_e^T \mathbf{D}\,\mathbf{F}_e - \mathbf{\Lambda}_p^T \mathbf{C}_e - \mathbf{C}_e \mathbf{\Lambda}_p, \tag{7.4b}$$

$$\dot{\mathbf{B}}_e = \mathbf{L}\,\mathbf{B}_e + \mathbf{B}_e \mathbf{L}^T - \mathbf{F}_e(\mathbf{\Lambda}_p + \mathbf{\Lambda}_p^T)\mathbf{F}_e^T. \tag{7.4c}$$

In particular, notice that unless $\mathbf{\Lambda}_p$ is specified in a special form, these evolution equations depend on $\mathbf{F}_e$, so that it is still necessary to integrate the evolution equation (7.4a) for $\mathbf{F}_e$ and there is no advantage of the evolution equations (7.4b,c). As a special case, $\mathbf{\Lambda}_p$ can be specified in the form

$$\mathbf{\Lambda}_p = \mathbf{F}_e^{-1} \mathbf{L}_p \mathbf{F}_e, \tag{7.5}$$

to obtain the theory of BESSELING [25] for elastically anisotropic response which focuses attention on an evolution equation for the elastic deformation

$$\dot{\mathbf{F}}_e = (\mathbf{L} - \mathbf{L}_p)\mathbf{F}_e. \tag{7.6}$$

Also, when $\mathbf{\Lambda}_p$ satisfies the condition

$$\mathbf{\Lambda}_p + \mathbf{\Lambda}_p^T = 2\mathbf{F}_e^{-1} \mathbf{D}_p \mathbf{F}_e^{-T}, \tag{7.7}$$

then the evolution equation (5.1) is consistent with the theories of ECKART [21] and LEONOV [22] for elastically isotropic response.

The advantages of the alternative method are not emphasized in the small deformation theory because the difference between the total deviatoric strain $\boldsymbol{\varepsilon}'$and the plastic strain $\boldsymbol{\varepsilon}_p$ (which is deviatoric) is measurable. Consequently, the arbitrariness of $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\varepsilon}_p$ which influences the material response can be easily removed. However, for large deformations, one standard approach requires the Cauchy stress $\mathbf{T}$ to be a function of both $\mathbf{F}$ and $\mathbf{F}_p$ in order to be properly invariant under the superposed rigid body motions. Consequently, the arbitrariness which is associated with the determination of the initial values of $\mathbf{F}$ and $\mathbf{F}_p$ cannot be removed in general and can cause an unphysical influence on the response of elastically anisotropic inelastic materials. Additional differences between the alternative theories reviewed in this paper and the more classical theories associated with (7.6) and (7.7) have been discussed in [13].

The alternative constitutive equations discussed here not only have the conceptual advantage that the required initial conditions can be measured; they also have a practical advantage for computations. In particular, it has been shown [20] that the formulation of Section 5 can be implemented into standard wave propagation codes by using the Cauchy stress $\mathbf{T}$ to determine the elastic deformation, so that there is no need to store $\mathbf{F}$ and $\mathbf{F}_p$ as history-dependent variables. Moreover, one standard approach for elastically anisotropic inelastic materials requires the calculation of the two tensors $\mathbf{F}$ and $\mathbf{F}_p$. In contrast, for the alternative equations of Section 6, the microstructural vectors $\mathbf{m}_i$ (9 independent quantities) can be stored instead of the two tensors $\mathbf{F}$ and $\mathbf{F}_p$ (18 independent quantities).

This paper has emphasized that the total strain associated with the current state of a material cannot be measured because it is dependent on an arbitrary choice of the reference configuration. Nevertheless, the total strain which is measured relative to a reference configuration which an experimenter specifies, is certainly a useful parameter for monitoring the history of total deformations of a specimen. This measured total strain is similar to time, in the sense that both are measured relative to an arbitrary reference state, and both they cannot appear explicitly in constitutive equations.

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