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Generalized proper states for anisotropic elastic materials

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THE MAIN AIM OF THIS PAPER is to determine all the unit stresses $\omega(\omega \cdot \omega = 1)$ for which the stored elastic energy $\Phi(\omega)$ has the local extrema in some classes of stresses. Our consideration is restricted to two classes: $\mathcal{K}_1$ – uniaxial tensions and then the directions for which the Young modulus assumes its extremal value are determined, and $\mathcal{K}_2$ – pure shears in physical space. The problem is then reduced to the determination of the planes of minimal and maximal shear modulus. The idea of a generalized proper state for Hooke's tensor is introduced. It is shown that a mathematical treatment of the considered problem comes down to the problem of the generalized proper elastic states for the compliance tensor **C**. The problem has been effectively solved for cubic symmetry.

1. Introduction

MOST MODERN MATERIALS are, or can be considered anisotropic. Composites have these properties due to the production technology, while the natural and biological bodies such as wood, bones, tissues, rock structures, can also be considered anisotropic.

Designation of the optimal structure needs some clear criteria for forming the properties of composite materials.

The aim of this paper is to provide clear criteria for controlling the properties of composite materials by a proper choice of the stiffness and /or compliance tensor at a given material point. The rules of this choice could be based upon the determination of the directions for which the Young modulus assumes its minimal and maximal values, as well as on the determination of the plane of minimal and maximal shear modulus, the Kirchhoff modulus.

It is convenient to make use of the qualitative description of the properties of the stiffness and the compliance tensors of anisotropic bodies developed by J. Rychlewski [1].

The problem considered is reduced to the problem of generalized proper elastic states for the compliance tensor. For cubic isotropy, the problem can be effectively solved. The solution obtained in the paper offers some new possibilities of approaching the problem of optimal formation of the internal structure of materials, eg. the direction of reinforcement of fibrous composites.

2. Formulation of the problem

We are discussing the classical materials, with linear elasticity, in which the infinitesimal strain $\boldsymbol{\varepsilon}$ causes the stress $\boldsymbol{\sigma}$ according to *Hooke's law*

$$\boldsymbol{\sigma} = \mathbf{S} \cdot \boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \mathbf{C} \cdot \boldsymbol{\sigma}, \tag{2.1}$$

where $\mathbf{S}$ is the *stiffness tensor* and $\mathbf{C}$ is the *compliance tensor*. They are connected by the relation

$$\mathbf{S}^{-1} = \mathbf{C}, \quad \mathbf{S} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{S} = \mathbb{I}_{\mathcal{S}}, \tag{2.2}$$

$$\mathbb{I}_{\mathcal{S}} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}.$$

The anisotropic form of Hooke's law written in indicial notation is given in the Appendix.

For the stress tensor $\boldsymbol{\sigma} \in \mathcal{S}$, the quadratic form

$$\Phi(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} \tag{2.3}$$

is the doubled work of stress $\boldsymbol{\sigma}$. It is also called the complementary energy function or stored elastic energy function.

PROBLEM

We are looking for all unit stress states $\boldsymbol{\omega} \in \mathcal{K} \subset S$ $(\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1)$ for which the stored elastic energy has a local extremum. It means that

$$\Phi(\boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \mathbf{C} \cdot \boldsymbol{\omega} = ext$$

for $\boldsymbol{\omega} \in \mathcal{K}$.

Our considerations are restricted to the two classes of stresses: $\mathcal{K}_1$ and $\mathcal{K}_2$. For these classes the solution of the problem is obtained in analytical form.

a) Uniaxial tensions $\boldsymbol{\sigma}$ in any direction $\mathbf{n}$ in the physical space are considered as the subspace $\mathcal{K}_1 \subset \mathcal{S}$

$$\mathcal{K}_1 : \{\boldsymbol{\omega} \in \mathcal{S};\ \boldsymbol{\omega} = \mathbf{n} \otimes \mathbf{n} \equiv \boldsymbol{\sigma}\}, \quad \mathbf{n} \cdot \mathbf{n} = 1. \tag{2.4}$$

In this case the stored elastic energy (2.3) may be rewritten as:

$$\Phi(\boldsymbol{\omega}) = \frac{1}{\lambda(\boldsymbol{\omega})} = \frac{1}{E(\boldsymbol{\sigma})} \tag{2.5}$$

and the local extremum of the *Young modulus* E is sought for.

b) Pure shears $\boldsymbol{\tau}$ in the plane $\mathbf{n}_1$, $\mathbf{n}_2$ in the physical space are forming the subspace $\mathcal{K}_2 \subset \mathcal{S}$

$$\begin{gathered} \mathcal{K}_2 \;:\; \{\boldsymbol{\omega} \in \mathcal{S};\;\; \boldsymbol{\omega} = \frac{\sqrt{2}}{2}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1) \equiv \boldsymbol{\tau}\}, \\ \mathbf{n}_1 \cdot \mathbf{n}_1 = \mathbf{n}_2 \cdot \mathbf{n}_2 = 1, \quad \mathbf{n}_1 \cdot \mathbf{n}_2 = 0. \end{gathered} \tag{2.6}$$

The stored elastic energy (2.3) for pure shears has the following form:

$$\Phi(\boldsymbol{\omega}) = \frac{1}{\lambda(\boldsymbol{\omega})} = \frac{1}{2G(\boldsymbol{\tau})} \tag{2.7}$$

and the local extremum of *shear modulus* G is looked for.

From the definition of tensors $\boldsymbol{\sigma}$ (2.4) and from the definition of tensor $\boldsymbol{\tau}$ (2.6) it is evident that the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ constitute the orbits, the group of rotations $\mathcal{O}$ in the space $\mathcal{S}$

$$\mathcal{K}_1 : \{\boldsymbol{\sigma} \in \mathcal{S};\;\; \operatorname{tr}\boldsymbol{\sigma} = 1, \;\; \operatorname{tr}\boldsymbol{\sigma}^2 = 1, \;\; \operatorname{tr}\boldsymbol{\sigma}^3 = 1\}, \tag{2.8}$$

$$\mathcal{K}_2 : \{\boldsymbol{\tau} \in \mathcal{S};\;\; \operatorname{tr}\boldsymbol{\tau} = 0, \;\; \operatorname{tr}\boldsymbol{\tau}^2 = 1, \;\; \operatorname{tr}\boldsymbol{\tau}^3 = 0\}. \tag{2.9}$$

3. Generalized proper states

Mathematical treatment of the above problem is the well-known Lagrange condition for the local extremum with constrains. The necessary extremum condition defines the generalized proper states problem for the compliance tensor $\mathbf{C}$.

We will call *Hooke's tensor* any Euclidean tensor of the fourth order $\mathbf{H}$ which realises a symmetric linear transformation of the space of symmetric second-order tensors $\mathcal{S}$ into itself. The space $\mathcal{S}$ is six-dimensional. Tensor $\mathbf{H}$ has the following internal symmetries:

$$H_{ijkl} = H_{jikl} = H_{klij} = H_{ijlk}. \tag{3.1}$$

DEFINITION 1. *For each Hooke's tensor* $\mathbf{H}$ *(3.1), the second order tensor* $\boldsymbol{\omega} \in \mathcal{S}$ $(\boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1)$ *is called* the generalized proper state *of this tensor if there exists such parameters* α, β *and* γ *that*

$$\mathbf{H} \cdot \boldsymbol{\omega} = \alpha(\mathbf{H}, \boldsymbol{\omega})\mathbf{1} + \beta(\mathbf{H}, \boldsymbol{\omega})\boldsymbol{\omega} + \gamma(\mathbf{H}, \boldsymbol{\omega})\boldsymbol{\omega}^2. \tag{3.2}$$

The scalar functions α, β and γ can be obtained from the set of the linear equations:

$$\begin{aligned}
\mathbf{1} \cdot \mathbf{H} \cdot \boldsymbol{\omega} &= 3\alpha + \operatorname{tr} \boldsymbol{\omega} \cdot \beta + \gamma, \\
\boldsymbol{\omega} \cdot H \cdot \boldsymbol{\omega} &= \operatorname{tr} \boldsymbol{\omega} \cdot \alpha + \beta + \operatorname{tr} \boldsymbol{\omega}^3 \cdot \gamma, \qquad (3.3)\\
\boldsymbol{\omega}^2 \cdot \mathbf{H} \cdot \boldsymbol{\omega} &= \alpha + \operatorname{tr} \boldsymbol{\omega}^3 \cdot \beta + \operatorname{tr} \boldsymbol{\omega}^4 \cdot \gamma.
\end{aligned}$$

When $\alpha = \gamma = 0$, the classical proper state problem is obtained

$$\mathbf{H} \cdot \boldsymbol{\omega} = \beta \boldsymbol{\omega}. \qquad (3.4)$$

If as a tensor $\mathbf{H}$ the tensor $\mathbf{C}$ is considered, then $\boldsymbol{\omega}$ is a proper elastic state for compliance tensor $\mathbf{C}$ and $\lambda = \frac{1}{\beta}$ is the Kelvin modulus (see J. Rychlewski [1]). The equation

$$\mathbf{C} \cdot \boldsymbol{\omega} = \frac{1}{\lambda} \boldsymbol{\omega} \qquad (3.5)$$

is the necessary condition for local extremum of the stored elastic energy (2.3) on unit sphere $\mathcal{K}$ in the stress space $\mathcal{S}$. It means that

$$\mathcal{K} : \{\boldsymbol{\omega} \in \mathcal{S}; \quad \boldsymbol{\omega} \cdot \boldsymbol{\omega} = 1\} \qquad (3.6)$$

and $\mathcal{K}_1 \subset \mathcal{K}$ and $\mathcal{K}_2 \subset \mathcal{K}$.

3.1. Uniaxial tension

Mathematical treatment of the local extemum problem for (2.5) by the Lagrange multipliers method leads to the following necessary condition for $\boldsymbol{\sigma} \in \mathcal{K}_1$ (2.8):

$$\mathbf{C} \cdot \boldsymbol{\sigma} = \frac{1}{2}(\mathbf{1} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma})\mathbf{1} + \frac{1}{2}(3\boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} - \mathbf{1} \cdot \mathbf{C} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}. \qquad (3.7)$$

It is not difficult to notice that the above equation has the form of the Eq.(3.2) for a generalized proper state when $\gamma = 0$.

From the definition of the class $\mathcal{K}_1$ (2.4) it follows that the direction $\mathbf{n}$ is a proper vector for $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{n} = \mathbf{n}. \qquad (3.8)$$

Substituting the relation (3.8) into (3.7) after multiplying by $\mathbf{n}$ we obtain the following result:

$$(\mathbf{C}\cdot\boldsymbol{\sigma})\cdot\mathbf{n} = (\boldsymbol{\sigma}\cdot\mathbf{C}\cdot\boldsymbol{\sigma})\mathbf{n}. \tag{3.9}$$

It means that

$$[\mathbf{C}\cdot(\mathbf{n}\otimes\mathbf{n})]\cdot\mathbf{n} = [(\mathbf{n}\otimes\mathbf{n})\cdot\mathbf{C}\cdot(\mathbf{n}\otimes\mathbf{n})]\,\mathbf{n}. \tag{3.10}$$

According to the paper by S. C. Cowin, M. M. Mehrabadi [2], condition (3.10) is the necessary condition for the vector $\mathbf{n}$ to be normal to a plane of symmetry of a material of given compliance $\mathbf{C}$.

Any material which is not entirely anisotropic, like for example crystals of triclinic system, was called by A. Blinowski and J. Rychlewski [3] a symetric elastic material. They proved that every symmetric elastic material has at least one plane of elastic symmetry. Hence every symmetric elastic material has at least one direction with extremal Young modulus.

3.2. Pure shear

We now apply the Lagrange method for searching the local extremum of stored elastic energy (2.7). The necessary condition for $\boldsymbol{\tau}\in\mathcal{K}_2$ (2.9) has the following form:

$$\mathbf{C}\cdot\boldsymbol{\tau} = (\mathbf{1}\cdot\mathbf{C}\cdot\boldsymbol{\tau})\mathbf{1} + (\boldsymbol{\tau}\cdot\mathbf{C}\cdot\boldsymbol{\tau})\,\boldsymbol{\tau} + 2(3\boldsymbol{\tau}^2\cdot\mathbf{C}\cdot\boldsymbol{\tau} - \mathbf{1}\cdot\mathbf{C}\cdot\boldsymbol{\tau})\,\boldsymbol{\tau}^2. \tag{3.11}$$

It is easy to see that a main direction of $\boldsymbol{\tau}$ is a main direction of $\mathbf{C}\cdot\boldsymbol{\tau}$, but not conversely. From the definition of $\boldsymbol{\tau}$ (2.6) we obtain that

$$\boldsymbol{\tau}^2 = \frac{1}{2}(\mathbf{n}_1\otimes\mathbf{n}_1 + \mathbf{n}_2\otimes\mathbf{n}_2).$$

For the plane tensors, the general description of the material can be significantly simplified. The same results obtained for the theory of plane elasticity are of self-contained value, more convenient for formulation and interpretation. For the two-dimensional case, the problem of local extrema of the stored elastic energy has been effectively solved in the paper by J. OSTROWSKA-MACIEJEWSKA, J. RYCHLEWSKI [4]. The solution obtained was discused in terms of energy limitations. The final solution depended strongly on the type and degree of anisotropy.

In a general three-dimensional case, the above problems are reduced, as it was shown, to the problem of generalized proper elastic states for the compliance tensor $\mathbf{C}$. The solution of the Eqs. (3.7) and (3.11) can be obtained only for some classes of symmetry.

4. Cubic isotropy

For quick orientation and to form an intuition in the presented approach, let us examine the simplest anisotropy, that is when the material has a cubic symmetry [7].

The spectral decomposition of compliance tensor **C** for cubic symmetry was given in the paper by J. OSTROWSKA-MACIEJEWSKA and J. RYCHLEWSKI [5] and has the form:

$$\mathbf{C} = \frac{1}{3\lambda_1}\mathbf{1}\otimes\mathbf{1} + \frac{1}{\lambda_2}\left(\mathbf{K} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right) + \frac{1}{\lambda_3}\left(\mathbb{I}_{L_S} - \mathbf{K}\right), \tag{4.1}$$

where λ_1, λ_2, λ_3 are the essentially different Kelvin moduli for the three mutually orthogonal subspaces of the proper elastic states $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$

$$\mathcal{S} = \mathcal{P}_1 \dot{+} \mathcal{P}_2 \dot{+} \mathcal{P}_3. \tag{4.2}$$

If the directions **k**, **l** and **m** coincide with the crystal axes for cubic isotropy (Fig. 1), the fourth-order tensor **K** may be represented in the form

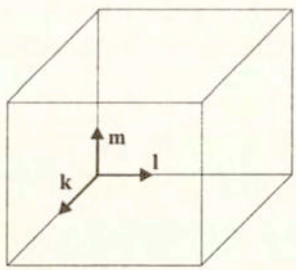

FIG. 1. Crystal axes **k**, **l**, **m**.

$$\mathbf{K} = \mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k} + \mathbf{l}\otimes\mathbf{l}\otimes\mathbf{l}\otimes\mathbf{l} + \mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}. \tag{4.3}$$

The space $\mathcal{P}_1$ is the one-dimensional subspace of spherical tensors:

$$\omega \sim \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \tag{4.4}$$

the space $\mathcal{P}_2$ is the two-dimensional subspace of deviators of diagonal form:

$$\omega \sim \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -u-v \end{pmatrix}, \tag{4.5}$$

and the space $\mathcal{P}_3$ is the three-dimensional subspace of deviators of extradiagonal form:

$$\boldsymbol{\omega} \sim \begin{pmatrix} 0 & p & r \\ p & 0 & q \\ r & q & 0 \end{pmatrix} \tag{4.6}$$

in the crystal axes.

From the spectral decomposition (4.1) and the definitions of $\boldsymbol{\sigma}$ (2.8) and $\boldsymbol{\tau}$ (2.9), we obtain that

$$\mathbf{C} \cdot \boldsymbol{\sigma} = \frac{1}{3}(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})\mathbf{1} + \frac{1}{\lambda_3}\boldsymbol{\sigma} + (\frac{1}{\lambda_2} - \frac{1}{\lambda_3})\mathbf{K} \cdot \boldsymbol{\sigma} \tag{4.7}$$

and

$$\mathbf{C} \cdot \boldsymbol{\tau} = \frac{1}{\lambda_3}\boldsymbol{\tau} + (\frac{1}{\lambda_2} - \frac{1}{\lambda_3})\mathbf{K} \cdot \boldsymbol{\tau}. \tag{4.8}$$

It is easy to see from (4.7) that if $\boldsymbol{\sigma}$ is a proper state of $\mathbf{K}$ (4.3) then it is not a proper state of $\mathbf{C}$. Quite a different situation is for $\boldsymbol{\tau}$ (4.8): each proper state of $\mathbf{K}$ is a proper state of $\mathbf{C}$.

The stored elastic energy (2.5) and (2.7) for both classes of stresses $\mathcal{K}_1$ (2.4) and $\mathcal{K}_2$ (2.6) now may be rewritten as:

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{E(\boldsymbol{\sigma})} = \boldsymbol{\sigma} \cdot \mathbf{C} \cdot \boldsymbol{\sigma} = \frac{1}{3}(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} + \frac{3}{\lambda_3}) + (\frac{1}{\lambda_2} - \frac{1}{\lambda_3})\boldsymbol{\sigma} \cdot \mathbf{K} \cdot \boldsymbol{\sigma}, \tag{4.9}$$

$$\Phi(\boldsymbol{\tau}) = \frac{1}{2G(\boldsymbol{\tau})} = \boldsymbol{\tau} \cdot \mathbf{C} \cdot \boldsymbol{\tau} = \frac{1}{\lambda_3} + (\frac{1}{\lambda_2} - \frac{1}{\lambda_3})\boldsymbol{\tau} \cdot \mathbf{K} \cdot \boldsymbol{\tau}. \tag{4.10}$$

Let us denote by $E_1 = E_2 = E_3 = \mathrm{E}$ the Young moduli for the directions of the axes of crystal $\mathbf{k}, \mathbf{l}$ and $\mathbf{m}$, by $G_{12} = G_{13} = G_{23} = \mathrm{G}$ the shear moduli, and by $\nu_{12} = \nu_{13} = \nu_{23} = \nu$ the Poisson coefficients for the planes connected with the axes of the crystal; then the relation between these two sets of material constants $\mathrm{E}, \mathrm{G}, \nu$ and λ_1, λ_2, λ_3 are as follows:

$$\lambda_1 = \frac{\mathrm{E}}{1 - 2\nu}, \quad \lambda_2 = \frac{\mathrm{E}}{1 + \nu}, \quad \lambda_3 = 2\mathrm{G} \tag{4.11}$$

or

$$\mathrm{E} = \frac{3\lambda_1\lambda_2}{2\lambda_1 + \lambda_2}, \quad \nu = \frac{\lambda_1 - \lambda_2}{2\lambda_1 + \lambda_2}, \quad \mathrm{G} = \frac{\lambda_3}{2}. \tag{4.12}$$

The anisotropy tensor $\mathbf{C}$ (4.1) decomposes the space $\mathcal{S}$ into three mutually orthogonal subspaces $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ (4.2). Any state of stress $\boldsymbol{\omega}$ can be projected on these subspaces and given as a sum of these portions. In the three-dimensional set of orthogonal coordinates connected with the subspaces of the proper states (4.2), the stored elastic energy (2.3) can be presented as a closed surface

$$\Phi(\boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \mathbf{C} \cdot \boldsymbol{\omega} = \frac{\cos^2\theta \cos^2\varphi}{\lambda_1} + \frac{\cos^2\theta \sin^2\varphi}{\lambda_2} + \frac{\sin^2\theta}{\lambda_3}. \tag{4.13}$$

In order to draw this surface some material of cubic isotropy must be chosen.

For copper which is a typical material of cubic symmetry, moduli λ_i are given in the paper S. Sutcliffe [6], and they are as follows:

$$\lambda_1 = 41.6, \quad \lambda_2 = 4.7, \quad \lambda_3 = 15. \tag{4.14}$$

The succession of eigenvalues for copper is typical of most cubic isotropy metals with the lowest eigenvalue associated with the two-dimensional eigenspace $\mathcal{P}_2$ (4.5). The surface of the stored elastic energy for copper is shown in Fig. 2.

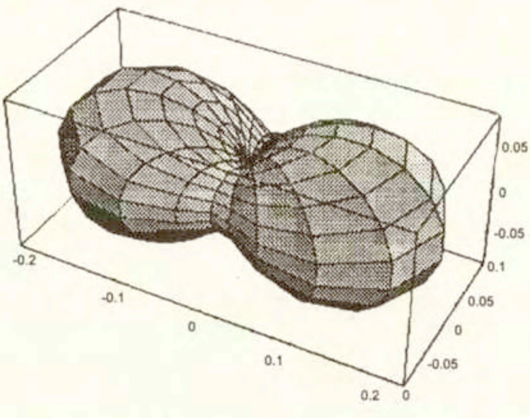

FIG. 2. Stored elastic energy in the space of proper elastic subspaces $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

According to the paper by J. RYCHLEWSKI [1], the local extrema of the stored elastic energy on a unit sphere (3.6) are located on the axes of coordinates. Intersection of the surface (4.13) by the cone $\boldsymbol{\omega} = \mathbf{n} \otimes \mathbf{n}$ gives us the class of stresses $\mathcal{K}_1$ (2.4), but intersection by the plane of deviators describes the class $\mathcal{K}_2$ (2.6).

The surface of the stored elastic energy for the class of stresses $\mathcal{K}_1$ (2.8) is described by the Eq.(2.5). Since

$$\frac{1}{E(\boldsymbol{\sigma})} = \frac{1}{E(\mathbf{n} \otimes \mathbf{n})} = \frac{1}{E(\mathbf{n})}, \tag{4.15}$$

then this surface may be drawn in the physical space. Using the crystal axes **k**, **l**, **m**, any direction **n** may be given as

$$\mathbf{n} = n_1\mathbf{k} + n_2\mathbf{l} + n_3\mathbf{m} \tag{4.16}$$

and the stress tensor $\boldsymbol{\sigma}$ (2.4) has the form

$$\begin{aligned}\boldsymbol{\sigma} = \mathbf{n}\otimes\mathbf{n} = n_1^2\mathbf{k}\otimes\mathbf{k} + n_2^2\mathbf{l}\otimes\mathbf{l} + n_3^2\mathbf{m}\otimes\mathbf{m} +\\ + n_1n_2(\mathbf{k}\otimes\mathbf{l} + \mathbf{l}\otimes\mathbf{k}) + n_1n_3(\mathbf{k}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{k}) + n_2n_3(\mathbf{l}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{l}).\end{aligned} \tag{4.17}$$

From the definition of tensor **K** (4.3) and the above form of $\boldsymbol{\sigma}$, it follows that

$$\boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma} = (\mathbf{n}\otimes\mathbf{n})\cdot\mathbf{K}\cdot(\mathbf{n}\otimes\mathbf{n}) = n_1^4 + n_2^4 + n_3^4 = 1 - 2(n_1^2n_2^2 + n_2^2n_3^2 + n_3^2n_1^2) \tag{4.18}$$

and the formula for the stored elastic energy (4.9) for uniaxial tension takes the form

$$\begin{aligned}\frac{1}{E(\mathbf{n})} &= \frac{2\lambda_1 + \lambda_2}{3\lambda_1\lambda_2} + 2\frac{\lambda_2 - \lambda_3}{\lambda_2\lambda_3}(n_1^2n_2^2 + n_2^2n_3^2 + n_3^2n_1^2)\\ &= \frac{2\lambda_1 + \lambda_2}{3\lambda_1\lambda_2} + \frac{\lambda_2 - \lambda_3}{2\lambda_2\lambda_3}(\cos^4\theta\sin^2 2\varphi + \sin^2 2\theta).\end{aligned} \tag{4.19}$$

For computational purposes, the eigenvalues for copper (4.14) were used. The surface (4.19) for copper is shown in the Fig. 3.

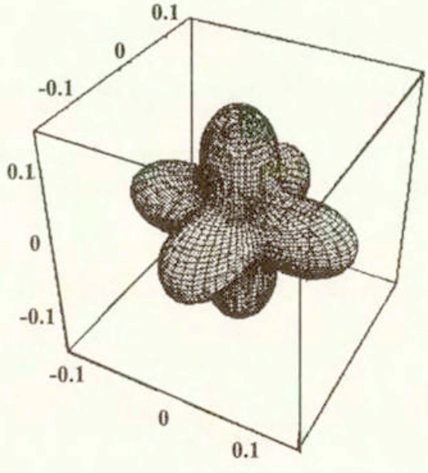

FIG. 3. Stored elastic energy under uniaxial tension in the physical space.

5. Local extrema of the Young modulus and the shear modulus for cubic isotropy.

In order to find the local extrema of the stored elastic energy for two classes of stresses $\mathcal{K}_1$ (2.4) and $\mathcal{K}_2$ (2.6), we have to come back to two equations for the generalized proper states of **C** (3.7) and (3.11).

For cubic isotropy, taking into account the spectral decomposition (4.1), the Eq.(3.7) and (3.11) may be rewritten as

(5.1) $$\mathbf{K}\cdot\boldsymbol{\sigma} = \frac{1}{2}(1-\boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma})\mathbf{1}+\frac{1}{2}(3\boldsymbol{\sigma}\cdot K\cdot\boldsymbol{\sigma}-1)\boldsymbol{\sigma},$$

(5.2) $$\mathbf{K}\cdot\boldsymbol{\tau} = (-2\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau})\mathbf{1}+(\boldsymbol{\tau}\cdot\mathbf{K}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+(6\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau})\boldsymbol{\tau}^2.$$

Please note that the obtained equations are the equations for generalised proper states of tensor **K** (4.3).

5.1. Young modulus

Let us begin with the class of stresses $\mathcal{K}_1$ (2.4) – uniaxial tension. Equation (5.1) is the necessery condition for the local extrema of Young's modulus. In order to determine the solution of the Eq. (5.1) we have to note that the tensor $\mathbf{K}\cdot\boldsymbol{\sigma}$ has a diagonal form in the crystal axes

(5.3) $$\mathbf{K}\cdot\boldsymbol{\sigma} = n_1^2\mathbf{k}\otimes\mathbf{k}+n_2^2\mathbf{l}\otimes\mathbf{l}+n_3^2\mathbf{m}\otimes\mathbf{m}.$$

Equation (5. 1) may be rewritten in the form

(5.4) $$\mathbf{K}\cdot\boldsymbol{\sigma}-\frac{1}{2}(1-\boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma})\,\mathbf{1} = \frac{1}{2}(3\boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma}-1)\boldsymbol{\sigma}.$$

The left-hand side of the Eq. (5.4) has a diagonal form because of (5.3). It means that the right-hand side has to be diagonal as well. It will happen if tensor $\boldsymbol{\sigma}$ has the diagonal form

(5.5) $$\boldsymbol{\sigma} = n_1^2\mathbf{k}\otimes\mathbf{k}+n_2^2\mathbf{l}\otimes\mathbf{l}+n_3^2\mathbf{m}\otimes\mathbf{m},$$

or when

(5.6) $$3\boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma} = 1.$$

a) If Eq. (5.5) is satisfied, then from (5.3) we have

(5.7) $$\mathbf{K}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}, \text{ and } \boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma} = \operatorname{tr}\boldsymbol{\sigma}^2 = 1.$$

It means that $\boldsymbol{\sigma}$ is the proper state of $\mathbf{K}$ and from (4.7) it follows that

$$\mathbf{C}\cdot\boldsymbol{\sigma} = \frac{1}{3}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\mathbf{1} + \frac{1}{\lambda_2}\boldsymbol{\sigma} \tag{5.8}$$

and $\boldsymbol{\sigma}$ is a generalized proper state of $\mathbf{C}$. See (3.2) with $\gamma = 0$.

The stored elastic energy (4.9) may be rewritten as

$$\Phi_a(\boldsymbol{\sigma}) = \frac{1}{E(\boldsymbol{\sigma})} = \frac{2\lambda_1 + \lambda_2}{3\lambda_1\lambda_2}. \tag{5.9}$$

The stress tensor $\boldsymbol{\sigma}$ (4.17) will have the form (5.5) if

$$n_1 n_2 = 0, \quad n_1 n_3 = 0, \quad n_2 n_3 = 0. \tag{5.10}$$

It means that the direction $\mathbf{n}$ has to coincide with the directions of the crystal axes

$$\begin{aligned} &\mathbf{n} = \mathbf{k}, \quad \boldsymbol{\sigma} = \mathbf{k}\otimes\mathbf{k}, \\ &\mathbf{n} = \mathbf{l}, \quad \boldsymbol{\sigma} = \mathbf{l}\otimes\mathbf{l}, \\ &\mathbf{n} = \mathbf{m}, \quad \boldsymbol{\sigma} = \mathbf{m}\otimes\mathbf{m}. \end{aligned} \tag{5.11}$$

Hence the Young modulus has the local extrema in the direction of the crystal axes.

b) In the case when the condition (5.6) is satified, then

$$\boldsymbol{\sigma}\cdot\mathbf{K}\cdot\boldsymbol{\sigma} = \frac{1}{3}, \tag{5.12}$$

and from Eq. (5.4) we conclude

$$\mathbf{K}\cdot\boldsymbol{\sigma} = \frac{1}{3}\mathbf{1} = \frac{1}{3}(\mathbf{k}\otimes\mathbf{k} + \mathbf{l}\otimes\mathbf{l} + \mathbf{m}\otimes\mathbf{m}). \tag{5.13}$$

According to (5.3), the above formula means that

$$n_1^2 = n_2^2 = n_3^2 = \frac{1}{3}. \tag{5.14}$$

Using (5.13), Eq. (4.7) may be rewritten as follows:

$$\mathbf{C}\cdot\boldsymbol{\sigma} = \frac{1}{3}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right)\mathbf{1} + \frac{1}{\lambda_3}\boldsymbol{\sigma}. \tag{5.15}$$

It is not difficult to notice that tensor $\boldsymbol{\sigma}$ is not a proper state for $\mathbf{C}$, but it is a generalized proper state of $\mathbf{C}$. Moreover, among eight possible forms, one of them may be presented as

$$\sigma = \frac{1}{3}(\mathbf{k}+\mathbf{l}+\mathbf{m}) \otimes (\mathbf{k}+\mathbf{l}+\mathbf{m}). \tag{5.16}$$

The direction $\mathbf{n}$ (4.16) is hence perpendicular to the octahedral plane.

The Young moduli have local extrema for these directions. The stored elastic energy (4.9) in this case is as follows

$$\Phi_b = \frac{1}{E(\sigma)} = \frac{2\lambda_1+\lambda_3}{3\lambda_1\lambda_3}. \tag{5.17}$$

If $\lambda_2 < \lambda_3$ (note that this situation occurs for copper (4.14)) then the following inequality is satisfied (Fig. 3)

$$\Phi_a(\sigma) > \Phi_b(\sigma). \tag{5.18}$$

5.2. Shear - Kirchhoff modulus

The necessary condition for the local extemum of the stored elastic energy for the class $\mathcal{K}_2$ (2.6) has the form (5.2). This fomula may be rewritten as

$$\mathbf{K}\cdot\boldsymbol{\tau}+2(\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau})\mathbf{1} = (\boldsymbol{\tau}\cdot\mathbf{K}\cdot\boldsymbol{\tau})\boldsymbol{\tau}+6(\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau})\boldsymbol{\tau}^2. \tag{5.19}$$

The left-hand side of the above equation has diagonal representation in the system of crystal axes. The right-hand side will have the diagonal form if

a) tensor $\boldsymbol{\tau}$ has a diagonal form

$$\boldsymbol{\tau} \sim \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}. \tag{5.20}$$

From the definition of tensor $\mathbf{K}$ (4.3) it follows that

$$\mathbf{K}\cdot\boldsymbol{\tau} = \boldsymbol{\tau}. \tag{5.21}$$

Tensor $\boldsymbol{\tau} \in \mathcal{K}_2$ (2.9). It means that the following conditions are satistfied:

$$\boldsymbol{\tau}\cdot\mathbf{K}\cdot\boldsymbol{\tau} = \operatorname{tr}\boldsymbol{\tau}^2 = 1, \qquad \boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau} = \operatorname{tr}\boldsymbol{\tau}^3 = 0. \tag{5.22}$$

Note that τ is a proper state of $\mathbf{K}$ (5.21) and it follows immediately from (4.8) that it is the proper state of $\mathbf{C}$

$$\mathbf{C}\cdot\boldsymbol{\tau} = \frac{1}{\lambda_2}\boldsymbol{\tau}. \tag{5.23}$$

Comparing (5.20) with (4.5) we note that $\boldsymbol{\tau}\in\mathcal{P}_2$. Because $\boldsymbol{\tau}\in\mathcal{K}_2$ (2.9), then the following equations

$$\det\boldsymbol{\tau} = \tau_{11}\cdot\tau_{22}\cdot\tau_{33} = 0, \tag{5.24}$$

$$\operatorname{tr}\boldsymbol{\tau} = \tau_{11}+\tau_{22}+\tau_{33} = 0, \tag{5.25}$$

have to be satisfied. It means that one and only one term at the diagonal must be equal to zero. Without any loss of generality we can assume, for instance, that

$$\boldsymbol{\tau}\sim\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{5.26}$$

Tensor $\boldsymbol{\tau}$ (5.20) will have the above representation if

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}(\mathbf{k}-\mathbf{l}), \qquad \mathbf{n}_2 = \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{l}); \tag{5.27}$$

then from the definition of $\boldsymbol{\tau}$ (2.6) we have

$$\boldsymbol{\tau} = \frac{1}{\sqrt{2}}(\mathbf{n}_1\otimes\mathbf{n}_2+\mathbf{n}_2\otimes\mathbf{n}_1) = \frac{1}{\sqrt{2}}(\mathbf{k}\otimes\mathbf{k}-\mathbf{l}\otimes\mathbf{l}). \tag{5.28}$$

The stored elastic energy (4.10) in the case when $\boldsymbol{\tau}$ has the representation (5.20) is the following:

$$\Phi_a(\boldsymbol{\tau}) = \frac{1}{2G(\boldsymbol{\tau})} = \frac{1}{\lambda_2}. \tag{5.29}$$

Hence the shear modulus has a local extremum in the plane described by the directions (5.27).

b) Coming back to Eq. (5.19) we see that the right-hand side of that equation will have the diagonal form if

$$\mathbf{K}\cdot\boldsymbol{\tau} = \mathbf{0}. \tag{5.30}$$

It means that tensor $\boldsymbol{\tau}$ has now an extradiagonal form in the system of crystal axes

$$\boldsymbol{\tau} \sim \begin{pmatrix} 0 & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & \tau_{23} \\ \tau_{13} & \tau_{23} & 0 \end{pmatrix}. \tag{5.31}$$

From (5.30) it follows that

$$\boldsymbol{\tau} \cdot \mathbf{K} \cdot \boldsymbol{\tau} = 0 \quad \text{and} \quad \boldsymbol{\tau}^2 \cdot \mathbf{K} \cdot \boldsymbol{\tau} = 0. \tag{5.32}$$

Substituting (5.30) to (4.8) we obtain that $\boldsymbol{\tau}$ is a proper state of $\mathbf{C}$, namely

$$\mathbf{C} \cdot \boldsymbol{\tau} = \frac{1}{\lambda_3} \boldsymbol{\tau} \tag{5.33}$$

and according to (4.10), the stored elastic energy is now as follows:

$$\Phi_b(\boldsymbol{\tau}) = \frac{1}{2G(\boldsymbol{\tau})} = \frac{1}{\lambda_3}. \tag{5.34}$$

Tensor $\boldsymbol{\tau}$ (5.31), in view of its extradiagonal form, belongs to $\mathcal{P}_3$ (4.6). Definition $\mathcal{K}_2$ (2.6) implies that

$$\det \boldsymbol{\tau} = 2\tau_{12} \cdot \tau_{13} \cdot \tau_{23} = 0. \tag{5.35}$$

It means that at least one of the terms in the matrix representation (5.31)has to be equal to zero. Let us assume that $\tau_{23} = 0$; then $\boldsymbol{\tau}$ has the following form:

$$\boldsymbol{\tau} \sim \begin{pmatrix} 0 & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & 0 \\ \tau_{13} & 0 & 0 \end{pmatrix}, \tag{5.36}$$

or may be rewritten as

$$\boldsymbol{\tau} = \tau_{12}(\mathbf{k} \otimes \mathbf{l} + \mathbf{l} \otimes \mathbf{k}) + \tau_{13}(\mathbf{k} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{k}). \tag{5.37}$$

From the definition of $\boldsymbol{\tau}$ (2.6) we obtain that $\boldsymbol{\tau}$ will have the form (5. 37) for

$$\mathbf{n}_1 = \mathbf{k}, \quad \text{and} \quad \mathbf{n}_2 = \sqrt{2}(\tau_{12}\mathbf{l} + \tau_{13}\mathbf{m}). \tag{5.38}$$

The terms τ_{12} and τ_{13} are not independent. Hence $\boldsymbol{\tau} \in \mathcal{K}_2$ and then

$$\operatorname{tr} \boldsymbol{\tau}^2 = 2(\tau_{12}^2 + \tau_{13}^2) = 1. \tag{5.39}$$

In the limiting cases we have

$$\begin{aligned}&\tau_{13} = 0, \quad \tau_{12}^2 = \frac{1}{2} \quad \text{and} \quad \mathbf{n}_1 = \mathbf{k},\ \mathbf{n}_2 = \pm\mathbf{l},\\ &\tau_{12} = 0, \quad \tau_{13}^2 = \frac{1}{2} \quad \text{and} \quad \mathbf{n}_1 = \mathbf{k},\ \mathbf{n}_2 = \pm\mathbf{m}.\end{aligned} \tag{5.40}$$

It means that there is the family of planes with the same value of shear modulus (5.34).

Changing the order of the crystal axes we will obtain other planes with extremal value (5.34) of the shear modulus.

c) The equation (5.19) may have another solution than the one mentioned above. Let us now assume that $\boldsymbol{\tau}$ has a full matrix representation in the system of crystal axes

$$\boldsymbol{\tau} \sim \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{pmatrix}. \tag{5.41}$$

According to the definition of $\mathcal{K}_2$ (2.9), the following equations have to be satisfied:

$$\begin{aligned}\operatorname{tr}\boldsymbol{\tau} &= \tau_{11} + \tau_{22} + \tau_{33} = 0,\\ \operatorname{tr}\boldsymbol{\tau}^2 &= \tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2 + 2\tau_{12}^2 + 2\tau_{13}^2 + 2\tau_{23}^2 = 1,\\ \frac{1}{3}\operatorname{tr}\boldsymbol{\tau}^3 &= \det\boldsymbol{\tau} = \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{13}\tau_{23} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{13}^2 - \tau_{33}\tau_{12}^2 = 0.\end{aligned} \tag{5.42}$$

Tensor $\mathbf{K}\cdot\boldsymbol{\tau}$ has a diagonal form

$$\mathbf{K}\cdot\boldsymbol{\tau} \sim \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}$$

and from this we see at once that

$$\operatorname{tr}(\mathbf{K}\cdot\boldsymbol{\tau}) = (\mathbf{1}\cdot\mathbf{K}\cdot\boldsymbol{\tau}) = \operatorname{tr}\boldsymbol{\tau} = 0$$

and

$$\boldsymbol{\tau}\cdot\mathbf{K}\cdot\boldsymbol{\tau} = \tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2, \tag{5.43}$$

$$(\mathbf{K}\cdot\boldsymbol{\tau})\cdot(\mathbf{K}\cdot\boldsymbol{\tau}) = \tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2. \tag{5.44}$$

Combining (5.44) with $(5.42)_2$ we conclude that

$$0 \leqslant \tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2 \leqslant 1. \tag{5.45}$$

Substituting (5.44) by (4.10) we can rewrite (4.10) as

$$\Phi_c(\boldsymbol{\tau}) = \frac{1}{2G(\boldsymbol{\tau})} = \boldsymbol{\tau}\cdot\mathbf{C}\cdot\boldsymbol{\tau} = \frac{1}{\lambda_3} + (\frac{1}{\lambda_2} - \frac{1}{\lambda_3})(\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2). \tag{5.46}$$

We may now turn to the assumption $\lambda_2 < \lambda_3$ (it is valid for copper (4.14)). In this case it is easy to see that

$$\frac{1}{\lambda_3} \leqslant \Phi_c(\boldsymbol{\tau}) \leqslant \frac{1}{\lambda_2}. \tag{5.47}$$

Substituting (5.29) and (5.34) by (5.47) we obtain

$$\Phi_b(\boldsymbol{\tau}) \leqslant \Phi_c(\boldsymbol{\tau}) \leqslant \Phi_a(\boldsymbol{\tau}). \tag{5.48}$$

When tensor $\boldsymbol{\tau}$ has the form (5.41), the equations (5.42) together with (5.19) form a set of nine equations for six components of $\boldsymbol{\tau}$ (5.41). It means that the set in general has no solution. From (5.19) it is easy to conclude that

$$(\mathbf{K}\cdot\boldsymbol{\tau})\cdot(\mathbf{K}\cdot\boldsymbol{\tau}) = (\boldsymbol{\tau}\cdot\mathbf{K}\cdot\boldsymbol{\tau})^2 + 6(\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau})^2. \tag{5.49}$$

Substituting (5.43) and (5.44) by (5.49) we get

$$3(\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau})^2 = (\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2)(\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2). \tag{5.50}$$

It means that

$$\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau} = 0$$

if and only if $\boldsymbol{\tau}$ is a proper state of $\mathbf{C}$. On the other hand, using the definition $\boldsymbol{\tau}^2$ and $\mathbf{K}\cdot\boldsymbol{\tau}$ and the property of $\boldsymbol{\tau}$ (5.42), we obtain

$$\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau} = \mathbf{2}(\tau_{11}\tau_{22}\tau_{33} - \tau_{12}\tau_{13}\tau_{23}). \tag{5.51}$$

There are solutions of (5.19) for which

$$\boldsymbol{\tau}^2\cdot\mathbf{K}\cdot\boldsymbol{\tau} \neq 0.$$

For instance, if we make the following assumption:

$$\tau_{23} = 0, \qquad \tau_{11}\tau_{22}\tau_{33} \neq 0$$

then we find from (5.19) and (5.42) that

$$\tau_{11}^2 = \tau_{22}^2 = \tau_{12}^2 = \frac{1}{8}, \quad \tau_{33}^2 = \frac{1}{2}, \quad \tau_{13} = \tau_{23} = 0, \tag{5.52}$$

$$\tau_{11} = \tau_{22}, \qquad \tau_{33} = -(\tau_{11} + \tau_{22}).$$

Substituting (5.52) by (5.43) we get

(5.53) $$\boldsymbol{\tau}\cdot\mathbf{K}\cdot\boldsymbol{\tau} = \tau_{11}^2+\tau_{22}^2+\tau_{33}^2 = \frac{3}{4}.$$

If tensor $\boldsymbol{\tau}$ is taken as

(5.54) $$\boldsymbol{\tau} \sim \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

then it satisfies (5.52) and the vectors $\mathbf{n}_1$, $\mathbf{n}_2$ (2.6) are as follows:

$$\mathbf{n}_1 = \frac{1}{2}(\mathbf{k}+\mathbf{l}+\sqrt{2}\mathbf{m}), \qquad \mathbf{n}_2 = \frac{1}{2}(\mathbf{k}+\mathbf{l}-\sqrt{2}\mathbf{m}).$$

The stored elastic energy (4.10) in this case has the form

(5.55) $$\Phi_c(\boldsymbol{\tau}) = \frac{1}{2G(\boldsymbol{\tau})} = \frac{\lambda_2+3\lambda_3}{4\lambda_2\lambda_3}.$$

It is easy to see that for copper

$$\Phi_b(\boldsymbol{\tau}) \leqslant \Phi_c(\boldsymbol{\tau}) \leqslant \Phi_a(\boldsymbol{\tau}),$$

and there are no local extrema of the stored elastic energy at this $\boldsymbol{\tau}$.

In this same manner we can consider the following assumption:

$$\tau_{33} = 0, \;\; \tau_{12}\cdot\tau_{13}\cdot\tau_{23} \neq 0.$$

An easy computation shows that there is no solution (5.19) of the above form.

The obtained results show that for cubic isotropy, the Young modulus has local extrema in the directions of the axes of crystal and the directions perpendicular to the octahedral planes. Uniaxial tension $\boldsymbol{\sigma} = \mathbf{n}\otimes\mathbf{n}$ is a generalized proper state of the compliance tensor $\mathbf{C}$. There is quite a different situation for the shear modulus. The Kirchhoff modulus has local extrema for some deviatoric proper states of $\mathbf{C}$ with the constraint $\det\boldsymbol{\tau} = 0$.

Appendix. Notation

For readers more accustomed to the usual Cartesian index notation, we shall add a convenient rephrasing of formulae:

$$\boldsymbol{\sigma} = \mathbf{S}\cdot\boldsymbol{\varepsilon} \leftrightarrow \sigma_{ij} = S_{ijkl}\varepsilon_{kl}, \quad \boldsymbol{\varepsilon} = \mathbf{C}\cdot\boldsymbol{\sigma} \leftrightarrow \varepsilon_{mn} = C_{mnij}\sigma_{ij},$$

$$\mathbf{S}\circ\mathbf{C} = \mathbf{C}\circ\mathbf{S} = \mathbb{I}_{\mathcal{S}} \leftrightarrow S_{ijkl}C_{klmn} = C_{ijkl}S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}),$$

$$\boldsymbol{\sigma}\cdot\mathbf{C}\cdot\boldsymbol{\sigma} \leftrightarrow C_{pqrs}\sigma_{pq}\sigma_{rs}, \qquad \boldsymbol{\omega}\cdot\boldsymbol{\omega} \leftrightarrow \omega_{ij}\omega_{ij},$$

$$\mathbf{1} \leftrightarrow \delta_{ij} \quad \text{(Kronecker's symbol)},$$

$$\mathbf{n}\otimes\mathbf{n} \leftrightarrow n_i n_j,$$

$$\mathbf{1}\cdot\mathbf{H}\cdot\boldsymbol{\omega} \leftrightarrow H_{pprs}\omega_{rs},$$

$$\mathbf{C}\cdot(\mathbf{n}\otimes\mathbf{n}) \leftrightarrow C_{ijkl}n_k n_l,$$

$$\boldsymbol{\tau}^2\cdot\mathbf{C}\cdot\boldsymbol{\tau} \leftrightarrow C_{ijkl}\tau_{im}\tau_{mj}\tau_{kl},$$

$$\operatorname{tr}\boldsymbol{\sigma} = \mathbf{1}\cdot\boldsymbol{\sigma} = \sigma_{pp}, \quad \operatorname{tr}\boldsymbol{\sigma}^2 = \sigma_{pq}\sigma_{pq}, \quad \operatorname{tr}\boldsymbol{\sigma}^3 = \sigma_{pq}\sigma_{qr}\sigma_{rp},$$

$$\begin{aligned}\mathbf{K} &= \mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}+\mathbf{l}\otimes\mathbf{l}\otimes\mathbf{l}\otimes\mathbf{l}+\mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m},\\ K_{pqrs} &= k_p k_q k_r k_s + l_p l_q l_r l_s + m_p m_q m_r m_s.\end{aligned}$$

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