Arch. Mech., **53**, 4-5, pp. 457–485, Warszawa 2001

Michell-like grillages and structures with locking

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THE PAPER GENERALIZES THE MICHELL theory of plane pseudo-continua to the anti-plane problems in which the loading is perpendicular to the plane of the structure. The starting point is the minimum compliance problem for a two-phase Kirchhoff plate. Upon relaxation, one of the materials can be degenerated to a void (or microvoids) and by imposing the condition of the volume being small, one arrives at the Michell-like problem for a locking plate. The locking locus B is determined explicitly; it tends to a square if the Poisson ratio tends to 1. In the last case the locking locus coincides with that used in the Rozvany-Prager theory of optimal grillages. A theory of perfectly-locking and elastic-locking plates and shells, not necessarily isotropic, is formulated. Dual extremum and existence theorems are also given.

1. Introduction

MICHELL STRUCTURES REPRESENT solutions to the following optimum design problem:

$$(\mathrm{P}_1) \qquad \inf\left\{ \int_\Omega \big(|\sigma_I| + |\sigma_{II}|\big)dx \quad \Big| \; \boldsymbol{\sigma} \in S_1(\Omega) \right\}.$$

Here σ_I, σ_{II}, $\sigma_I \geq \sigma_{II}$, are principal stresses of $\boldsymbol{\sigma}$ and $S_1(\Omega)$ stands for the set of statically admissible stresses within a bounded plane domain Ω. This domain is parametrized by a Cartesian system (x_1, x_2); $x = (x_1, x_2) \in \Omega$. The loading of density $\mathbf{p}=\mathbf{p}(\mathrm{x})$ is applied on $\Gamma_1 \subset \partial\Omega$. Thus

$$S_1(\Omega) = \{\sigma \in L^2(\mathbf{\Omega}), \mathbb{E}_2^s) \mid \int_\Omega \sigma^{\alpha\beta} v_{\alpha,\beta}\, dx = \int_{\Gamma_1} \mathbf{p}\cdot\mathbf{v}\, ds \quad \forall\, \mathbf{v} \in V_1(\Omega)\},$$

and

$$V_1(\Omega) = \{\mathbf{v} \in H^1(\Omega)^2 \quad \mid \mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega \setminus \Gamma_1\}.$$

$\mathbb{E}_2^s$ represents the space of symmetric 2×2 matrices and $H^1(\Omega)^2 = [H^1(\Omega)]^2$. The problem (P_1) can be found in ROZVANY [17, p.48] and was developed by STRANG and KOHN [18]. The problem dual to (P_1) is equivalent to the original MICHELL setting [14].

A natural counterpart of the above plane elasticity problem for the antiplane case, where the structure should carry a loading perpendicular to its plane, reads

$$(\mathrm{P}_2) \qquad \inf\left\{ \int_\Omega (|M_I| + |M_{II}|)dx \quad \mid \mathbf{M} \in \ S_2(\Omega) \right\}$$

where $S_2(\Omega)$ represents a set of statically admissible moments $\mathbf{M}$ or

$$S_2(\Omega) = \{\mathbf{M} \in L^2(\Omega; \mathbb{E}_2^s) \mid \int_\Omega M^{\alpha\beta} v_{,\alpha\beta}\, dx + \int_{\Gamma_1} (Q^0 v - M^0 \frac{\partial v}{\partial \mathbf{n}}) ds = 0 \ \forall v \in V_2(\Omega)\}$$

where

$$V_2(\Omega) = \left\{ v \in H^2(\Omega) | \quad v = 0, \quad \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma_1 \right\}.$$

Here $\mathbf{n}$ is a unit vector outward normal to $\partial\Omega$. The boundary loading Q^0, M^0 is assumed here along $\Gamma_1 \subset \Omega$ and the plate is clamped on $\Gamma_0 = \partial\Omega \setminus \Gamma_1$.

Several exact solutions to the problem (P_2) have been found by Prager, Rozvany and their collaborators, see ROZVANY[17].

In 1993 ALLAIRE and KOHN [1] showed that the problem (P_1) can be obtained by admitting the volume of a structure to be smaller and smaller in the minimum compliance shape design within the linear elasticity framework. This passage to a limit leads to the formulation (P_1) free of any elastic characteristics, see Remark 28.4.2 in LEWIŃSKI and TELEGA [11].

The above results suggest that the problem (P_2) can be achieved by imposing the condition of the volume being small in the shape design problem for a thin plate. But this is not the case. The present authors proved in Sec. 26.9 of [11] that the condition of the volume being small results in a new formulation

$$(\mathrm{P}_3) \qquad \inf\left\{ \int_\Omega \left[\frac{1-\nu}{1+\nu} (\mathrm{tr}\mathbf{M})^2 + (|M_I| + |M_{II}|)^2 \right]^{\frac{1}{2}} dx |\ \mathbf{M} \in S_2(\Omega) \right\},$$

where ν represents the Poisson ratio. It is seen that only for $\nu = 1$ the problem (P_3) assumes the form (P_2) Note, however, that in both the formulations (P_2) and (P_3) the integrand is of linear growth.

All the available analytical and numerical solutions to the problems (P_1), (P_2) are based on the formulations dual to them. The formulation dual to (P_1) reads, see STRANG and KOHN [18],

$$(\mathrm{P}_1^*) \qquad \sup\{ \int_{\Gamma_1} \mathbf{p} \cdot \mathbf{v}\, ds \quad | \boldsymbol{\epsilon}(\mathbf{v}) \in B_1 \}$$

where

$$B_1 = \{ \boldsymbol{\epsilon} \in \mathbb{E}_s^2 \Big| \quad |\epsilon_I| \le 1, \quad |\epsilon_{II}| \le 1 \}$$

and $\boldsymbol{\epsilon}(\mathbf{v}) = (\epsilon_{\alpha\beta}(\mathbf{v}))$, where

$$(1.1) \qquad \epsilon_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right).$$

Thus B_1 is a square in the $(\epsilon_I, \epsilon_{II})$-plane. By analogy, the problem (P_2^*) assumes the form

$$(\mathrm{P}_2^*) \qquad \sup\{ \int_{\Gamma_1} (Q^\circ v - M^\circ \frac{\partial v}{\partial \mathbf{n}}) ds \quad | \quad \boldsymbol{\kappa}(v) \in B_2 \}$$

with

$$B_2 = \{ \boldsymbol{\kappa} \in \mathbb{E}_s^2 \Big| \quad |\kappa_I| \le 1, \quad |\kappa_{II}| \le 1 \},$$

and

$$(1.2) \qquad \kappa_{\alpha\beta}(v) = -\frac{\partial^2 v}{\partial x_\alpha \partial x_\beta}.$$

The aim of the present paper is to derive the dual problem (P_3^*). Since the integrand in (P_3) is of linear growth, it is clear that the dual formulation should be similar to (P_2^*) but with the set B_2 replaced with a new set B. Just this convex set B will be explicitly constructed.

The problem (P_3^*) constructed in this manner can be viewed as a locking problem, see ČYRAS [4], DEMENGEL AND SUQUET [7] , JEMIOŁO and TELEGA [9]. The aim of the present paper is also to exploit the locking nature of the problem (P_3) thus showing its new physical interpretation and proving its well-posedness. This naturally leads to perfectly-locking and elastic-locking plates. We will also propose a theory of such shells. Our considerations imply that in the problem (P_1) the quantities $(\sigma^{\alpha\beta})$ denote not the stress tensor but the stress rate tensor, a fact usually overlooked in the relevant literature. Similarly in (P_2) and (P_3) the quantity $\mathbf{M} = (M^{\alpha\beta})$ is to be viewed as the rate of couple resultants. Our first results on plates of small volume were announced in [22], [20]. The results formulated in these two contributions need refinements, particularly the form of B.

2. Two-phase layout optimization of thin elastic plates

The classical layout problem for thin elastic plates consists in looking for an in-plane optimal distribution of *two isotropic materials*, of prescribed volumes, that realizes minimum of the total compliance. To be more specific, let us assume that the α-th material is characterized by the bulk and shear moduli $\hat{k}_\alpha, \hat{\mu}_\alpha, \alpha = 1, 2$. Let the thickness of the plate h be fixed. Thus the bending stiffness tensor of the α-th phase assumes the form

$$\mathbf{D}_\alpha = 2k_\alpha \mathbf{I}_1 + 2\mu_\alpha \mathbf{I}_2, \tag{2.1}$$

where

$$k_\alpha = \frac{h^3}{12}\hat{k}_\alpha, \quad \mu_\alpha = \frac{h^3}{12}\hat{\mu}_\alpha \tag{2.2}$$

and the tensors $\mathbf{I}_\alpha$ are defined by

$$I_1^{\alpha\beta\lambda\mu} = \frac{1}{2}\delta^{\alpha\beta}\delta^{\lambda\mu}, \qquad I_2^{\alpha\beta\lambda\mu} = \frac{1}{2}(\delta^{\alpha\lambda}\delta^{\beta\mu} + \delta^{\alpha\mu}\delta^{\beta\lambda} - \delta^{\alpha\beta}\delta^{\lambda\mu}). \tag{2.3}$$

The compliance tensor $\mathbf{C}_\alpha = \mathbf{D}_\alpha^{-1}$ is represented by

$$\mathbf{C}_\alpha(x) = \frac{1}{2}K_\alpha(x)\mathbf{I}_1 + \frac{1}{2}L_\alpha(x)\mathbf{I}_2 \tag{2.4}$$

where

$$K_\alpha = 1/k_\alpha, \qquad L_\alpha = 1/\mu_\alpha. \tag{2.5}$$

For simplicity, the assumption of ordering is usually assumed

$$k_2 > k_1, \qquad \mu_2 > \mu_1, \tag{2.6}$$

hence the terms $\mathbf{D}_2 - \mathbf{D}_1$ and $\mathbf{C}_1 - \mathbf{C}_2$ are positive definite and invertible. Assume that the α-th material covers the domain Ω_α, hence $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}, \Omega_1 \cap \Omega_2 = \varnothing$. Let $\chi_\alpha(x) = \chi_{\Omega_\alpha}(x)$ be the characteristic function of Ω_α. The bending stiffness tensor has the form

$$\mathbf{D}(x) = \chi_1(x)\mathbf{D}_1 + \chi_2(x)\mathbf{D}_2. \tag{2.7}$$

Let us impose the isoperimetric condition

$$|\Omega_2| = \int_\Omega \chi_2(x)dx = c_2 \tag{2.8}$$

and $|\Omega_1| = |\Omega| - c_2$. Assume that the plate is subjected along Γ_1 to the transverse boundary loading $Q^0\ (s)$ and the boundary moment $M^0\ (s)$, (s) being the natural parameter of $\Gamma = \partial\Omega$.

Let us define the linear form, being the loading functional, by

$$f(v) = \int_{\Gamma_1} \left(M_n^0 \left(-\frac{\partial v}{\partial \mathbf{n}} \right) + Q^0 v \right) ds. \tag{2.9}$$

One can also assume that $\Gamma_1 = \partial\Omega$ and then we additionally assume that

$$f(v) = 0 \qquad \forall \mathrm{v} \in \mathcal{R}$$

with

$$\mathcal{R} = \{v| \qquad v = v_\circ + \alpha_1 x_1 + \alpha_2 x_2\}, \qquad v_\circ, \alpha_1, \alpha_2, \in \mathbb{R}.$$

Note that $\boldsymbol{\kappa}(v) = \mathbf{0}$ for $v \in \mathcal{R}$; here $\boldsymbol{\kappa}(v) = (\kappa_{\alpha\beta}(v))$, see Eq.(1.2). Let us define the bilinear form

$$a(w, v) = \int_\Omega \kappa_{\alpha\beta}(w) D^{\alpha\beta\lambda\mu}(x) \kappa_{\lambda\mu}(v) dx, \tag{2.10}$$

with $\mathbf{D}$ being given by (2.7). Let w_{χ_2} be a solution to the equilibrium problem

$$(\mathrm{P}_{\chi_2}) \qquad \Big| \quad a(w_{\chi_2}, v) = f(v) \qquad \forall v \in V_2(\Omega).$$

The minimum compliance problem reads:

Find $\bar{\chi}_2$ such that

$$\text{(P)} \qquad f(w_{\bar{\chi}_2}) = \min\{f(w_{\chi_2}) |\ w_{\chi_2} \text{ solves } (P_{\chi_2}) \text{ with} \int_\Omega \chi_2(x)dx = c_2\}.$$

It is well-known that the above problem needs a relaxation, see LIPTON [12], TELEGA and LEWIŃSKI [21], LEWIŃSKI and TELEGA [11]. The relaxation means replacing:

$$\chi_\alpha \text{ by } m_\alpha \in L^\infty(\Omega; [0,1]) \quad \text{and } \mathbf{D} \text{ by } \mathbf{D}_h,$$

where $\mathbf{D}_h$ represents the effective bending stiffness tensor of a composite plate in which the area fractions of both materials are equal to m_1 and m_2 , respectively. Moreover, the isoperimetric condition (2.8) is replaced by

$$\int_\Omega m_2(x)dx = c_2. \tag{2.11}$$

The stiffness tensor $\mathbf{D}_h$ is determined by the formula of homogenization. Moreover, one can prove that $\mathbf{D}_h$ is fully characterized by periodic composites, see RAITUMS [15] and LEWIŃSKI and TELEGA [11]. This feature makes the formula for $\mathbf{D}_h$ explicit and, consequently, makes it possible to introduce the relaxed formulation ($\tilde{\text{P}}$) of (P) in a constructive manner. Since all details of posing the relaxed problem ($\tilde{\text{P}}$) can be found in LEWIŃSKI and TELEGA [11, Secs. 26.2, 26.3] there is no need to rewrite the details. It is sufficient to recall that in the first step we apply the Castigliano theorem and then reformulate the problem by the translation method of GIBIANSKY and CHERKAEV, see [11].

The relaxed formulation of the minimum compliance problem for two-phase thin plates assumes eventually the form

$$(\tilde{\text{P}}) \qquad \min \{F(m_2) + \lambda \int_\Omega m_2(x)dx \,|\, m_2 \in L^\infty\,(\Omega; [0,1])\}.$$

Here λ is a multiplier associated with the isoperimetric condition (2.11) and

$$F(m_2) = 2 \min_{\mathbf{M} \in S_2(\Omega)} \int_\Omega \mathcal{W}^*(\mathbf{M}(x), m_2(x))dx. \tag{2.12}$$

The potential $\mathcal{W}^*$ is defined as follows:

$$\mathcal{W}^*(\mathbf{M}, m_2) = \begin{cases} \dfrac{1}{4} I^2(\mathbf{M}) H(\zeta_M) & \text{if} \quad I(\mathbf{M}) \neq 0, \\ \dfrac{1}{4} \{L\}_m II^2(\mathbf{M}) & \text{if} \quad I(\mathbf{M}) = 0. \end{cases} \tag{2.13}$$

The following notation has been introduced:

$$I(\mathbf{M}) = \frac{1}{\sqrt{2}}\operatorname{tr}(\mathbf{M}), \quad II(\mathbf{M}) = \frac{1}{\sqrt{2}}[(\operatorname{tr}(\mathbf{M}))^2 - 4\det\mathbf{M}]^{1/2}, \tag{2.14}$$

$$\{L\}_m = (m_1 L_1^{-1} + m_2 L_2^{-1})^{-1}, \tag{2.15}$$

$$\zeta_M = \frac{II(\mathbf{M})}{|I(\mathbf{M})|} \tag{2.16}$$

or

$$\zeta_M = \frac{|M_I - M_{II}|}{|M_I + M_{II}|}.$$

The function $H(\zeta)$ is defined as follows:

$$H(\zeta) = \begin{cases} H_L(\zeta) & \text{if} \quad \zeta \in [0, \zeta_2], \\ H_i(\zeta) & \text{if} \quad \zeta \in [\zeta_2, \zeta_1], \\ H_R(\zeta) & \text{if} \quad \zeta \geq \zeta_1. \end{cases} \tag{2.17}$$

Here

$$\zeta_1 = \frac{[L]_m \triangle K}{[K]_m \triangle L}, \quad \zeta_2 = \frac{m_2 \triangle K}{[K]_m + L_2}, \tag{2.18}$$

and

$$[f]_m = m_1 f_2 + m_2 f_1, \quad \triangle f = |f_2 - f_1|, \quad f \in \{K, L\}. \tag{2.19}$$

Moreover,

$$H_L(\zeta) = a_L + c_L\zeta^2, \quad H_R(\zeta) = a_R + c_R\zeta^2,$$

$$\begin{aligned} H_i(\zeta) &= H_L(\zeta) + A_L(\zeta - \zeta_2)^2, \\ H_i(\zeta) &= H_R(\zeta) + A_R(\zeta - \zeta_1)^2, \end{aligned} \tag{2.20}$$

where

$$(2.21)\quad a_L = \frac{< K >_m L_2 + K_1 K_2}{L_2 + [K]_m}, \quad a_R = \{K\}_m, \quad c_L = L_2, \quad c_R = \{L\}_m,$$

and

$$(2.22)\quad \begin{aligned} A_L &= \frac{m_1 \triangle L (L_2 + [K]_m)}{[K]_m + [L]_m}, \\ A_R &= \frac{m_1 m_2 (\triangle L)^2 [K]_m}{[L]_m ([K]_m + [L]_m)}, \end{aligned}$$

and

$$< f >_m = m_1 f_1 + m_2 f_2, \quad f \in \{K, L\}.$$

Let us note that (2.12) may be viewed as an equilibrium problem of a nonlinearly elastic plate with a smooth potential $\mathcal{W}^*$.

3. Shape design of a thin plate

The notion of *shape design* means that only one material is at our disposal and we should arrange a given amount of this material to form the stiffest plate. Thus it suffices to put $k_1 = 0$ and $\mu_1 = 0$ into the problem ($\tilde{\mathrm{P}}$) and observe that this substitution is admissible. Then we obtain

$$(3.1)\quad \mathcal{W}^*(\mathbf{M}, m_2) = \begin{cases} \frac{1}{4} I^2(\mathbf{M}) H_0(\zeta_M) & \text{if} \quad I(\mathbf{M}) \neq 0, \\ \frac{1}{4m_2} L_2 II^2(\mathbf{M}) & \text{if} \quad I(\mathbf{M}) = 0, \end{cases}$$

where

$$(3.2)\quad H_0(\zeta) = \begin{cases} \frac{1}{m_2}(K_2 + m_1 L_2) + L_2 \zeta^2 & \text{if} \quad \zeta \in [0,1], \\ \frac{1}{m_2}(K_2 + L_2 \zeta^2) & \text{if} \quad \zeta \geq 1. \end{cases}$$

Note that the conditions

$$(3.3)\quad \zeta_M \in [0,1], \quad \zeta_M \geq 1,$$

are equivalent to

$$(3.4)\quad \det \mathbf{M} \geq 0, \quad \det \mathbf{M} \leq 0.$$

The function $H(\zeta)$ was smooth but $H_0(\zeta)$ is not. It is continuous but has a cusp at $\zeta = 1$.

The formula (3.1) can be expressed by a concise formula

$$\mathcal{W}^*(\mathbf{M}, m_2) = \mathcal{W}_0^*(\mathbf{M}) + \frac{1-m_2}{4m_2}[K_2 I^2(\mathbf{M}) + L_2 u(\mathbf{M})] \tag{3.5}$$

where $\mathcal{W}_0^*(\mathbf{M})$ given by

$$\mathcal{W}_0^*(\mathbf{M}) = \frac{1}{4}K_2 I^2(\mathbf{M}) + \frac{1}{4}L_2 II^2(\mathbf{M}), \tag{3.6}$$

represents the potential of a virgin plate and the function $u(\mathbf{M})$ is defined by

$$u(\mathbf{M}) = \begin{cases} II^2(\mathbf{M}) & \text{if} \quad \det(\mathbf{M}) \le 0, \\ I^2(\mathbf{M}) & \text{if} \quad \det(\mathbf{M}) \ge 0, \end{cases} \tag{3.7}$$

or

$$u(\mathbf{M}) = \frac{1}{2}(|M_I| + |M_{II}|)^2. \tag{3.8}$$

Thus the expression(3.5) can be rearranged to the form

$$\mathcal{W}^*(\mathbf{M}, m_2) = \mathcal{W}_0^*(\mathbf{M}) + \frac{1-m_2}{2m_2} g(\mathbf{M}), \tag{3.9}$$

with

$$g(\mathbf{M}) = \frac{1}{4}K_2(M_I + M_{II})^2 + \frac{1}{4}L_2(|M_I| + |M_{II}|)^2. \tag{3.10}$$

Let us put (3.9) into $(\tilde{P})$ and interchange the order of minima. One finds

$$\min_{\mathbf{M} \in S_2(\Omega)} \int_\Omega F_\lambda(\mathbf{M})dx \tag{3.11}$$

where

$$F_\lambda(\mathbf{M}) = \min_{0 \le m_2 \le 1}[2\mathcal{W}^*(\mathbf{M}, m_2) + \lambda m_2]. \tag{3.12}$$

Minimization over m_2 can be performed analytically. Finally we arrive at

$$F_\lambda(\mathbf{M}) = 2\mathcal{W}_0^*(\mathbf{M}) + \begin{cases} 2[\lambda g(\mathbf{M})]^{1/2} - g(\mathbf{M}) & \text{if} \quad g(\mathbf{M}) \le \lambda, \\ \lambda & \text{if} \quad g(\mathbf{M}) \ge \lambda. \end{cases} \tag{3.13}$$

The minimization problem (3.11) with the integrand of the form (3.13) is well-posed; its solution exists, see LEWIŃSKI and TELEGA [11, Sec.26.7].

4. Shape design of thin plates of small volume

The relaxed formulation (3.11) of the shape design problem of thin plates can serve as a starting point for finding the shape design formulation of plates of very small volume. The compliance minimization assumes a peculiar form. In such a case the material tries to carry the loading by very thin, continuously distributed ribs.

4.1. Compliance minimization problem

The solutions to the problem (3.11) become exteremely interesting if the given amount of the material is very small. In such a case the optimum plates degenerate to grillages forming ribbed structures or fans. The condition of small volume implies that the multiplier λ is very large. Thus the region $g(\mathbf{M}) > \lambda$ will be absent and for large values of λ, the main term of $F_\lambda(\mathbf{M})$ has the form

$$F_\lambda(\mathbf{M}) = 2\lambda^{1/2}[g(\mathbf{M})]^{1/2}. \tag{4.1}$$

Since only one material is at our disposal, we shall simplify the notation by writing $k_2 = k, \quad \mu_2 = \mu, \quad \nu_2 = \nu, \quad K_2 = K, \quad L_2 = L.$

Consequently, the problem (3.11) assumes the form

$$(\hat{\mathrm{P}}) \qquad \inf_{\mathbf{M}\in S_2(\Omega)} \int_\Omega [K(M_I + M_{II})^2 + L(|M_I| + M_{II}|)^2]^{1/2} dx.$$

We set

$$G(\mathbf{M}) = [K(M_I + M_{II})^2 + L(|M_I| + |M_{II}|)^2]^{1/2}, \quad \mathbf{M} \in \mathbb{E}_2^s.$$

A straightforward calculation shows that

$$G_\infty(\mathbf{M}) = G(\mathbf{M}), \ \mathbf{M} \in \mathbb{E}_2^s,$$

where G_∞ denotes the so-called recession function of G defined by, cf. LEWIŃSKI and TELEGA [11], ROCKAFELLAR [16],

$$G(\mathbf{M}) := \lim_{t\to\infty} \frac{1}{t} G(t\mathbf{M}).$$

It means that the stiffest plate of small volume exhibits perfectly-locking behaviour. Since $K/L = \mu/k$ and $\nu = (k-\mu)/(k+\mu)$ one finds: $K/L = (1-\nu)/(1+\nu)$ and the problem ($\hat{\mathrm{P}}$) assumes the form (P_3) mentioned in the Introduction.

Let us consider the level lines of the integrand of $(\hat{\mathrm{P}})$ in the plane (M_I, M_{II}). Let us introduce the polar representation

$$M_I = r\cos\vartheta, \quad M_{II} = r\sin\vartheta \tag{4.2}$$

into the condition

$$\left[\frac{1-\nu}{1+\nu}(M_I + M_{II})^2 + (|M_I| + |M_{II}|)^2\right]^{1/2} = \text{const.} \tag{4.3}$$

Hence we find

$$M_I = M_0 \frac{\cos\vartheta}{\left[1 + \dfrac{1-\nu}{1+\nu}(1+\sin 2\vartheta) + |\sin 2\vartheta|\right]^{1/2}}, \tag{4.4}$$

$$M_{II} = M_0 \frac{\sin\vartheta}{\left[1 + \dfrac{1-\nu}{1+\nu}(1+\sin 2\vartheta) + |\sin 2\vartheta|\right]^{1/2}},$$

and M_0 is a constant. Assume that $M_0 = 1$. For $\nu = 1$ the contour (4.4) forms a square, of the side $\sqrt{2}$, rotated by $\pi/4$, see Fig.1. For each $\nu \in [0,1]$ the contours in the quarters $M_I M_{II} > 0$ are straight lines parallel to the line: $M_I + M_{II} = 1$.

For $\nu = 0$ the contour (4.4), in the quarters $M_I M_{II} < 0$, becomes a circle of radius $\sqrt{2}/2$, see Fig.1.

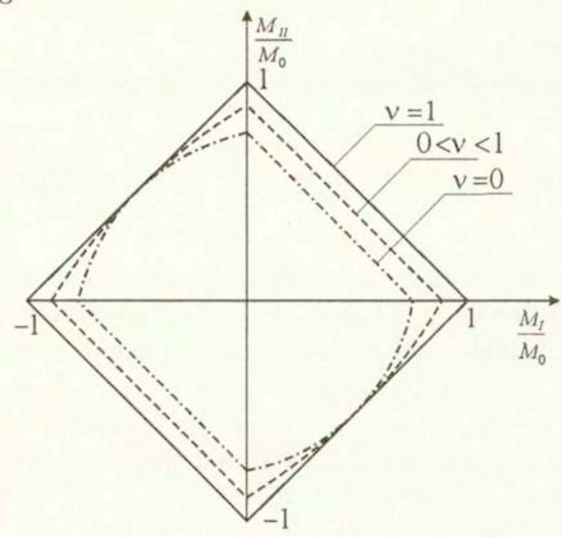

FIG. 1. Level lines of the integrand of problem $(\hat{\mathrm{P}})$

4.2. Dual formulation of $(\hat{\text{P}})$

It is highly convenient to rearrange the problem $(\hat{\text{P}})$ to its dual form involving kinematic variables. Note first that $\hat{\text{P}}$ represents a locking problem in which $\mathbf{M}$ is not a moment field but rather a moment rate.

Let us disclose the equilibrium equation concealed in $S_2(\Omega)$ in a standard manner:

$$\inf_{\mathbf{M}\in L^2(\Omega;\mathbb{E}^s_2)} \sup_{v\in V_2} \left\{ \int_\Omega \{[K(M_I+M_{II})^2 + L(|M_I|+|M_{II}|)^2]^{1/2}\}dx - \int_\Omega \mathbf{M} : \boldsymbol{\kappa}(v)dx + f(v) \right\} \tag{4.5}$$

where $\mathbf{M} : \boldsymbol{\kappa}(v) = M^{\alpha\beta}\kappa_{\alpha\beta}$. Note that v plays simultaneously two roles: it is a trial field of the variational equilibrium equation and the Lagrangian multiplier. The operations inf and sup can be formally interchanged and thus we arrive at

$$\sup_{v\in V_2(\Omega)} \{f(v) + \int_\Omega R(\boldsymbol{\kappa}(v))dx\} \tag{4.6}$$

with

$$R(\boldsymbol{\kappa}) = \inf_{\mathbf{M}\in\mathbb{E}^s_2} \{[K(M_I+M_{II})^2 + L(|M_I|+|M_{II}|)^2]^{1/2} - \mathbf{M} : \boldsymbol{\kappa}\}. \tag{4.7}$$

Since the integrand appearing in the problem $(\hat{\text{P}})$ is of linear growth therefore inf $\hat{\text{P}}$ is not, in general, attained in $S_2(\Omega)$. The problem $(\hat{\text{P}})$ is convex but the functional

$$J(\mathbf{M}) = \int_\Omega [K(M_I+M_{II})^2 + L(|M_I|+|M_{II}|)^2]^{1/2}dx$$

is not lower semicontinuous over $L^2(\Omega,\mathbb{E}^s_2)$ Consequently, minimax theorems expounded in EKELAND and TEMAM [8] are not applicable to our case. However, one can apply the duality theory presented in EKELAND and TEMAM [8] directly to the problem $(\hat{\text{P}})$ and then, after standard calculation, we get (4.6) and (4.7). In this manner the interchange of inf and sup operations is justified. Note that the integrand of $(\hat{\text{P}})$ is of linear growth and the dual function $R(\boldsymbol{\kappa})$ must have the following form, cf. ROCKAFELLAR [16],

$$R(\boldsymbol{\kappa}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{\kappa}\in B, \\ -\infty & \text{otherwise.} \end{cases} \tag{4.8}$$

Here B is a convex set containing $\boldsymbol{\kappa} = \mathbf{0}$ in its interior. This set, defining the locking locus, can be explicitly constructed. The construction is the subject of the next section.

Thus the problem (4.6) takes the form

$$(\hat{\mathrm{P}}^*) \qquad \sup\{f(v) | v \in V_2(\Omega), \quad \boldsymbol{\kappa}(v(x)) \in B \quad \text{for a.e.} \quad x \in \Omega\}.$$

Existence theorems, for the problems $\hat{\mathrm{P}}$ and $\hat{\mathrm{P}}^*$, will be formulated in Sec.5.

4.3. Geometry of the set B

The aim of this section is to analyze the problem (4.7) and find the explicit form of the set B.

STEP 1. We show first that minimum in (4.7) is attained by a matrix $\mathbf{M}$ of principal directions (x_α^M) coinciding with the principal directions (x_α^κ) of the tensor $\boldsymbol{\kappa}$. We consider three coordinate systems: $(x_1, x_2), (x_1^\kappa, x_2^\kappa), (x_1^M, x_2^M)$, see Fig. 2, where the angles α_κ and α_M are depicted.

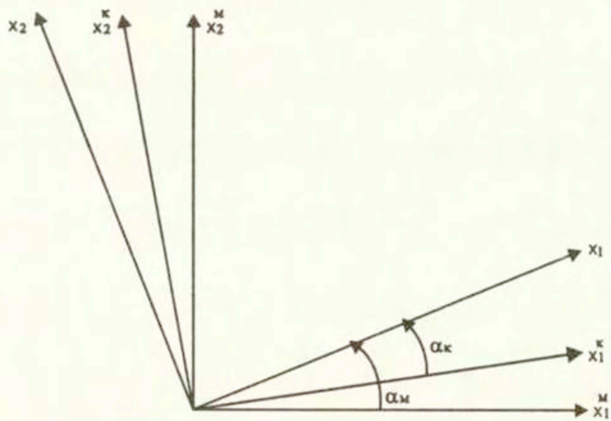

FIG. 2. The axes x_α versus principal directions of $\boldsymbol{\kappa}$ and $\mathbf{M}$

The components of $\boldsymbol{\kappa}$ are represented by the classical formulae

$$\kappa_{11} = \frac{1}{2}[(\kappa_I + \kappa_{II}) + (\kappa_1 - \kappa_{II}) \cos 2\alpha_\kappa],$$

$$\kappa_{22} = \frac{1}{2}[(\kappa_I + \kappa_{II}) - (\kappa_1 - \kappa_{II}) \cos 2\alpha_\kappa], \tag{4.9}$$

$$\kappa_{12} = -\frac{1}{2}(\kappa_I - \kappa_{II}) \sin 2\alpha_\kappa,$$

and the components M^{11}, M^{22}, M^{12} are represented similarly, in terms of M_I, M_{II} and α_M. We assume: $\kappa_I \geq \kappa_{II}, M_I \geq M_{II}$. We calculate

$$\text{(4.10)} \quad \mathbf{M} : \boldsymbol{\kappa} = M^{\alpha\beta}\kappa_{\alpha\beta} = \kappa_I M_I + \kappa_{II} M_{II} - \sin^2(\alpha_M - \alpha_\kappa)(M_I - M_{II})(\kappa_I - \kappa_{II}).$$

Thus (4.7) can be expressed as follows:

$$\text{(4.11)} \quad R(\boldsymbol{\kappa}) = \min_{M_I, M_{II}} \Big\{ \min_{\alpha_M}\{[K(M_I + M_{II})^2 + L(|M_I| + |M_{II}|)^2]^{1/2} - (\kappa_I M_I + \kappa_{II} M_{II}) + \sin^2(\alpha_M - \alpha_\kappa)(M_I - M_{II})(\kappa_I - \kappa_{II})\} \Big\}.$$

Minimum over α_M is attained for $\alpha_M = \alpha_\kappa$ since $(M_I - M_{II})(\kappa_I - \kappa_{II}) > 0$. Thus the principal directions of $\mathbf{M}$ coincide with the principal directions of $\boldsymbol{\kappa}$.

STEP 2. The function $R(\boldsymbol{\kappa})$ can be put in the form

$$\text{(4.12)} \qquad R(\boldsymbol{\kappa}) = \min_{\bar{x}, \bar{y} \in \mathbb{R}} \{[K(\bar{x} + \bar{y})^2 + L(|\bar{x}| + |\bar{y}|)^2]^{1/2} - \bar{x}\kappa_I - \bar{y}\kappa_{II}\}$$

since now it is unimportant that M_I, M_{II} in (4.11) are principal values of a certain matrix of $\mathbb{E}_2^s$.

Let us change the variables

$$\text{(4.13)} \qquad \sqrt{L}\,\bar{x} = x, \quad \sqrt{L}\,\bar{y} = y, \quad \kappa_I/\sqrt{L} = \varkappa_I, \quad \kappa_{II}/\sqrt{L} = \varkappa_{II},$$

to obtain

$$\text{(4.14)} \qquad R(\sqrt{L}\varkappa) = \min_{x, y \in \mathbb{R}} \{[\gamma(x + y)^2 + (|x| + |y|)^2]^{1/2} - x\varkappa_I - y\varkappa_{II}\},$$

where

$$\text{(4.15)} \qquad \gamma = K/L, \quad \gamma = \frac{1 - \nu}{1 + \nu}, \quad \nu \in [0, 1].$$

Let us introduce the polar representation

$$\text{(4.16)} \qquad \begin{aligned} x &= r\cos\vartheta, \quad y = r\sin\vartheta, \quad r \geq 0, \\ \varkappa_I &= k\cos\varphi, \quad \varkappa_{II} = k\sin\varphi, \quad k \geq 0. \end{aligned}$$

Let $R(\sqrt{L}\varkappa) = \tilde{R}(k, \varphi)$,where

$$\text{(4.17)} \qquad \tilde{R}(k, \varphi) = \min_{\substack{r \geq 0 \\ \vartheta \in \mathbb{R}}} r\{[1 + \gamma(1 + \sin 2\vartheta) + |\sin 2\vartheta|]^{1/2} - k\cos(\vartheta - \varphi)\}$$

Further we introduce an auxiliary function

$$h(\varphi) = \min_{\vartheta\in\mathbb{R}} \frac{[1+\gamma(1+\sin 2\vartheta)+|\sin 2\vartheta|]^{1/2}}{|\cos(\vartheta-\varphi)|}. \tag{4.18}$$

Here γ is treated as a parameter. We note that

$$\tilde{R}(k,\varphi) = \begin{cases} 0 & \text{if} \quad k \le h(\varphi)|, \\ -\infty & \text{otherwise.} \end{cases} \tag{4.19}$$

Consequently,

$$R(\boldsymbol{\kappa}) = \begin{cases} 0 & \text{if} \quad ||\boldsymbol{\kappa}|| \le \sqrt{L}\, h(\varphi) \\ -\infty & \text{otherwise,} \end{cases} \tag{4.20}$$

where $||\boldsymbol{\kappa}|| = \sqrt{(\kappa_I)^2 + (\kappa_{II})^2}$ and φ=arctg(κ_{II}/κ_I). Thus we have

$$R(\boldsymbol{\kappa}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{\kappa} \in B \\ -\infty & \text{otherwise,} \end{cases} \tag{4.21}$$

and the locking locus observed in the plane (κ_I, κ_{II}), denoted by $\tilde{B}$, has the form

$$\tilde{B} = \{(\kappa_I, \kappa_{II}) \;\Big|\; \| \boldsymbol{\kappa} \| \le r(\varphi), \quad \varphi = \text{arctg}(\frac{\kappa_{II}}{\kappa_I})\} \tag{4.22}$$

and

$$r(\varphi) = \sqrt{L}\, h(\varphi), \quad r(\varphi) = \sqrt{L+K}\tilde{h}(\varphi), \tag{4.23}$$

with

$$\tilde{h}(\varphi) = \Big(\frac{L}{L+K}\Big)^{1/2} h(\varphi). \tag{4.24}$$

The function $\tilde{h}(\varphi)$ assumes the form (a detailed derivation is given in the Appendix)

$$\tilde{h}(\varphi) = \begin{cases} \dfrac{1}{\sin\varphi} & \text{if} \quad \dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{2} + \beta, \\ \left[\dfrac{1-\nu^2}{1+\nu\sin 2\varphi}\right]^{1/2} & \text{if} \quad \dfrac{\pi}{2} + \beta \le \varphi \le \dfrac{3}{4}\pi, \end{cases} \tag{4.25}$$

where tg $\beta = \nu$ or $\beta = \text{arctg}\nu$.

Note that for $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2} + \beta$ the contour of $\tilde{B}$ is a line $\kappa_{II} = \sqrt{L+K}$. For greater φ the contour becomes a curve. For $\varphi = \frac{3}{4}\pi$ we have $\tilde{h} = \sqrt{1+\nu}$. For all $\nu \in[0,1]$ the relevant locking locus has a corner at $\kappa_I = \kappa_{II}$ and is smooth for $\varphi = \frac{\pi}{2} + \beta$. Indeed, one can check that

$$\tag{4.26} \begin{aligned} \tilde{h}\left((\frac{\pi}{2}+\beta)^-\right) &= \tilde{h}\left((\frac{\pi}{2}+\beta)^+\right) = \sqrt{1+\nu^2}, \\ \frac{d\tilde{h}}{d\varphi}\left((\frac{\pi}{2}+\beta)^-\right) &= \frac{d\tilde{h}}{d\varphi}\left((\frac{\pi}{2}+\beta)^+\right) = \nu\sqrt{1+\nu^2}, \end{aligned}$$

which confirms that stitching at $\varphi = \frac{\pi}{2} + \beta$ is smooth. For the limit case $\nu = 0$ we have $\beta = 0$ and hence

$$\tag{4.27} \tilde{h}(\varphi) = \begin{cases} \dfrac{1}{\sin\varphi} & \text{if} \quad \dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{2}, \\ 1 & \text{if} \quad \dfrac{\pi}{2} \le \varphi \le \dfrac{3}{4}\pi. \end{cases}$$

For the limit case $\nu = 1$ we get

$$\tag{4.28} \tilde{h}(\varphi) = \frac{1}{\sin\varphi}, \quad \frac{\pi}{4} \le \varphi \le \frac{3}{4}\pi.$$

Those two extreme shapes of $\tilde{B}$ are depicted in Fig.3, where $\tilde{\kappa}_\alpha = \kappa_\alpha(L+K)^{-1/2}, \alpha = I, II$.

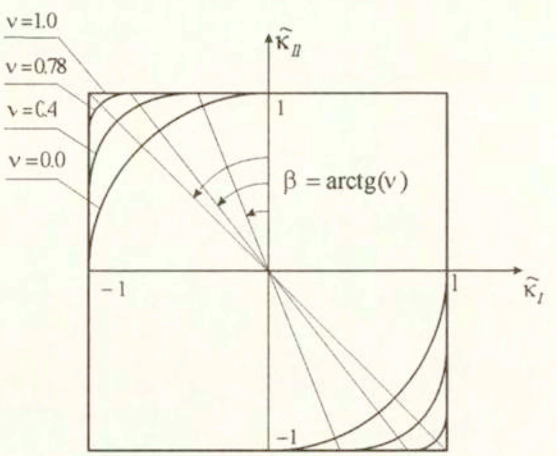

FIG. 3. Shapes of the locking loci $\tilde{B}$ for various values of ν

The contours of Figs. 1 and 3 are reciprocal. We note that for $\nu = 0$ the arcs of a circle transform into the same arcs. For $\nu = 1$ we note that the corners in Fig.1 transform into the sides of the square and vice versa, a well-known result of the duality theory. Transformation of the intermediate contours for $0 < \nu < 1$ is more complicated, but is still found analytically.

A correct qualitative characterization of the set B has already been given in TELEGA *et al.* [22]. However, the contours of $\tilde{B}$ for det $\kappa < 0$ were found there incorrectly. Figure 3 constitutes a refinement of Fig. 1 in [22, 20].

4.4. Mutually dual constitutive relations

The locking locus $\tilde{B}$ is bounded, convex and closed in the space of principal strain measures κ_I, κ_{II}. Similar properties characterize the locking locus B in the space of strain measures $\kappa_{\alpha\beta}$, since $I_B = G^*$. Moreover, $\boldsymbol{\kappa} = \mathbf{0}$ lies in the interior of B ($\mathbf{0} \in \operatorname{int} B$). The constitutive relation for the problem ($\hat{\mathrm{P}}^*$) of Sec. 4.2 can be put in the subdifferential form

$$\mathbf{M} \in \partial I_B(\boldsymbol{\kappa}), \tag{4.29}$$

where I_B denotes the indicator function of the set B, see EKELAND and TEMAM [8], ROCKAFELLAR [16],

$$I_B(\boldsymbol{\kappa}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{\kappa} \in B, \\ \infty & \text{otherwise.} \end{cases} \tag{4.30}$$

The inverse constitutive relationship is given by

$$\boldsymbol{\kappa} \in \partial\, \mathrm{G}(\mathbf{M}). \tag{4.31}$$

In (4.29) and (4.31) ∂ denotes subdifferentiation, cf.[16]. We recall that the function $G(\mathbf{M})$ is the integrand of the functional appearing in the problem ($\hat{\mathrm{P}}$), cf. Sec. 4.1 Moreover, by Eq.(3.10), we write $G(\mathbf{M}) = 2\sqrt{g(\mathbf{M})}$. Since λ in Eq.(4.1) is positive, therefore we may write

$$\boldsymbol{\kappa} \in \partial F_\lambda(\mathbf{M}) \qquad <=> \qquad \boldsymbol{\kappa} \in \partial G(\mathbf{M}).$$

Elementary, we also have, cf [16],

$$G(\mathbf{M}) = \sup\left\{\mathbf{M} : \boldsymbol{\kappa} | \boldsymbol{\kappa} \in B\right\}, \tag{4.32}$$

which links both the formulations ($\hat{\mathrm{P}}$) and $(\hat{\mathrm{P}})^*$.

Let us pass now to the basic properties of the function G. They are given by

(i) $$G(\mathbf{0}) = 0.$$

(ii) There exist constants $C_1 > C_0 > 0$ such that

$$\forall \mathbf{M} \in \mathbb{E}_2^s, \quad C_0|\mathbf{M}| \le G(\mathbf{M}) \le C_1(1 + |\mathbf{M}|).$$

(iii) G is positively homogeneous, i.e. ,

$$G(a\mathbf{M}) = aG(\mathbf{M}) \text{ if } a > 0.$$

Here $|\mathbf{M}| = \sum_{\beta,\alpha=1}^{2} M^{\alpha\beta} M^{\alpha\beta}$.

The proof of properties (i)-(iii) is similar to the procedure used in perfect plasticity for the study of the dissipation density, cf. LEWIŃSKI and TELEGA [11], TEMAM [23].

In the existence study which is the subject of the next section, a particular form of locking locus is not required. General assumptions on a locking locus, still denoted by B, are

(a) $B \subset \mathbb{E}_s^2$ is convex and closed.

(b) There exist $0 < r_1 < r_2 < +\infty$ such that $K(0, r_1) \subset B \subset K(0, r_2)$ where $K(0, r)$ denotes the ball with the centre at zero and with radius r.

The sets B and $\tilde{B}$ derived in Sec.4.3 satisfy (a) and (b). In the general case, the support function G of B is determined by the relation of type (4.32).

5. Existence theorems

In the case of plates with locking extremal problems cannot be formulated in standard manner, as for elastic plates. Let us first solve the problem of existence of transverse displacements. The primal problem is formulated as follows, cf. the problem $(\hat{\mathrm{P}})^*$,

$(\mathcal{P})$ $$\sup\{f(w)\,|w \in U,\ \kappa(w(x)) \in B \text{ a.e. } x \in \Omega\}$$

where

$$U = \{w \in V(\Omega)| w = \tilde{u}a_1,\ \partial w/\partial \mathbf{n} = \tilde{u}a_2 \text{ on } \Gamma_1\}$$

Here a_α, $\alpha = 1, 2$, are prescribed functions of $s \in \Gamma_1$ whilst $\tilde{u}$, possibly depending on s, denotes the intensity of generalized boundary displacements $(w, \partial w/\partial \mathbf{n})$. One parameter problem is a specific case, where $m = \tilde{u} \in \mathbb{R}$ is simply a load parameter. In the functional f, Q^0 and M_n^0 are to be viewed as fields of weight multipliers, cf. ČYRAS [4].

THEOREM 1. *Under the assumptions (a) and (b), the maximization problem $(\mathcal{P})$ possesses a solution $\bar{w} \in U$ provided that*

$$\int_{\Gamma_1} (Q^0 a_1 - M_n^0 a_2)\, ds \neq 0. \tag{5.1}$$

P r o o f. The proof follows the approach used in Demengel and Suquet [7], Demengel [6] for 3D problems. Therefore it suffices to sketch the proof for the plates with locking. Applying the duality theory we prove

$$\sup \mathcal{P} = \inf \mathcal{P}^*, \tag{5.2}$$

where

$$(\mathcal{P}^*) \quad \inf\Big\{ \int_{\Omega} G(\mathbf{M})\, dx | \mathbf{M} \in L^2(\Omega, \mathbb{E}_2^s),\ \mathrm{divdiv}\mathbf{M} = 0 \ \text{ a.e. in } \Omega;$$
$$Q = Q^0, M_n = M_n^0 \quad \text{on } \Gamma_1 \Big\},$$

where $M_n = M^{\alpha\beta} n_\alpha n_\alpha$ and Q is the rate of the KIRCHHOFF shear force. To simplify the remaining part of the proof we consider the case of one parameter loading: $w = mw^0$ and $\partial w/\partial \mathbf{n} = mw^1$ on Γ_1. We set, see the formula (5.9) below,

$$U_{ad}(m) = \{ w \in H^2(\Omega) | w = 0, \quad \partial w/\partial \mathbf{n} = 0 \quad \text{on} \quad \Gamma_0;$$
$$w = mw^0, \quad \partial w/\partial \mathbf{n} = mw^1 \quad \text{on } \Gamma_1 \}, \tag{5.3}$$

$$\mathcal{S}_0(\Omega) = \{ \mathbf{M} \in \mathcal{Z}(\Omega, \mathbb{E}_2^s) |\ \mathrm{divdiv}\mathbf{M} = 0 \ \text{ a.e. in } \Omega \}, \tag{5.4}$$

$$B_g = \{ w \in H^2(\Omega) | \quad \kappa(w(x)) \in B \ \text{ a.e. } x \in \Omega \}. \tag{5.5}$$

The locking limit analysis problems assume now the following form:

$$(\mathrm{Q}) \qquad \sup \{ m | B_g \cap U_{ad}(m) \neq \varnothing \},$$

$$(\mathrm{Q}^*) \qquad \inf \Big\{ \int_{\Omega} G(\mathbf{M})\, dx | \mathbf{M} \in \mathcal{S}_0,\ \int_{\Gamma_1} (Qw^0 - M_n w^1) ds = 1 \Big\}.$$

It can be shown that

$$\sup Q = \inf Q^*.$$

Under the assumption (5.1), which now means that there exists $\tilde{\mathbf{M}} \in \mathcal{S}_0$ such that

$$\int_{\Gamma_1} (\tilde{Q}w^0 - \tilde{M}_n w^1)ds \neq 0, \tag{5.6}$$

we have

$$m_l = \inf Q^* = \sup Q < +\infty. \tag{5.7}$$

Here m_l is the locking limit load. Indeed, it can be shown that if (5.6) is not satisfied then $\inf Q^* = \sup Q = +\infty$. To assess the equality between $\sup Q$ and $\inf Q^*$ one can apply a penalty method, cf. DEMENGEL and SUQUET [7].To this end we introduce the following perturbed problem:

$$(\mathrm{Q}_\delta) \qquad \inf\left\{-m + \frac{1}{\delta} d(\boldsymbol{\kappa}(w), B_d)\,|m \in \mathbb{R}, w \in U_{ad}(m)\right\},$$

where

$$B_d = \left\{\boldsymbol{\varepsilon} \in L^2(\Omega, \mathbb{E}_s^2) | \boldsymbol{\varepsilon}(x) \in B \text{ a.e.}\, x \in \Omega\right\}.$$

The dual problem means evaluating

(Q_δ^*)

$$\sup\left\{-\int_\Omega G(\mathbf{M})\,ds | \mathbf{M} \in \mathcal{S}_0, \int_{\Gamma_1} (Qw^0 - M_n w^1)\,ds = 1, ||\mathbf{M}||_{L^2(\Omega, \mathbb{E}_2^s)} \leq \frac{1}{\delta}\right\}.$$

Under (5.6) we have

(i) $\inf Q_\delta = \sup Q_\delta^* > -\infty$,

(ii) Q_δ admits a solution,

(iii) $-\inf Q_\delta$ converges to $\sup Q$.

The functional

$$J_\delta(m, w) = -m + \frac{1}{\delta} d(\boldsymbol{\kappa}(w), B_d)$$

is coercive on $H^2(\Omega) \cap U_{ad}(m)$. Consequently, there exists at least one solution (m_δ, w_δ) to the problem (Q_δ). The sequence $\{(m_\delta, w_\delta)\}_{\delta>0}$ is bounded in $\mathbb{R} \times$

$H^2(\Omega)$ and we can extract a subsequence $\{(m_{\delta'}, w_{\delta'})\}_{\delta'>0}$ such that $m_{\delta'} \to \bar{m}$ in $\mathbb{R}$, $w_{\delta'} \rightharpoonup \bar{w}$ weakly in $H^2(\Omega)$ when $\delta' \to 0$. Finally

$$\limsup(-\inf Q_{\delta'}) = \liminf(-\inf Q_{\delta'}) = \sup Q = \inf Q^* = \bar{m} = m_l.$$

REMARK 1. In fact, the function $\bar{w}$ solving the primal problem $(\mathcal{P})$ or (Q) belongs to a nonreflexive Banach space, cf. DEMENGEL [6],

$$V^\infty(\Omega) = \{v \in L^\infty(\Omega) | \kappa_{\alpha\beta}(v) \in L^\infty(\Omega)\}.$$

□

Let us pass to the study of dual problem (Q*). We introduce the following nonreflexive Banach space:

$$\mathcal{S}(\Omega) = \{\mathbf{M} \in \mathcal{Z}(\Omega, \mathbb{E}_2^s) | \operatorname{divdiv}\mathbf{M} \in \mathbb{M}_b(\Omega)\}, \tag{5.8}$$

where $\mathbb{M}_b(\Omega)$ denotes the space of bounded measures on Ω and, cf. DEMENGEL [6],

$$\mathcal{Z}(\Omega, \mathbb{E}_2^s) = \{\mathbf{M} \in \mathbb{M}_b(\Omega, \mathbb{E}_2^s) | \operatorname{div}\mathbf{M} \in L^2(\Omega)^2\}. \tag{5.9}$$

Now the function G is a convex function of the measure, cf. BOUCHITTÉ and VALADIER [3]. We also introduce the relaxed problem

$$\text{(RQ*)} \qquad \inf\left\{ \int_\Omega G(\mathbf{M})\,dx + m_l \left| \int_{\Gamma_1} (Qw^o - M_n w^1)\,ds - 1 \right|,\ \mathbf{M} \in \mathcal{S}(\Omega),\ \operatorname{divdiv}\mathbf{M} = 0 \text{ in } \Omega \right\}.$$

THEOREM 2. *From each minimizing sequence of* (Q*) *or* (RQ*) *one can extract a subsequence weak-* convergent in* $\mathbb{M}_b(\Omega, \mathbb{E}_2^s)$ *to a solution of* (RQ*).

To prove the last theorem it suffices to follow the approach elaborated by DEMENGEL [6].

REMARK 2. Let γ be a sufficiently smooth curve in Ω, for instance dividing Ω into subdomains Ω^+ and Ω^- such that $\Omega = \Omega^+ \cup \Omega^- \cup \gamma$. If $\mathbf{M} \in \mathcal{S}(\Omega)$, there exists a couple $(\mathbf{M}_1, \xi) \in L^2(\Omega, \mathbb{E}_2^s) \times HB(\Omega)$ such that, cf. DEMENGEL [6],

(i) $\mathbf{M} = \mathbf{M}_1 + (\operatorname{cof} \nabla^2 \xi)^t$,

(ii) the mass of $\mathbf{M}$ on γ takes the form $\left[\frac{\partial\xi}{\partial\mathbf{n}}\right]\mathbf{t}\otimes\mathbf{t}$, and

$$M_n = \frac{\partial^2\xi}{\partial s^2} - \frac{1}{R}\frac{\partial\xi}{\partial\mathbf{n}}, \qquad [M_n] = \frac{1}{R}\left[\frac{\partial\xi}{\partial\mathbf{n}}\right], \tag{5.10}$$

$$M^{\alpha\beta}n_\alpha t_\beta = \frac{\partial}{\partial s}\left(\frac{\partial\xi}{\partial\mathbf{n}}\right) + \frac{1}{R}\frac{\partial\xi}{\partial s}, \qquad [M^{\alpha\beta}n_\alpha t_\beta] = \left[\frac{\partial}{\partial s}\left(\frac{\partial\xi}{\partial\mathbf{n}}\right)\right]. \tag{5.11}$$

Here R is the curvature radius of γ and $\mathbf{t}$ denotes a unit tangent vector, and [q] denotes the jump of q across γ. Obviously, $(\mathrm{cof}\,\mathbf{A})^t$ is the transpose matrix of the cofactor matrix of $\mathbf{A}$, cf. Dacorogna [5]. If γ is a line interval then $[M_n] = 0$. Further, if R is bounded then $[M_n] \in L^1(\gamma)$. Particularly, γ can be a part of Γ. Anyway, the moment stress rate tensor $\bar{\mathbf{M}}$ solving problem (RQ*) exhibits discontinuities.

6. Theory of perfectly-locking thin plates and shells

Let $S \subset \mathbb{R}^3$ denote a sufficiently regular middle surface of a thin shell [2]. Its deformation is determined by the displacement vector $(\mathbf{u}, w)$, where $\mathbf{u} = (u_a)$. By $\boldsymbol{\gamma} = (\gamma_{\alpha\beta})$ and $\boldsymbol{\rho} = (\rho_{\alpha\beta})$ we denote the linear deformation measures. The locking locus is now contained in the space $\mathbb{E}^2_s \times \mathbb{E}^2_s$, i.e. $B \subset \mathbb{E}^2_s \times \mathbb{E}^2_s$. For instance, once the locking condition

$$l(x, \boldsymbol{\gamma}, \boldsymbol{\rho}) \le 0, \quad x = (x_i) \in S, \quad i = 1, 2, 3, \tag{6.1}$$

is known, then

$$B(x) = \{(\boldsymbol{\gamma}, \boldsymbol{\rho}) \in \mathbb{E}^2_s \times \mathbb{E}^2_s \mid l(x, \boldsymbol{\gamma}, \boldsymbol{\rho}) \le 0, \ x \in S\}. \tag{6.2}$$

The strains measures the for linear Koiter's shell are given by Eqs.(6.6) below.

We observe that no assumption of isotropy is required. A specific case of B for thin plates has been given in Sec.4.3. For homogeneous materials B does not depend on x. General assumptions on B are specified by:

(A_1) $B(x) \subset \mathbb{E}^2_s \times \mathbb{E}^2_s$ is convex and closed.

(A_2) There exist $0 < r_1 < r_2 < +\infty$ such that, $K(0, r_1) \subset B \subset K(0, r_2)$, where $K(0, r)$ denotes the ball with the centre at zero (in $\mathbb{E}^2_s \times \mathbb{E}^2_s$) and with radius r.

The constitutive relationship is assumed in the subdifferential form, cf. (4.29),

$$(\mathbf{N}, \mathbf{M}) \in \partial I_{B(x)}(\boldsymbol{\gamma}, \boldsymbol{\rho}). \tag{6.3}$$

As we already know, for perfectly-locking shells $\mathbf{N}, \mathbf{M}$ denote the rates of the stress resultants and couple resultants, respectively.

The support function of B is now given by, cf. Eq.(4.32),

$$G(x, \mathbf{N}, \mathbf{M}) = \sup\{N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\rho_{\alpha\beta} | (\boldsymbol{\gamma}, \boldsymbol{\rho}) \in B(x)\}. \tag{6.4}$$

The function G is a counterpart of the density of plastic dissipation. Now, however, it does not describe the dissipation since the locking response is reversible. The constitutive relationship inverse to (6.3) is given by

$$(\boldsymbol{\gamma}, \boldsymbol{\rho}) \in \partial G(x, \mathbf{N}, \mathbf{M}), \;\; x \in S. \tag{6.5}$$

The function $G(x, ., .)$ has linear growth and satisfies:

(i) $G(x, \mathbf{0}, \mathbf{0}) = 0$,

(ii) there exist constants $C_1 > C_0 > 0$ such that

$$\forall\; (\mathbf{N}, \mathbf{M}) \in \mathbb{E}^s_2 \times \mathbb{E}^s_2, C_0(|\mathbf{N}| + |\mathbf{M}|) \leq G(x, \mathbf{N}, \mathbf{M}) \leq C_1(1 + |\mathbf{N}| + |\mathbf{M}|),$$

(iii) $G(x, ., .)$ is positively homogeneous

$$G(x, a\mathbf{N}, a\mathbf{M}) = aG(x, \mathbf{N}, \mathbf{M}) \text{ if } a > 0.$$

Dependence of B on $x \in S$ is not arbitrary. After BOUCHITTÉ and VALADIER [3] we make the following assumption, cf. also Remark 13.2.1 in LEWIŃSKI and TELEGA [11],

$$\forall \boldsymbol{\phi} \in C_0(S, \mathbb{E}^2_s \times \mathbb{E}^2_s), \boldsymbol{\phi}(x) \in B(x) \text{ almost everywhere} \Rightarrow \boldsymbol{\phi}(x) \in B(x) \text{ everywhere.}$$

Locking limit analysis

Let the boundary ∂S of the shell middle surface S consist of two disjoint parts denoted by ∂S_0 and ∂S_1. For KOITER's shells the strain measures are, cf.[2],

$$\begin{aligned} \gamma_{\alpha\beta}(\mathbf{u}, w) &= \frac{1}{2}(u_{\alpha||\beta} + u_{\beta||\alpha}) - b_{\alpha\beta}w, \\ \rho_{\alpha\beta}(\mathbf{u}, w) &= -w\,\|_{\alpha\beta} - b^{\gamma}_{\alpha||\beta}u_{\gamma} - b^{\gamma}_{\alpha}u_{\gamma||\beta} - b^{\gamma}_{\beta}u_{\gamma||\alpha} + c_{\alpha\beta}w. \end{aligned} \tag{6.6}$$

The meaning of (6.6) is standard in the linear shell theory. Now we introduce the spaces of kinematically admissible fields and of statically admissible fields as follows, cf. (5.3-5.5),

$$\mathcal{U}_{ad}(m) = \Big\{(\mathbf{u}, w) \in \left[H^1(S)\right]^2 \times H^2(S) \mid \mathbf{u} = m\mathbf{u}^0,\ w = mw^0,\ \frac{\partial w}{\partial \mathbf{n}} = mw^1 \text{ on } \partial S_1\Big\},$$

$$\mathcal{B}_g = \Big\{(\mathbf{u}, w) \in \left[H^1(S)\right]^2 \times H^2(S) | \ [\boldsymbol{\gamma}(\mathbf{u}(x), w(x)),\ \boldsymbol{\rho}(\mathbf{u}(x), w(x))] \in B(x), \text{a.e. } x \in S\Big\},$$

$$\mathcal{S}_o(S) = \{(\mathbf{N}, \mathbf{M}) \in L^2(S, \mathbb{E}^s_2) \times L^2(S, \mathbb{E}^s_2)|$$

$$eq_1 := N^{\alpha\beta}{}_{\|\beta} - 2b^\alpha_\sigma M^{\sigma\beta}{}_{\|\beta} - b^\alpha_{\sigma\|\beta} M^{\sigma\beta} = 0,$$

$$eq_2 := b_{\alpha\beta} N^{\alpha\beta} + M^{\alpha\beta}_{\|\alpha\beta} - c_{\alpha\beta} M^{\alpha\beta} = 0 \text{ in } S,$$

$$\tilde{N}^a = (N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta}) n_\beta - b^\alpha_\beta M^{\lambda\beta} n_\lambda = 0 \text{ on } \partial S_0,$$

$$T = M^{\alpha\beta}_{\|\beta} n_\alpha + \partial/\partial s(M^{\alpha\beta} t_\alpha n_\beta) = 0,\ M_n = M^{\alpha\beta} n_\alpha n_\beta = 0 \text{ on } \partial S_o\Big\}.$$

The locking limit analysis is formulated in the form of two dual problems:

$$(\mathcal{R}) \qquad \sup\{m | B_g \cap U_{ad}(m) \neq \varnothing\}$$

$$(\mathcal{R}^*) \qquad \inf\Big\{\int_S G(x, \mathbf{N}, \mathbf{M}) dS | (\mathbf{N}, \mathbf{M}) \in \mathcal{S}_o(S),\ \int_{\partial S_1} (\tilde{N}^a u^0_\alpha + T w^0 - M_n w^1) ds = 1\Big\}.$$

Similarly to the previous section we get

$$m_l = \inf \mathcal{R}^* = \sup \mathcal{R} < +\infty,$$

where m_l denotes the locking limit load multiplier.

7. Elastic-perfectly locking thin shells

Let us introduce a model of such shells, being a counterpart of the deformational theory of plasticity. Now we have

$$\mathbf{N} = \mathbf{N}_e + \mathbf{N}_l, \quad \mathbf{M} = \mathbf{M}_e + \mathbf{M}_l. \tag{7.1}$$

Here the subscripts e, l denote the elastic and locking parts, respectively. For the elastic parts linear elastic behavior is assumed. The locking parts obey the following rule, cf. (6.3),

$$(\mathbf{N}_l, \mathbf{M}_l) \in \partial I_{B(x)}(\boldsymbol{\gamma}, \boldsymbol{\rho}). \tag{7.2}$$

The locking locus $B(\mathrm{x})$ has the properties (A_1) and (A_2) specified in Sec.6. We observe, however, that in contrast to plasticity, only some components of $(\mathbf{N}(x), \mathbf{M}(x))$ can be constrained by $B(x)$. The density of the strain energy (elastic-locking potential) is given by

$$j^*(x, \boldsymbol{\gamma}, \boldsymbol{\rho}) = \frac{1}{2}(A^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + D^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}\rho_{\lambda\mu}) + I_{B(x)}(\boldsymbol{\gamma}, \boldsymbol{\rho}), \tag{7.3}$$

where $(\boldsymbol{\gamma}, \boldsymbol{\rho}) \in \mathbb{E}^2_s \times \mathbb{E}^2_s$. Its conjugate (dual) function $j(x, ., .)$ satisfies, cf. LEWIŃSKI and TELEGA [11], TEMAM [23],

$$\exists\, \tilde{c}_0, \tilde{c}_1 > 0,\ \tilde{c}_0(|\mathbf{N}| + |\mathbf{M}| - 1) \le j(x, \mathbf{N}, \mathbf{M}) \le \tilde{c}_1(|\mathbf{N}| + |\mathbf{M}| + 1), \tag{7.4}$$

where $(\mathbf{N}, \mathbf{M}) \in \mathbb{E}^s_2 \times \mathbb{E}^s_2$. The loading functional is assumed in the following form :

$$\hat{F}(\mathbf{u}, w) = \int\limits_S (q^\alpha u_\alpha + qw)dS + \int\limits_{\partial S_0} \left(N^0_\alpha u_\alpha + T^0 w - M^0_n \frac{\partial w}{\partial \mathbf{n}} \right) ds. \tag{7.5}$$

We are now in a position to formulate a pair of dual extremum principles characterizing the displacements and generalized stresses in the elastic-locking shell.

$$(\mathcal{P}_1) \qquad \inf \left\{ \int\limits_S j^*(x, \boldsymbol{\gamma}(\mathbf{u}, w),\ \boldsymbol{\rho}(\mathbf{u}, w))dS - \hat{F}(\mathbf{u}, w) | (\mathbf{u}, w) \in \mathcal{U}_{ad} \right\},$$

$(\mathcal{P}^*)$

$$\sup \left\{ -\int\limits_S j(x, \mathbf{N}, \mathbf{M})dS + \int\limits_{\partial S_1} (\tilde{N}^a \hat{u}_\alpha + T\hat{w} - M_n \hat{w}^1)\, ds | (\mathbf{N}, \mathbf{M}) \in \mathcal{S}_{ad} \right\}.$$

Here

$$\mathcal{U}_{ad} = \left\{ (\mathbf{u}, w) \in [H^1(S)]^2 \times H^2(S) | \mathbf{u} = \hat{\mathbf{u}},\ w = \hat{w},\ \partial w / \partial \mathbf{n} = \hat{w}^1 \quad \text{on} \quad \partial S_1 \right\}$$

$$\mathcal{S}_{ad} = \Big\{ (\mathbf{N}, \mathbf{M}) \in L^2(S, \mathbb{E}^s_2) \times L^2(S, \mathbb{E}^s_2) | eq_1 + \mathbf{q} = \mathbf{0},\ eq_2 + q = 0 \text{ in } S;$$
$$\tilde{N}^\alpha = N^0_\alpha,\ T = T^0,\ M_n = M^0_n \text{ on } \partial S_0 \Big\}.$$

Problem $(\mathcal{P}_1)$ admits a solution provided that $B_g \cap \mathcal{U}_{ad} \neq \varnothing$ It means that the loading functional $\hat{F}$ cannot be arbitrary.

REMARK 3. The model of elastic-locking plates is a specific case of the model proposed for shells.

8. Final remarks

We have shown that the shape optimization problem of plates of small volume leads to plates with locking. This fact has been established for isotropic plates. Whether a similar statement holds for anisotropic plates and shells, not necessarily isotropic, remains an open problem. Nevertheless, a general theory of perfectly-locking and elastic-locking plates and shells has been proposed. The elastic-locking model is a counterpart of the deformational theory of plasticity.

Acknowledgement

The work was supported by the Polish Committee for Scientific Research (KBN) through the grant No 7 T07A 04318.

Appendix

Te aim of the Appendix is to derive the formula (4.25) defining the contour of the set $\tilde{B}$. To find $h(\varphi)$ given by (4.18) we represent it in the form

$$h(\varphi) = \left[\min_{\vartheta} f_{\varphi}(\vartheta)\right]^{\frac{1}{2}}, \tag{A.1}$$

where

$$f_{\varphi}(\vartheta) = \frac{1 + \gamma(1 + \sin 2\vartheta) + |\sin 2\vartheta|}{\cos^2(\vartheta - \varphi)}. \tag{A.2}$$

Let us show that the lines

$$\kappa_I = \pm\kappa_{II} \tag{A.3}$$

are symmetry axes of $\tilde{B}$. It is sufficient to show that

$$h(\varphi_1) = h(\varphi_2), \quad h(\varphi_3) = h(\varphi_4), \tag{A.4}$$

for $\varphi_2 = \dfrac{\pi}{2} - \varphi_1$, $\varphi_1 \in [0, \pi/2]$, and for $\varphi_4 = \pi - \left(\varphi_3 - \dfrac{\pi}{2}\right)$, $\varphi_3 \in \left[\dfrac{\pi}{2}, \pi\right]$.

We have

$$h\left(\frac{\pi}{2}-\varphi\right)=\left[\min_{\vartheta} f_{\frac{\pi}{2}-\varphi}(\vartheta)\right]^{1/2}.$$

Note that

$$f_{\frac{\pi}{2}-\varphi}(\vartheta)=\frac{1+\gamma(1+\sin 2\vartheta')+|\sin 2\vartheta'|}{\cos^2(\vartheta'-\varphi)},$$

where $\vartheta'=\frac{\pi}{2}-\vartheta$. Hence $f_{\frac{\pi}{2}-\varphi}(\vartheta)=f_\varphi(\vartheta')$. Thus

$$\min_{\vartheta} f_{\frac{\pi}{2}-\varphi}(\vartheta)=\min_{\vartheta'} f_\varphi(\vartheta')=\min_{\vartheta} f_\varphi(\vartheta).$$

Therefore, $h(\frac{\pi}{2}-\varphi)=h(\varphi)$, which proves (A4). We conclude that in order to find the contour of $\tilde{B}$, it is suffcient to find this contour for $\varphi\in\left[\frac{\pi}{4},\frac{3}{4}\pi\right]$.

We proceed further to find the contour of $\tilde{B}$ for $\varphi\in\left[\frac{\pi}{4},\frac{3}{4}\pi\right]$. We note that the formula (A2) can be rearranged to the form

$$f_\varphi(\vartheta)=\hat{f}_\varphi(\mathrm{tg}\vartheta), \tag{A.5}$$

$$\hat{f}_\varphi(x)=\frac{1+\gamma}{\sin^2\varphi}\,g_\varphi(x). \tag{A.6}$$

with

$$g_\varphi(x)=\begin{cases}\dfrac{1-2\nu x+x^2}{(x+\mathrm{ctg}\varphi)^2} & \text{if} \quad x\le 0,\\[2ex] \left(\dfrac{1+x}{x+\mathrm{ctg}\varphi}\right)^2 & \text{if} \quad x\ge 0.\end{cases} \tag{A.7}$$

The results above lead to

$$h(\varphi)=\frac{\sqrt{1+\gamma}}{\sin\varphi}\left[\min_{x\in\mathbb{R}} g_\varphi(x)\right]^{1/2} \tag{A.8}$$

and the variable ϑ is no longer necessary. To find $\min\limits_{x} g_\varphi(x)$ we shall make use of two elementary results:

(a) $$\min_{x\ge 0}\left|\frac{x+1}{x+\mathrm{ctg}\varphi}\right|=1 \quad \text{if} \quad \varphi\in\left[\frac{\pi}{4},\frac{3}{4}\pi\right],$$

(b) $$\min\frac{1-2bx+x^2}{(x-p)^2}=\frac{1-b^2}{1-2bp+p^2} \quad \text{if} \quad b<1,\ p\in\mathbb{R}.$$

The minimum in (b) is attained at

$$x_0 = \frac{1 - bp}{b - p}, \tag{A.9}$$

To find $\min_x g(x)$ one should consider the case of $x_0 \le 0$ and

$$\frac{1 - b^2}{1 - 2bp + p^2} \le 1, \tag{A.10}$$

for $b = \nu$, $p = -\text{ctg}\varphi$. Let us note that

$$1 - \frac{1 - b^2}{1 - 2bp + p^2} = \frac{(b - p)^2}{1 - 2bp + p^2}.$$

Thus the condition (A10) is satisfied provided that $x_0 \le 0$. Let $\Psi = \varphi - \pi/2$. Then $\text{ctg}\varphi = -\text{tg}\Psi$.

The condition $x_0 \le 0$ implies

$$\beta \le \Psi \le \frac{\pi}{2} - \beta \tag{A.11}$$

where $\beta = \text{arctg}\nu$ and for such φ the minimum in (b) is attained. Thus the final result has the form

$$\min_{x \in \mathbb{R}} g_\varphi(x) = \begin{cases} 1 & \text{if } \dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{2} + \beta, \\[2ex] \dfrac{1 - \nu^2}{1 + 2\nu\text{ctg}\varphi + \text{ctg}^2\varphi} & \text{if } \dfrac{\pi}{2} + \beta \le \varphi \le \dfrac{3}{4}\pi. \end{cases} \tag{A.12}$$

Now we use (A.8) and come back to (4.23), (4.24) to find $r(\varphi) = \sqrt{L + K}\,\tilde{h}(\varphi)$, where $\tilde{h}(\varphi)$ is given by (4.25).

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Received March 23, 2001.