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On the determination of residual stress distribution in plane elasticity

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PROBLEM OF THE DISTRIBUTION of incompatibilities in elastic solid in a quasi-plane stress state is discussed. It is assumed that the stress distribution can be measured at the boundary of a simply connected subregion of the body. Residual stress field must satisfy static equilibrium equations but, in general, the corresponding elastic strain field does not satisfy strain compatibility conditions. Taking the measured values of the stress components at the boundary of the region as the external boundary tractions and solving the corresponding boundary value problem of elasticity (including strain compatibility conditions), one can obtain a unique stress field, which, in general, differs from the actual one. It is reasonable to treat their difference as the residual stress field assigned to the region under consideration. Obviously, the values of residual stress defined in such a way (at given point of the region) depend on the choice of the region and do not depend on the external loading of the elastic body. Fourier series integral technique of determining such residual stress fields for simply connected circular regions are proposed. Some quantitative integral characteristics of residual stress fields are discussed.

1. Introduction

CONCEPTS OF THE RESIDUAL stresses of the first, second and third kind are widely used for the description of the state of polycrystalline metals. Despite the lack of rigorous definitions of these concepts there are no misunderstandings between the material engineering scientists owing to improper use of these notions corresponding to three scales of research: sub-micro- scale (electron microscope), micro-scale (optical microscope), and macro-scale (according to the newer terminology – micro-, meso- and macro-scale). This subdivision corresponds to the physical levels of the structure: defects in crystal lattice, granular structure of the real polycrystalline metals and alloys and the whole manufactured structure. Outside of these contexts these notions lose their meaning. Any considerations such as: "The residual stresses of the second kind are those, which are equilibrated within a structural element" have no meaning as long as the "structural element" is not defined rigorously and one does not know how the term "equilibrate" should be understood – any static stress field in any region fulfills the equilibrium equations.

To describe residual stresses in any material, no matter: crystalline, amorphous or composite, one needs well-defined quantitative characteristics making it possible to compare the intensities and space distribution of the residual stress fields in any solid. In the present paper we consider the simplest case of the self-stressed body: a two-dimensional, linearly elastic isotropic homogeneous continuous medium. The author believes that such a model yields a fairly good approximate description of the plane subsurface layer in an unloaded 3-dimensional specimen. In our consideration we shall avoid the use of advanced geometric methods of description such as non-Euclidean material connection as well as the use of the concept of dislocation density tensor. For further considerations we shall adopt the Cartesian tensor notation and the summation convention.

2. Basic concepts

Let us consider a plane elastic state (no matter whether plane stress or strain) of the linearly elastic isotropic homogeneous body. In the absence of body forces (we shall assume they vanishing in all further considerations) one can introduce scalar parameters of incompatibility in terms of second derivatives of both elastic strain and stress field:

$$K_{(\varepsilon)} = \varepsilon_{ij,kl} e_{ik} e_{jl}, \tag{2.1}$$

$$K_{(\sigma)} = \sigma_{ii,kk}, \tag{2.2}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ denote elastic strain and stress tensor respectively, while e_{ij} denotes representation of the unit skew-symmetric tensor ($e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$ in Cartesian co-ordinates). In the absence of residual stress both these quantities vanish. The conditions of vanishing of expressions (2.1) and (2.2) are equivalent under the above assumptions. In general (e.g. for the case material inhomogeneity), only vanishing of (2.1) ensures the local lack of residual stress.[1)]

If all the components of the two-dimensional stress (elastic strain) field were known exactly and were smooth enough, the appropriate area or contour integrals of the quantities defined by Eq. (2.1) and/or (2.2) could be used as good scalar measures of the residual stress associated with a chosen region. In reality,

[1)] This remains true also in the case of the unloaded surface of half-space (quasi-plane stress state). Moreover, simple count of the number of known fields at the surface shows that no other compatibility condition can be formulated in terms of these fields quantities which can be measured at the surface i.e. without knowledge of the normal components of the gradients of the stress components and/or strain tensors at the surface.

however, taking into account all the difficulties associated with the stress and/or elastic strain measurements, one can hardly expect to obtain sets of data dense enough to calculate reliable values of the second derivatives of the stress components. Moreover, the history of inelastic deformations producing the fields of residual stress is often restricted to some subdomains, thus even discontinuous stress fields may appear.

For better comprehension of the difficulties we shall consider three elementary, textbook examples of residual stress fields defined on the circular discs of radius r_2 in plane states in the absence of body forces and boundary tractions (c.f. [1]-[4]).

Example 1

Edge dislocation is introduced into the center of a circular region $0 \leqslant \rho \leqslant r_2$, internal region $0 \leqslant \rho \leqslant r_1$ is removed and disc of unstressed material is placed instead, ρ denotes polar co-ordinate (Fig. 1):

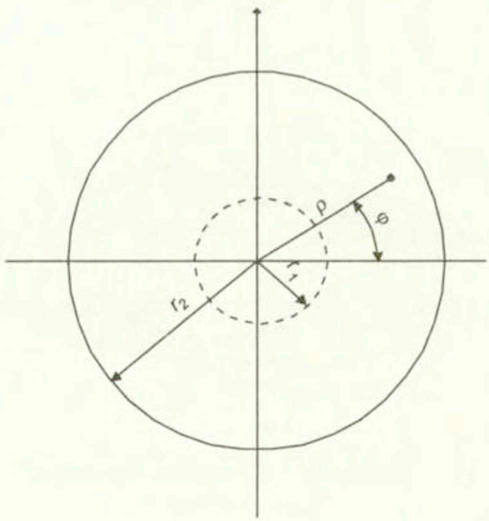

Fig. 1. Geometric scheme of the plane body.

The Airy stress function is written as

(2.3)
$$F(\rho,\varphi) = \begin{cases} -A\left(\rho \ln \dfrac{\rho}{r_2} - \dfrac{1}{2}\dfrac{\rho^3}{r_1^2 + r_2^2} + \dfrac{1}{2}\dfrac{r_1^2 r_2^2}{r_1^2 + r_2^2}\dfrac{1}{\rho}\right)\sin\varphi & \text{for } r_1 < \rho \leq r_2, \\ 0 & \text{for } 0 < \rho \leq r_1, \end{cases}$$

where A is an arbitrary constant (for plane strain state $A = \mu b/[2\pi(1-\nu)]$, where μ is the Lamé constant, ν denotes Poisson's ratio, b is a the length of the Burgers vector). For the stress field one obtains:

$$\sigma_{\rho\rho} \equiv \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2} = \begin{cases} A\dfrac{\left(\rho^2 - r_2^2\right)\left(\rho^2 - r_1^2\right)}{\rho^3\left(r_1^2 + r_2^2\right)} \sin\varphi & \text{for } r_1 < \rho \leqslant r_2, \\ 0 & \text{for } 0 \leqslant \rho \leqslant r_1, \end{cases}$$

$$\text{(2.4)} \quad \sigma_{\varphi\varphi} \equiv \frac{\partial^2 F}{\partial \rho^2} = \begin{cases} A\dfrac{3\rho^4 - \left(r_1^2 + r_2^2\right)\rho^2 - r_1^2 r_2^2}{\rho^3\left(r_1^2 + r_2^2\right)} \sin\varphi & \text{for } r_1 < \rho \leqslant r_2, \\ 0 & \text{for } 0 \leqslant \rho \leqslant r_1, \end{cases}$$

(compare Fig. 2).

$$\sigma_{\rho\varphi} \equiv -\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial F}{\partial \varphi}\right) = \begin{cases} A\dfrac{\left(\rho^2 - r_2^2\right)\left(\rho^2 - r_1^2\right)}{\rho^3\left(r_1^2 + r_2^2\right)} \cos\varphi & \text{for } r_1 < \rho \leqslant r_2, \\ 0 & \text{for } 0 \leqslant \rho \leqslant r_1. \end{cases}$$

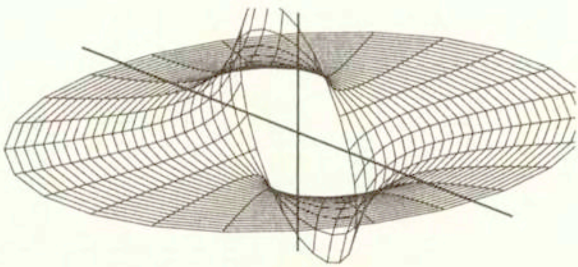

FIG. 2. An example of the residual stress distribution-transversal stress $\sigma_{\varphi\varphi}(r,\varphi)$arround edge dislocation.

Example 2

Angle disclination in the outer ring $r_1 < \rho \leqslant r_2$ and unstressed material in the internal disc $0 \leqslant \rho \leqslant r_1$:

Airy stress function

$$\text{(2.5)} \quad F(\rho,\varphi) = \begin{cases} \dfrac{1}{2}B\left(\rho^2 - \dfrac{r_2^2\alpha^2}{1-\alpha^2}\ln\alpha^2\right)\left(\ln\dfrac{\rho^2}{r_2^2} + \dfrac{\alpha^2}{1-\alpha^2}\ln\alpha^2 - 1\right) \\ \qquad\qquad \text{for } r_1 < \rho \leq r_2, \\ 0 \qquad \text{for } 0 \leqslant \rho \leqslant r_1, \end{cases}$$

where $\alpha \equiv r_1/r_2$, B is an arbitrary constant (for plane stress conditions $B = \frac{E\alpha}{8\pi}$, where α denotes the opening angle of disclination, E denotes the Young modulus).

Stress fields:

$$\sigma_{\rho\rho} = \begin{cases} B\left[\left(1-\dfrac{r_2^2}{\rho^2}\right)\dfrac{\alpha^2}{1-\alpha^2}\ln\alpha^2 + \ln\dfrac{\rho^2}{r_2^2}\right] & \text{for} \quad r_1 < \rho \leqslant r_2, \\ 0 & \text{for} \quad 0 \leqslant \rho \leqslant r_1, \end{cases}$$

(2.6) $$\sigma_{\varphi\varphi} = \begin{cases} B\left[\left(1+\dfrac{r_2^2}{\rho^2}\right)\dfrac{\alpha^2}{1-\alpha^2}\ln\alpha^2 + \ln\dfrac{\rho^2}{r_2^2} + 2\right] & \text{for} \quad r_1 < \rho \leq r_2, \\ 0 & \text{for} \quad 0 \leqslant \rho \leqslant r_1, \end{cases}$$

$$\sigma_{\rho\varphi} = 0.$$

Example 3

Internal disc of the diameter $r_1 + \delta$, $\delta << (r_2 - r_1)$ inserted into initially unstrained outer ring $r_1 < \rho \leqslant r_2$ of the same material.

Airy stress function

(2.7) $$F(\rho, \varphi) = \begin{cases} \dfrac{E\delta}{2r_1}\left(\dfrac{1}{2}\alpha^2\rho^2 - r_1^2\ln\dfrac{\rho}{r_1}\right) & \text{for} \quad r_1 < \rho \leqslant r_2, \\ \dfrac{E\delta}{2r_1}\left(\alpha^2 - 1\right)\rho^2 & \text{for} \quad 0 \leqslant \rho \leqslant r_1, \end{cases}$$

where E denotes the Young modulus. Stress fields are expressed as follows:

$$\sigma_{\rho\rho} = \begin{cases} \dfrac{E\delta}{2r_1}\left(\alpha^2 - \dfrac{r_1^2}{\rho^2}\right) & \text{for } r_1 < \rho \leqslant r_2, \\ \dfrac{E\delta}{2r_1}\left(\alpha^2 - 1\right) & \text{for } 0 \leqslant \rho \leqslant r_1, \end{cases}$$

(2.8) $$\sigma_{\varphi\varphi} = \begin{cases} \dfrac{E\delta}{2r_1}\left(\alpha^2 + \dfrac{r_1^2}{\rho^2}\right) & \text{for} \quad r_1 < \rho \leqslant r_2, \\ \dfrac{E\delta}{2r_1}\left(\alpha^2 - 1\right) & \text{for } 0 \leqslant \rho \leqslant r_1, \end{cases}$$

$$\sigma_{\rho\varphi} = 0.$$

All three examples specified above are by no means "exotic", one can easily point out real production techniques leading to such residual stress fields or their linear combinations. It is not difficult to notice that in all three cases the incompatibility parameters defined by Eq. (2.1) and (2.2) vanish almost everywhere, except for the set of null surface measure – the contour $\Sigma : \{\rho = r_1\}$, where they are not defined. It is not difficult to notice that: any simply connected subbody Ω that doesn't intersect contour Σ is not a self-stressed body in the following sense: *if one removed at the boundary $\partial\Omega$ all contact forces of interaction with the remainder part of the body, then the stress field would vanish at all points of Ω.*

The above examples distinctly show that it would be impractical to look for such additive characteristics of the residual stress which can be obtained by integration of some density functions (e.g. such as defined by Eq. (2.1) and (2.2)) over the whole body or its part[2)]. In the course of further considerations we shall look for the characteristics of residual stress fields assigned rather to the domains of the body than to its individual points.

In accordance with the formulated above notion of the body free of residual stress we shall specify for further consideration an operational procedure of subdivision of the two-dimensional actual stress field $\sigma_A(\mathbf{x})$ on the simply connected sub-domain Ω of the body bounded by the closed contour $\partial\Omega$, into two parts: *residual* and *induced*[3)] .

1. Choose a material region Ω, measure the components of the actual stress tensor $\sigma_A(s)$ at the contour $\partial\Omega$ and find the contact forces $\mathbf{t}(s)$ of interaction of the sub-body Ω with the rest of the body across $\partial\Omega$, s denoting a parameter defining the closed curve $\partial\Omega : \{\mathbf{x} = \mathbf{x}(s)\}$.

2. Solve the plane elastic boundary value problem for initially unstrained body of the same material occupying region Ω and loaded with tractions $\mathbf{t}(s)$ at $\partial\Omega$, obtaining the *induced stress field* $\sigma_I(\mathbf{x})$defined on Ω.

3. Subtract on the region Ω the field $\sigma_I(\mathbf{x})$ from $\sigma_A(\mathbf{x})$ obtaining the *intrinsic residual stress field* $\sigma_R(\Omega, \mathbf{x})$ associated with the region Ω

$$\sigma_R(\Omega, \mathbf{x}) \stackrel{\text{df}}{=} \sigma_A(\mathbf{x}) - \sigma_I(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.9}$$

Note that:

- Value of intrinsic residual stress field at a given point $\mathbf{x}$ essentially depends on the choice of the region Ω.
- From the superposition principle of linear elasticity it follows immediately that for given region Ω, the field $\sigma_R(\Omega, \mathbf{x})$ is insensitive to the change of

2) Such an approach, if possible at all, would lead us to rather inconvenient considerations concerning second derivatives of distributions (generalized functions).

3) Such an approach would be equivalent to the commonly used definition of the residual stress in terms of unloading, if the process of unloading were always purely elastic.

the external load applied to the body **outside** Ω, even if it causes inelastic deformations (up to the disintegration of the body), provided the material **inside** Ω remains in elastic state.

- Taking into account that real plastic deformations usually exceed considerably these which can be attained within elastic limits, one can expect that any inelastic deformation **inside** Ω may drastically change its intrinsic residual stress field.
- Residual stress field may not obey the limit state conditions, *actual* stress field is the only one, which must fulfill them. If this is the case, then a path of global elastic unloading of the body may not exist.

For practical applications it would be difficult to obtain the complete information about the *whole* intrinsic residual stress field for *all* parts of the body within reasonable accuracy limits. It is quite thinkable to perform measurements of the stress or elastic strain along a chosen curve, while performing such measurements for sufficiently dense set of internal points of two-dimensional region may turn out to be unreasonably time and/or money consuming. Some proposals of the choice of a limited number of parameters describing residual stress fields will be discussed in the next sections of the present paper, the problem however remains to be open for further investigations.

3. Circular domains

In this section we shall consider the simplest example of intrinsic residual stress distribution – residual stress fields in plane elastic state for the circular simply connected domains of the isotropic homogeneous material. All considerations of this section will be presented in terms of polar co-ordinates $\{\rho, \varphi\}$.

According to the E. Goursat theorem (c.f. [1] p. 322) any biharmonic function $F(x, y)$ defined in a simply connected region bounded by a smooth curve can be represented in the following form:

$$F(x, y) = Re[\bar{z} f(z) + g(z)] \tag{3.1}$$

where $f(z)$ and $g(z)$ are holomorphic functions of the complex variable $z = x + iy$, $\bar{z} \equiv x - iy$. Making use of the uniqueness of the power expansion of the holomorphic function and representing z^n as $\rho^n(\cos n\varphi + i \sin n\varphi)$, one can rewrite relation (10) in the following form:

$$F\left(\rho, \varphi\right) = \sum_{n=0}^{\infty} \left[\left(C_{(2)_n} \rho^{n+2} + C_{(0)_n} \rho^n\right) \cos n\varphi \right. \\ \left. + \left(S_{(2)_n} \rho^{n+2} + S_{(0)_n} \rho^n\right) \sin n\varphi. \tag{3.2}$$

Taking $F(\rho,\varphi)$ as the Airy stress function and using well-known relations

$$(3.3)\qquad \begin{aligned} \sigma_{\rho\rho} &= \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2}, \\ \sigma_{\varphi\varphi} &= \frac{\partial^2 F}{\partial \rho^2}, \\ \sigma_{\rho\varphi} &= -\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial F}{\partial \varphi}\right), \end{aligned}$$

one obtains the following stress field:

$$(3.4)\quad \sigma_{\rho\rho} = 2C_{(2)0} - \sum_{n=1}^{\infty}\Big[C_{(2)n}\Big(n-2\Big)\Big(n+1\Big)\rho^n + C_{(0)n}\Big(n-1\Big)\rho^{n-2}\Big]\cos n\varphi - \sum_{n=1}^{\infty}\Big[S_{(2)n}\Big(n-2\Big)\Big(n+1\Big)\rho^n + S_{(0)n}n\Big(n-1\Big)\rho^{n-2}\Big]\sin n\varphi,$$

$$(3.5)\quad \sigma_{\varphi\varphi} = 2C_{(2)0} + \sum_{n=1}^{\infty}\Big[C_{(2)n}\Big(n+2\Big)\Big(n+1\Big)\rho^n + C_{(0)n}\Big(n-1\Big)\rho^{n+2}\Big]\cos n\varphi + \sum_{n=1}^{\infty}\Big[S_{(2)n}\Big(n-2\Big)\Big(n+1\Big)\rho^n + S_{(0)n}n\Big(n-1\Big)\rho^{n-2}\Big]\sin n\varphi,$$

$$(3.6)\quad \sigma_{\rho\varphi} = \sum_{n=1}^{\infty}\left[C_{(2)n}n\,(n+1)\,\rho^n + C_{(0)n}n\,(n-1)\,\rho^{n-2}\right]\sin n\varphi - \sum_{n=1}^{\infty}\left[S_{(2)n}n\,(n+1)\,\rho^n + S_{(0)n}n\,(n-1)\,\rho^{n-2}\right]\cos n\varphi.$$

This stress field corresponds to the stress distribution in a circular, initially unstressed disc Ω of radius R loaded at the boundary $\partial\Omega : \{\rho = R\}$ by the normal stress $t_n(\varphi)$ and the tangent stress $t_\tau(\varphi)$, where the following relations must be satisfied:

$$(3.7)\qquad \begin{aligned} t_n(\varphi) &= \sigma_{\rho\rho}(R,\varphi), \\ t_\tau(\varphi) &= \sigma_{\rho\varphi}(R,\varphi), \end{aligned}$$

For the time being we shall assume, omitting technical details, that we are able to measure these quantities along the arbitrarily chosen circular contour $\partial\Omega$ inside the two-dimensional region occupied by the whole body.

Denoting:

$$
\begin{aligned}
N_{Cn} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{\rho\rho} \cos n\varphi \, d\varphi, \qquad & N_{Sn} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{\rho\rho} \sin n\varphi \, d\varphi, \\
T_{Cn} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{\rho\varphi} \cos n\varphi \, d\varphi, \qquad & T_{Sn} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{\rho\varphi} \sin n\varphi \, d\varphi, \\
P_{Cn} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{\varphi\varphi} \cos n\varphi \, d\varphi, \qquad & P_{Sn} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_{\varphi\varphi} \sin n\varphi \, d\varphi,
\end{aligned}
\tag{3.8}
$$

where $\sigma_{\rho\rho}(\varphi)$, $\sigma_{\rho\varphi}(\varphi)$ and $\sigma_{\varphi\varphi}(\varphi)$ are the values of stress measured at the contour, and making use of (13) and (15), one obtains: for $n = 0$: $N_{C0} = 4C_{(2)0}$, and

$$
\begin{aligned}
N_{Cn} &= -n(n-1)\; C_{(0)n} R^{n-2} - (n+1)(n-2)\; C_{(2)n} R^{n}, \\
N_{Sn} &= -n(n-1)\; S_{(0)n} R^{n-2} - (n+1)(n+2)\; S_{(2)n} R^{n}, \qquad \text{for } n \geq 1. \\
T_{Cn} &= -n(n-1)\; S_{(0)n} R^{n-2} - n(n+1)\; S_{(2)n} R^{n}, \\
T_{Sn} &= n(n-1)\; C_{(0)n} R^{n-2} + n(n+1)\; C_{(2)n} R^{n}.
\end{aligned}
\tag{3.9}
$$

Equations (3.9) can be solved with respect to $C_{(0)n}$, $C_{(2)n}$, $S_{(0)n}$ and $S_{(2)n}$ as follows[4)]:

$$
\begin{aligned}
C_{(0)n} &= -\frac{nN_{(C)n} + (n-2)\,T_{(S)n}}{2R^{n-2}n(n-1)}, \quad & C_{(2)n} &= \frac{N_{(C)n} + T_{(S)n}}{2R^{n}(n+1)}, \\
S_{(0)n} &= \frac{-nN_{(S)n} + (n-2)\,T_{(C)n}}{2R^{n-2}n(n-1)}, \quad & S_{(2)n} &= \frac{N_{(S)n} - T_{(C)n}}{2R^{n}(n+1)}.
\end{aligned}
\tag{3.10}
$$

Substituting values (3.10) into expressions (3.4), (3.5) and (3.6) we are able to express the formulae for the induced stress field $\sigma_I(\rho,\varphi)$ inside the circular region in terms of the Fourier coefficients of the tractions applied at the boundary:

[4)] Expressions for $C_{(0)1}$ and $S_{(0)1}$ are not defined by Eq. (19); it is not difficult to notice however that these quantities can be taken arbitrarily since they don't contribute to the expressions for stress components.

$$(3.11)\quad \sigma_{I\rho\rho} = \frac{N_{(C)0}}{2}$$

$$+\sum_{n=1}^{\infty}\left[\frac{nN_{(C)n}+\left(n-2\right)T_{(S)n}}{2R^{n-2}}\rho^{n-2}-\frac{\left(N_{(C)n}+T_{(S)n}\right)\left(n-2\right)}{2R^{n}}\rho^{n}\right]\cos n\varphi$$

$$+\sum_{n=1}^{\infty}\left[\frac{nN_{(S)n}-\left(n-2\right)T_{(C)n}}{2R^{n-2}}\rho^{n-2}-\frac{\left(N_{(S)n}+T_{(C)n}\right)\left(n-2\right)}{2R^{n}}\rho^{n}\right]\sin n\varphi,$$

$$(3.12)\quad \sigma_{I\varphi\varphi} = \frac{N_{(C)0}}{2}$$

$$-\sum_{n=1}^{\infty}\left[\frac{nN_{(C)n}+(n-2)\,T_{(S)n}}{2R^{n-2}}\rho^{n-2}-\frac{\left(N_{(C)n}+T_{(S)n}\right)(n+2)}{2R^{n}}\rho^{n}\right]\cos n\varphi$$

$$-\sum_{n=1}^{\infty}\left[\frac{nN_{(S)n}-(n-2)\,T_{(C)n}}{2R^{n-2}}\rho^{n-2}-\frac{\left(N_{(S)n}+T_{(C)n}\right)(n+2)}{2R^{n}}\rho^{n}\right]\sin n\varphi$$

$$(3.13)\quad \sigma_{I\rho\varphi} =$$

$$-\sum_{n=1}^{\infty}\left[\frac{nN_{(C)n}+(n-2)\,T_{(S)n}}{2R^{n-2}}\rho^{n-2}-\frac{\left(N_{(C)n}+T_{(S)n}\right)n}{2R^{n}}\rho^{n}\right]\sin n\varphi$$

$$+\sum_{n=1}^{\infty}\left[\frac{nN_{(S)n}-(n-2)\,T_{(C)n}}{2R^{n-2}}\rho^{n-2}-\frac{\left(N_{(S)n}+T_{(C)n}\right)n}{2R^{n}}\rho^{n}\right]\cos n\varphi.$$

Subtracting these values from the measured actual values of stress components at an arbitrary internal point $\mathbf{x}$ of the Ω domain, one can easily obtain the values of residual stress component $\sigma_R(\Omega,\mathbf{x})$at this point associated with Ω. If $\sigma_R(\Omega,\mathbf{x})$ vanishes at all points of Ω (within the accuracy limits of the measuring techniques) then one may consider domain Ω as being free from residual stress. Obviously this remains true with respect to any subdomain $\Omega_i \subset \Omega$. On the contrary, for a larger domain $\Omega_e \supset \Omega$ it may occur that $\sigma_R(\Omega,\mathbf{x})$ does not vanish (compare *Example* 3). It is quite reasonable to consider the radius of such a domain for which the first non-vanishing residual stress field appears as the lower threshold of the range of residual stress. More detailed consideration on the estimate of the residual stress range we shall present in the next section.

For the time being we shall point out an interesting property of the induced stress fields defined on circular regions. Combining Eq. (3.11) and (3.12) and taking $\rho = R$ one obtains:

$$\frac{\sigma_{I\rho\rho} - \sigma_{I\varphi\varphi}}{2} = -\sum_{n=1}^{\infty} T_{(S)n} \cos n\varphi + \sum_{n=1}^{\infty} T_{(C)n} \sin n\varphi. \tag{3.14}$$

Thus, taking into account that at the boundary $\sigma_{I\rho\rho} = \sigma_{A\rho\rho}$ and $\sigma_{I\rho\varphi} = \sigma_{A\rho\varphi}$ we conclude that the only component of residual stress non-vanishing at the boundary can be expressed in terms of actual stress components as follows:

$$\sigma_{R\varphi\varphi} = \sigma_{A\varphi\varphi} - \sigma_{A\rho\rho} - 2\sum_{n=1}^{\infty} T_{(S)n} \cos n\varphi + 2\sum_{n=1}^{\infty} T_{(C)n} \sin n\varphi. \tag{3.15}$$

In the absence of residual stress, the right-hand side of Eq. (3.15) vanishes identically (compare Eqs. (3.4) – (3.6)). It is not difficult, however, to show an example of residual stress field non-vanishing in Ω for which this expression is equal to zero at the entire boundary of Ω[5)]. Thus vanishing of the right-hand side of Eq. (3.15) at the boundary of some domain Ω of radius R is the necessary but not a sufficient condition of the absence of residual stress inside Ω.

Note that the value of $\sigma_{R\varphi\varphi}(\varphi)$ given by (3.15) describes the transversal component of residual stress at the circular contour of radius R **only for residual stress field associated with the circular region Ω of the same radius R**. Taking any larger concentric region Ω_1 of radius $R_1 > R$, one can use relation (3.12) and calculate $\sigma_{R\varphi\varphi}(\rho, \varphi)$ and then subtract it from $\sigma_{A\varphi\varphi}(\rho, \varphi)$ for $\rho = R$, obtaining (at the circle of radius R) the transversal component of **residual stress field associated with Ω_1**. In general **these quantities are different**[6)] . In order to avoid possible misunderstandings, let us introduce new symbol $\kappa(R, \varphi)$ denoting **the value of residual transversal stress associated with the circular domain Ω of radius R taken at the boundary of the domain** (determined by the right-hand side of Eq. (24) for $\rho = R$).

Let us consider a domain Ω_0 of radius R_0. Assume that $\kappa(R, \varphi)$ vanishes for every circular domain Ω of radius $R \leqslant R_0$, concentric with Ω_0. We shall prove that such stress field is entirely load-induced and doesn't include residual constituent in Ω_0.

[5)]One may take an arbitrary self-stressed body and surround it with unstressed material, then the right-hand side of Eq. (3.15) would identically vanish for any contour $\partial\Omega$ surrounding (but not touching) initial, self-stressed body, while the residual stress field associated with Ω doesn't vanish.

[6)]It will turn out to be clear from the foregoing considerations that they are equal if the expression (24) vanishes for all concentric domains $\Omega'$of intermediate radii $R < R' \leqslant R_1$.

Let us represent the stress function $\Phi(\rho,\varphi)$ as the following Fourier series:

$$\Phi(\rho,\varphi) = f_0(\rho) + \sum_{n=1}^{\infty} f_n(\rho)\cos n\varphi + \sum_{n=1}^{\infty} g_n(\rho)\sin n\varphi. \tag{3.16}$$

Calculating the stress components (using relations (3.3)) and substituting the results into Eq. (3.15), one obtains the following relation:

$$\begin{aligned}\kappa(R,\varphi) = \Bigg[&\frac{d^2 f_0(\rho)}{d\rho^2} - \frac{1}{\rho}\frac{df_0(\rho)}{d\rho}\frac{d^2 f_0(\rho)}{d\rho^2} - \frac{1}{\rho}\frac{df_0(\rho)}{d\rho}\\ &+ \sum_{n=1}^{\infty}\left(\frac{d^2 f_n(\rho)}{d\rho^2} - \frac{2n+1}{\rho}\frac{df_n(\rho)}{d\rho} + \frac{n(n+1)}{\rho^2}f_n(\rho)\right)\cos n\varphi\\ &+ \sum_{n=1}^{\infty}\left(\frac{d^2 g_n(\rho)}{d\rho^2} - \frac{2n+1}{\rho}\frac{dg_n(\rho)}{d\rho} + \frac{n(n+1)}{\rho^2}g_n(\rho)\right)\sin n\varphi\Bigg]_{\rho=R}.\end{aligned} \tag{3.17}$$

The right-hand side of Eq. (3.17) vanishes for every $R \leqslant R_0$ if and only if

$$\begin{aligned}&\frac{d^2 f_0(\rho)}{d\rho^2} - \frac{1}{\rho}\frac{df_0(\rho)}{d\rho} = 0,\\ &\left.\begin{aligned}&\frac{d^2 f_n(\rho)}{d\rho^2} - \frac{2n+1}{\rho}\frac{df_n(\rho)}{d\rho} + \frac{n(n+1)}{\rho^2}f_n(\rho) = 0,\\ &\frac{d^2 g_n(\rho)}{d\rho^2} - \frac{2n+1}{\rho}\frac{dg_n(\rho)}{d\rho} + \frac{n(n+1)}{\rho^2}g_n(\rho) = 0,\end{aligned}\right\}\ \text{for all } n = 1,2,3,\ldots\end{aligned} \tag{3.18}$$

i.e. when

$$\begin{aligned}&f_0(\rho) = A_0 + B_0\rho^2,\\ &\left.\begin{aligned}&f_n(\rho) = A_n\rho^n + B_n\rho^{n+2},\\ &g_n(\rho) = C_n\rho^n + D_n\rho^{n+2},\end{aligned}\right\}\ \text{for all } n = 1,2,3,\ldots\end{aligned} \tag{3.19}$$

where A_n, B_n, C_n, D_n are arbitrary constants. This means, however that the stress function $\Phi(\rho,\varphi)$ is biharmonic in Ω_0 (compare (3.2)). Therefore the entire stress field in Ω_0 is induced by the external load. QED.

The last result suggests that the intensity of $\kappa(R,\varphi)$ can serve as a convenient scalar characteristics of the magnitude and range of reach of residual stress fields.[7)]

[7)] Let us understand here all these notions in accordance with their colloquial meaning; we shall try to make them more precise later on.

4. Concluding propositions

Note that applying our operational procedure of determination of residual stress (compare Eq. (2.9)) beginning with the largest possible circle and gradually reducing its diameter, we act as "peeling" successively the original body (possibly "hardened" in advance to exclude inelastic behavior due to stress redistribution) and observing the resulting changes of the stress field. If for some range of R between R_1 and R_2, $(R_1 < R_2)$, our "stripping" does not change essentially the residual stress distribution, we would be inclined to claim that we do not observe significant internal stress of the range (scale) between R_1 and R_2 (on the level of intuitive understanding of all terms involved). On the contrary, if acting in such a way we change drastically the residual stress field magnitude and/or distribution, we would rather consider the *contribution of the residual stress of such a scale to the total stress distribution* as significant.

Let us try to express quantitatively these intuitive notions. To this end we shall define an integral characteristics of the residual stress distributions associated with the sequence of concentric circular regions:

$$I_1^2(R) \equiv \frac{1}{2\pi}\int_0^{2\pi} \kappa^2(R,\varphi)d\varphi. \tag{4.1}$$

We shall call $I_1(R)$ the *boundary intensity of residual stress*. It describes the mean value of the norm of residual stress tensor field (associated with the circular domain of radius R) calculated at the outline of the domain (circle of radius R).

For the description of the scale of reach of a residual stress field as well as its "coarseness" (or smoothness), not only the information on the intensity of residual stress can be useful – we have already seen that in some special cases the boundary residual stress intensity can vanish locally for large R while the specimen may be highly stressed inside. Also the "rate of change" of the boundary residual stress intensity during our imaginary process of "peeling" can be important. Thus we introduce another function of the domain radius R:

$$I_2(R) \equiv \frac{dI_1(R)}{dR}. \tag{4.2}$$

Before proceeding further in the description of residual stress distributions in terms of $I_1(R)$ and $I_2(R)$, we should try to examine their behavior in the case of the examples exposed in Sec. 2.

Ad *Example* 1

For the transversal component of residual stress in initial domain of radius r_2 one has:

$$\text{(4.3)}\qquad \sigma_{R\varphi\varphi}(\rho,\varphi)=\begin{cases} A\dfrac{3\rho^4-\left(r_1^2+r_2^2\right)\rho^2-r_1^2r_2^2}{\rho^3\left(r_1^2+r_2^2\right)}\sin\varphi & \text{for } r_1<\rho\leqslant r_2,\\ 0 & \text{for } 0\leqslant\rho\leqslant r_1,\end{cases}$$

while

$$\text{(4.4)}\qquad \kappa(R,\varphi)=\begin{cases} A\dfrac{2\left(R^2-r_1^2\right)}{R\left(R^2+r_1^2\right)}\sin\varphi & \text{for } r_1<R\leqslant r_2,\\ 0 & \text{for } 0<R\leqslant r_1,\end{cases}$$

thus:

$$\text{(4.5)}\qquad \begin{cases} \left[A\dfrac{\sqrt{2}\left(R^2-r_1^2\right)}{R\left(R^2+r_1^2\right)}\right] & \text{for } r_1<R\leqslant r_2,\\ 0 & \text{for } 0<R\leqslant r_1,\end{cases}$$

and

$$\text{(4.6)}\qquad I_2(R)=\begin{cases} -A\dfrac{\sqrt{2}\left(R^4-4R^2r_1^2-r_1^4\right)}{\left(R^2+r_1^2\right)^2R^2} & \text{for } \mathrm{r}_1<\mathrm{R}\leqslant\mathrm{r}_2\\ 0 & \text{for } 0<\mathrm{R}\leqslant\mathrm{r}_1\end{cases}$$

(compare Fig. 3.)

Ad *Example 2*

$$\text{(4.7)}\qquad \sigma_{\varphi\varphi}(\rho,\varphi)=\sigma_{\varphi\varphi}(\rho,)$$

$$=\begin{cases} 2B\left[\left(\alpha^2+\dfrac{r_1^2}{\rho^2}\right)\dfrac{\ln\alpha}{1-\alpha^2}+\ln\dfrac{\rho}{r_1}+\ln\alpha+1\right] & \text{for } r_1\leqslant\rho<r_2,\\ 0 & \text{for } 0\leq\rho<r_1,\end{cases}$$

$$\text{(4.8)}\qquad \kappa(R,\varphi)=\kappa(R)=\begin{cases} 2B\left(\dfrac{2r_1^2}{R^2-r_1^2}\ln\dfrac{r_1}{R}+1\right) & \text{for } r_1,\leqslant R<r_2,\\ 0 & \text{for } 0\leqslant R<r_1,\end{cases}$$

$$\text{(4.9)}\qquad I_1(R)=|\kappa(R)|=\begin{cases} 2|B|\left(\dfrac{2r_1^2}{R^2-r_1^2}\ln\dfrac{r_1}{R}+1\right) & \text{for } r_1\leqslant R<r_2,\\ 0 & \text{for } 0\leqslant R<r_1,\end{cases}$$

$$(4.10)\quad I_2(R) = \begin{cases} \left| \dfrac{-4|B|r_1^2}{R\left(R^2 - r_1^2\right)} \left(\dfrac{2R^2}{R^2 - r_1^2} \ln \dfrac{r_1}{R} + 1 \right) \right| & \text{for} \quad r_1 \leqslant R < r_2, \\ 0 & \text{for} \quad 0 \leqslant R < r_1. \end{cases}$$

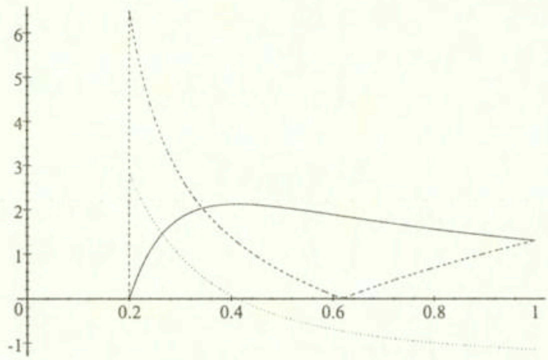

FIG. 3. Intensity of transversal stress $\sigma_{(i)\varphi\varphi}(\rho) \equiv \sqrt{\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \sigma_{\varphi\varphi}^2\left(r, \varphi\right) d\varphi}$ - dashed line, intensity of boundary residual stress $I_1(\rho)$ - solid line, derivative of the intensity of boundary residual stress $\frac{dI_1(R)}{dR}$ - dotted line.

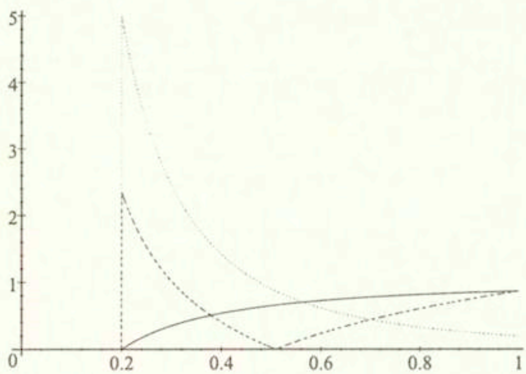

FIG. 4. Absolute value of transversal stress $|\sigma_{\varphi\varphi}(\rho)|$ - dashed line, intensity of boundary residual stress $I_1(\rho)$ - solid line, derivative of the intensity of boundary residual stress $\frac{dI_1(R)}{dR}$ - dotted line.

Ad *Example* 3

(4.11) $$\sigma_{\varphi\varphi}(\rho,\varphi) = \sigma_{\varphi\varphi}(\rho) = \begin{cases} \dfrac{E\delta}{2r_1}\left(\alpha^2 + \dfrac{r_1^2}{\rho^2}\right) & \text{for } r_1 < \rho \leqslant r_2 \\ \dfrac{E\delta}{2r_1}\left(\alpha^2 - 1\right) & \text{for } 0 \leqslant \rho \leqslant r_1 \end{cases}$$

(4.12) $$\kappa(R,\varphi) = \kappa(R) = \begin{cases} \dfrac{E\delta\, r_1}{R^2} & \text{for } r_1 < \rho \leqslant r_2 \\ 0 & \text{for } 0 \leqslant \rho \leqslant r_1 \end{cases}$$

(4.13) $$I_1(R) = |\kappa(R)| = \begin{cases} \dfrac{E|\delta|\, r_1}{R^2} & \text{for } r_1 < \rho \leqslant r_2 \\ 0 & \text{for } 0 \leqslant \rho \leqslant r_1 \end{cases}$$

(4.14) $$I_2(R) = \begin{cases} -\dfrac{2E|\delta|\, r_1}{R^3} & \text{for } r_1 < \rho \leqslant r_2 \\ 0 & \text{for } 0 \leqslant \rho \leqslant r_1 \end{cases}$$

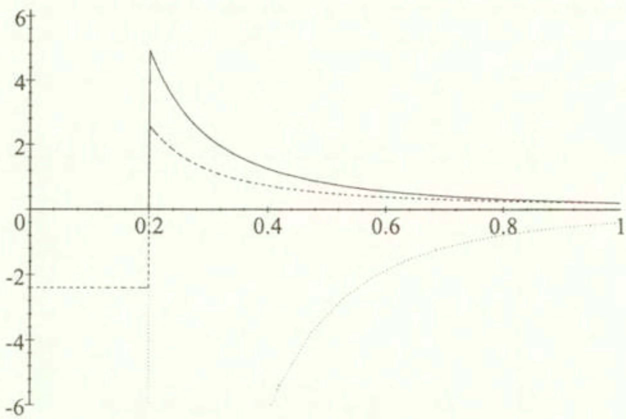

FIG. 5. Transversal stress $\sigma_{\varphi\varphi}(\rho)$ - dashed line, intensity of boundary residual stress $I_1(\rho)$ - solid line, derivative of the intensity of boundary residual stress $\frac{dI_1(R)}{dR}$ - dotted line.

Inspecting the curves describing the boundary residual stress intensity and their "rates" as the functions of the radius R, one can hardly point out (at least concerning the examples under consideration) any characteristic numbers pointing out the scale of reach of the residual stress fields. Only in Fig. 3 at $R \cong 0.4$ one can observe a maximum of the boundary residual stress intensity. Figure 4 suggests that the whole specimen is involved in generation of residual stress. The curves in Fig. 5 indicate that the influence of the defect decreases rapidly with the distance from central inclusion. In the last case one can arbitrarily take some value of the stress intensity, say 10% of the maximal value, and consider the distance at which the stress intensity becomes lower than this threshold as the range of reach of the residual stress field caused by the inclusion. The proposed quantitative characteristics of the internal stress scale do not pretend to be the only possible, many other similar propositions can be discussed as well. Generally speaking, rather the information of the whole distribution profile of the boundary stress intensity and its gradient gives more or less complete information. Thus we should admit that our original goal consisting in defining a single parameter describing the reach of the residual stress has not been fully achieved. Instead we have proposed the way of description of the two-dimensional field of residual stress using functions of one variable obtained by the procedure based on the measurements along the contours only. In author's opinion the most important result of the considerations presented above consists in drastic reduction of the density of the measured points and the accuracy of measurements needed for the non-destructive evaluation of residual stress fields. It was achieved due to replacement of the differential operations by the integration along contours. It should be mentioned again, here, that the method proposed is insensitive to the external loading, making thus possible the interpretation of the results of *in situ* strain measurements on the elements of working construction. All the considerations assume the linearly elastic behavior of the material, the problem of the applicability of the results to real engineering situations is out of the scope of the present considerations and must be separately considered for each individual case.

Generalization of the considerations to the (plane) regions of arbitrary shape seems to be thinkable, while the other ways of generalization such as three-dimensional approach, anisotropy, inhomogeneity etc. do not seem to be straightforward.

As the last point of our consideration it is worthwhile to discuss briefly some technical details. At the present state of art of the stress measurement techniques, determination of all necessary data in the framework of the present approach may turn out to be unreasonably expensive and time-consuming. It is, however, the author's hope that the development of the automatic measurement procedures may change this situation in the near future. The fast computation methods based on the Fourier analysis are quite well advanced nowadays thus any thinkable amount of experimental data may be easily processed. The scarcity and low accuracy of the experimental data combined with the powerful data processing capabilities may result in obtaining some "artifacts" such as (non-existing in reality) short range stress fields obtained in calculations, due to the experimental data scattering. Thus, certain precautions must be recommended, e.g. the number of the Fourier terms taken into considerations must be much less than the number of measuring points at the circular boundary.

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