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Nonstationary two-phase flow through elastic porous medium

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THE AIM OF THIS CONTRIBUTION is to derive macroscopic equations governing the dynamic flow of two immiscible viscous fluids through an elastic microperiodic porous medium. To this end homogenization methods were employed. The procedure used can be justified by the method of two-scale convergence. Passage to the stationary case and illustrative example were also provided.

1. Introduction

IN THE PAPERS [14, 15] we studied one-phase nonstationary flow of incompressible viscous Stokes fluid through a linear elastic microporous medium. The present paper is aimed at examining a similar problem in the case of two different immiscible Stokesian fluids. The new phenomenon which has to be taken into account is the surface tension on interfaces between the two fluids. The model of surface tension we use is the one proposed by LEVICH and KRYLOV [61], cf. also [41, 58]. The fluids are described by the Stokes equations with different viscosities. The solid phase is made of a linear anisotropic elastic material.

The macroscopic equations and Darcy's law, the last being nonlocal in time (see also [3]), are obtained by using the homogenization methods. Similar problem was studied by Auriault *et al.* [6] in the case of *harmonic* unsaturated flow and different from our scaling on interfaces between the fluids.

In our case, the passage to the stationary two-phase flow through a porous medium yields the equations similar to those previously derived by SAINT JEAN PAULIN and TAOUS [74] provided that the skeleton is rigid, see also [8].

The time-dependent permeabilities involve an implicit dependence on the matrix structure, saturation, interfacial tension and fluid densities. This dependence is inherently present in the local problems which allow to determine the formula for the extended Darcy law. Consequently, at the expense of simplifications it seems possible to recover the approach exploiting the notion of relative permeabilities, cf. [20, 41, 58]. We observe that if we can predict the permeability for single-phase flow through a porous material from digitized information obtained from cross-sections of the rocks, then we may also be able to obtain estimates of traditional relative permeability as well when two fluids phases are present, cf. BLAIR and BERRYMAN [17].

We observe that problems of fluid transport through porous media, including multi-phase flows are important in many branches of chemical technology, biomechanics and geological research. For instance, soft tissues are porous materials, cf. [57].

The plan of the paper is as follows. In Sec. 2 a rather intensive, though not exhaustive, overview of previous results on two-phase flows through porous media is compiled. Formulation of the flow problem to be studied is given in Secs. 3 and 4. Homogenization is performed in Sec. 5. Among the results obtained in the present paper the most important are the macroscopic equations of Biot-type involving the nonlocal in time extended Darcy's law. The formal homogenization procedure is justified in Sec. 6 by the method of two-scale convergence. Passage to the stationary case is performed in Sec. 7. An illustrative example of stationary flow through bundles of tubes is performed in Sec. 8. In Appendix A main points of the asymptotic analysis are derived.

2. Overview of previous results on two-phase flow through porous media

In essence, there are four approaches to modelling the behaviour of porous media filled with fluids. The first approach is in fact macroscopic from the very beginning, though microscopic information is somewhat hidden, see COUSSY [35].

The second approach is typical for mixture theories, see DE BOER [18], KUBIK *et al.* [60], SIMON [77] and the references therein. It seems that MORLAND [68] was the first who used the volume fraction concept in connection with the mixture theory. According to ZIENKIEWICZ *et al.* [85], the mixture theory introduces some arbitrariness in the selection of various parameters. GANESAN and BRENNER [47] critically reviewed the convential mixture theory, and other theories where spatial averaging is used. According to these authors, we cite ([47], p.736): "Our identification of appropriate macrofields is based upon their rigorous *physical, scale-invariant* definitions rather than upon simple *ad hoc* volume averaging of the corresponding microfields". Consequently the paper contains quite a lot of definitions. The generalized Darcy's law has the form similar to the one given on p. 419 of [58] and is a particular case of the law rigorously derived by SAINT-JEAN PAULIN and TAOUS [74], cf. also Sec. 5 of our paper.

The third approach, often used in the mechanics of porous media, relies on volume and surface averaging, see BEAR and BACHMAT [10], BRENNER and EDWARDS [28], FRAS and BENET [46], GRAY and HASSANIZADEH [49, 50, 51, 52] DU PLESSIS [72], WHITAKER [81]. Similarly to micromechanics, one introduces the concept of representative elementary volume (REV). With every microscopic field $\varphi(x,t)$, a macroscopic field defined at every point x of a domain $\Omega \subset \mathbb{R}^3$ by the average of φ in the translation V of REV is centered at $x \in \Omega$. For a density φ, the macroscopic field is defined by

$$\langle\varphi\rangle(x,t) = \frac{1}{|V|}\int\limits_V \varphi(y,t)\,dy(x). \tag{2.1}$$

For a quantity φ_s on a surface S, the average is written as follows:

$$\langle\varphi_s\rangle(x,t) = \frac{1}{|V|}\int\limits_{V\cap S} \varphi_s(y,t)\,dS(x). \tag{2.2}$$

Such averaging is used to derive macroscopic balance equations. The system of balance equations requires a sufficient number of equations such that all unknowns can be determined. It is accomplished by providing equations of state and constitutive relations [66, 81]. It is worth noticing that in a comprehensive paper by MILLER *et al.* [66], the following topics related to multiphase flow and transport modelling were discussed: balance equations, constitutive relationships for both pressure-saturation-conductivity and interphase mass transfer, and stochastic and computional issues. Experimental observations were also reported. As an evolving approach, a Lattice Boltzmann (LB) method to simulate multiphase flow at the pore scale was discussed, see also ADLER and THOVERT [2], KRAFCZYK *et al.* [59] and the references therein. We observe that Lattice Gas (LG) and LB models were originally developed for single-phase flows. One major interest of these techniques is their potential ability to cope with interfaces [2].

For more information on multi-phase, and particularly two-phase flows, where various physical approaches were used, the reader is referred to the books by CUSHMAN [37], DULLIEN [41], KAVIANY [58] and SAHIMI [76], and the papers by ALLAIRE and KOKH [5], DARTLEY and RUTH [40], BENNETHUM and GIORGI [11], BLAIR and BERRYMAN [17], CHAVENT *et al.* [30], CONSTANTINIDES and PAYATAKES [32, 33, 34] CHRISTAKOS *et al.* [31], DALE [38, 39], EKRANN and DALE [42], EKRANN *et al.* [43], GRAY and HASSANIZADEH [49, 50], HASSANIZADEH and GRAY [54], HARTER and YEH [53], MONTEMAGNO and GRAY [67], RUDMAN [73], THIGPEN and BERRYMAN , TZIMAS *et al.* [79], VALAVANIDES *et al.* [80], YARIN and HESTRONI [82], ZANOTTI and CARBONELL [83], ZHANG and PROSPETTI [84]. The approach used by CUSHMAN [37] to modelling flow and swelling in hierarchical porous media is extremely complicated, similar in spirit of the modelling used by GRAY [48], GRAY and HASSANIZADEH [49], HASSANIZADEH and GRAY [54].

The books by LEWIS and SCHREFLER [62] and ZIENKIEWICZ *et al.* [85] present the basis of modern computational approaches to various practical problems of geomechanics. In each of these books an important role is attributed to the derivation of macroscopic equations by using averaging techniques.

Less familiar in the mechanics of porous media seems to be the ganglion dynamics, cf. [32, 33, 34, 80]. Experimental observations of steady-state flow of water and oil through planar and nonplanar chamber-and-throat pore networks etched in glass have shown that over broad ranges of values of the main dimensionless parameters (capillary number, viscosity ratio, water saturation) the oil is disconnected in the form of ganglia. In [32, 33, 34, 80] a new approach for the study of two-phase flow processes in porous media on mesoscopic scale, when ganglion dynamics is the main flow regime, has been developed.

We observe that in the study of flows through porous media, mainly the transport problems are investigated. The solid component is then obviously undeformable.

We proceed to the fourth important approach used in modelling flows through porous media, both deformable and undeformable, cf. BIELSKI and TELEGA [13], the books by BOURGEAT *et al.* [19], CROLLET and EL HATRI [36], HORNUNG [56], ENE and POLIŠEVSKI [44], PANFILOV [71], SANCHEZ-PALENCIA [75]. For instance, it is now clear that Darcy's law and its extensions can be derived by using homogenization methods, see also Sec.5 of the present paper. A characteristic feature of various homogenization approaches is that a small parameter $\varepsilon > 0$ is introduced. The homogenization procedure consists in a formal (the method of multiple scales) or rigorous passage with ε to zero (G- and H-convergence, Γ-convergence, two-scale convergence), see the Appendix A by ALLAIRE to the book [56].

Surprisingly, the author of a comprehensive paper [18] who pursued the developments of the porous media theory from the middle of the 18th century until 1996, mentioned no contribution to this theory by rapidly developing field of homogenization.

From the physical viewpoint one can distinguish two classes of homogenization problem related to two-phase flows, similarly to one-phase flows.

The first class comprises problems where micro-macro approach is used. Then one arrives at Darcy's law and its generalizations, cf. AURIAULT - SANCHEZ-PALENCIA [7, 8], BERNABÉ [12], ENE and POLIŠEVSKI [44], FIRDAOUSS *et al.* [45], HORNUNG [56], MARUSIĆ - PALOKA [63] and MIKELIĆ [64]. In the case of elastic porous media one additionally obtains Biot-type equations, cf. AURIAULT *et al.* [6], BIELSKI *et al.* [14, 15, 16], BURRIDGE and KELLER [29], and the references therein. The second class comprises a meso-macro approach, where the quantities like porosity, permeability and dispersion are assumed to be microperiodic, cf. BOURGEAT [20, 21], BOURGEAT and HIDAMI [23], BOURGEAT and MIKELIĆ [27], BOURGEAT and PANFILOV [22], BOURGEAT *et al.* [25, 26], MIKELIĆ [64]. MIKELIĆ and PAOLI [65] derived the Buckley-Leverett system for a two-phase immiscible incompressible flow through a thin slab, starting from the incompressible Navier-Stokes system governing a flow of two fluids separated by a free boundary and disregarding surface tension.

In what concerns homogenization of two-phase flow through randomly heterogeneous media, only two papers are known to us [9, 25], cf. also [24, 31]. In these papers the authors considered incompressible two-phase flow in heterogeneous reservoirs with randomly distributed inhomogeneities, that is in media with permeability and porosity being stationary random fields.

3. Notations and basic relations

We assume that the porous skeleton reveals a micro-periodic structure. The basic cell Y has a form of a cube and consists of three parts (three disjoint open sets): Y_S, Y_A and Y_B, where the subscript S denotes the solid part, and A and B stand for the fluid parts. We also set $Y_L = Y_A \cup Y_B \cup \Gamma_{AB}$, where Γ_{AB} stands for the interface between Y_A and Y_B.

The interface between the sets Y_L and Y_S is denoted by Γ. We observe that the surface of liquid-liquid contact Γ_{AB}, is in general unknown, and $\Gamma_{AB} = \Gamma_{AB}(t)$.

The porous medium is identified with a bounded set $\overline{\Omega} \subset \mathbb{R}^3$, where Ω is a sufficiently regular domain. The domain Ω is assumed to possess an εY - periodic structure: $\Omega = \Omega^\varepsilon_S \cup \Omega^\varepsilon_L$ where Ω^ε_L denotes the part occupied by the liquid, and $\Omega^\varepsilon_S = \Omega \backslash \overline{\Omega}^\varepsilon_L$. We have $\overline{\Omega} = \overline{\Omega}^\varepsilon_S \cup \overline{\Omega}^\varepsilon_A \cup \overline{\Omega}^\varepsilon_B$.

A small parameter ε $(0 < \varepsilon < 1)$ characterizes the microstructure of the porous medium considered. Namely $\varepsilon = l/\mathcal{L}$, and $l, \mathcal{L}$ are typical length scales associated with the dimension of micropores and with length of waves contributing to the considered transient phenomenon. To obtain the macroscopic relationships (homogenization) we pass with ε to zero.

The subsets Ω^ε_S, Ω^ε_A and Ω^ε_B correspond to the parts of Ω occupied by the solid and two immiscible fluids. We have $\overline{\Omega} = \overline{\Omega}^\varepsilon_S \cup \overline{\Omega}^\varepsilon_A \cup \overline{\Omega}^\varepsilon_B$, $\qquad \Gamma^\varepsilon_{AB} = \partial\Omega^\varepsilon_A \cap \partial\Omega^\varepsilon_B$. We set

$$\langle(\cdot)\rangle = \frac{1}{|Y|}\int_Y (\cdot)\, dy, \qquad \langle(\cdot)\rangle_\alpha = \frac{1}{|Y|}\int_{Y_\alpha} (\cdot)\, dy, \qquad \alpha = S, L, A, B. \tag{3.1}$$

Note that the surfaces of the solid phase and the A-liquid and B-liquid phases are given by

$$\partial Y_S = \Gamma \cup P_S,\ \ \partial Y_A = \Gamma \cup \Gamma_{AB} \cup P_A,\ \ \partial Y_B = \Gamma \cup \Gamma_{AB} \cup P_B \ \text{ and } \ \partial Y_L = \Gamma \cup P_L.$$

Here P_S, P_A, P_B and P_L denote the cross-sections of the Y_S, Y_A, Y_B and Y_L with the faces of the basic cell Y. Obviously we have $\Gamma = \Gamma_A \cup \Gamma_B$.

The porosity f is defined as the volume fraction of the liquid in the considered solid-liquid medium. We have

$$f_\alpha = \frac{|Y_\alpha|}{|Y|}, \qquad f = f_A + f_B = \frac{|Y_L|}{|Y|}, \quad 1 = f_S + f_A + f_B = f_S + f. \tag{3.2}$$

We note that f_A and f_B depend on time. By $\mathbf{n}^\alpha$ we denote the exterior unit normal to $\Omega^\varepsilon_\alpha$. The summation convention is used, unless otherwise stated.

4. Basic equations for the motion of porous medium with biphasic fluid

Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ denote the displacement field in the solid, $\mathbf{v}^A = \mathbf{v}^A(\mathbf{x}, t)$ and $\mathbf{v}^B = \mathbf{v}^B(\mathbf{x}, t)$ denote the velocity in the liquids A and B, $p^A = p^A(\mathbf{x}, t)$ and $p^B = p^B(\mathbf{x}, t)$ being the respective pressures in both fluid phases.

Further, let ρ^S, ρ^A and ρ^B denote the densities of phases S, A and B. We assume that these quantities do not exhibit the microstructure. The viscosities of both fluids A and B are denoted by η^A_{ijmn} and η^B_{ijmn}, respectively.

For a fixed $\varepsilon > 0$ all the relevant quantities that exhibit the microstructure are denoted with the superscript ε. The fields $\mathbf{u}^\varepsilon$, $\mathbf{v}^{A\varepsilon}$, $\mathbf{v}^{B\varepsilon}$ $p^{A\varepsilon}$ and $p^{B\varepsilon}$ satisfy the following equations:

$$\begin{aligned}
\rho^S \ddot{u}^\varepsilon_i &= \partial_{x_j}\left(a^\varepsilon_{ijmn}\partial_{x_n} u^\varepsilon_m\right) + F^S_i && \text{in } \ \Omega_S,\\
\rho^A \dot{v}^{A\varepsilon}_i &= \partial_{x_j}\left(-p^{A\varepsilon}\delta_{ij} + \varepsilon^2 \eta^{A\varepsilon}_{ijmn}\partial_{x_n} v^{A\varepsilon}_m\right) + F^A_i && \text{in } \ \Omega_A,\\
\partial_{x_i} v^{A\varepsilon}_i &= 0 && \text{in } \ \Omega_A,\\
\rho^B \dot{v}^{B\varepsilon}_i &= \partial_{x_j}\left(-p^{B\varepsilon}\delta_{ij} + \varepsilon^2 \eta^{B\varepsilon}_{ijmn}\partial_{x_n} v^{B\varepsilon}_m\right) + F^B_i && \text{in } \ \Omega_B,\\
\partial_{x_i} v^{B\varepsilon}_i &= 0 && \text{in } \ \Omega_B.
\end{aligned} \tag{4.1}$$

On the known solid-liquid interface Γ^ε and on the unknown liquid-liquid interface Γ^ε_{AB} the jump conditions are

$$\begin{aligned}
&[\![S^\varepsilon_{ij}]\!]\, n_j = 0 \qquad v^{A\varepsilon}_i|_A = \dot{u}^\varepsilon_i|_S \qquad v^{B\varepsilon}_i|_B = \dot{u}^\varepsilon_i|_S \quad \text{on} \ \ \Gamma^\varepsilon,\\
&\left[\!\!\left[\left(-p^\varepsilon\delta_{ij} + \varepsilon^2\eta^\varepsilon_{ijmn}\partial_{x_n} v^\varepsilon_m\right)\right]\!\!\right]_{AB} n_j = \varepsilon\sigma H^\varepsilon_{ij} n_j \qquad [\![v^\varepsilon_i]\!] = 0 \quad \text{on} \ \ \Gamma^\varepsilon_{AB},
\end{aligned} \tag{4.2}$$

where

$$(4.3)\qquad S^{\varepsilon}_{ij} = \begin{cases} a^{\varepsilon}_{ijmn}\partial_{x_n}u^{\varepsilon}_m & \text{in } \Omega^{\varepsilon}_S, \\ -p^{A\varepsilon}\delta_{ij} + \varepsilon^2\eta^{A\varepsilon}_{ijmn}\partial_{x_n}v^{A\varepsilon}_m & \text{in } \Omega^{\varepsilon}_A, \\ -p^{B\varepsilon}\delta_{ij} + \varepsilon^2\eta^{B\varepsilon}_{ijmn}\partial_{x_n}v^{B\varepsilon}_m & \text{in } \Omega^{\varepsilon}_B. \end{cases}$$

Here, σ denotes the coefficient of surface tension and H^{ε}_{ij} is the curvature tensor of the surface $\Gamma^{\varepsilon}_{AB}$. In Eqs. $(4.1)_{2,4}$ and $(4.2)_4$ the following rescaling is introduced, cf. Appendix B,

$$\eta^{\varepsilon}_{ijmn} \rightsquigarrow \varepsilon^2\, \eta^{\varepsilon}_{ijmn} \quad \text{and} \quad \sigma \rightsquigarrow \varepsilon\, \sigma.$$

The following interpretation can be ascribed to the quantity $\mathbf{H} = (H_{ij})$. We perform the decomposition in normal and tangential parts as follows:

$$\sigma\mathbf{H}\mathbf{n} = \sigma H_{\mathbf{n}}\mathbf{n} + \sigma\mathbf{H}_{\tau},$$

where

$$H_{\mathbf{n}} \equiv H_{ij}n_i n_j \equiv H,$$

and

$$\mathbf{H}_{\tau} \equiv \mathbf{H}\mathbf{n} - H_{\mathbf{n}}\mathbf{n}.$$

We assume that

$$\sigma\mathbf{H}_{\tau} = \sigma\frac{1}{\sigma}\frac{\partial\sigma}{\partial\boldsymbol{\tau}}\boldsymbol{\tau} = \frac{\partial\sigma}{\partial\boldsymbol{\tau}}\boldsymbol{\tau}.$$

According to the method of two-scale asymptotic expansions, we assume the following expansions for the scalar field p^{ε} and vector field $\mathbf{u}^{\varepsilon}$, cf. [56, 57],

$$(4.4)\qquad \begin{gathered} p^{\alpha\varepsilon} = p^{\alpha(0)}(\mathbf{x},\mathbf{y},t) + \varepsilon p^{\alpha(1)}(\mathbf{x},\mathbf{y},t) + \varepsilon^2 p^{\alpha(2)}(\mathbf{x},\mathbf{y},t) + \ldots \\ \mathbf{y} = \mathbf{x}/\varepsilon,\ (\alpha = A, B) \\ u^{\varepsilon}_i = u^{(0)}_i(\mathbf{x},\mathbf{y},t) + \varepsilon u^{(1)}_i(\mathbf{x},\mathbf{y},t) + \varepsilon^2 u^{(2)}_i(\mathbf{x},\mathbf{y},t) + \ldots \qquad \mathbf{y} = \mathbf{x}/\varepsilon, \end{gathered}$$

as well as similar expansions for $\mathbf{v}^{\mathbf{A}\varepsilon}$ and $\mathbf{v}^{\mathbf{B}\varepsilon}$.

The initial conditions are specified by

$$(4.5)\quad \mathbf{v}^{\alpha\varepsilon}(x,0) = \mathbf{0} \ \text{ in } \ \Omega^{\varepsilon}_{\alpha}, \quad \alpha = A, B; \ \mathbf{u}^{\varepsilon}(x,0) = \mathbf{0}, \quad \dot{\mathbf{u}}^{\varepsilon}(x,0) = \mathbf{0} \ \text{ in } \ \Omega^{\varepsilon}_S.$$

5. Results of homogenization

More detailed analysis related to the homogenization procedure is performed in Appendix A. In the present section we are going to present the final results.

The analysis of terms of the order ε^{-2} carried out in Appendix A yields, cf. (A.16),

$$u_i^{(0)} = u_i^{(0)}(\mathbf{x}, t), \tag{5.1}$$

and from (A.19) we get

$$p^{A(0)} = p^{B(0)} \equiv p^{(0)} = p^{(0)}(\mathbf{x}, t) \qquad \text{for } \mathbf{x} \in \Omega. \tag{5.2}$$

Consequently, we obtain, cf.(A.9),

$$\left[a_{ijmn}\left(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\right)\right] n_j^S = - p^{(0)}\, n_i^S \qquad \text{on} \quad \Gamma = \Gamma_A \cup \Gamma_B. \tag{5.3}$$

The set of equations (A.17) is satisfied provided that

$$u_m^{(1)}(\mathbf{x}, \mathbf{y}, t) = A_m^{(pq)}(\mathbf{x}, \mathbf{y}, t)\, \partial_{x_q} u_p^{(0)}(\mathbf{x}, t) + P_m(\mathbf{x}, (\mathbf{y}, t)\, p^{(0)}(\mathbf{x}, t) \quad \text{in } Y_S \tag{5.4}$$

and the functions $A_m^{(pq)}$ and P_m are Y-periodic solutions to the following local equations on Y_S:

$$\begin{aligned} &\partial_{y_j}\left[a_{ijpq} + a_{ijmn}\partial_{y_n} A_m^{(pq)}\right] = 0, \\ &\partial_{y_j}\left(a_{ijmn}\partial_{y_n} P_m + \delta_{ij}\right) = 0. \end{aligned} \tag{5.5}$$

By using the expression (5.4) and Eqs.(A.9) we get

$$\begin{aligned} &\left(a_{ijpq} + a_{ijmn}\partial_{y_n} A_m^{(pq)}\right) n_j^S = 0 \qquad \text{on } \Gamma, \\ &\left(a_{ijmn}\partial_{y_n} P_m + \delta_{ij}\right) n_j^S = 0 \qquad \text{on } \Gamma. \end{aligned} \tag{5.6}$$

5.1. Macroscopic constitutive relations

Applying asymptotic expansions to constitutive relation (4.3), comparing the terms linked with ε^0 and exploiting (5.2) we get

$$S_{ij}^{(0)} = \begin{cases} a_{ijmn}\left(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\right) & \text{in} \quad Y_S, \\ -p^{(0)}\delta_{ij} & \text{in} \quad Y_L = Y_A \cup Y_B. \end{cases} \tag{5.7}$$

We observe that according to the definitions of averages (3.1) we have

$$\langle S_{ij}^{(0)} \rangle = \langle S_{ij}^{(0)} \rangle_S + \langle S_{ij}^{(0)} \rangle_A + \langle S_{ij}^{(0)} \rangle_B, \tag{5.8}$$

or

$$\langle S_{ij}^{(0)} \rangle = \langle S_{ij}^{(0)} \rangle_S + \langle S_{ij}^{(0)} \rangle_L. \tag{5.9}$$

Using Eqs.(5.4) and (5.7) we get

$$\langle S_{ij}^{(0)} \rangle = a_{ijpq}^{h} \partial_{x_q} u_p^{(0)} + \Big(\langle a_{ijmn} \partial_{y_n} P_m \rangle_S - f \delta_{ij} \Big) p^{(0)}, \tag{5.10}$$

where

$$a_{ijpq}^{h} = \langle a_{ijpq} + a_{ijmn} \partial_{y_n} A_m^{(pq)} \rangle_S. \tag{5.11}$$

The functions $A_m^{(pq)}$ and P_m have to be determined from the local equations (5.5) and (5.6).

5.2. Mechanics of porous medium with biphasic liquid

From Eqs. (A.1) and (*A*.2), by comparing the terms linked with ε^0, we get

$$\begin{aligned} \rho^S \ddot{u}_i^{(0)} &= F_i^S + \partial_{x_j} \Big[a_{ijmn} \Big(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)} \Big) \Big] \\ &\qquad + \partial_{y_j} \Big[a_{ijmn} \Big(\partial_{x_n} u_m^{(1)} + \partial_{y_n} u_m^{(2)} \Big) \Big] \qquad \text{in } Y_S, \\ \rho^A \dot{v}_i^{A(0)} &= F_i^A - \Big(\partial_{x_i} p^{(0)} + \partial_{y_i} p^{A(1)} \Big) + \partial_{y_j} \Big(\eta_{ijmn}^A \partial_{y_n} v_m^{A(0)} \Big) \qquad \text{in } Y_A, \\ \rho^B \dot{v}_i^{B(0)} &= F_i^B - \Big(\partial_{x_i} p^{(0)} + \partial_{y_i} p^{B(1)} \Big) + \partial_{y_j} \Big(\eta_{ijmn}^B \partial_{y_n} v_m^{B(0)} \Big) \qquad \text{in } Y_B. \end{aligned} \tag{5.12}$$

On the other hand, the terms linked with ε^1 in the interface condition (A.4) lead to the relations

$$\begin{aligned} \Big[a_{ijmn} \Big(\partial_{x_n} u_m^{(1)} + \partial_{y_n} u_m^{(2)} \Big) \Big] n_j \Big|_S &= \Big(p^{A(1)} \delta_{ij} - \eta_{ijmn}^A \partial_{y_n} v_m^{A(0)} \Big) n_j \Big|_A, \\ \Big[a_{ijmn} \Big(\partial_{x_n} u_m^{(1)} + \partial_{y_n} u_m^{(2)} \Big) \Big] n_j \Big|_S &= \Big(p^{B(1)} \delta_{ij} - \eta_{ijmn}^B \partial_{y_n} v_m^{B(0)} \Big) n_j \Big|_B. \end{aligned} \tag{5.13}$$

Integration of Eq. $(5.12)_1$ over Y_S, and $(5.12)_2$ and $(5.12)_3$ over Y_A and Y_B yields

$$(5.14)\quad \begin{aligned} (1-f)\rho^S \ddot{u}_i^{(0)} = {} & \partial_{x_j} \langle a_{ijmn}\Big(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\Big)\rangle_S \\ & + \int\limits_{\partial Y_S} \Big[a_{ijmn}\Big(\partial_{x_n} u_m^{(1)} + \partial_{y_n} u_m^{(2)}\Big)\Big] n_j\, dS \quad \text{in } \Omega, \\ f_A \rho^A \langle \dot{v}_i^{A(0)}\rangle_A = {} & f_A F_i^A - f_A \partial_{x_i} p^{(0)} \\ & + \int\limits_{\partial Y_A} \Big[-p^{A(1)}\delta_{ij} + \eta^A_{ijmn}\partial_{y_n} v_m^{A(0)}\Big] n_j\, dS \quad \text{in } \Omega, \\ f_B \rho^B \langle \dot{v}_i^{B(0)}\rangle_B = {} & f_B F_i^B - f_B \partial_{x_i} p^{(0)} \\ & + \int\limits_{\partial Y_B} \Big[-p^{B(1)}\delta_{ij} + \eta^B_{ijmn}\partial_{y_n} v_m^{B(0)}\Big] n_j\, dS \quad \text{in } \Omega. \end{aligned}$$

Adding Eqs.(5.14), using the interface relations (5.13) and taking into account Eqs. (5.7) and (5.8) we obtain

$$(5.15)\quad \begin{aligned} (1-f)\rho^S \ddot{u}_i^{(0)} + \rho^A \langle \dot{v}_i^{A(0)}\rangle_A + \rho^B \langle \dot{v}_i^{B(0)}\rangle_B \\ = \partial_{x_j}\langle S_{ij}^{(0)}\rangle + (1-f)F_i^S + f_A F_i^A + f_B F_i^B. \end{aligned}$$

This is the macroscopic equation of motion for the porous medium filled with the biphasic fluid.

5.3. Flow of biphasic fluid in porous medium

Equation (5.12) yield the following subsystem of equations posed in $\Omega \times Y_L$ describing the behaviour of the liquid part of the system:

$$(5.16)\quad \begin{aligned} \rho^A \dot{v}_i^{A(0)} &= F_i^A - \partial_{x_i} p^{(0)} - \partial_{y_i} p^{A(1)} + \partial_{y_j}\Big(\eta^A_{ijmn}\partial_{y_n} v_m^{A(0)}\Big), \\ \rho^B \dot{v}_i^{B(0)} &= F_i^B - \partial_{x_i} p^{(0)} - \partial_{y_i} p^{B(1)} + \partial_{y_j}\Big(\eta^B_{ijmn}\partial_{y_n} v_m^{B(0)}\Big), \end{aligned}$$

which can be rewritten as

$$(5.17)\quad \begin{aligned} \rho^A \dot{w}_i^{A(0)} &= F_i^A - \rho^A \ddot{u}_i^{(0)} - \partial_{x_i} p^{(0)} - \partial_{y_i} p^{A(1)} + \partial_{y_j}\Big(\eta^A_{ijmn}\partial_{y_n} w_m^{B(0)}\Big), \\ \rho^B \dot{w}_i^{B(0)} &= F_i^B - \rho^B \ddot{u}_i^{(0)} - \partial_{x_i} p^{(0)} - \partial_{y_i} p^{B(1)} + \partial_{y_j}\Big(\eta^B_{ijmn}\partial_{y_n} w_m^{B(0)}\Big) \end{aligned}$$

or

$$
\tag{5.18}
\begin{aligned}
\rho^A \dot{w}_i^{A(0)} &= \Phi_i^{A(0)} - \partial_{y_i} p^{A(1)} + \partial_{y_j}\left(\eta_{ijmn}^A \partial_{y_n} v_m^{A(0)}\right),\\
\rho^B \dot{w}_i^{B(0)} &= \Phi_i^{B(0)} - \partial_{y_i} p^{B(1)} + \partial_{y_j}\left(\eta_{ijmn}^B \partial_{y_n} v_m^{B(0)}\right).
\end{aligned}
$$

Here the relative velocities are defined by

$$
\tag{5.19}
w_i^{A(0)} \equiv v_i^{A(0)} - \dot{u}_i^{(0)} \qquad \text{and} \qquad w_i^{B(0)} \equiv v_i^{B(0)} - \dot{u}_i^{(0)};
$$

moreover,

$$
\tag{5.20}
\begin{aligned}
\Phi_i^{A(0)} &= \Phi_i^{A(0)}(\mathbf{x}, t) \equiv F_i^A - \partial_{x_i} p^{(0)} - \rho^A \ddot{u}_i^{(0)},\\
\Phi_i^{B(0)} &= \Phi_i^{B(0)}(\mathbf{x}, t) \equiv F_i^B - \partial_{x_i} p^{(0)} - \rho^B \ddot{u}_i^{(0)}.
\end{aligned}
$$

Since $u_i^{(0)} = u_i^{(0)}(\mathbf{x}, t)$, therefore, cf. Eq. (5.1),

$$
\partial_{y_n} w_m^{A(0)} = \partial_{y_n} v_m^{A(0)},
$$

and similarly for $w_m^{B(0)}$ and $v_m^{B(0)}$. Note that in virtue of (A.10), we may write

$$
\dot{u}_i^{(0)}\Big|_S = v_i^{A(0)}\Big|_A \quad \text{on } \Gamma_A \qquad \text{and} \qquad \dot{u}_i^{(0)}\Big|_S = v_i^{B(0)}\Big|_B \quad \text{on } \Gamma_B.
$$

Hence

$$
\tag{5.21}
\mathbf{w}^{A(0)} = \mathbf{0} \qquad \text{and} \qquad \mathbf{w}^{B(0)} = \mathbf{0} \quad \text{on } \Gamma.
$$

To satisfy Eqs.(5.18) we set

$$
\tag{5.22}
\begin{aligned}
p^{A(1)} &= \Phi_m^{A(0)} \gamma_m^{AA}(\mathbf{y}, t) + \Phi_m^{B(0)} \gamma_m^{AB}(\mathbf{y}, t) + \alpha^A(\mathbf{y}, t),\\
p^{B(1)} &= \Phi_m^{A(0)} \gamma_m^{BA}(\mathbf{y}, t) + \Phi_m^{B(0)} \gamma_m^{BB}(\mathbf{y}, t) + \alpha^B(\mathbf{y}, t).
\end{aligned}
$$

The local functions $\gamma_m^{AB}(\mathbf{y}, t)$, $\alpha^A(\mathbf{y}, t)$ *etc.* are to be found. Here and further on, no summation over the capital indices.

Then Eqs.(5.18), posed in $\Omega \times Y_L$, take the form

$$
\tag{5.23}
\begin{aligned}
\rho^A \dot{w}_i^{A(0)} &= \Phi_m^{A(0)}(\delta_{im} - \partial_{y_i}\gamma_m^{AA}) - \Phi_m^{B(0)} \partial_{y_i}\gamma_m^{AB}\\
&\qquad \partial_{y_i}\alpha_i^A + \partial_{y_j}\left(\eta_{ijmn}^A \partial_{y_n} w_m^{A(0)}\right),\\
\rho^B \dot{w}_i^{B(0)} &= \Phi_m^{B(0)}(\delta_{im} - \partial_{y_i}\gamma_m^{BB}) - \Phi_m^{A(0)} \partial_{y_i}\gamma_m^{BA} - \partial_{y_i}\alpha_i^B\\
&\qquad + \partial_{y_j}\left(\eta_{ijmn}^B \partial_{y_n} w_m^{B(0)}\right).
\end{aligned}
$$

These equations are satisfied provided that $\mathbf{w}^{A(0)}$ and $\mathbf{w}^{B(0)}$ are given by the following time convolutions:

$$
\begin{aligned}
w_m^{A(0)} &= w_m^{A(0)}(\mathbf{x},\mathbf{y},t) = \frac{1}{\varrho^A}\int_0^t \Big[\Phi_s^{A(0)}(\mathbf{x},\tau)\,\chi_{ms}^{AA}(\mathbf{y},t-\tau) \\
&\qquad + \Phi_s^{B(0)}(\mathbf{x},\tau)\,\chi_{ms}^{AB}(\mathbf{y},t-\tau)\Big]d\tau + \beta_m^A(\mathbf{y},t), \\
w_m^{B(0)} &= w_m^{B(0)}(\mathbf{x},\mathbf{y},t) = \frac{1}{\varrho^B}\int_0^t \Big[\Phi_s^{A(0)}(\mathbf{x},\tau)\,\chi_{ms}^{BA}(\mathbf{y},t-\tau) \\
&\qquad + \Phi_s^{B(0)}(\mathbf{x},\tau)\,\chi_{ms}^{BB}(\mathbf{y},t-\tau)\Big]d\tau + \beta_m^B(\mathbf{y},t).
\end{aligned}
\tag{5.24}
$$

The functions $\gamma_n^{\alpha\beta} = \gamma_n^{\alpha\beta}(\mathbf{y},t)$ and $\chi_{ms}^{\alpha\beta} = \chi_{ms}^{\alpha\beta}(\mathbf{y},t)$, $\alpha,\beta \in \{A,B\}$ are Y-periodic solutions to the following local problems, cf. the last subsection of Appendix A,

$$
\begin{aligned}
&\rho^A \frac{d}{dt}\chi_{is}^{AA}(\mathbf{y},t) - \Big(\delta_{is} - \partial_{y_i}\gamma_s^{AA}(\mathbf{y},t)\Big) - \partial_{y_j}\Big(\eta_{ijmn}^A \partial_{y_n}\chi_{ms}^{AA}\Big) = 0 \\
&\qquad \mathbf{y} \in Y_A \times (0,T), \\
&\rho^A \frac{d}{dt}\chi_{is}^{AB}(\mathbf{y},t) + \partial_{y_i}\gamma_s^{AB}(\mathbf{y},t) - \partial_{y_j}\Big(\eta_{ijmn}^A \partial_{y_n}\chi_{ms}^{AB}\Big) = 0 \\
&\qquad \mathbf{y} \in Y_A \times (0,T), \\
&\rho^B \frac{d}{dt}\chi_{is}^{BA}(\mathbf{y},t) + \partial_{y_i}\gamma_s^{BA}(\mathbf{y},t) - \partial_{y_j}\Big(\eta_{ijmn}^B \partial_{y_n}\chi_{ms}^{BA}\Big) = 0 \\
&\qquad \mathbf{y} \in Y_B \times (0,T), \\
&\rho^B \frac{d}{dt}\chi_{is}^{BB}(\mathbf{y},t) - \Big(\delta_{is} - \partial_{y_i}\gamma_s^{BB}(\mathbf{y},t)\Big) - \partial_{y_j}\Big(\eta_{ijmn}^B \partial_{y_n}\chi_{ms}^{BB}\Big) = 0 \\
&\qquad \mathbf{y} \in Y_B
\end{aligned}
\tag{5.25}
$$

$$
\frac{\partial \chi_{ki}^{\alpha\beta}}{\partial y_i} = 0 \qquad \text{where} \quad \alpha,\beta = A,B. \tag{5.26}
$$

The interface conditions completing Eqs. (5.25), (5.26) are given by formulae (A.23) and (A.27). After averaging of (5.24) we get

$$(5.27)\quad \left\langle w_i^{A(0)}\right\rangle_A + \left\langle w_i^{B(0)}\right\rangle_B$$

$$= \left\langle \frac{1}{\rho^A}\int_0^t \Big[\Phi_s^A(\mathbf{x},\tau)\chi_{is}^{AA}(\mathbf{y},t-\tau) + \Phi_s^B(\mathbf{x},\tau)\chi_{is}^{AB}(\mathbf{y},t-\tau)\Big]d\tau\right\rangle_A$$

$$+ \left\langle \frac{1}{\rho^B}\int_0^t \Big[\Phi_s^A(\mathbf{x},\tau)\chi_{is}^{BA}(\mathbf{y},t-\tau) + \Phi_s^B(\mathbf{x},\tau)\chi_{is}^{BB}(\mathbf{y},t-\tau)\Big]d\tau\right\rangle_B$$

$$+ \left\langle \int_0^t \beta_i^A(y,t)\,dt\right\rangle_A + \left\langle \int_0^t \beta_i^B(y,t)dt\right\rangle_B.$$

The functions $\boldsymbol{\beta}^\alpha$ are solutions to Eqs. (A.24)–(A.27). In more elaborate models of two-phase flows comprising surface tension on the interface fluid-solid the last two integral vanish. After appropriate substitutions we have finally

$$(5.28)\quad \langle v_i^{(0)}\rangle_L - \dot{u}_i^{(0)} = \Big\langle \int_0^t \Big[\chi_{is}^{AA}(\mathbf{y},\tau)\,\Big(F_s^A - \rho^A\ddot{u}_s^{(0)} - \partial_{x_s}p^{(0)}\Big)(\mathbf{x},t-\tau)$$

$$+\chi_{is}^{AB}(\mathbf{y},\tau)\,\Big(F_s^B - \rho^B\ddot{u}_s^{(0)}\Big) - \partial_{x_s}p^{(0)}\Big)(\mathbf{x},t-\tau)\Big]\,d\tau\Big\rangle_A$$

$$+ \Big\langle \int_0^t \Big[\chi_{is}^{BA}(\mathbf{y},\tau)\,\Big(F_s^A - \rho^A\ddot{u}_s^{(0)} - \partial_{x_s}p^{(0)}\Big)(\mathbf{x},t-\tau)$$

$$+\chi_{is}^{BB}(\mathbf{y},\tau)\,\Big(F_s^B - \rho^B\ddot{u}_s^{(0)}\Big) - \partial_{x_s}p^{(0)}\Big)(\mathbf{x},t-\tau)\Big]\,d\tau\Big\rangle_B$$

$$+ \Big\langle \int_0^t \beta_i^A(y,t)\,dt\Big\rangle_A + \Big\langle \int_0^t \beta_i^B(y,t)\,dt\Big\rangle_B.$$

This is the Darcy equation describing the flow of biphasic liquid in a porous deformable body.

5.4. Consolidation equation

At $\varepsilon^{(0)}$ Eq. (A.3) yields for the Y_A part of the elementary cube

$$(5.29)\qquad \partial_{x_i}v_i^{A(0)}(\mathbf{x},\mathbf{y},t) + \partial_{y_i}v_i^{A(1)}(\mathbf{x},\mathbf{y},t) = 0,$$

and similarly for the Y_B part. Hence, after averaging,

$$\partial_{x_i}\langle v_i^{A(0)}\rangle_A = -\langle \partial_{y_i} v_i^{A(1)}\rangle_A \qquad \text{and} \qquad \partial_{x_i}\langle v_i^{B(0)}\rangle_A = -\langle \partial_{y_i} v_i^{B(1)}\rangle_B$$

or

$$\partial_{x_i}\langle v_i^{A(0)}(\mathbf{x},\mathbf{y},t)\rangle_A = -\frac{1}{|Y|}\int_{\partial Y_A} v_i^{A(1)}(\mathbf{x},\mathbf{y},t)\; n_i^A dS,$$

(5.30)

$$\partial_{x_i}\langle v_i^{B(0)}(\mathbf{x},\mathbf{y},t)\rangle_B = -\frac{1}{|Y|}\int_{\partial Y_B} v_i^{B(1)}(\mathbf{x},\mathbf{y},t)\; n_i^B dS.$$

Hence

$$(5.31)\quad \partial_{x_i}\left[\left\langle v_i^{A(0)}(\mathbf{x},\mathbf{y},t)\rangle_A + \langle v_i^{B(0)}(\mathbf{x},\mathbf{y},t)\right\rangle_B\right] = \frac{1}{|Y|}\Big[\int_{\partial Y_A} v_i^{A(1)} n_i^A(\mathbf{x},\mathbf{y},t)dS + \int_{\partial Y_B} v_i^{B(1)} n_i^B(\mathbf{x},\mathbf{y},t)dS\Big].$$

In virtue of (A.10) and (A.11) we have also

$$\partial_{x_i}\left[\left\langle v_i^{A(0)}(\mathbf{x},\mathbf{y},t)\rangle_A + \left\langle v_i^{B(0)}(\mathbf{x},\mathbf{y},t)\right\rangle_B\right] = \frac{1}{|Y|}\int_{\partial Y_S} \dot{u}_i^{(1)}(\mathbf{x},\mathbf{y},t)\; n_i^S dS.$$

Since, from the definition of averages (3.1)

$$\langle v_i^{A(0)}(\mathbf{x},\mathbf{y},t)\rangle_A + \langle v_i^{B(0)}(\mathbf{x},\mathbf{y},t)\rangle_B = \langle v_i^{(0)}(\mathbf{x},\mathbf{y},t)\rangle_L,$$

therefore by (5.4) we have

$$(5.32)\quad \partial_{x_i}\langle v_i^{(0)}\rangle_L = \frac{1}{|Y|}\int_{\partial Y_S} \frac{d}{dt}\Big[A_i^{(pq)}(\mathbf{x},\mathbf{y},t)\,\partial_{x_q} u_p^{(0)}(\mathbf{x},t) + P_i(\mathbf{x},t)\,p^{(0)}(\mathbf{x},t)\Big]\; n_i^S dA.$$

As result, after use of the divergence theorem, we obtain

$$(5.33)\quad \partial_{x_i}\langle v_i^{(0)}\rangle_L = \frac{d}{dt}\Big[\Big\langle \partial_{y_m} A_m^{(pq)}(\mathbf{x},\mathbf{y},t)\Big\rangle_S \partial_{x_q} u_p^{(0)}(\mathbf{x},t) + \Big\langle \partial_{y_m} P_m(\mathbf{x},t)\Big\rangle_S p^{(0)}(\mathbf{x},t)\Big]$$

where $\langle v_i^{(0)}\rangle_L$ is given by the Darcy law (5. 28). This is the result of Biot's type for nonstationary processes of seepage of incompressible biphasic liquid, identical with those for monophasic flow, cf. [14].

6. Justification of the asymptotic analysis by the two-scale convergence

The aim of this section is to rigorously justify the results obtained by the formal method of two-scale asymptotic expansions. To this end, we exploit the notion of the two-scale convergence, cf. [4, 69, 70]. Consider the following system of equations:

$$\begin{gathered}
\rho_S \ddot{\mathbf{u}}^\varepsilon_S(x,t) = \operatorname{div}(\mathbf{a}^\varepsilon \mathbf{e}(\mathbf{u}^\varepsilon(x,t))) + F^S(x,t) \qquad \text{in } \Omega^\varepsilon_S \times (0,T),\\
\rho_A \dot{\mathbf{v}}^\varepsilon_A(x,t) = \mu_A \varepsilon^2 \Delta \mathbf{v}^\varepsilon_A(x,t) - \nabla p^\varepsilon_A(x,t) + F^A(x,t) \quad \text{in } \Omega_A \times (0,T),\\
\rho_B \dot{\mathbf{v}}^\varepsilon_B(x,t) = \mu_B \varepsilon^2 \Delta \mathbf{v}^\varepsilon_B(x,t) - \nabla p^\varepsilon_B(x,t) + F^B(x,t) \quad \text{in } \Omega_B \times (0,T),\\
\operatorname{div} \mathbf{v}^\varepsilon_A = 0 \quad \text{in } \Omega^\varepsilon_A \times (0,T),\\
\operatorname{div} \mathbf{v}^\varepsilon_B = 0 \quad \text{in } \Omega^\varepsilon_B \times (0,T),\\
\mathbf{a}^\varepsilon \mathbf{e}(\mathbf{u}^\varepsilon)\mathbf{n} = (-p^\varepsilon_B \mathbf{1} + \varepsilon^2 \mathbf{e}(\mathbf{v}^\varepsilon_B))\mathbf{n} \quad \text{on } \Gamma^\varepsilon_{SB} \times (0,T),\\
[\![-p^\varepsilon \mathbf{1} + \varepsilon^2 \mu \mathbf{e}(\mathbf{v}^\varepsilon)]\!]_{AB}\mathbf{n} = \sigma H^\varepsilon \mathbf{n} \quad \text{on } \Gamma^\varepsilon_{AB} \times (0,T),\\
\dot{\mathbf{u}}^\varepsilon(x,t) = \mathbf{v}^\varepsilon_B(x,t) \quad \text{on } \Gamma^\varepsilon_{SB} \times (0,T),\\
\mathbf{v}^\varepsilon_A(x,t) = \mathbf{v}^\varepsilon_B(x,t) \quad \text{on } \Gamma^\varepsilon_{AB} \times (0,T),
\end{gathered} \tag{6.1}$$

and the initial conditions

$$\mathbf{u}^\varepsilon(x,0) = \mathbf{u}_0(x) \quad \text{in } \Omega^\varepsilon_S, \tag{6.2}$$

$$\dot{\mathbf{u}}^\varepsilon(x,0) = \mathbf{u}_1(x) \quad \text{in } \Omega^\varepsilon_S, \tag{6.3}$$

$$\mathbf{v}^\alpha(x,0) = \mathbf{v}^\alpha_0(x) \quad \text{in } \Omega^\alpha;\ \alpha = A, B. \tag{6.4}$$

We assume here that the function $H^\varepsilon(x) = H(\frac{x}{\varepsilon})$ is an εY-periodic function defined on the surface Γ^ε_{AB}. We can then formulate the following theorem.

THEOREM. *The sequence* $\{\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon_A, \mathbf{v}^\varepsilon_B, p^\varepsilon_A, p^\varepsilon_B\}_{\varepsilon>0}$ *of solutions of the system* (6.1)–(6.12) *is two-scale convergent to the solution*

$$(\mathbf{u}^{(0)}(x,t), \mathbf{v}^{(0)}_A(x,y,t), \mathbf{v}^{(0)}_B(x,y,t), p^{(0)}_A(x,t), p^{(0)}_B(x,t))$$

of the two-scale homogenized problem

$$(6.5)\quad (1-f)\rho^S\ddot{\mathbf{u}}^{(0)} + \rho^A\langle\dot{\mathbf{v}}_A^{(0)}\rangle_{Y_A} + \rho^B\langle\dot{\mathbf{v}}_B^{(0)}\rangle_{Y_B}$$

$$= \operatorname{div}_x \int\limits_{Y_S} \left[\mathbf{a}(\nabla_x\mathbf{u}^{(0)} + \nabla_y\mathbf{u}^{(1)})\right] dy - \int\limits_{Y_A} \nabla_y p_A^{(1)}(x,y,t)\, dy$$

$$- \int\limits_{Y_B} \nabla_y p_B^{(1)}(x,y,t)\, dy - f_A\nabla_x p_A^{(0)}(x,t), -f_B\nabla_x p_B^{(0)}(x,t)$$

$$+ (1-f)F^S(x,t) + f_A F^A(x,t) + f_B F^B(x,t) \quad \text{in } \Omega\times(0,T),$$

$$(6.6)\qquad \operatorname{div}_y[\mathbf{a}(\nabla_x\mathbf{u}^{(0)} + \nabla_y\mathbf{u}^{(1)})] = 0 \quad \text{in } \Omega\times Y_S\times(0,T),$$

$$(6.7)\quad \rho^A\dot{\mathbf{v}}_A^{\varepsilon}(x,y,t) = \mu_A\Delta_y\mathbf{v}_A^{(0)}(x,y,t)$$

$$- \nabla_x p_A^{(0)}(x,t) - \nabla_y p_A^{(1)}(x,y,t) + F^A(x,t) \quad \text{in } \Omega\times Y_A\times(0,T),$$

$$(6.8)\quad \rho^B\dot{\mathbf{v}}_B^{(0)}(x,y,t) = \mu_B\Delta\mathbf{v}_B^{(0)}(x,y,t) - \nabla_x p_B^{(0)}(x,t)$$

$$- \nabla_y p_B^{(1)}(x,y,t) + F^B(x,t) \text{ in } \Omega\times Y_B\times(0,T),$$

$$(6.9)\qquad \operatorname{div}_y\mathbf{v}_A^{(0)} = 0 \quad \text{in } \Omega\times Y_A\times(0,T),$$

$$(6.10)\qquad \operatorname{div}_y\mathbf{v}_B^{(0)} = 0 \quad \text{in } \Omega\times Y_B\times(0,T),$$

$$(6.11)\qquad \operatorname{div}_x\int\limits_{Y_A}\mathbf{v}_A^{(0)}(x,y,t)\, dy = \int\limits_{Y_A}\operatorname{div}_y\mathbf{v}(1)_A(x,y,t),$$

$$(6.12)\qquad \operatorname{div}_x\int\limits_{Y_B}\mathbf{v}_B^{(0)}(x,y,t)\, dy = \int\limits_{Y_B}\operatorname{div}_y\mathbf{v}_B^{(1)}(x,y,t),$$

$$(6.13)\qquad [\![p^{(0)}]\!] = 0 \quad \text{on } \Gamma_{AB}\times(0,T),$$

$$(6.14)\qquad [\![p^{(1)}\mathbf{1} + 2\mu\mathbf{e}(\mathbf{v}^{(0)})]\!]\mathbf{n} = \sigma H^{(1)}\mathbf{n} \quad \text{on } \Gamma_{AB}\times(0,T).$$

Sketch of the proof. By a standard procedure we show that there exists a constant $c > 0$ independent of ε such that

$$\|\mathbf{u}^{\varepsilon}\|_{H^1(\Omega)^3} \le c, \ \|\mathbf{v}^{\varepsilon}\|_{L^2(\Omega)^3} \le c, \ \varepsilon\|\nabla\mathbf{v}^{\varepsilon}\|_{L^2(\Omega,\mathbb{E}^N)} \le c,$$

where $\mathbb{E}^N$ stands for the space of $N \times N$ matrices. For the definitions and properties of the Lebesgue and Sobolev spaces the reader is referred to ADAMS [1]. The two-scale limits of these sequences satisfy the following properties, cf. [4],

$$\operatorname{div}_y \mathbf{v}^{(0)A}(x,y,t) = 0 \qquad \text{in} \qquad \Omega \times Y_B \times (0,T),$$

$$\operatorname{div}_x \mathbf{v}_A^{(0)}(x,y,t) + \operatorname{div}_y \mathbf{v}_A^{(1)}(x,y,t) = 0 \text{ in } \Omega \times Y_A \times (0,T),$$

$$\nabla_y \mathbf{u}^{(0)}(x,y,t) = 0, \qquad \text{or} \qquad \mathbf{u}^{(0)} = \mathbf{u}^{(0)}(x,t), \ t \in (0,T), \ x \in \Omega.$$

Similar equation is satisfied by the velocity $\mathbf{v}_B$ in $(0,T) \times \Omega \times Y_B$. The next conclusion, deduced from the two-scaled convergence, is that

$$\nabla_y p_A^{(0)}(x,y,t) = 0, \text{ or } p_A^{(0)} = p_A^{(0)}(x,t) \ (x,t) \in \Omega \times (0,T).$$

Similarly, for the pressure $p_B^{(0)}$ we write

$$\nabla_y p_B^{(0)}(x,y,t) = 0, \text{ or } p_B^{(0)} = p_B^{(0)}(x,t) \ (x,t) \in \Omega \times (0,T).$$

Let $\chi_S^{\varepsilon}, \chi_A^{\varepsilon}, \chi_B^{\varepsilon}$ be the characteristic functions and let $\phi = \phi(t) \in C^{\infty}(0,T)$ be such that $\phi(0) = \dot{\phi}(T) = 0$. We also take a test function $\mathbf{\Phi}^{\varepsilon}(x)$ such that

$$\mathbf{\Phi}^{\varepsilon}(x) = \boldsymbol{\eta}(x) + \varepsilon\boldsymbol{\vartheta}(x, \frac{x}{\varepsilon}),$$

where $\boldsymbol{\eta} \in \mathcal{D}(\Omega)^3$ and $\boldsymbol{\vartheta}(x,y) \in \mathcal{D}[\Omega; C_{per}^{\infty}(Y)]^3$. Now the proof, rather lengthy, is an extension of the proof derived in [14] for a one-phase flow. However, difficulties which arise are due to the fact that interface $\Gamma^{\varepsilon}{}_{AB}$ depends on time. We hope to present a detailed proof elsewhere.

7. Passage to the stationary case

To perform this passage we shall first reformulate the local problems and Darcy's law. To this end we set: $\boldsymbol{\chi}^{\alpha\beta} = \dot{\boldsymbol{\psi}}^{\alpha\beta}$, $\alpha, \beta \in \{A, B\}$, cf. [14],

PROBLEM P_1

Find the functions $\boldsymbol{\chi}^{\alpha\beta}$ and $\gamma^{AB(k)}$such that

$$\varrho^A \dot{\boldsymbol{\chi}}^{AA(k)}(y,t) = \eta^A \Delta_{yy}\boldsymbol{\chi}^{AA(k)}(y,t) - \nabla\gamma^{AA(k)}(y,t) + \mathbf{e}_k, \text{ in } Y_A \times (0,T),$$

$$\varrho^A\dot{\chi}^{AB(k)}(y,t) = \eta^A\Delta_{yy}\chi^{ABk)}(y,t) - \nabla\gamma^{ABk)}(y,t), \text{ in } Y_A \times (0,T),$$

$$\varrho^B\dot{\chi}^{BB(k)}(y,t) = \eta^B\Delta_{yy}\chi^{BB(k)}(y,t) - \nabla\gamma^{BB(k)}(y,t) + \mathbf{e}_k, \text{ in } Y_B \times (0,T),$$

$$\varrho^B\dot{\chi}^{BA(k)}(y,t) = \eta^B\Delta_{yy}\chi^{BA(k)}(y,t) - \nabla\gamma^{BA(k)}(y,t), \text{ in } Y_B \times (0,T),$$

$$\mathrm{div}\psi^{AA(k)} = \mathrm{div}\psi^{AB(k)} = 0, \quad \text{in } Y_A \times (0,T);$$

$$\mathrm{div}\psi^{BB(k)} = \mathrm{div}\psi^{BA(k)} = 0, \quad \text{in } Y_B \times (0,T);$$

$$\eta^A\mathbf{e}^y(\psi^{AA(k)})\mathbf{n} - \gamma^{AA(k)}\mathbf{n} = \eta^B\mathbf{e}^y(\psi^{BA(k)})\mathbf{n} - \gamma^{BA(k)}\mathbf{n}, \qquad \text{on } \Gamma_{AB},$$

$$\eta^A\mathbf{e}^y(\psi^{AB(k)})\mathbf{n} - \gamma^{AB(k)}\mathbf{n} = \eta^B\mathbf{e}^y(\psi^{BB(k)})\mathbf{n} - \gamma^{BB(k)}\mathbf{n}, \qquad \text{on } \Gamma_{AB},$$

$$\psi^{AA(k)} - \psi^{BA(k)} = 0, \ \psi^{AB(k)} - \psi^{BB(k)} = 0 \qquad \text{on } \Gamma_{AB}.$$

Moreover, homogeneous initial conditions and no-slip condition on the fluid-solid interface are assumed. Let us pass to the formulation of the second local problem.

LOCAL PROBLEM P_2

Find functions β^α and π^α such that

$$\varrho^\alpha\dot{\beta}^\alpha = \eta^\alpha\Delta_{yy}\beta^\alpha - \nabla\pi^\alpha, \quad \text{in } Y_\alpha \times (0,T),$$

$$\mathrm{div}_y\beta^\alpha = 0 \quad \text{in } Y_\alpha \times (0,T),$$

$$\beta^A - \beta^B = 0 \quad \text{on} \quad \Gamma_{AB},$$

$$(\eta^A\mathbf{e}^y(\beta^A) - \pi^A\mathbf{I})\mathbf{n} = (\eta^B\mathbf{e}^y(\beta^B) - \pi^B\mathbf{I})\mathbf{n} + \sigma H\mathbf{n}, \text{ on } \Gamma_{AB},$$

$$\beta^A = \beta^B = \mathbf{0} \quad \text{on} \quad \Gamma.$$

This system is completed by the homogeneous initial conditions. Then Darcy's law takes the form, cf. (5.27), where instead of χ we write $\dot{\psi}$. This law can be written as follows:

$$(7.1)\quad \langle\bar{\mathbf{v}}^0\rangle_L = \frac{1}{\varrho^A}\int_0^t \left\langle\dot{\psi}^{AA(k)}(y,s)\right\rangle_A (F_k^A - \varrho^A\ddot{\mathbf{u}}_k^{(0)} - \frac{\partial p^{A(0)}}{\partial x_k})(x,t-s)\,ds$$

$$+ \frac{1}{\varrho^A}\int_0^t \left\langle\dot{\psi}^{AB(k)}(y,s)\right\rangle_A (F_k^B - \varrho^B\ddot{\mathbf{u}}_k^{(0)} - \frac{\partial p^{B(0)}}{\partial x_k})(x,t-s)\,ds$$

$$+ \frac{1}{\varrho^B}\int_0^t \left\langle\dot{\psi}^{BB(k)}(y,s)\right\rangle_B (F_k^B - \varrho^B\ddot{\mathbf{u}}_k^{(0)} - \frac{\partial p^{B(0)}}{\partial x_k})(x,t-s)\,ds$$

$$+ \frac{1}{\varrho^B}\int_0^t \left\langle\dot{\psi}^{BA(k)}(y,s)\right\rangle_B (F_k^A - \varrho^A\ddot{\mathbf{u}}_k^{(0)} - \frac{\partial p^{A(0)}}{\partial x_k})(x,t-s)\,ds$$

$$+ \int_0^t \left\langle\dot{\beta}^A(y,t-s)\right\rangle_A ds + \int_0^t \left\langle\dot{\beta}^B(y,t-s)\right\rangle_B ds.$$

It may be rewritten as follows:

$$(7.2)\quad \langle\bar{\mathbf{v}}^0\rangle_L = \int_0^t \left\{\left[\frac{1}{\varrho^A}\left\langle\dot{\psi}^{AA(k)}(y,s)\right\rangle_A\right.\right.$$

$$\left.\left. + \frac{1}{\varrho^B}\left\langle\dot{\psi}^{BA(k)}(y,s)\right\rangle_B\right] (F_k^A - \varrho^A\ddot{\mathbf{u}}_k^{(0)} - \frac{\partial p^{A(0)}}{\partial x_k})(x,t-s)\right\}ds$$

$$\int_0^t \left\{(\frac{1}{\varrho^A}\left\langle\dot{\psi}^{AB(k)}(y,s)\right\rangle_A\right.$$

$$\left. + \frac{1}{\varrho^B}\left\langle\dot{\psi}^{BB(k)}(y,s)\right\rangle_B\right] (F_k^B - \varrho^B\ddot{\mathbf{u}}_k^{(0)} - \frac{\partial p^{B(0)}}{\partial x_k})(x,t-s)\}\,ds$$

$$+ \int_0^t \left\langle\dot{\beta}^A(y,t-s)\right\rangle_A ds + \int_0^t \left\langle\dot{\beta}^B(y,t-s)\right\rangle_B ds.$$

The passage to the stationary case is enforced by assuming that the forcing terms $\mathbf{F}^A - \varrho^A\ddot{\mathbf{u}}^{(0)} - \nabla p^{A(0)}$ and $\mathbf{F}^B - \varrho^B\ddot{\mathbf{u}}^{(0)} - \nabla p^{B(0)}$ are time-independent. Next we have to pass with time to infinity, cf. (BIELSKI and TELEGA [13]), HEYWOOD [55]. Consequently, since $\ddot{\mathbf{u}}^{(0)} = \mathbf{0}$ we get

$$(7.3)\quad \langle \bar{\mathbf{v}}^0 \rangle_L = \Big\{ \int_0^t \Big[\frac{1}{\varrho^A} \Big\langle \dot{\psi}^{AA(k)}(y,s) \Big\rangle_A + \frac{1}{\varrho^B} \Big\langle \dot{\psi}^{BA(k)}(y,s) \Big\rangle_B \Big] ds \Big\} \Big(F_k^A - \frac{\partial p^{A(0)}}{\partial x_k} \Big)\Big(x\Big) + \Big\{ \int_0^t \Big[\frac{1}{\varrho^A} \Big\langle \dot{\psi}^{AB(k)}(y,s)| \Big\rangle_A + \frac{1}{\varrho^B} |\Big\langle \dot{\psi}^{BB(k)}(y,s) \Big\rangle_B \Big] ds \Big\} \Big(F_k^B - \frac{\partial p^{B(0)}}{\partial x_k} \Big)\Big(x\Big) + \int_0^t \Big\langle \dot{\mathbf{W}}^{\mathbf{A}}(y, t-s) \Big\rangle_A ds + \int_0^t \Big\langle \dot{\mathbf{W}}^{\mathbf{B}}(y, t-s) \Big\rangle_B ds.$$

Consider the first integral in the r.h.s of the last relation. We have

$$(7.4)\quad \int_0^t A_{ij}^A\Big(t-s\Big)\, ds = \frac{1}{\varrho_A |Y|} \int_{Y_A} \Big(\int_0^t \frac{\partial \psi^{AA(i)}(t-s,y)}{\partial s}\, ds \Big) \cdot \mathbf{e}_j\, dy + \frac{1}{\varrho_B |Y|} \int_{Y_B} \Big(\int_0^t \frac{\partial \psi^{BA(i)}(t-s,y)}{\partial s}\, ds \Big) \cdot \mathbf{e}_j\, dy.$$

Here A_{ij}^A is a part of the permeability tensor that corresponds to the forcing term $\mathbf{F}^A - \nabla_x p^{A(0)}$, i.e.,

$$\langle \mathbf{v}_A \rangle_A = \int_0^t A_{ij}(t-s)(\mathbf{F}^A - \nabla p^{A(0)}\, ds.$$

Performing the integration by parts in Eq. (7.4) we get

$$(7.5)\quad \int_0^t A_{ij}^A(t-s)\, ds = \frac{1}{\varrho^A |Y|} \int_{Y_A} \chi^{AA(i)}(t,y) \cdot \mathbf{e}_j\, dy + \frac{1}{\varrho^B |Y|} \int_{Y_B} \psi^{BA(i)}(t,y) \cdot \mathbf{e}_j\, dy.$$

Letting t to tend to infinity we find

$$(7.6)\quad \lim_{t\to\infty}\int_0^t A^A_{ij}(t-s)\,ds = \frac{1}{\varrho^A|Y|}\int_{Y_A}\psi^{AA(i)}_\infty(y)\cdot\mathbf{e}_j\,dy + \frac{1}{\varrho_B|Y|}\int_{Y_B}\psi^{BA(i)}_\infty(y)\cdot\mathbf{e}_j\,dy = K^A_{ij}.$$

Similarly, the second integral on the r.h.s. of Eq. (7.3) yields

$$\lim_{t\to\infty}\int_0^t A^A_{ij}(t-s)\,ds = \lim_{t\to\infty}\frac{1}{\varrho^A|Y|}\int_{Y_B}(\psi^{AA(i)}(t,y)\cdot\mathbf{e}_j\,dy + \frac{1}{\varrho^B|Y|}\int_{Y_B}\psi^{BA(i)}(t,y)\cdot\mathbf{e}_j\,dy = K^B_{ij}.$$

The third integral on the r.h.s. of Eq. (7.3) gives

$$(7.7)\quad \int_0^t \frac{1}{\varrho^A|Y|}\Big(\int_{Y_A}\frac{\partial\beta^A(y,t-s)}{\partial s}\,dy\Big)\,ds = \frac{1}{\varrho^A|Y|}\int_{Y_A}\Big(\int_0^t\frac{\partial\beta^A(y,t-s)}{\partial s}\,ds\Big)\,dy = \frac{1}{\varrho^A|Y|}\int_{Y_A}\Big[-\beta^A(y,0)+\beta^A(y,t)\Big]\,dy.$$

By the initial condition we write $\beta(y,0)=0$. Moreover, $\lim_{t\to\infty}\mathbf{W}^A(y,t)$ exists. We denote it by $\beta^A_\infty(y)$, i.e.,

$$\lim_{t\to\infty}\beta^A(y,t)=\beta^A_\infty(y).$$

Finally, Darcy's law for the stationary two-phase flow assumes the following form

$$\langle\bar{\mathbf{v}}^{(0)}\rangle = K^A_{ij}(F^A_j-\frac{\partial p^{A(0)}}{\partial x_j}) + K^B_{ij}(F^B_j-\frac{\partial p^{B(0)}}{\partial x_j}) + \frac{1}{|Y|}\int_{Y_A}\beta^A(y)\,dy + \frac{1}{|Y|}\int_{Y_B}\beta^B(y)\,dy.$$

This law is similar to the one derived in [74] for two-phase flow through undeformable porous medium provided that our symbols ψ are indentified with $\mathbf{V}$ in [74] and β_∞ with $\mathbf{W}$. The local problems for the determination of functions $\mathbf{V}$, $\mathbf{W}$, etc., are specified and studied in [74].

8. An example

As an example we consider one-dimensional stationary flow of two-phase liquid through a bundle of tubes in the direction z in the cylindrical coordinate system. We assume that the local problem has a cylindrical symmetry. Hence only the velocity in the z-direction does not vanish. For the sake of simplicity we assume that the fluid A occupies the pipe of radius R_A and is surrounded by the fluid B which forms a ring of the external radius R_B. Let us pass to the study of local problems derived from (7.6). For simplicity we wrrite $\boldsymbol{\chi}^{\alpha\beta}$ instead of $\boldsymbol{\chi}^{\alpha\beta}_{\infty}$; obviously now $\boldsymbol{\chi}^{\alpha\beta} = \boldsymbol{\psi}^{\alpha\beta}$.

LOCAL PROBLEM P$_1$

$$\begin{aligned}\eta_A \frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi^{AA}(r)}{dr}\right) &= \frac{d\gamma^{AA}(z)}{dz} + 1 = 0, \text{ for } 0 \le r \le R_A,\\ \eta_B \frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi^{BA}(r)}{dr}\right) &= \frac{d\gamma^{BA}(z)}{dz} = 0, \text{ for } R_A \le r \le R_B.\end{aligned} \tag{8.1}$$

The interface conditions are now specified by

$$\begin{gathered}\chi^{AA}(R_A) = \chi^{BA}(R_A),\ \eta_A \frac{d\chi^{AA}}{dr}(R_A) = \eta_B \frac{d\chi^{BA}}{dr}(R_A),\ \chi^{AA}(R_B) = 0\\ \gamma^{AA} = \gamma^{BA} \quad \text{for } r = R_A.\end{gathered} \tag{8.2}$$

Similar problems have to be formulated for $\chi^{BB}, \chi^{AB}, \gamma^{BB}$ and γ^{AB}.
The solution ofproblemP$_1$ is given by

$$\begin{aligned}\chi^{AA}(r) &= \frac{1}{4\eta_A}(r^2 - R_A^2) + \frac{1}{2\eta_B}R_A^2 \ln\frac{R_A}{R_B} \quad \text{for } 0 \le r \le R_A\\ \chi^{BA}(r) &= \frac{1}{2\eta_B}R_A^2 \ln\frac{r}{R_B}, \quad \text{for } R_A \le r \le R_B.\end{aligned} \tag{8.3}$$

The local functions χ^{BB} and χ^{AB} are specified by

$$\begin{aligned}\chi^{AB}(r) &= \frac{1}{4\eta_B}(R_A^2 - R_B^2) - \frac{1}{2\eta_B}R_A^2 \ln\frac{R_A}{R_B}, \quad \text{if } 0 \le r \le R_A,\\ \chi^{BB}(r) &= \frac{1}{4\eta_B}(r^2 - R_B^2) - \frac{1}{2\eta_B}R_A^2 \ln\frac{r}{R_B}, \quad \text{if } R_A \le r \le R_B.\end{aligned} \tag{8.4}$$

LOCAL PROBLEM P$_2$

Now $\boldsymbol{\beta}^A = (0, 0, \beta^A(r))$ and we have

$$\eta_A \frac{1}{r}\frac{d}{dr}\left(r\frac{d\beta^A(r)}{dr}\right) = \frac{d\pi^A(z)}{dz} = 0, \quad \text{for } 0 \le r \le R_A. \tag{8.5}$$

Similarly β^B is a solution to

$$\eta_B \frac{1}{r}\frac{d}{dr}\left(r\frac{d\beta^B(r)}{dr}\right) = \frac{d\pi^B(z)}{dz} = 0, \quad \text{for } R_A \le r \le R_B. \tag{8.6}$$

The boundary and interface conditions are:

$$\begin{gathered}\beta^A(R_A) = \beta^B(R_A), \ \ \eta_A \frac{d\beta^A}{dr}(R_A) = \eta_B \frac{d\beta^B}{dr}(R_A), \ \ \beta^B(R_B) = 0,\\ \pi^A = \pi^B + \frac{2\sigma}{R_A}, \quad \text{for } r = R_A.\end{gathered} \tag{8.7}$$

The solution of the local problem P_2 is trivial,

$$\beta^A(r) = \beta^B(r) = 0. \tag{8.8}$$

According to formula (7.6) Darcy's law assumes now the form

$$\begin{aligned}\left\langle v^{(0)}\right\rangle_L - f\dot{u} = \left\{\frac{\pi}{8\eta_A|Y|}R_A^4 + \frac{\pi}{2\eta_B|Y|}R_A^2\left(R_A^2 - R_B^2\right)\left(\frac{1}{2} - \ln R_B\right)\right\}\left(F_3^A - \frac{\partial p^{A(0)}}{\partial x_3}\right)\\ + \frac{\pi}{8\eta_B|Y|}\left(R_A^2 - R_B^2\right)^2\left(F_3^B - \frac{\partial p^{B(0)}}{\partial x_3}\right).\end{aligned}$$

We observe that a similar stationary flow was investigated by ZANOTTI and CARBONELL [83]. However, an explicit form of the Darcy law was not given by these authors. Moreover, homogenization was not used.

9. Final remarks

Essential feature of viscous flows through microperiodic porous media studied in the present paper and in [14, 15] is a linear elastic solid phase and Stokesian fluids. A challenging problem is to derive macroscopic flow for Navier-Stokes or other nonlinear fluids and more involved solid phases. We think of hyperelastic and inelastic phases. For instance, one can think of the solid phase made of elastic Ogden's material or elastic-plastic phases. Also, thermal flows are worth of examination.

Another open problem is to extend our results to the case of porous media with random distribution of micropores, cf. Sec. 2 of our paper for a brief discussion of simpler problems.

Darcy's law for nonstationary flows is nonlocal in time, also in the case of one-phase flow, cf. [14, 15]. Such a law is complicated and in general not directly applicable. Hence the need for elaboration of reliable approximate methods, the problem well known in micromechanics.

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Appendix A. Asymptotic analysis

Equations of motion$(4.1)_{1,2}$ and $(4.1)_3$ after substitution of expansions (4.4) take the form

$$\text{(A.1)}\quad \rho^S\left[\ddot{u}_i^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon\ddot{u}_i^{(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2\ddot{u}_i^{(2)}(\mathbf{x},\mathbf{y},t)+...\right]$$
$$=\left(\frac{\partial}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\left\{a_{ijmn}\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left[u_m^{(0)}(\mathbf{x},\mathbf{y},t)\right.\right.$$
$$\left.\left.+\varepsilon u_m^{(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2u_m^{(2)}(\mathbf{x},\mathbf{y},t)+...\right]\right\}+F_i^S,$$

$$\text{(A.2)}\quad \rho^A\left[\dot{v}_i^{A(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon\dot{v}_i^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2\dot{v}_i^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]$$
$$=\left(\frac{\partial}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\left\{\left[p^{A(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon p^{A(1)}(\mathbf{x},\mathbf{y},t)\right.\right.$$
$$\left.+\varepsilon^2p^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]\delta_{ij}+\varepsilon^2\eta^A_{ijmn}\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left[v_m^{A(0)}(\mathbf{x},\mathbf{y},t)\right.$$
$$\left.\left.+\varepsilon v_m^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2v_m^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]\right\}+F_i^A,$$

and

$$\text{(A.3)}\quad \left(\frac{\partial}{\partial x_i}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\left[v_i^{A(0)}(\mathbf{x},\mathbf{y},t)\right.$$
$$\left.+\varepsilon v_i^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2v_i^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]=0.$$

Similar relations held for $\mathbf{v}^B$, only the index A should be replaced by B. The interface conditions (4.2) read

$$\text{(A.4)}\quad a_{ijmn}\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left[u_m^{(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon u_m^{(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2 u_m^{(2)}(\mathbf{x},\mathbf{y},t)+...\right]n_j$$
$$=\Big\{-\left[p^{A(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon p^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2 p^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]\delta_{ij}$$
$$+\varepsilon^2\eta^A_{ijmn}\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left[v_m^{A(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon v_m^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2 v_m^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]\Big\}n_j \quad \text{on } \Gamma,$$

$$\text{(A.5)}\quad \begin{aligned}&\left(\dot u_i^{(0)}+\varepsilon\dot u_i^{(1)}+\varepsilon^2\dot u_i^{(2)}+\cdots\right)\Big|_S=\left(v_i^{A(0)}+\varepsilon v_i^{A(1)}+\varepsilon^2 v_i^{A(2)}+\cdots\right)\Big|_A \quad \text{on } \Gamma_A\\ &\left(\dot u_i^{(0)}+\varepsilon\dot u_i^{(1)}+\varepsilon^2\dot u_i^{(2)}+\cdots\right)\Big|_S=\left(v_i^{B(0)}+\varepsilon v_i^{B(1)}+\varepsilon^2 v_i^{B(2)}+\cdots\right)\Big|_B \quad \text{on } \Gamma_B\end{aligned}$$

$$\text{(A.6)}\quad \left(v_i^{A(0)}+\varepsilon v_i^{A(1)}+\varepsilon^2 v_i^{A(2)}+\cdots\right)\Big|_A=\left(v_i^{B(0)}+\varepsilon v_i^{B(1)}+\varepsilon^2 v_i^{B(2)}+\cdots\right)\Big|_B \quad \text{on } \Gamma_{AB}$$

and

$$\text{(A.7)}\quad \Big\{-\left[p^{A(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon p^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2 p^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]\delta_{ij}$$
$$+\varepsilon^2\eta^A_{ijmn}\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left[v_m^{A(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon v_m^{A(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2 v_m^{A(2)}(\mathbf{x},\mathbf{y},t)+...\right]\Big\}n_j=\Big\{-\left[p^{B(0)}(\mathbf{x},\mathbf{y},t)+\varepsilon p^{B(1)}(\mathbf{x},\mathbf{y},t)\right.$$
$$\left.+\varepsilon^2 p^{B(2)}(\mathbf{x},\mathbf{y},t)+...\right]\delta_{ij}+\varepsilon^2\eta^B_{ijmn}\left(\frac{\partial}{\partial x_n}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left[v_m^{B(0)}(\mathbf{x},\mathbf{y},t)\right.$$
$$\left.+\varepsilon v_m^{B(1)}(\mathbf{x},\mathbf{y},t)+\varepsilon^2 v_m^{B(2)}(\mathbf{x},\mathbf{y},t)+...\right]\Big\}n_j+\varepsilon\sigma H_{ij}n_j \quad \text{on } \Gamma_{AB}.$$

Analysis of terms of different orders of ε in equations at interfaces

From the interface condition (A.4) we find that at ε^{-1} appears the term

$$a_{ijmn}\partial_{y_n}u_m^{(0)}\, n_j = 0 \qquad \text{on } \Gamma \tag{A.8}$$

while the terms linked with ε^0 give

$$\begin{aligned} &\left[a_{ijmn}\left(\partial_{x_n}u_m^{(0)} + \partial_{y_n}u_m^{(1)}\right)\right]n_j^S = -p^{A(0)}\, n_i^S \qquad \text{on} \quad \Gamma_A,\\ &\left[a_{ijmn}\left(\partial_{x_n}u_m^{(0)} + \partial_{y_n}u_m^{(1)}\right)\right]n_j^S = -p^{B(0)}\, n_i^S \qquad \text{on} \quad \Gamma_B. \end{aligned} \tag{A.9}$$

Comparing the terms at different orders of ε in (A.5) we get

$$\begin{aligned} &\dot{u}_i^{(0)}\Big|_S = v_i^{A(0)}\Big|_A, \quad \dot{u}_i^{(1)}\Big|_S = v_i^{A(1)}\Big|_A, \quad \dot{u}_i^{(2)}\Big|_S = v_i^{A(2)}\Big|_A \ \ etc. \ \text{on}\,\Gamma_A,\\ &\dot{u}_i^{(0)}\Big|_S = v_i^{B(0)}\Big|_B, \quad \dot{u}_i^{(1)}\Big|_S = v_i^{B(1)}\Big|_B, \quad \dot{u}_i^{(2)}\Big|_S = v_i^{B(2)}\Big|_B \ \ etc. \ \text{on}\,\Gamma_B, \end{aligned} \tag{A.10}$$

and similarly in (A.6)

$$v_i^{A(0)}\Big|_A = v_i^{B(0)}\Big|_B \ \ v_i^{A(1)}\Big|_A = v_i^{B(1)}\Big|_B \ \ v_i^{A(2)}\Big|_A = v_i^{B(2)}\Big|_B \ \ etc. \ \text{on}\ \Gamma_{AB}. \tag{A.11}$$

The terms at ε^0 in (A.7) give

$$p^{A(0)}(\mathbf{x},\mathbf{y},t) = p^{B(0)}(\mathbf{x},\mathbf{y},t) \qquad \text{on } \Gamma_{AB} \tag{A.12}$$

and at ε^1

$$\begin{aligned} &-\left[p^{A(1)}(\mathbf{x},\mathbf{y},t)\delta_{ij} + \eta^A_{ijmn}\ \partial y_n v_m^{A(0)}(\mathbf{x},\mathbf{y},t)\right]n_j\\ &= -\left[p^{B(1)}(\mathbf{x},\mathbf{y},t)\delta_{ij} + \eta^B_{ijmn}\ \partial y_n v_m^{A(0)}(\mathbf{x},\mathbf{y},t)\right]n_j + \sigma H_{ij}n_j \quad \text{on } \Gamma_{AB}. \end{aligned} \tag{A.13}$$

Analysis of terms of order ε^{-2} in equations of motion

Now we get

$$\partial_{y_j}\left[a_{ijmn}\partial_{y_n}u_m^{(0)}\right] = 0. \tag{A.14}$$

Multiplying (A.14) by $u_i^{(0)}$ and integrating by parts we get

$$\int\limits_{\partial Y_S} a_{ijmn}\partial_{y_n}u_m^{(0)}u_i^{(0)}n_j dA - \int\limits_{Y_S} a_{ijmn}(\partial_{y_j}u_i^{(0)})\partial_{y_n}u_m^{(0)} d\mathbf{y} = 0. \tag{A.15}$$

By virtue of the interface condition (A.8), the surface integral in the last equation vanishes and the rest implies, by positive definiteness of a_{ijmn}, that $\mathbf{u}$ depends on $\mathbf{x}$ and t only

$$u_i^{(0)} = u_i^{(0)}(\mathbf{x}, t) \tag{A.16}$$

but does not depend on $\mathbf{y}$.

Analysis of terms of order ε^{-1} in equations of motion

From Eq. (A.1) we get the following relation for $\mathbf{x} \in \Omega$ and $\mathbf{y} \in Y_S$:

$$\partial_{y_j}\left[a_{ijmn}\left(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\right)\right] = 0. \tag{A.17}$$

In turn, Eq. (A.2) for the fluid A and its counterpart for the fluid B give for $\mathbf{x} \in \Omega$ and $\mathbf{y} \in Y_L$

$$\partial_{y_i} p^{A(0)} = 0 \qquad \mathbf{y} \in Y_A \quad \text{and} \quad \partial_{y_i} p^{B(0)} = 0 \qquad \mathbf{y} \in Y_B. \tag{A.18}$$

Hence, taking account of (A.12) we obtain

$$p^{A(0)} = p^{B(0)} \equiv p^{(0)} = p^{(0)}(\mathbf{x}, t) \qquad \text{for } \mathbf{x} \in \Omega. \tag{A.19}$$

The incompressibility equation (A.3) yields

$$\partial_{y_i} v_i^{A(0)} = 0 \qquad \mathbf{y} \in Y_A \quad \text{and} \quad \partial_{y_i} v_i^{B(0)} = 0 \qquad \mathbf{y} \in Y_B \tag{A.20}$$

what means that the fields $\mathbf{v}^{A(0)}$ and $\mathbf{v}^{B(0)}$ are divergence-free with respect to for $\mathbf{y} \in Y_A$ and $\mathbf{y} \in Y_B$, respectively.

Local function for the flow problem

From (5.24) we get by the time differentiation

$$\begin{aligned}
\dot{w}_m^{A(0)} &= \frac{1}{\varrho^A} \int_0^t \Big[\Phi_s^A(\mathbf{x}, \tau) \, \frac{d}{dt} \, \chi_{ms}^{AA}(\mathbf{y}, t-\tau) \\
&\quad + \Phi_s^B(\mathbf{x}, \tau) \, \frac{d}{dt} \, \chi_{ms}^{AB}(\mathbf{y}, t-\tau)\Big] d\tau + \dot{\beta}_m^A(\mathbf{y}, t), \\
\dot{w}_m^{B(0)} &= \frac{1}{\varrho^B} \int_0^t \Big[\Phi_s^A(\mathbf{x}, \tau) \, \frac{d}{dt} \, \chi_{ms}^{BA}(\mathbf{y}, t-\tau) \\
&\quad + \Phi_s^B(\mathbf{x}, \tau) \, \frac{d}{dt} \, \chi_{ms}^{BB}(\mathbf{y}, t-\tau)\Big] d\tau + \dot{\beta}_m^B(\mathbf{y}, t),
\end{aligned} \tag{A.21}$$

where the following zero initial conditions for local functions were used in agreement with the initial condition (5.21) for the relative velocities $\mathbf{w}^A$ and $\mathbf{w}^B$,

$$\chi^{AA}_{ms}(\mathbf{y},0) = 0, \ \chi^{AB}_{ms}(\mathbf{y},0) = 0, \ \chi^{BA}_{ms}(\mathbf{y},0) = 0, \ \chi^{BB}_{ms}(\mathbf{y},0) = 0. \tag{A.22}$$

Applying (5.24) to the interface condition (A.11) we find on Γ_{AB}

$$\begin{aligned} \chi^{AA}_{ms}(\mathbf{y},t)|_A &= \chi^{BA}_{ms}(\mathbf{y},t)|_B, \\ \chi^{AB}_{ms}(\mathbf{y},t)|_A &= \chi^{BB}_{ms}(\mathbf{y},t)|_B, \\ \beta^{A}_{m}(\mathbf{y},t)|_A &= \beta^{B}_{m}(\mathbf{y},t)|_B. \end{aligned} \tag{A.23}$$

Substituting (A.21) and (5.22) we obtain, apart from the system (5.25), the following local equations

$$\begin{aligned} &\rho^A \frac{d}{dt}\beta^A_i(\mathbf{y},t) + \partial_{y_i}\alpha^A_i(\mathbf{y},t) - \partial_{y_j}\left(\eta^A_{ijmn}\partial_{y_n}\beta^A_m\right) = 0 \qquad \mathbf{y} \in Y_A, \\ &\rho^B \frac{d}{dt}\beta^B_i(\mathbf{y},t) + \partial_{y_i}\alpha^B_i(\mathbf{y},t) - \partial_{y_j}\left(\eta^B_{ijmn}\partial_{y_n}\beta^B_m\right) = 0 \qquad \mathbf{y} \in Y_B. \end{aligned} \tag{A.24}$$

Substituting (5.24) into (A.20) with taking into account the definitions (5.19) and the propriety (A.16), we obtain

$$\begin{aligned} &\partial_{y_i}\chi^{AA}_{ik}(\mathbf{y},t) = 0, \qquad \partial_{y_i}\chi^{AB}_{ik}(\mathbf{y},t) = 0, \\ &\partial_{y_i}\chi^{BA}_{ik}(\mathbf{y},t) = 0, \qquad \partial_{y_i}\chi^{BB}_{ik}(\mathbf{y},t) = 0. \end{aligned} \tag{A.25}$$

and

$$\partial_{y_i}\beta^A_i(\mathbf{y},t) = 0, \qquad \partial_{y_i}\beta^B_i(\mathbf{y},t) = 0. \tag{A.26}$$

Finally, substitution of (5.22) and (5.24) into (A.13) yields the following conditions on the interface Γ_{AB} :

$$\begin{aligned} &\left[-\gamma^{AA}_i(\tau)\delta_{ij}\eta^A_{ijmn}\partial_{y_n}\chi^{AA}_{ms}(t-\tau)\right] n_j|_A \\ &\qquad\qquad = \left[-\gamma^{BA}_i(\tau)\,\delta_{ij}\eta^B_{ijmn}\partial_{y_n}\chi^{BA}_{ms}(t-\tau)\right] n_j|_B, \\ &\left[-\gamma^{AB}_i(\tau)\,\delta_{ij}\eta^A_{ijmn}\partial_{y_n}\chi^{AB}_{ms}(t-\tau)\right] n_j|_A \\ &\qquad\qquad = \left[-\gamma^{BB}_i(\tau)\delta_{ij}\eta^B_{ijmn}\partial_{y_n}\chi^{BB}_{ms}(t-\tau)\right] n_j|_B, \\ &\left[-\alpha^A_i(\tau)\delta_{ij}\eta^A_{ijmn}\partial_{y_n}\beta^A_m(t-\tau)\right] n_j|_A \\ &\qquad\qquad = \left[-\alpha^B_i(\tau)\delta_{ij}\eta^B_{ijmn}\partial_{y_n}\beta^B_m(t-\tau) + \sigma H_{ij}\right] n_j|_A. \end{aligned} \tag{A.27}$$

Appendix B. Scalling

According to Auriault *et al.* [6], one considers a relation Q between the pressure term ∇p and the viscosity term: $\eta \triangle \mathbf{v}$

$$Q \equiv \frac{|\nabla p|}{|\eta \triangle \mathbf{v}|}. \tag{B.1}$$

Those terms should be of the same order in the Stokes equation if the pressure and viscosity effect were meaningful in considered phenonemon

$$|\nabla p| = \mathcal{O}(|\eta \triangle \mathbf{v}|). \tag{B.2}$$

The estimation of the quotient (B. 1) may be written as

$$Q = \mathcal{O}\left(\frac{\frac{p}{L}}{\eta \frac{v}{L^2}}\right) = \mathcal{O}\left(\frac{p}{L}\frac{L^2}{\eta v}\right). \tag{B.3}$$

Since the pressure changes in the macroscopic scale, we have

$$\nabla p = \mathcal{O}\left(\frac{p}{L}\right). \tag{B.4}$$

However, the liquid velocity changes in the pore scale, thus we write

$$|\eta \triangle \mathbf{v}| = \mathcal{O}\left(\eta \frac{v}{l^2}\right). \tag{B.5}$$

Hence, instead of (B. 3) we obtain

$$\frac{p}{L} = \mathcal{O}\left(\eta \frac{v}{l^2}\right), \tag{B.6}$$

and

$$Q = \mathcal{O}\left(\frac{p}{L}\eta \frac{v}{L^2}\right) = \mathcal{O}\left(\frac{p}{L}\eta \frac{v^2}{l^2}\frac{l^2}{L^2}\right) = \mathcal{O}(\varepsilon^2). \tag{B.7}$$

The second scaling $\sigma \rightsquigarrow \varepsilon\sigma$ can be proved similarly. To this end we observe that now of the l.h.s of $(4.2)_2$, only the first derivative is present.

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References

1. R. A. ADAMS, *Sobolev spaces*, Academic Press, New York 1975.

2. P. M. ADLER, J.-F. THOVERT, *Real porous media: local geometry and macroscopic properties*, Appl. Mech. Reviews, **51**, 537-585, 1998.

3. G. ALLAIRE, *Homogenization of the unsteady Stokes equations in porous media*, [in]: Progress in partial differential equations - calculus of variations, applications, C. BANDLE, J. BEMELMANS, M. CHIPOT, J. SAINT JEAN PAULIN[Eds.], pp.109–123, Longman 1991.

4. G. ALLAIRE, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., **23**, 1482–1518, 1992.

5. G. ALLAIRE, S. KOKH, *Simulating interfaces in two-phase flows, Preprint,* Laboratoire d'Analyse Numérique, Université Paris VI, 2000.

6. J.-L. AURIAULT, O. LEBAIGUE, G. BONNET, *Dynamics of two immiscible fluids flowing through deformable porous media*, Transp. Porous Media, **4**, 105–128, 1989.

7. J.-L. AURIAULT, E. SANCHEZ-PALENCIA, *Étude du comportement macroscopique d'un milieu poreux saturé déformable*, J. de Mécanique, **16**, 575–603, 1977.

8. J. L. AURIAULT, E. SANCHEZ-PALENCIA, *Remarques sur la loi de Darcy pour les écoulements biphasiques en milieu poreux*, J. de Méc. Théorique et Appl., (Special Issue), pp. 141–156, 1986.

9. A. BADEA, A. BOURGEAT, *Homogenization of two-phase flow through randomly heterogeneous porous media*, : Mathematical modelling of flow through porous media, A.P. BOURGEAT, C. CARASSO, S. LUCKHAUS, A. MIKELIĆ [Eds.], pp. 44-58, World Scientific, Singapore 1995.

10. J. BEAR, Y. BACHMAT, *Introduction to modelling of transport phenomena in porous media,* Kluwer Academic Publishers, Dordrecht 1991.

11. L. S. BENNETHUM, T. GIORGI, *Generalized Forchheimer equation for two-phase flow based on hybrid mixture thery*, Transp. Porous Media, **26**, 261-275, 1997.

12. Y. BERNABÉ, *On the measurement of permeability in anisotropic rocks*, in: Fault mechanics and transport properties of rocks, B. EVANS and T.-F. WONG [Eds.], pp. 147-167, Academic Press, London 1992.

13. W. BIELSKI, J. J. TELEGA, *Effective properties of geomaterials: rocks and porous media,* Publications of the Inst. of Geophys. Pol. Ac. Sci., **A – 26(285)**, Warszawa 1997.

14. W. R. BIELSKI, J. J. TELEGA, R. WOJNAR, *Macroscopic equations for nonstationary flow of Stokesian fluid through porous elastic skeleton*, Arch. Mech., **51**, 243–274, 1999.

15. W. R. BIELSKI, J. J. TELEGA, R. WOJNAR, *Nonstationary flow of a viscous fluid through a porous elastic medium: asymptotic analysis and two-scale convergence*, Mech. Res. Comm., **26**, 619–628, 1999.

16. W. R. BIELSKI, J. J. TELEGA, R. WOJNAR, *Nonstationary flow of diphasic viscous fluid through porous deformable medium*, Comput. Geosci., **26** [in press].

17. S. C. BLAIR, J. G. BERRYMAN, *Permeability and relative permeability in rocks*, in: Fault mechanics and transport properties of rocks, B. EVANS and T.-F. WONG [Ed.], pp.169–186, Academic Press, London 19.

18. R. DE BOER *Highlights in the historical development of the porous media theory: toward a consistent macroscopic theory*, Appl. Mech. Reviews, **49**, 201-262, 1996.

19. A. P. BOURGEAT, C. CARASSO, S. LUCKHAUS, A. MIKELIĆ [Eds.], *Mathematical modelling of flow through porous media,* World Scientific, Singapore 1995.

20. A. BOURGEAT, *Homogenized behavior of two-phase flows in naturally fractured reservoirs with uniform fractures distribution*, Comput. Meth. Appl. Mech. Eng., **47**, 205-216, 1984.

21. A. BOURGEAT *Two-phase flow, in: Homogenization and porous media,* U. HORNUNG [Ed]. pp. 95–127, Springer-Verlag, New York 1997.

22. A. BOURGEAT, M. PANFILOV *Effective two-phase flow through highly heterogeneous media: capillary nonequilbrium effects*, Comput. Geosci., **2**, 191-215, 1998.

23. A. BOURGEAT, A. HIDAMI, *Effective model of two-phase flow in a porous medium made of different rock types*, Applicable Anal., **58**, 1-29, 1995.

24. A. BOURGEAT, A. MIKELIĆ, S. WRIGHT, *Stochastic two-scale convergence in the mean and applications*, J. Reine Angew. Math., **456**, 19-51, 1994.

25. A. BOURGEAT, S. M. KOZLOV, A. MIKELIĆ, *Effective equations of two-phase flow in random media*, Calc. Variations, **3**, 385-406, 1995.

26. A. BOURGEAT, S. LUCKHAUS, A. MIKELIĆ, *Convergence of the homogenization process for a double-porosity model of immiscible two-phase flow,* SIAM J. Math. Anal., **27**, 1520-1543, 1996.

27. A. BOURGEAT, A. MIKELIĆ, *Homogenization of two-phase immiscible flow in one-dimensional porous medium*, Asymptotic Anal., **9**, 359-380, 1994.

28. H. BRENNER, D. A. EDWARDS, *Macrotransport processes,* Butterworth-Heinemann, Boston 1993.

29. R. BURRIDGE, J. B. KELLER, *Poroelasticity derived equations derived from microstructure*, J. Acoust. Soc. Am., **70**, 1140–1146, 1981.

30. G. CHAVENT, J. JAFFRÉ, S. JAN-JÉGOU, *Estimation of relative permeabilities in three-phase flow in porous media*, Inverse Probl., **15**, 33-39, 1999.

31. G. CHRISTAKOS, D. T. HRISTOPULOS, X. LI, *Multiphase flow in heterogeneous media from a stochastic differential geometry viewpoint*, Water Resources Res., **34**, 93-102, 1998.

32. G. N. CONSTANTINIDES, A. C. PAYATAKES, *A three-dimensional network model for consolidated porous media. Basic studies*, Chem. Eng. Comm., **81**, 55-81, 1989.

33. G. N. CONSTANTINIDES, A. C. PAYATAKES, *Network simulation of steady-state two-phase flow in consolidated porous media*, AIChE J., **42**, 369-382, 1996.

34. G. N. CONSTANTINIDES, A. C. PAYATAKES, *Effects of precursor wetting films in immiscible displacement through porous media*, Transp. Porous Media, **38**, 291-317, 2000.

35. O. COUSSY, *Mécanique des milieux poreux,* Editions Technip, Paris 1991.

36. J. M. CROLET, M. EL HATRI [Eds.], *Recent advances in problems of flow and transport in porous media,* Kluwer Academic Publishers, Dordrecht 1998.

37. J. H. CUSHMAN, *The physics of fluids in hierarchical porous media: angstroms to miles,* Kluwer Academic Publishers, Dordrecht 1997.

38. M. DALE, *Averaging of relative permeability in composite cores*, [in:] The mathematics of oil recovery [Ed.], P. R. KING, pp. 649-683, Clarendon Press, Oxford 1992.

39. M. DALE, *Darcy equations in heterogeneous porous media: interface conditions and uniqueness of periodic solutions*, Working paper from Stavanger College, School of Science and Technology, 43/1998.

40. J. T. DARTLEY, D. W. RUTH, *Relative permeabilility analysis of tube bundle models*, Transp. Porous Media, **36**, 161-187, 1999.

41. F. A. L. DULLIEN, *Porous media: fluid transport and pore structure*, Academic Press, San Diego 1992.

42. S. EKRANN, M. DALE, *Averaging of relative permeability in heterogeneous reservoirs*, in: The mathematics of oil recovery, P. R. KING, pp. 173-198, Clarendon Press, Oxford 1992.

43. EKRANN, M. DALE, K. LANGAAS, J. MYKKELTVEIT, *Capilary limit effective two-phase properties for 3D media*, SPE Int., Proc. European 3-D Reservoir Modelling Conf., Stavanger, Norway, 16-17 April 1996, pp. 119-129.

44. H. I. ENE, D. POLIŠEVSKI, *Thermal flow in porous media*, D. Reidel Publishing Company, Dordrecht 1987.

45. M. FIRDAOUSS, J.-L. GUERMOND, P. LE QUÉRE, *Nonlinear corrections to Darcy's law at low Reynolds numbers*, J. Fluid Mech., **343**, 331-350, 1997

46. G. FRAS, J. C. BENET, *Physical approach to averaging theorems on phase interfaces in a dispersed multiphase medium*, Transp. Porous Media, **15**, 209-227, 1994.

47. V. GANESAN, H. BRENNER, *A diffuse interface model of two-phase flow in porous media*, Proc. R. Soc. London, A **456**, 731-803, 2000.

48. W. G. GRAY, *Elements of a systematic procedure for the derivation of macroscale conservation equations for multi-phase flow in porous media*,in: Kinetic and continuum theories of granular and porous media,[Eds.] K. HUTTER AND K. WILMAŃSKI, pp. 67-129, Springer-Verlag, Wien-New York 1999.

49. W. G. GRAY, S. M. HASSANIZADEH, *Averaging theorems and averaged equations for transport of interface properties in multiphase systems*, Int. J. Multiphase Flow, **15**, 81-95, 1989.

50. W. G. GRAY, S. M. HASSANIZADEH, *Paradoxes and realities in unsaturated flow theory*, Water Resources Res., **27**, 1847-1854, 1991.

51. W. G. GRAY, S. M. HASSANIZADEH, *Unsaturated flow theory including interfacial phenomena*, Water Resources Res., **27**, 1855-1863, 1991.

52. W. G. GRAY, S. M. HASSANIZADEH, *Macroscale continuum mechanics for multiphase porous media flow including phases, interfaces, common lines, and common points*, Adv. Water Resources, **21**, 261-281, 1998.

53. T. HARTER, T. C. J. YEH, *Flow in unsaturated random porous media, nonlinear numerical analysis and comparison to analytical stochastic models*, Adv. Water Resources, **22**, 257-272, 1998.

54. S. M. HASSANIZADEH, W. G. GRAY, *Recent advances in theories of two-phase flow in porous media*, [in:] Fluid transport in porous media, P. DU PLESSIS[Ed.], pp.105–160, Computational Mechanics Publications, Southampton 1997.

55. J.G. HEYWOOD, *On convergence to steady state of solutions of parabolic equations in unbounded domains*, J. Diff. Eqs., **20**, 336–355, 1978.

56. U. HORNUNG [Ed.], *Homogenization and porous media*, Springer-Verlag, New York, 1997.

57. S. JEMIOŁO, J.J. TELEGA, *Modelling hyperelastic behaviour of soft tissues, Part I. Isotropy*, Eng. Trans., 2001, in press.

58. M. KAVIANY, *Principles of heat transfer in porous media*, Springer-Verlag, New York 1991.

59. M. KRAFCZYK, M. SCHULZ, E. RANK, *Lattice-gas simulations of two-phase flow in porous media*, Comm. Numer. Meth. Eng., **14**, 709-717, 1998.

60. J. KUBIK, M. CIESZKO, M. KACZMAREK, *Foundations of dynamics of saturated porous media*,in Polish,Biblioteka Mechaniki Stosowanej, Instytut Podstawowych Problemów Techniki PAN, ZTUREK, Warszawa 2000.

61. V.G. LEVICH, V. S. KRYLOV, *Surface-tension-driven phenomena*, Ann. Rev. Fluid Mech., **1**, 293–316, 1969.

62. R. W. LEWIS, B. A. SCHREFLER, *The finite element method in the static and dynamic deformation and consolidation of porous media*, John Wiley&Sons, Chichester 1998.

63. E. MARUŠIĆ-PALOKA, A. MIKELIĆ, *A derivation of a nonlinear filtration law including the inertia effects via homogenization*, Nonlinear Anal., **42**, 97-137, 2000.

64. A. MIKELIĆ, *A convergence theorem for homogenization of two-phase miscible flow through fractured reservoirs with uniform structure distribution*, Applicable Anal., **32**, 203-214, 1989.

65. A. MIKELIĆ, L. PAOLI, *On the derivation of the Buckley-Leverett model from the two fluid Navier-Stokes equation in a thin domain*, Comput. Geosci., **1**, 59-83, 1997.

66. C. T. MILLER, G. CHRISTAKOS, P. T. IMHOFF, J. F. C. MCBRIDE, J. A. PEDIT, J. A. TRANGENSTEIN, *Multiphase flow and transport modeling in heterogeneous porous media: challenges and approaches*, Advances in Water Resources, **21**, 77-120, 1998.

67. C. D. MONTEMAGNO, W. G. GRAY, *Photoluminescent volumetric imaging: A technique for the exploration of multiphase flow and transport in porous media*, Geoph. Res. Letters, **22**, 425-428, 1995.

68. L. W. MORLAND, *A simple constitutive theory for fluid-saturated porous solid*, J. Geoph. Res., **77**, 890–900, 1972.

69. G. NGUETSUNG, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., **20**, 608–623, 1989.

70. G. NGUETSUNG, *Asymptotic analysis for a stiff variational problem arising in mechanics*, SIAM J. Math. Anal., **21**, 1394–1414, 1990.

71. M. PANFILOV, *Macroscale models of flow through highly heterogeneous porous media*, Kluwer Academic Publishers, Dordrecht 2000.

72. P. DU PLESSIS[Ed.], *Fluid transport in porous media*, Computational Mechanics Publications, Southampton-Boston 1997.

73. M. RUDMAN, *One-field equations for two-phase flows*, J. Austral. Math. Soc., B**39**, 149-170, 1997.

74. J. SAINT JEAN PAULIN, K. TAOUS, *A generalized Darcy law. Homogenization of a diphasic flow problem in a porous medium*, Ric. Mat., **40**, 1991, 223–241.

75. E. SANCHEZ-PALENCIA, *Non-homogeneous media and vibration theory*, Springer, Berlin 1980.

76. M. SAHIMI, *Flow and transport in porous media and fractured rock: from classical methods to modern approaches*, VCH, Weinheim - New York, 1995.

77. B. R. SIMON, *Multiphase poroelastic finite element models for soft tissue structures*, Appl. Mech. Reviews, **45**, 191-218, 1992.

78. L. THIGGEN, J. G. BERRYMAN, *Mechanics of porous elastic materials containing multiphase fluid*, Int. J. Eng. Sci., **23**, 1203-1214, 1985.

79. G. C. TZIMAS, T. MATSUNRA, D. G. AVRAAM, W. VAN DER BRUGGHEN, G. N. CONSTANTINIDES, A. C. PAYATAKES, *The combined effect of the viscosity ratio and the wettability during forced imbibition through nonplanar porous media*, J. Colloid Interf. Sci., **189**, 27-36, 1997.

80. M. S. VALAVANIDES, G. N. CONSTANTINIDES, A. C. PAYATAKES, *Mechanistic model of steady-state two-phase flow in porous media based on ganglion dynamics*, Transp. Porous Media, **30**, 267-299, 1998.

81. S. WHITAKER, *The method of volume averaging*, Kluwer Academic Publishers, Dordrecht 1999.

82. L. P. YARIN, G. HETSRONI, *Turbulence intensity in dilute two-phase flows, - Part 1. Effect of particle-size distribution on the turbulence of the carrier fluid*, Int. J. Multiphase Flow, **20**, 1-15, 1994; *Part 2. Temperature fluctuations in particle-laden dilute flows, ibid.*, 17-25; *Part 3. ibid., 27-44.*

83. F. ZANOTTI, R. G. CARBONELL, *Development of transport equations for multiphase systems - I. General developments for two phase systems, Chem. Eng. Sci.,* **39**, *263-278, 1984; II. Application to one-dimensional axi-symmetric flows of two phases, ibid., 279-297; III. Application to heat transfer in packed beds, ibid., 299-311.*

84. D. Z. ZHANG, A. PROSPERETTI, *Averaged equations for inviscid disperse two-phase flow, J. Fluid Mech.,* **267**, *185-219, 1994.*

85. O. C. ZIENKIEWICZ, A. H. C. CHAN, M. PASTOR, B. A. SCHREFLER, T. SHIOMI, *Computational geomechanics with special reference to earthquake engineering, John Wiley&Sons, Chichester 1999.*

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