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Constitutive relations for dynamic material instability at finite deformation

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THIS PAPER AIMS to present a mathematically consistent formulation of the second gradient dependence in the constitutive equations for material instability phenomena in case of finite deformations. Thus the set of fundamental equations of the solid continuum (the kinematic equations, the Cauchy equations of motion and the constitutive equations) should also be written for finite deformations. Two basic properties are required: the existence and regular propagation of waves and the generic behavior at the loss of stability. Firstly, the wave dynamics is studied. To encounter the second gradient effects, we should use the third order waves here. Secondly, the system of fundamental equations completed with initial and boundary value conditions forms a dynamical system. Then, identifying material stability with Lapunov stability of a state of the continuus body, the loss of stability should be one of the two basic types of instabilities of dynamical systems: a static or a dynamic bifurcation. These instability modes should be strictly different for a generic dynamical system.

1. Introduction

IN THE RECENT YEARS, the study of material instability problems received an increasing interest [8, 13]. However, most of the investigations published dealt with small deformations and static or quasi-static loading conditions. To perform works encounter also dynamic effects with high rate loadings we need appropriate constitutive equations. Such materials were studied by postulating the existence of a (second order) acceleration wave with finite wave speed [2].

Unfortunately, the constitutive theory based on the second order waves cannot include such cases of non-locality as the so-called second gradient materials [16] being widely used for numerical investigations of post-localization. The effect of inclusion of second gradient terms and the difficulties in dynamic studies can be described by applying the theory of dynamical systems [5, 6]. Dynamical systems are widely used for (Lapunov) stability investigations of various cases. Quite recently we obtained results for the forms of possible constitutive relations by combining the acceleration wave dynamics and the theory of dynamical systems [4].

The aim of the paper is to find constitutive formulation for non-local material instability problems. The definition of material stability/instability is based on the Lapunov stability concept of the theory of dynamical systems (see [5] for the details) for this reason we call it "dynamic material instability". We assume for the solid body that a generalized wave exists in the derivative of the acceleration field [7] and this singular surface propagates forwards and backwards with finite velocities. From that assumption, the conditions are obtained for the second order derivatives of the variables of the constitutive equations. Additionally we assume that the loss of stability should be a generic [1] one in terms of the theory of dynamical systems [15], which is essential in dealing with instability problems. There are two main points at this topic. The one is quite practical: a numerical solution of the material instability problems in non-generic case may suffer serious technical difficulties (loss of convergence, mesh sensitivity [8] etc.). The other is of theoretical significance. By modeling physical phenomena we should obtain a set of equations which is typical (or generic), that is, differs only a little from the "exact unknown mathematical model". This modeling concept is treated in details by FARKAS [10].

All the studies are performed for finite deformations. The resulting constitutive equations are suitable for solving material stability/instability problems with large deformations.

The second section presents the set of fundamental equations of the solid continuum at large deformation. It consists of the Cauchy equtions of motion, the kinematic equation (for large displacements) and the constitutive equations. Constitutive formulation should satisfy the so-called Axiom of Objectivity, that is, should be form-invariant under arbitrary rigid motions of the spatial frame of reference and a constant shift of the origin in time. As a special case, this requirement includes the invariance under Galilean transformation (see for example [9] or [12]). Such physically objective quantities are the Lie derivative of the stress gradient tensor, the Lie derivative of the (Euler) strain gradient tensor and the second covariant derivative of the stress and strain tensors.

In the next section, a dynamical wave study of constitutive equations is applied based on the existence of a third order generalized wave with regular propagation properties. Here the wave speed equation [2] derived from the fundamental equations implies conditions (Wave Dynamical Condition, WDC) for the existence of the required waves.

In the fourth section, as an application of wave dynamical theory, we perform a material instability investigation for finite displacements with an appropriate constitutive equation in uniaxial case. In this section, the wave speed equation is a scalar third order algebraic equation and should have real nonzero solutions

[2]. By using dynamical systems theory we should be able to study a generic behavior (as it is defined in the theory of dynamical systems [5]) at the loss of stability because of the aforementioned general modeling concept of physical phenomena. Thus we obtain additional conditions. There are two different ways for the loss of stability of a dynamical system [15]. These are the so-called static and dynamic bifurcations and should be strictly different phenomena (Dynamical Systems Condition, DSC). Thus the system of fundamental equations (especially the constitutive equation) should meet both the WDC and DSC requirements.

2. The basic equations for large deformation

If we would like to have constitutive equations containing the second (physically objective) derivatives of the stress and strain tensors, we should need besides the conventional equations of motion

$$t^{kp}_{;p} + q^k = \rho \dot{v}^k, \quad t^{kp} = t^{pk}, \tag{2.1}$$

also the equation of motion for the Lie derivative of the stress tensor [3, 7]

$$\left(L_v t^{kp}\right)_{;p} + \left(t^{qp} v^k_{;q} + t^{kp} v^s_{;s}\right)_{;p} + \dot{q}^k - q^s v^k_{;s} = \rho\left(\ddot{v}^k - \dot{v}^s v^k_{;s} - \dot{v}^k v^s_{;s}\right), \tag{2.2}$$

where

$$L_v t^{kp} \equiv \dot{t}^{kp} - t^{ks} v^p_{;s} - t^{ps} v^k_{;s}$$

denotes the Lie derivative of the stress tensor. Here, in (2.1), (2.2) and in all further equations and expressions, Roman indices run from 1 to 3.

For finite deformation

$$v_{ij} = L_v\left(a_{ij}\right), \tag{2.3}$$

where

$$L_v a_{ij} \equiv \dot{a}_{ij} + a_{ip} v^p_{;j} + a_{pj} v^p_{;i}.$$

Then the kinematic equation is written for the Lie derivative of the deformation rate tensor

$$\begin{aligned} L_v\left(v_{ij}\right) = \ddot{a}_{ij} + a_{ik}\dot{v}^k_{;j} + a_{kj}\dot{v}^k_{;i} + 2\dot{a}_{ik} v^k_{;j} \\ + 2\dot{a}_{kj} v^k_{;i} + a_{ik} v^k_{;\ell} v^\ell_{;j} + a_{kj} v^k_{;\ell} v^\ell_{;i} + 2a_{\ell k} v^k_{;j} v^\ell_{;i} \end{aligned} \tag{2.4}$$

The notations are: q^k denotes the body force, ρ is the mass density, $X^K_{,p}$ is the deformation gradient, g_{pq}, G_{KL} are metric tensors in the current and the initial configurations, v^i and $v^i_{;j}$ are velocity and velocity gradient, v_{ij} is the deformation rate tensor. Cauchy stress tensor is denoted by t^{pk}, and

$$a_{ik} = \frac{1}{2}\left(g_{ik} - X^K_{,i} X^L_{,k} G_{KL}\right)$$

denotes the Euler strain tensor, respectively. Semicolon means covariant derivative and an overdot indicates material time derivative:

$$\dot{v}^i = \frac{\partial v^i}{\partial \tau} + v^k v^i_{;k},$$

where τ denotes time. Remark that brackets used at the Lie derivative preserve upper and lower indices as in (2.3); we use them to show clearly for which variable it is applied. (For example $L_v\left(t^{kp}{}_{;\ell}\right)$ is the Lie derivative of the covariant derivative of the stress tensor t^{kp} and not the covariant derivative of the Lie derivative.) Assume that the constitutive equation has the form

$$(2.5) \qquad f_\alpha\left(L_v\left(t^{kp}{}_{;\ell}\right), L_v\left(a_{ij;\ell}\right), t^{kp}{}_{;\ell m}, a_{ij;\ell m}\right) = 0,$$

where $\alpha = 1, 2, \ldots, 6$. We use physically objective quantities such as

- the Lie derivative of the stress gradient tensor

$$L_v\left(t^{kp}{}_{;\ell}\right) = \left(t^{kp}{}_{;\ell}\right)^{\cdot} - t^{qp}{}_{;\ell} v^k_{;q} - t^{kq}{}_{;\ell} v^p_{;q} + t^{kp}{}_{;q} v^q_{;\ell},$$

- the Lie derivative of the (Euler) strain gradient tensor,

$$L_v\left(a_{ij;k}\right) = \left(a_{ij;k}\right)^{\cdot} + a_{qj;k} v^q_{;i} + a_{iq;k} v^q_{;j} + a_{ij;q} v^q_{;k},$$

- the second covariant derivative of the stress tensor $t^{kp}{}_{;\ell m}$,
- the second covariant derivative of the strain $a_{ij;\ell m}$.

In the set of equations (2.2), (2.4) and (2.5), the number of scalar variables and equations are the same thus it can be considered the set of fundamental equations. Remark that the continuity equation for ρ can also be introduced, but it is not necessary for the following calculations. Moreover,the dissipation inequality should also be satisfied

$$(2.6) \qquad \dot{s} = {}_D t^{ij} L_v\left(a_{ij}\right) > 0,$$

where s denotes entropy and ${}_D t^{ij}$ is the dissipative part of the stress tensor. Introducing also the reversible part ${}_E t^{ij}$ we have

$$t^{ij} = {}_E t^{ij} + {}_D t^{ij},$$

and by introducing the elasticity tensor $C^{ijk\ell}$

$$_D t^{ij} = t^{ij} - C^{ijk\ell} a_{k\ell} .$$

Thus (2.6) implies

$$t^{ij} L_v (a_{ij}) - C^{ijk\ell} a_{k\ell} L_v (a_{ij}) > 0. \tag{2.7}$$

3. A wave dynamical theory of constitutive equations

As promised in the Introduction, assume that there are jumps in the second derivatives of stress and strain and in the third derivative of acceleration fields along surface $\varphi(x^i, \tau) = 0$, that is,

$$\begin{aligned}
\left[\left(t^{kp}{}_{;\ell}\right)^{\cdot}\right] &= \gamma^{kp}(\varphi_4 + \nu^n \varphi_n)\varphi_\ell,\\
[(a_{ij;k})^{\cdot}] &= \alpha_{ij}(\varphi_4 + \nu^n \varphi_n)\varphi_k,\\
\left[\left(v^k{}_{;q}\right)^{\cdot}\right] &= -\nu^k(\varphi_4 + \nu^n \varphi_n)\varphi_q,\\
\left[\ddot{v}^k\right] &= \nu^k(\varphi_4 + \nu^n \varphi_n)^2,
\end{aligned} \tag{3.1}$$

where jumps are denoted by [ ] and $\varphi_k = \frac{\partial \varphi}{\partial x^k}$ and $\varphi_4 = \frac{\partial \varphi}{\partial \tau}$. There are no jumps in other derivatives. The dynamic, kinematic and constitutive compatibility conditions can be obained by using (3. 1).

The wave speed can be expressed as

$$c = -\frac{\varphi_4 + \nu^n \varphi_n}{\sqrt{g^{k\ell}\varphi_k \varphi_\ell}},$$

and the unit normal vector of the wave front reads $n_p = \frac{\varphi_p}{\sqrt{g^{k\ell}\varphi_k\varphi_\ell}}$.
The set of the dynamical compatibility conditions are

$$\begin{aligned}
\left[\left(L_v t^{kp}\right)_{;p}\right] + \left[\left(t^{qp} v^k_{;q} + t^{kp} v^s_{;s}\right)_{;p}\right] + \left[\dot{q}^k + q^k v^s_{;s} - q^s v^k_{;s}\right]&\\
= \left[\rho\left(\ddot{v}^k - \dot{v}^s v^k_{;s}\right)\right]&,
\end{aligned} \tag{3.2}$$

that is

$$\left[\left(L_v t^{kp}\right)_{;p}\right] + t^{qp}\left[v^k_{;qp}\right] + t^{kp}\left[v^s_{;sp}\right] = \rho\left[\ddot{v}^k\right] \tag{3.3}$$

the kinematic compatibility conditions are (for large displacements)

$$(3.4)\quad [L_v(v_{ij})] = [\ddot{a}_{ij}] + \left[a_{ik}\dot{v}^k_{;j}\right] + \left[a_{kj}\dot{v}^k_{;i}\right] + \left[2\dot{a}_{ik}v^k_{;j} + 2\dot{a}_{kj}v^k_{;i} + a_{ik}v^k_{;\ell}v^\ell_{;j} + a_{kj}v^k_{;\ell}v^\ell_{;i} + a_{\ell k}v^k_{;j}v^\ell_{;i}\right],$$

that is

$$(3.5)\qquad [\dot{v}_{ij}] = [\ddot{a}_{ij}] + a_{ik}\left[\dot{v}^k_{;j}\right] + a_{kj}\left[\dot{v}^k_{;i}\right].$$

The constitutive compatibility conditions are

$$(3.6)\quad f_\alpha\left(\mathring{L}_v\left({t^{kp}}_{;\ell}\right) + \left[L_v\left({t^{kp}}_{;\ell}\right)\right], \mathring{L}_v(a_{ij;\ell}) + [L_v(a_{ij;\ell})], {\mathring{t}^{kp}}_{;\ell m} + \left[{t^{kp}}_{;\ell m}\right], \mathring{a}_{ij;\ell m} + [a_{ij;\ell m}]\right) - \mathring{f}_\alpha = 0,$$

where circle over a symbol denotes its value in front of the surface $\varphi(x^i,\tau) = 0$. For example, $\mathring{f}_\alpha$ is the value of function f in front of the surface $\varphi(x^i,\tau) = 0$.

By using the set of equations (3. 1), (3. 2), (3. 4) and (3. 6), we can introduce kinematic, dynamic and constitutive compatibility conditions for such generalized waves [2]. These conditions lead to the wave propagation equation

$$(3.7)\qquad \frac{\partial f_\alpha}{\partial\varphi_4}\varphi_4 + \frac{\partial f_\alpha}{\partial\varphi_r}\varphi_r = 0.$$

Introducing notations

$$T_{\alpha kp}{}^{\ell} \equiv \frac{\partial f_\alpha}{\partial L_v\left({t^{kp}}_{;\ell}\right)},\quad A^{ijk}_\alpha \equiv \frac{\partial f_\alpha}{\partial L_v(a_{ij;k})},$$

$$S_{\alpha kp}{}^{ms} \equiv \frac{\partial f_\alpha}{\partial {t^{kp}}_{;ms}},\quad E^{ijhz}_\alpha \equiv \frac{\partial f_\alpha}{\partial a_{ij;hz}},$$

equation (3. 7) takes the form

$$(3.8)\quad \Big\{2\rho T_{\alpha kp}{}^{\ell} n_\ell c^3 - 2\rho S_{\alpha kp}{}^{ms} n_m n_s c^2 - A^{ijr}_\alpha n_r n_p\left((2a_{ik} - g_{ik})n_j + (2a_{kj} - g_{kj})n_i\right)c + E^{ijhz}_\alpha n_h n_z n_p\left((2a_{ik} - g_{ik})n_j + (2a_{kj} - g_{kj})n_i\right)\Big\}\gamma^{kp} = 0.$$

For $\gamma^{kp} \neq 0$, Eq. (3.8) should have a solution, thus the polynomial of matrices in brackets {...} should have a zero determinant

$$\det\{\ldots\} = 0. \tag{3.9}$$

Equation (3.9) is called the wave speed equation which implies Wave Dynamical Conditions (WDC) for the existence of the required waves (for details see [2]). These conditions should be added to (2.7).

By using the dynamical systems theory and assuming a generic behavior (as it is defined in the theory of dynamical systems [5]), at the loss of stability we obtain additional Dynamical Systems Condition (DSC). Thus in studying the system of fundamental equations, we should consider (2.7) and both the WDC and DSC. The next part shows how DSC can be formulated in a uniaxial case.

4. Material instability and dynamical systems

Now we perform a material instability investigation of state S^0 of the solid body by considering finite displacements in uniaxial case with an appropriate constitutive equation

$$L_v(t_{,x}) + K_1 L_v(a_{,x}) + K_2 t_{,xx} + K_3 a_{,xx} = 0, \tag{4.1}$$

where partial derivatives of a function g are denoted by $g_{,x} = \frac{\partial g}{\partial x}$, or $g_{,\tau} = \frac{\partial g}{\partial \tau}$ and coefficients K_1, K_2, K_3 are considered to be piecewise constant. Now at S^0, Eq. (2.7) has the form

$$(t_0 - Ea_0)\frac{\partial a_0}{\partial \tau} > 0, \tag{4.2}$$

where E denotes the Young modulus, as usual. Then WDC means that the scalar third-order algebraic equation

$$\rho c^3 - \rho K_2 c^2 - K_1(2a-1)c + K_3(2a-1) = 0 \tag{4.3}$$

should have real nonzero solutions ([2]). Assume that S^0 is described by values a_0, t_0, v_0 of the field variables. Then such values should satisfy the system of fundamental equations formed by (4.1) and the uniaxial forms of (3.2) and (3.4)

$$\dot{v} = \frac{1}{\rho}t_{,x}, \quad \dot{a} = v_{,x} - 2av_{,x}. \tag{4.4}$$

Lapunov stability investigates the response of a mechanical system for sufficiently small perturbations, thus the perturbed quantities $a_0+\Delta a, t_0+\Delta t, v_0+\Delta v$ should be substituted into (4.1) and (4.4). Having done the necessary calculations and by linearizing the set of equations at S^0, a system of differential equations is obtained for the perturbations

$$\begin{aligned} v_{,\tau\tau} &= C_1 v + C_2 a_{,x} + C_3 a + C_4 v_{,x} + C_5 v_{,xx} + C_6 a_{,xx} + C_7 v_{,x\tau}, \\ a_{,\tau} &= D_1 v + D_2 a_{,x} + D_3 a + D_4 v_{,x}, \end{aligned} \tag{4.5}$$

where Δ is omitted for the sake of simplicity, and the following notations are used:

$$C_1 = -2v_{0,x\tau} - 2v_{0,xx}v_0, C_2 = \frac{2K_1}{\rho}v_{0,x},$$

$$C_3 = \frac{2K_1}{\rho}v_{0,xx}, C_4 = \frac{2K_1}{\rho}a_{0,x} - K_2,$$

$$C_5 = v_0^2 - \frac{K_1}{\rho} + \frac{2K_1}{\rho}a_0, C_6 = -\frac{K_3}{\rho}, C_7 = 2v_0,$$

$$D_1 = -a_{0,x}, D_2 = -v_0, D_3 = -2v_{0,x}, D_4 = -2a_0 + 1.$$

Introducing new variables $y_1 = a, y_2 = v, y_3 = v_{,\tau}$ and vector $y = [y_1, y_2, y_3]$, a dynamical system can be added to (4.5) [5]

$$\frac{\partial}{\partial \tau} y = \begin{bmatrix} H_1 & H_2 & 0 \\ 0 & 0 & 1 \\ H_3 & H_4 & H_5 \end{bmatrix} y, \tag{4.6}$$

where operators $H_1 = D_2\frac{\partial}{\partial x} + D_3, H_2 = D_4\frac{\partial}{\partial x} + D_1, H_3 = C_6\frac{\partial^2}{\partial x^2} + C_2\frac{\partial}{\partial x} + C_3, H_4 = C_5\frac{\partial^2}{\partial x^2} + C_4\frac{\partial}{\partial x} + C_1, H_5 = C_7\frac{\partial}{\partial x}$.

The characteristic equation of (4.6) reads

$$\lambda y = \begin{bmatrix} H_1 & H_2 & 0 \\ 0 & 0 & 1 \\ H_3 & H_4 & H_5 \end{bmatrix} y. \tag{4.7}$$

and the linear Lapunov stability condition of state S^0 is: $\operatorname{Re}\lambda \le 0$ for all eigenvalues of (4.7). Stability boundary is at $\operatorname{Re}\lambda = 0$. The loss of stability can be classified as a static bifurcation (or divergence) type instability ($\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda = 0$), or a dynamic one ($\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda \neq 0$) [5]. Determination of the eigenvalues of (4.7) requires the solution of a boundary value problem, which may cause serious difficulties and needs numerical computations.

To remain at analytic methods, we should perform simplifications: the use of small periodic perturbations. While stability is considered here as a local property of state, the small perturbation technique is quite obvious, but not the periodicity of perturbations. It is really a restriction, but used widely in engineering literature of the linear case [16]. (A detailed study on that restriction is presented in [5].) While perturbations are small, $a_\tau = v_x$ and then Eq. (4.5) can be transformed into the velocity field,

$$\begin{aligned} v_{\tau\tau\tau} &= C_1 v_\tau + C_2 v_{xx} + C_3 v_x + C_4 v_{x\tau} + C_5 v_{xx\tau} + C_6 v_{xxx} + C_7 v_{x\tau\tau} \\ v_{x\tau} &= D_1 v_\tau + D_2 v_{xx} + D_3 v_x + D_4 v_{x\tau}. \end{aligned} \tag{4.8}$$

By assuming periodic perturbations

$$v = \exp(i\omega x) \tag{4.9}$$

in a similar way as it was done in the general case with (4.7), the characteristic equation yields a set of algebraic equations

$$\begin{aligned} \lambda^3 &= C_1\lambda - C_2\omega^2 - C_5\omega^2\lambda^2, \\ 0 &= C_3 + C_4\lambda - C_6\omega^2 + C_7\lambda^2 \\ 0 &= D_1\lambda - D_2\omega^2, \\ \lambda &= D_3 + D_4\lambda, \end{aligned} \tag{4.10}$$

and the static bifurcation condition is the existence of a $\lambda = 0$ solution of (4.10). Then we obtain the following relations:

$$D_3 = 0, \iff \frac{\partial v_0}{\partial x} = 0, \tag{4.11}$$

$$D_2 = 0, \iff v_0 = 0, \tag{4.12}$$

$$C_2 = 0, \iff K_2\frac{\partial v_0}{\partial x} = 0, \tag{4.13}$$

and finally, the equations

$$C_3 = 0, \iff K_1\frac{\partial^2 v_0}{\partial x^2} = 0, \tag{4.14}$$

and

$$C_6 = 0, \iff K_3 = 0, \tag{4.15}$$

or

$$C_3 - C_6\omega^2 = 0, \iff 2K_1\frac{\partial^2 v_0}{\partial x^2} + K_3\omega^2 = 0, \tag{4.16}$$

should be satisfied. Obviously (4.11) implies (4.13) thus there is a static bifurcation if

A : (4. 11), (4. 12), (4. 14) and (4. 15), or

B : (4. 11), (4. 12) and (4. 16) are valid.

Now let us return to wave dynamics. Case **A** does not meet the WDC: there is a zero wave speed solution c of (4. 3) because of (4. 15). In the classical material instability concept [13] it means localization. On the other hand, if (4. 15) holds the constitutive equation (4. 1) has no second strain gradient-dependent term, which corresponds to the fact that there is a stationary singular surface (a localization zone of zero width). Thus we have exactly the classical result of Rice [13]. However, in case **B**, from (4. 16) we obtain

$$\omega^2 = -\frac{2K_1}{K_3}\frac{\partial^2 v_0}{\partial x^2},$$

if $\frac{2K_1}{K_3}\frac{\partial^2 v_0}{\partial x^2} < 0$. This means that there is a critical eigenfunction to the zero eigenvalue, that is, we have a critical periodic perturbation (4. 9)

$$v_{cr} = \exp\left(ix\sqrt{-\frac{2K_1}{K_3}\frac{\partial^2 v_0}{\partial x^2}}\right),$$

at which state S^0 unergoes a static bifurcation.

Let us now study the dynamic bifurcation case. Then we need for $\lambda^2 < 0$, a solution of (4. 10). The corresponding conditions are (4. 11), (4. 12) and

$$C_5 = 0 \iff v_0^2 - \frac{K_1}{\rho} + \frac{2K_1}{\rho}a_0 = 0, \tag{4.17}$$

$$C_4 = 0 \iff \frac{2K_1}{\rho}a_{0,x} - K_2 = 0, \tag{4.18}$$

$$D_1 = 0 \iff a_{0,x} = 0, \tag{4.19}$$

$$D_4 = 1 \iff a_0 = 0. \tag{4.20}$$

Then from (4. 17), (4.12) and (4. 20)

$$K_1 = 0, \tag{4.21}$$

and from (4. 18) and (4. 19)

$$K_2 = 0. \tag{4.22}$$

Moreover from the second equation of (4.10) and (4.18) with (4.12) we obtain (4.15)

$$K_3 = 0.$$

Finally, from the first equation of (4.10) substituting (4.12), (4.11) and (4.17), we have

(4.23)
$$\lambda^2 = -2\frac{\partial^2 v_0}{\partial x \partial \tau},$$

thus there is a dynamic bifurcation if conditions (4.11), (4.12), (4.15), (4.19), (4.20), (4.21), (4.22) are satisfied and

(4.24)
$$\frac{\partial^2 v_0}{\partial x \partial \tau} > 0.$$

Additionally, if (4.20) is substituted into the dissipation inequality, the condition

$$t_0 \frac{\partial a_0}{\partial \tau} > 0$$

is obtained. Unfortunately this is not a generic dynamic bifurcation. We can easily see that (4.21) implies (4.14); consequently, dynamic bifurcation is coexistent with a static bifurcation of case **A**. Moreover, if (4.15), (4.21) and (4.22) are valid, Eq. (4.3) has a zero solution $c = 0$, thus neither WDC nor DSC are satisfied.

As a summary of this section we find that costitutive equation (4.1) can only be used for the description of the static bifurcation-type loss of stability. In case **B** both WDC and DSC are valid. When we disregard WDC, even case **A** can be accepted, if at least one of the conditions (4.19), (4.20), (4.22), or (4.24) fails, because then no coexistent dynamic bifurcation is present and we may speak about a stationary discontinuity as the instability phenomenon.

5. Summary

In the nonlinear case of finite deformations, by using a second order constitutive equation of form (4.1), both **A** and **B** types of static bifurcation instability are generic in the sense of dynamical systems theory because there is no coexistent dynamic bifurcation. Moreover, we could preserve the nice property of the linear study of second strain gradient-dependent material: the dimension of the critical eigenspace at the static bifurcation type loss of stability remains finite (case **B**). This result shows that in a post-localization study even now we should use constitutive equations including second strain gradient dependence. When we neglect this term (case **A**) we cannot find a unique critical eigenfunction but a "stationary discontinuity": the jump (discontinuity surface in the higher derivatives of the field variables) stops at the conditions of instability. Now the material instability condition and the critical eigenfunctions (if they exist) are explicitly dependent on the values of the field variables at the state under consideration. Of course they do depend implicitly on the material properties (K_1, K_2, K_3) because the values of the field variables at state S^0 are determined by solving the whole set of fundamental equations. Unfortunately such constitutive equation cause a seriously ungeneric behavior at the dynamic bifurcation instability. It does not exist as a distinct type of instability, because the necessary conditions of a dynamic bifurcation are sufficient for a static one.

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