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Possible constitutive equations of micropolar solids

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RECENTLY THE MICROPOLAR continuum is used for investigation of some problems of solids. There exist several theories of micropolar continuum; We will study two types of them. The simplest one is Cosserat's continuum while the Eringen's model is more complicated. If the solid is not elastic then the constitutive equations are not generally known. In the paper, a method will be shown for the determination of possible constitutive equation in case of inelastic solids.

1. Introduction

THE MICROPOLAR SOLID MODEL is now frequently used in mechanics, for example for investigation of localization, material instability, plastic bodies, granular materials etc. The first micropolar model was the Cosserat continuum being the simplest model, which extends classical elasticity to a generalised form, supplementing the displacement vector field with the rotation vector field and the stress with the couple-stress tensor. Then the equations of geometry and motion contain more unknown functions than in the classical model. The rotation vector is not an independent vector; consequently, the antisymmetric part of the stress and the symmetric part of the couple-stress remain undetermined [1,2].

Eringen introduced a general theory of a nonlinear microelastic continuum. This theory in a special case contains the Cosserat continuum. Presently, there exist several approaches to the formulation of a micropolar continuum. Moreover, we should also deal with the constitutive equation of micropolar continuum because it exhibits not only the elastic behaviour [1,2].

We would like to investigate the constitutive relations or equations assuming that the second order wave exists in the micropolar solid. Function $\varphi(x_1, x_2, x_3, x_4)$, which describes the wave surface, satisfies a system of nonlinear partial differential equations. This system of partial differential equations results from the constitutive compatibility conditions. The system of partial differential equations has a solution if the Poisson-bracket is zero. It is the necessary condition of existence of the second order waves. The sufficient condition is that the speed of propagation of wave c cannot be infinite. The necessary and sufficient conditions yield the general restriction concerning to the constitutive equations [3,4].

The second order waves are the acceleration and spin wave. When the solid is isotropic, both longitudinal and transversal waves are possible. There are four direct and reverse waves. First of all we will investigate the constitutive equations in case of the simplest theory. We use rectangular Cartesian coordinates in the following equations.

2. Possible constitutive equations of a Cosserat continuum

2.1. General equations

The fundamental equations of Cosserat continuum are equations of motion, kinematic and constitutive equations, that is, [1]

$$t_{ji,j} + q_i = \rho \dot{\nu}_i, \tag{2.1}$$

$$m_{\ell k,\ell} + \in_{kmn} t_{mn} + \ell_k = \tilde{I} \dot{w}_k, \tag{2.2}$$

$$\dot{\gamma}_{ij} = \nu_{j,i} + \varepsilon_{jik} w_k, \tag{2.3}$$

$$\dot{\kappa}_{k\ell} = w_{\ell,k}, \tag{2.4}$$

$$f_\alpha \left(\gamma_{rs,\hat{i}}\,, t_{rs,\hat{i}}\,, \kappa_{pq,\hat{j}}\,, m_{pq,\hat{j}}\,, \gamma_{rs}\,, t_{rs}\,, \kappa_{pq}\,, m_{pq} \right) = 0, \tag{2.5}$$

$$\alpha = 1, 2, ...18\,; \;\; r, s, p, q... = 1, 2, 3, \;\; \hat{i}, \hat{j} = 1, 2, 3, 4,$$

where t_{ji}, $m_{\ell k}$ and q_i, ℓ_k are stress, couple-stress and body forces, couple body forces and ρ, $\tilde{I}$ are mass and inertia moment densities, γ_{ij} and $\kappa_{k\ell}$ the asymmetric strain and the torsion tensor and w_i is the rotation vector, ε_{kmn} is Levi-Civita's symbol, $\dot{v}_i$ is the acceleration vector. In all expressions of the paper we use index notation for the partial derivative of tensors like stress $t_{ji,j} \equiv \frac{\partial t_{ji}}{\partial x^j}$ etc. Indices with hat $\hat{i}$ are equal to 1,2,3,4, while if there are no hats, $q = 1, 2, 3$. Space coordinates are denoted by x_1, x_2, x_3 and $x_4 \equiv t$ is the time. Constitutive equations (2.5) are not generally well known, and for this reason we will look for their possible form.

2.2. Conditions of jumps on the acceleration wave surface

In case of acceleration waves, functions $t_{ji}, m_{\ell k}, \gamma_{ij}, \kappa_{k\ell}, v_i$ and w_k are continuous, however their derivatives have jumps on the wave surface.

Let the acceleration wave surface be $\varphi\,(x_1, x_2, x_3, x_4) = \text{const}$ or with $x_{\hat{i}}$ it is $\varphi\,(x_{\hat{i}}) = \text{const} \quad \left(\hat{i} = 1, 2, ...4\right)$. When, for example, acceleration $\dot{v}_i$ in front and behind the wave surface is denoted by $\dot{v}_i^-$ and $\dot{v}_i^+$, then the jump is $\dot{v}_i^+ - \dot{v}_i^- \equiv [\dot{v}_i]$. The jumps can be written as

$$[t_{ji,j}] \equiv \mu_{ji}\varphi_j\,,\ [\dot{\nu}_i] \equiv \beta_i\varphi_4\,,\ [\dot{\gamma}_{ij}] \equiv \Gamma_{ij}\varphi_4\,,\ \ \varphi_{\hat{i}} \equiv \frac{\partial\varphi}{\partial x^{\hat{i}}}\,,$$

$$[m_{\ell k,\ell}] \equiv \lambda_{\ell k}\varphi_\ell,\ [\dot{w}_k] \equiv \eta_k\varphi_4,\ [\dot{\kappa}_{k\ell}] \equiv \Omega_{k\ell}\varphi_4.$$

The coefficients of derivatives φ denote the appropriate amplitudes of the waves. Function φ and the amplitudes should satisfy three kinds of compatibility conditions. These are the dynamic compatibility conditions

$$\mu_{ji}\varphi_j = \rho\,\beta_i\,\varphi_4, \tag{2.6}$$

$$\lambda_{\ell k}\varphi_\ell = \tilde{I}\eta_k\varphi_4, \tag{2.7}$$

the kinematic compatibility conditions

$$\Gamma_{ij}\varphi_4 = \beta_j\varphi_i, \tag{2.8}$$

$$\Omega_{k\ell}\varphi_4 = \eta_\ell\varphi_k \tag{2.9}$$

and

$$f_\alpha\left(\gamma^\circ_{r,s,\hat{i}} + \Gamma_{rs}\varphi_{\hat{i}}\,,\ t^\circ_{rs,\hat{i}} + \mu_{rs}\varphi_{\hat{i}}\,, ...\right) - f^\circ_\alpha = 0 \tag{2.10}$$

$(\alpha = 1, 2, 3..., 18)$ are the constitutive compatibility conditions [3].

2.3. Constitutive equations

2.3.1. Possible constitutive equation when the stresses and couple stresses depend on the strain and torsion. Now the constitutive equations are

$$t_{ji} = f_{ji}\left(\gamma_{rs}\,,\ \kappa_{pq}\right),$$

$$m_{\ell k} = g_{\ell k}\left(\gamma_{rs}\,,\ \kappa_{pq}\right).$$

Let us form the time-derivatives of these equations. We obtain

$$(2.11)\qquad \dot t_{ji} = \frac{\partial f_{ji}}{\partial \gamma_{rs}}\dot\gamma_{rs} + \frac{\partial f_{ji}}{\partial \kappa_{pq}}\dot\kappa_{pq} \equiv A_{jirs}\dot\gamma_{rs} + B_{jk\ell}\dot\kappa_{k\ell},$$

$$(2.12)\qquad \dot m_{\ell k} = \frac{\partial g_{\ell k}}{\partial \gamma_{rs}}\dot\gamma_{rs} + \frac{\partial g_{k\ell}}{\partial \kappa_{pq}}\dot\kappa_{pq} \equiv D_{\ell krs}\dot\gamma_{rs} + E_{\ell kpq}\dot\kappa_{pq}.$$

The constitutive compatibility conditions are obtained from equations (2.11) and (2.12), have the form

$$(2.13)\qquad \mu_{ji} = A_{jirs}\Gamma_{rs} + B_{jik\ell}\Omega_{k\ell},$$

$$(2.14)\qquad \lambda_{\ell k} = D_{\ell krs}\Gamma_{rs} + E_{\ell kpq}\Omega_{pq}.$$

Using the unit normal vector n_i of the wave front and the wave speed c , that is,

$$n_i \equiv \frac{\varphi_i}{\sqrt{\varphi_j\varphi_j}}, \qquad c \equiv -\frac{\varphi_4}{\sqrt{\varphi_j\varphi_j}},$$

we obtain the wave propagation equation from the dynamic, kinematic and constitutive equations [3,4]:

$$(2.15)\qquad \left(A_{jirs}n_j n_r - \rho c^2\delta_{si}\right)\beta_s + B_{jik\ell}n_j n_\ell \eta_k = 0,$$

$$(2.16)\qquad D_{qprs}n_q n_r\beta_s + \left(E_{qpk\ell}n_q n_\ell - \tilde I c^2\delta_{pk}\right)\eta_k = 0,$$

or in invariant (or matrix) form:

$$(2.17)\qquad \left(\underline{\underline{A}} - pc^2\underline{\underline{I}}\right)\cdot\underline{\beta} + \underline{\underline{B}}\cdot\underline{\eta} = \underline{0},$$

$$(2.18)\qquad \underline{\underline{D}}\cdot\underline{\beta} + \left(\underline{\underline{E}} - \tilde I c^2\underline{\underline{I}}\right)\cdot\underline{\eta} = \underline{0}.$$

The acceleration and spin waves exist, therefore the amplitudes $\underline{\beta}$ and $\underline{\eta}$ cannot be zero. Thus the corresponding determinant should be equal to zero, namely

$$(2.19)\qquad \det\left[\underline{\underline{D}} - \left(\underline{\underline{E}} - \tilde I c^2\underline{\underline{I}}\right)\cdot\underline{\underline{B}}^{-1}\left(\underline{\underline{A}} - \rho c^2\underline{\underline{I}}\right)\right] = 0,$$

if $\det\left(\underline{\underline{B}}\right) \neq 0$. Equation (2.19) is wave speed equation, which has at least four positive roots c . These roots are the propagation speeds of the wave.

Additional special cases are the following:

CASE (A1). The couple-stress tensor does not depend on strain γ_{rs}, that is $\frac{\partial g_{\ell k}}{\partial \gamma_{rs}} \equiv 0$ if $\underline{\underline{D}} \equiv 0$. Now we obtain two equations for the wave speed: one for the acceleration wave the other for the spin wave. They are

$$\det\left(\underline{\underline{A}} - \rho c^2 \underline{\underline{I}}\right) = 0, \tag{2.20}$$

and

$$\det\left(\underline{\underline{E}} - \tilde{I} c^2 \underline{\underline{I}}\right) = 0. \tag{2.21}$$

The acceleration and the spin waves are independent of each other.

CASE (A2). The couple-stress is independent of the strain and the stress is independent of the torsion tensor, that is

$$\frac{\partial g_{\ell k}}{\partial \gamma_{rs}} \equiv 0 \quad \text{or} \quad \underline{\underline{D}} \equiv 0 \quad \text{and} \quad \frac{\partial f_{ji}}{\partial \kappa_{k\ell}} \equiv 0 \quad \text{or} \quad \underline{\underline{B}} = \underline{\underline{0}} \,.$$

The wave speed equations are (2.20) and (2.21) again. These equations can be obtained from (2.17) and (2.18).

CASE (A3). Assume that the inertia moment $\tilde{I}$ is very small. We obtain from Eqs. (2.17) and (2.18)

$$\underline{\eta} = -\underline{\underline{E}}^{-1} \cdot \underline{\underline{D}} \cdot \underline{\beta} \quad \text{therefore} \quad \tilde{I} \equiv 0$$

and the wave speed equation

$$\det\left[\underline{\underline{A}} - \underline{\underline{B}} \cdot \underline{\underline{E}}^{-1} \cdot \underline{\underline{D}} - \rho c^2 \underline{\underline{I}}\right] = 0. \tag{2.22}$$

The spin wave propagates together with the acceleration wave.

2.3.2. The stress and couple-stress depend on strain and torsion and also on their rates. The possible constitutive equations are

$$\dot{t}_{ji} = F_{ji}\left(\dot{\gamma}_{rs}\,, \dot{\kappa}_{pq}\,, \gamma_{rs}\,, \kappa_{pq}\right), \tag{2.23}$$

$$\dot{m}_{\ell k} = G_{\ell k}\left(\dot{\gamma}_{rs}\,, \dot{\kappa}_{pq}\,, \gamma_{rs}\,, \kappa_{pq}\right). \tag{2.24}$$

Now the constitutive compatibility conditions are

$$\mu_{ji}\varphi_4 = F_{ji}\left(\dot{\gamma}^{\circ}_{rs} + \Gamma_{rs}\varphi_4\,, \dot{\kappa}^{\circ}_{pq} + \Omega_{pq}\varphi_4\,, \gamma_{rs}\,, \kappa_{pq}\right) - F^{\circ}_{ji}, \tag{2.25}$$

$$\lambda_{\ell k}\varphi_4 = G_{\ell k}\left(\dot{\gamma}^{\circ}_{rs} + \Gamma_{rs}\varphi_4\,, \dot{\kappa}^{\circ}_{pq} + \Omega_{pq}\varphi_4\,, \gamma_{rs}\,, \kappa_{pq}\right) - G^{\circ}_{\ell k}, \tag{2.26}$$

where γ°_{rs} and κ°_{pq} are the strain and torsion rate in front of the wave surface. The constitutive compatibility equations have the form of a system of first order nonlinear partial differential equations. Considering its characteristic equations and using the dynamic and kinematic compatibility conditions, we obtain the wave propagation equations [3,4]

$$\mu_{ji}\varphi_4 = F_{ji}\left(\dot{\gamma}^{\circ}_{rs} + \Gamma_{rs}\varphi_4\,,\, \dot{\kappa}^{\circ}_{pq} + \Omega_{pq}\varphi_4\,,\, \gamma_{rs}\,,\, \kappa_{pq}\right) - F^{\circ}_{ji}, \tag{2.27}$$

$$\lambda_{\ell k}\varphi_4 = G_{\ell k}\left(\dot{\gamma}^{\circ}_{rs} + \Gamma_{rs}\varphi_4\,,\, \dot{\kappa}^{\circ}_{pq} + \Omega_{pq}\varphi_4\,,\, \gamma_{rs}\,,\, \kappa_{pq}\right) - G^{\circ}_{\ell k}, \tag{2.28}$$

or in invariant form

$$\left(\underline{\underline{A}}^{*} - \rho\, c^2 \underline{\underline{I}}\right) \cdot \underline{\beta} + \underline{\underline{B}}^{*} \cdot \underline{\eta} = \underline{0}, \tag{2.29}$$

$$\underline{\underline{D}}^{*} \cdot \underline{\beta} + \left(\underline{\underline{E}}^{*} - \tilde{I} c^2 \underline{\underline{I}}\right) \cdot \underline{\eta} = \underline{0}. \tag{2.30}$$

Equations (2.29) and (2.30) are similar as (2.17) and (2.18).

We can speak about cases A1, A2 and A3 similarly as it was done in Sec. 2.3.1. in cases B1, B2 and B3, and the conclusions are similar, too.

2.3.3. An important conclusion of the general constitutive equations [3,4]. We return to the general constitutive equations, (2.5) and the compatibility conditions (2.10). Constitutive compatibility conditions (2.10) form a system of first order nonlinear partial differential equations with respect to $\varphi\left(x_{\hat{i}}\right)$. This system has a solution if the Poisson bracket is equal to zero, that is if

$$\left(\frac{\partial f_{\alpha}}{\partial \varphi_{\hat{i}}}\frac{\partial f_{\beta}}{\partial x_{\hat{i}}} - \frac{\partial f_{\alpha}}{\partial x_{\hat{i}}}\frac{\partial f_{\beta}}{\partial \varphi_{\hat{i}}}\right) = 0. \tag{2.31}$$

It is satisfied if the constitutive equations are written in form

$$f_{\alpha}\left(\tau_{rs\hat{i}}\,,\, \vartheta_{pq\hat{j}}\,,\, \gamma_{rs}....m_{pq}\right) = 0,$$

where $\tau_{rs,\hat{i}}$ and $\vartheta_{pq,\hat{j}}$ mean connections among $t_{rs,\hat{i}}\,,\, \gamma_{k\ell,\hat{j}}\,,\, m_{pq,\hat{j}}\,,\kappa_{k\ell,\hat{j}}$

$$\tau_{rs\hat{i}} \equiv t_{rs,\hat{i}} + H_{rsk\ell\hat{i}\hat{j}}\,\gamma_{k\ell,\hat{j}},$$
$$\vartheta_{pq\hat{j}} \equiv m_{pq,\hat{j}} + L_{pqk\ell\hat{j}\hat{i}}\,\kappa_{k\ell,\hat{j}}$$

if the next equations satisfy

$$\delta_{\hat{j}\hat{p}}\mu_{rs,\hat{i}} + H_{k\ell rs\hat{j}\hat{p}}\,\Gamma_{k\ell,\hat{i}} = 0,$$
$$\delta_{\hat{j}\hat{p}}\lambda_{rs,\hat{i}} + L_{k\ell rs\hat{j}\hat{p}}\,\Omega_{k\ell,\hat{i}} = 0$$

thus we may reduce the number of tensorial variables in equations (2.5) from 8 to 6.

3. Basic variables and connections of Eringen's continuum

The Eringen continuum consists of macroelements. A macroelement is divided into microelements. The microelements form a nonpolar continuum. Let $\bar{x}_k$ be the particle of a microelement and x_k – the mass centre of the macroelement in the spatial configuration. Similarly, $\overline{X}_K$ and X_K are the particle and the centre of mass in the reference configuration. The followings define vectors $\overline{p}_k$ and $\overline{P}_K$ in the spatial and reference configurations:

$$\overline{x}_k = x_k + \eta_{kK}\overline{P}_K$$

and

$$\overline{X}_K = X_K + \zeta_{Kk}\overline{P}_k\,.$$

The geometrical relations between the two configurations are written in the form:

$$d\overline{x}_k = \left(x_{k,K} + \eta_{kL,K}\overline{P}_L\right) dX_K + \eta_{kK}d\overline{P}_K,$$

$$d\overline{X}_K = \left(X_{K,k} + \zeta_{K\ell}d\overline{p}_\ell\right) dx_k + \zeta_{Kk}d\overline{p}_k,$$

the deformation gradient and microdeformation gradient are

$$x_{k,K} = \left(\delta_{LK} + U_{L,K}\right)\ \delta_{kL}$$

and

$$\eta_{kK} = \left(\delta_{LK} + \psi_{LK}\right)\ \delta_{kL}\,,$$

or their inverse tensors

$$X_{K,k} = \left(\delta_{\ell k} - u_{\ell,k}\right)\ \delta_{K\ell}$$

and

$$\zeta_{Kk} = \left(\delta_{\ell k} - \psi_{\ell k}\right)\ \delta_{K\ell},$$

where u_ℓ, U_L and $\psi_{\ell k}$, Ψ_{LK} are the displacement vector and the microdisplacement gradient in the spatial and reference configurations. Let us denote the Euler's strain by $\gamma_{k\ell}$ and the first and second microdeformation tensors by $\kappa_{k\ell}$ and $\gamma_{k\ell m}$. According to [2], they can be expressed by the following formulae:

$$2\gamma_{k\ell} = u_{k,\ell} + u_{\ell,k} - u_{m,k}u_{m,\ell},$$

$$\kappa_{k\ell} = \psi_{k\ell} + u_{\ell,k} - u_{m,k}\psi_{m\ell},$$

$$\gamma_{k\ell m} = -\psi_{k\ell,m} + u_{n,k}\psi_{n\ell,m}.$$

From these formulae we obtain the kinematic equations by differentiating with respect to time, that is,

$$\dot{\gamma}_{k\ell} = \nu_{k\ell} - \left(\gamma_{km}\nu_{m,\ell} + \gamma_{m\ell}\nu_{m,k}\right), \tag{3.1}$$

(3.2) $$\dot{\kappa}_{k\ell} = \alpha_{k\ell} + \nu_{\ell,k} - (\kappa_{km}\alpha_{m\ell} + \kappa_{m\ell}\nu_{m,k}),$$

(3.3) $$\dot{\gamma}_{k\ell m} = -\alpha_{k\ell,m} + \kappa_{kr}\alpha_{r\ell,m} - (\gamma_{k\ell r}\nu_{r,m} + \gamma_{krm}\alpha_{r\ell} + \gamma_{r\ell m}\nu_{r,k}),$$

where $v_{k,\ell}$ is the velocity gradient and $\alpha_{k\ell} = \dot{\eta}_{kK}\zeta_{K\ell}$ is microrotation tensor.

Let us interpret the densities of internal force system in both macro- and microstructures. Denote by dV and $d\overline{V}$ the volumes of macro- and microelements. These volumes are bounded by closed surfaces dA and $d\overline{A}$. If $\bar{t}_{k\ell}$ is the microstress on $d\overline{A}_k$ then $t_{k\ell}$ is the macrostress on dA_k, that is, $\int\limits_{dA} \bar{t}_{k\ell} d\overline{A}_k = t_{k\ell} dA_k$. Similarly we can introduce

$$\int\limits_{dV} \overline{q}_k d\overline{V} = q_k dV,$$

$$\int\limits_{dA} \bar{t}_{k\ell}\overline{p}_m d\overline{A_k} = \lambda_{k\ell m} dA_k\,, \quad \int\limits_{dV} \overline{q}_k \overline{p}_m d\overline{V} = \ell_{km} dV,$$

$$\int\limits_{dV} \overline{s}_{m\ell} d\overline{V} = s_{m\ell} dV\,, \quad s_{m\ell} = s_{\ell m},$$

for body force q_k, couple-stress $\lambda_{k\ell m}$, couple body force ℓ_{km} and microstress $s_{m\ell}$ on dV.

Equations of motion are [2]

(3.4) $$t_{k\ell,k} + q_\ell = \rho\dot{\nu}_\ell,$$

(3.5) $$t_{k\ell} - s_{k\ell} + \lambda_{p\ell m,p} + \ell_{\ell k} = \rho\dot{\pi}_{\ell k},$$

where $\dot{v}_k$ is the acceleration, $\rho\, i_{nm}$ is Euler's inertia tensor and $\dot{\pi}_{kn}$ – the spin tensor. Following [2], they are expressed by the formulae

$$\int\limits_{dV} \overline{\rho}\,\overline{p}_m \overline{p}_n d\overline{V} = \rho\, i_{mn} dV,$$

$$\dot{\pi}_{kn} \equiv i_{mn}(\dot{\alpha}_{km} + \alpha_{k\ell}\alpha_{\ell m}).$$

We look for the constitutive equations in the following form

(3.6) $$t_{k\ell} = f_{k\ell}(\gamma_{pq}, \kappa_{pq}, \gamma_{pqr}),$$

(3.7) $$\lambda_{k\ell m} = g_{k\ell m}(\gamma_{pq}, \kappa_{pq}, \gamma_{pqr}),$$

(3.8) $$s_{k\ell} = k_{k\ell}\left(\gamma_{pq}, \kappa_{pq}, \gamma_{pqr}\right).$$

We use the material time derivative of the system of equations (3.6) - (3.7), that is,

$$\dot{t}_{k\ell} = \frac{\partial f_{k\ell}}{\partial \gamma_{pq}}\dot{\gamma}_{pq} + \frac{\partial f_{k\ell}}{\partial \kappa_{pq}}\dot{\kappa}_{pq} + \frac{\partial f_{k\ell}}{\partial \gamma_{pqr}}\dot{\gamma}_{pqr} \equiv F_{k\ell pq}\dot{\gamma}_{rq} + A_{k\ell pq}\dot{\kappa}_{pq} + B_{k\ell pqr}\dot{\gamma}_{pqr}$$

$$\dot{\lambda}_{k\ell m} = \frac{\partial g_{k\ell m}}{\partial \gamma_{pq}}\dot{\gamma}_{pq} + \frac{\partial g_{k\ell m}}{\partial \kappa_{pq}}\dot{\kappa}_{pq} + \frac{\partial g_{k\ell m}}{\partial \gamma_{pqr}}\dot{\gamma}_{pqr}$$
$$\equiv H_{k\ell mpq}\dot{\gamma}_{rq} + D_{k\ell mpq}\dot{\kappa}_{pq} + E_{k\ell mpqr}\dot{\gamma}_{pqr}.$$

We determine the microstress $s_{m\ell}$ from (3.5) by taking its symmetric part.

The jumps of the deformations and internal force system and their derivatives on wave surface φ can be written as

$$[\gamma_{k\ell}] = 0, \quad [\kappa_{k\ell}] = 0, \quad [\gamma_{k\ell m}] = 0, \quad [\alpha_{k\ell}] = 0, \quad [\psi_{m\ell}] = 0,$$

$$[\psi_{k\ell,m}] = 0, \quad [t_{k\ell}] = 0, \quad [s_{k\ell}] = 0, \quad [\lambda_{k\ell m}] = 0,$$
$$[\dot{\gamma}_{k\ell}] = \Gamma_{k\ell}\varphi_4, \quad [\dot{\kappa}_{k\ell}] = \Omega_{k\ell}\varphi_4, \quad [\dot{\gamma}_{k\ell m}] = \vartheta_{k\ell m}\varphi_4,$$
$$[\dot{\alpha}_{k\ell}] = \sigma_{k\ell}\varphi_4, \quad [t_{k\ell,k}] = \mu_{k\ell}\varphi_k, \quad [\lambda_{p\ell m,p}] = \Lambda_{p\ell m}\varphi_p.$$

The dynamic, kinematic and constitutive compatibility equations are

$$\mu_{k\ell}\varphi_k = \rho\beta_\ell\varphi_4,$$

$$\Lambda_{k\ell m}\varphi_k = \rho\, i_{nm}\sigma_{\ell n}\varphi_4,$$
$$2\Gamma_{k\ell}\varphi_4 = aL_{k\ell m}\beta_m,$$
$$\Omega_{k\ell m}\varphi_4 = aM_{k\ell m}\beta_m,$$
$$\vartheta_{k\ell m}\varphi_4 = -a\left(M_{mkp}\sigma_{p\ell} + N_{mk\ell r}\beta_r\right),$$
$$\mu_{pq} = A_{pqk\ell}\Omega_{k\ell} + B_{pqk\ell}\vartheta_{k\ell m} + F_{pqk\ell}\Gamma_{k\ell},$$
$$\Lambda_{pqr} = D_{pqrk\ell}\Omega_{k\ell} + E_{pqrk\ell m}\vartheta_{k\ell m} + H_{pqrk\ell}\Gamma_{k\ell},$$

where

$$L_{k\ell m} \equiv \left(\delta_{km} - 2\gamma_{km}\right)n_\ell + \left(\delta_{\ell m} - 2\gamma_{\ell m}\right)n_k,$$

$$M_{k\ell m} \equiv n_k\left(\delta_{m\ell} - \kappa_{m\ell}\right), \quad N_{mk\ell r} \equiv n_m\gamma_{k\ell r} + n_k\gamma_{r\ell m},$$

$$a \equiv \left(\varphi_j\varphi_j\right)^{\frac{1}{2}}.$$

We use equations of motion (3.4), (3.5), kinematic equations (3.1) - (3.3) and the constitutive equations in the form (3.9), (3.10). The wave propagation equations are

$$\left\{n_p\left(A_{pqk\ell}M_{k\ell r} - B_{pqk\ell m}N_{mk\ell r} + \frac{1}{2}F_{pqk\ell}L_{k\ell r}\right) - \rho c^2\delta_{qr}\right\}\beta_r - n_p B_{pqk\ell m}N_{mkr}\sigma_{r\ell} = 0,$$

$$n_p\left(D_{pqrk\ell}M_{k\ell s} - E_{pqrk\ell m}N_{mk\ell s} + \frac{1}{2}H_{pqrk\ell}L_{k\ell s}\right)\beta_s - \left(n_p B_{pqrk\ell m}N_{mks} + \rho i_{\ell r}\delta_{sq}\right)\sigma_{\ell s} = 0.$$

The wave propagation equations in invariant form are

$$\left(\underline{\underline{A}}^{**} - \rho c^2 \underline{\underline{I}}\right)\cdot\underline{\beta} - \underline{\underline{\underline{B}}}^{**}\cdot\cdot\underline{\underline{\sigma}} = \underline{0},$$

$$\underline{\underline{\underline{D}}}^{**}\cdot\underline{\beta} - \left(\underline{\underline{\underline{E}}}^{**} + \rho c^2 \underline{\underline{\underline{J}}}\right)\cdot\cdot\underline{\underline{\sigma}} = \underline{0}.$$

These equations are similar to (2.17) and (2.18) if $\underline{\underline{\underline{G}}}^{**} = 0$ or $\underline{\underline{\underline{E}}}^{**} + \rho c^2 \underline{\underline{\underline{J}}} \approx \underline{\underline{\underline{E}}}^{**}$. These can be investigated as it was already shown.

The wave propagation equations are a system of linear homogeneous algebraic equations for $\underline{\beta}$ and $\underline{\underline{\sigma}}$. The determinant of the system is equal to zero. This equation is called the wave speed equation, which has at least eight real roots c with different signs [3]. The existence of the roots results from the constitutive equations.

4. Conclusions

These investigations show the general behaviour of the possible constitutive equations. It follows that the investigations are similar for the Cosserat continuum and the special Eringen continuum. However, in case of the Eringen continuum, the determination of a more general internal force system is possible. In both cases the investigation is based on the existence and regular (nonzero wave-speed) propagation of acceleration and speed waves. Such waves may be coupled or independent for various constitutive equations.

The results may also be useful for experimental works: they show what kind of data of wave propagation should be measured to obtain numerical values for the constitutive equations.

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