Brief Notes

On the application of a work postulate to frictional contact

R. E. JONES ([1])   and   P. PAPADOPOULOS

Department of Mechanical Engineering, University of California, Berkeley, CA 94720-1740, USA

THE WORK POSTULATE of Naghdi and Trapp is applied to a frictional contact interface to derive an inequality restricting the relation between slip traction and slip direction.

Key words: work inequality, contact mechanics, friction.

1. Introduction

THE QUASI-THERMODYNAMIC POSTULATE of NAGHDI and TRAPP [1] has been employed extensively in deriving restrictions to the constitutive laws of elastic-plastic materials [2, 3, 4]. The postulate is an extension to finite deformations of an earlier hypothesis by ILYUSHIN [5] concerning the work done in a closed cycle of homogeneous deformation. In this short paper, it is shown that the work postulate is applicable and relevant to frictional contact (when formulated in a plasticity-like setting) and gives rise to a physically meaningful restriction of the constitutive law for the frictional tractions. This finding serves to further demonstrate the wide-ranging significance of the postulate.

2. Background

Consider two bodies which occupy open regions Ω^α, $\alpha = 1, 2$. Under quasi-static conditions, the motion χ^α of each body is governed by the equilibrium equation

$$\text{div}\mathbf{T}^\alpha + \rho^\alpha \mathbf{b}^\alpha = \mathbf{0} \qquad (\text{no sum on } \alpha) ,$$

where $\mathbf{T}^\alpha$ denotes the Cauchy stress, ρ^α the mass density, and $\mathbf{b}^\alpha$ the body force. The traction vector $\mathbf{t}^\alpha$ on the smooth boundary surface $\partial\Omega^\alpha$ with outward unit normal $\mathbf{n}^\alpha$ is related to the Cauchy stress $\mathbf{T}^\alpha$ by $\mathbf{t}^\alpha = \mathbf{T}^\alpha \mathbf{n}^\alpha$. The vector $\mathbf{t}^\alpha$ can

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([1]) Currently at Sandia National Laboratories, Livermore, CA

be uniquely decomposed as $\mathbf{t}^\alpha = -p^\alpha \mathbf{n}^\alpha + \boldsymbol{\tau}^\alpha$, where $p^\alpha \geq 0$ is the pressure and $\boldsymbol{\tau}^\alpha$ is the tangential traction.

The principle of impenetrability stipulates that $\Omega^\alpha \cap \Omega^\beta = \varnothing$, where $\beta = \mathrm{mod}(\alpha, 2) + 1$. On the contact surface $C = \partial\Omega^\alpha \cap \partial\Omega^\beta$, impenetrability is enforced by p^α, interpreted here as a Lagrange multiplier field. Additionally, the smoothness of $\partial\Omega^\alpha$ implies that $\mathbf{n}^\alpha = -\mathbf{n}^\beta$ on C, so that the traction fields on the two bodies must satisfy the linear momentum balance in the form

$$(2.1) \qquad\qquad\qquad \mathbf{t}^\beta = -\mathbf{t}^\alpha .$$

A yield-like function Υ, dependent on $\{p^\alpha, \boldsymbol{\tau}^\alpha\}$, determines the regions of stick and slip as

$$C_{\mathrm{stick}}^\alpha = \{\mathbf{x}^\alpha \in C \mid \Upsilon < 0\}, \qquad C_{\mathrm{slip}}^\alpha = \{\mathbf{x}^\alpha \in C \mid \Upsilon = 0\} .$$

The equation $\Upsilon = 0$ defines a surface with closed projections on the $\boldsymbol{\tau}^\alpha$-plane for all $p^\alpha \geq 0$. On $C_{\mathrm{stick}}^\alpha$, the jump in velocity $[\![\mathbf{v}]\!]^\alpha$, defined as

$$(2.2) \qquad\qquad\qquad [\![\mathbf{v}]\!]^\alpha = \mathbf{v}^\beta - \mathbf{v}^\alpha ,$$

vanishes and $\boldsymbol{\tau}^\alpha$ acts as a Lagrange multiplier to enforce stick. On C_{slip}^α, the tangential traction is constitutively determined by a function $\boldsymbol{\tau}$ which is assumed to depend on p^α and the relative slip direction $\mathbf{d}^\alpha = \dfrac{[\![\mathbf{v}]\!]^\alpha}{\|[\![\mathbf{v}]\!]^\alpha\|}$. Invariance under superposed rigid body motions implies that

$$(2.3) \qquad\qquad\qquad \mathbf{Q}\boldsymbol{\tau}(p^\alpha, \mathbf{d}^\alpha) = \boldsymbol{\tau}(p^\alpha, \mathbf{Q}\mathbf{d}^\alpha) ,$$

for all proper orthogonal $\mathbf{Q}$.

3. Application of a work postulate

The work postulate of Naghdi and Trapp [1] states that the external work done on a body undergoing a smooth and closed cycle of spatially homogeneous deformation is non-negative. For a cycle over the time interval $[t_1, t_2]$ the postulate implies that

$$(3.1) \qquad \int_{t_1}^{t_2} \left[\int_{\partial\Omega^\alpha} \mathbf{t}^\alpha \cdot \mathbf{v}^\alpha \, da + \int_{\Omega^\alpha} \rho^\alpha \mathbf{b}^\alpha \cdot \mathbf{v}^\alpha \, dv \right] dt \geq 0 .$$

Recall that homogeneous deformation maps material points $\mathbf{X}^\alpha$ to $\mathbf{x}^\alpha$, according to

$$(3.2) \qquad\qquad\qquad \mathbf{x}^\alpha = \mathbf{F}^\alpha \mathbf{X}^\alpha + \mathbf{c}^\alpha ,$$

where $\mathbf{F}^\alpha$ denotes the deformation gradient. Since the cycle of deformation is assumed to be closed, it follows that $\mathbf{F}^\alpha(t_1) = \mathbf{F}^\alpha(t_2)$ and $\mathbf{c}^\alpha(t_1) = \mathbf{c}^\alpha(t_2)$.

With regard to the work postulate, note that forces on the frictional interface C are external to both Ω^α and Ω^β but internal to the union $\Omega^\alpha \cup \Omega^\beta$. Consequently, if the postulate in the form (3.1) is applied to $\Omega^\alpha \cup \Omega^\beta$ and the corresponding inequalities for Ω^α and Ω^β are subtracted, it follows that

$$\int_{t_1}^{t_2} \left[\int_C (\mathbf{t}^\alpha \cdot \mathbf{v}^\alpha + \mathbf{t}^\beta \cdot \mathbf{v}^\beta)\, da \right] dt \geq 0 \,.$$

Taking into account (2.1), (2.2), and that impenetrability and stick are workless constraints, the preceding inequality can be also written as

$$(3.3) \qquad \int_{t_1}^{t_2} \left[\int_{C_{\text{slip}}^\alpha} \boldsymbol{\tau}^\alpha \cdot [\![\mathbf{v}]\!]^\alpha \, da \right] dt \leq 0 \,.$$

For the purpose of obtaining constitutive restrictions on $\boldsymbol{\tau}$, consider the contact between a homogeneous deformable body and a flat, rigid and stationary foundation. In particular, assume that in its stress-free, undeformed state ($t = t_1$) the body is a rectangular parallelepiped. For convenience, take a fixed Cartesian basis $\{\mathbf{e}_i\}$ on the surface of the rigid foundation and let $\mathbf{e}_3$ be the outward normal to this surface. Consequently, the contact surface C^α at $t = t_1$ is defined by $X_3^\alpha = 0$. Also, taking into account the homogeneity of the motion, it is clear that the deformable body will remain a parallelepiped. For notational brevity, the superscripts α and β are omitted in the remainder of this note and all quantities are implicitly referred to the deformable body.

In order for the contact to persist, it is sufficient that the normal component of the relative velocity on $X_3 = 0$ vanish. Recalling (2.2) and (3.2), it follows that

$$(3.4) \qquad\qquad [\![\mathbf{v}]\!] = -(\dot{\mathbf{F}}\mathbf{X} + \dot{\mathbf{c}}) \,,$$

hence $[\![\mathbf{v}]\!] \cdot \mathbf{e}_3 = 0$ leads to $\dot{F}_{3\gamma} = 0$ ($\gamma = 1, 2$), and $\dot{c}_3 = 0$. The inner integrand in (3.3) is independent of position if the effected motion is such that:

(a) The velocity jump $[\![\mathbf{v}]\!]$ on the interface is uniform;

(b) The surface traction $\mathbf{t}$ on the interface is uniform.

Condition (a) immediately implies a state of uniform stick or slip on C. In either case, Eq. (3.4) yields $\dot{F}_{i\gamma} = 0$, thus $F_{i\gamma}$ are constant throughout the

homogeneous cycle. It follows that the deformation gradient, relative to the configuration at $t = t_1$, must be of the form

$$(3.5) \qquad \mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + F_{i3}\,\mathbf{e}_i \otimes \mathbf{e}_3 \,,$$

where $F_{33}(t) > 0$ for all t. Condition (b) is satisfied if the deformation gives rise to homogeneous stress, thus resulting in uniform traction on any flat surface such as C. This is the case when the homogeneous body is also assumed to be Cauchy-elastic, i.e., $\mathbf{T} = \mathbf{T}(\mathbf{F})$.[2] Existence of a non-empty intersection of the regions

$$\{p = -T_{33}(F_{i3}), \quad \boldsymbol{\tau} = T_{\gamma 3}(F_{i3})\mathbf{e}_\gamma, \quad \forall\, F_{i3} \mid F_{33} > 0\}$$

and

$$\{p, \boldsymbol{\tau} \mid \Upsilon(p, \boldsymbol{\tau}) \le 0\}$$

in the neighborhood of $p = 0$, $\boldsymbol{\tau} = \mathbf{0}$ is tacitly assumed, as is the controllability of motions of the type (3.5).

Now, examine a homogeneous cycle of deformation of the form (3.5) starting at $t = t_1$, in which p and $\boldsymbol{\tau}$ increase until $\Upsilon = 0$ at a time $t = t_a$. At that instant, slip is initiated on C and, by fixing $\mathbf{F}$, the body begins to translate rigidly with homogeneous relative velocity $[\![\mathbf{v}]\!] = -\dot{\mathbf{c}}$, constant slipping direction $\bar{\mathbf{d}} = -\dfrac{\dot{\mathbf{c}}}{\|\dot{\mathbf{c}}\|}$, and constant pressure $\bar{p}$. At time t_b, after the body has slipped a distance $|L|$, unloading is effected smoothly so that the body instantaneously returns to stick. Subsequently, through a reverse process, the body is returned to its initial configuration, with slip in the opposite direction occurring during the interval $[t_c, t_d]$. For the given cycle, with the aid of (2.3), inequality (3.3) reduces to

$$\int\limits_{t_a}^{t_b} \boldsymbol{\tau}(\bar{p}, \bar{\mathbf{d}}) \cdot \|\dot{\mathbf{c}}\|\, \bar{\mathbf{d}}\, dt + \int\limits_{t_c}^{t_d} \boldsymbol{\tau}(\bar{p}, -\bar{\mathbf{d}}) \cdot \|\dot{\mathbf{c}}\|\, (-\bar{\mathbf{d}})dt = \boldsymbol{\tau}(\bar{p}, \bar{\mathbf{d}}) \cdot \bar{\mathbf{d}}2|L| \le 0,$$

which requires

$$\boldsymbol{\tau} \cdot \mathbf{d} \le 0\,.$$

Therefore, the Naghdi-Trapp postulate implies that the tangential traction $\boldsymbol{\tau}$ must oppose the slip direction $\mathbf{d}$, as is commonly assumed, and places a corresponding restriction on the constitutive function $\boldsymbol{\tau}$.

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[2] This constitutive choice is made in order to render friction the sole source of dissipation. Since the friction law and the bulk material response are uncoupled, no loss in generality results from this assumption.

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