

Thermomechanically consistent formulations of the standard linear solid using fractional derivatives

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WE STUDY THE THERMOMECHANICAL properties of a frequently used fractional generalisation of the standard linear solid. Its mathematical structure arises from an ordinary linear differential equation between stress and strain when replacing the first order time rates by fractional derivatives of the order $0 \leq \alpha, \beta < 1$. If the parameters α and β are not further restricted, the model leads to an unphysical behaviour. In the case of harmonic deformations the dissipation modulus can become negative. This corresponds to a negative entropy production and violates the second law of thermodynamics. Then we propose two generalisations of the standard linear solid which are based on a so-called thermodynamically consistent fractional rheological element. It possesses a non-negative free energy and rate of dissipation for arbitrary deformation processes and is compatible with the second law of thermodynamics. The differential equations between stress and strain of the proposed generalisations contain also fractional derivatives of different orders but both the dynamic moduli and the relaxation spectra are non-negative functions of their arguments. No restrictions on the material parameters are required.

Notations

$\varepsilon, \varepsilon_0$	strain, strain amplitude
σ, σ_0	stress, stress amplitude
ω	angular frequency
G^*, G', G''	complex modulus, storage modulus and dissipation modulus
δ	specific dissipation
ψ, ψ_{mean}	free energy, temporal mean value of the free energy
ρ	mass density
$\nu, H(\nu)$	relaxation frequency, relaxation spectrum
w_d	dissipated energy per loading cycle
$d^\alpha f / dt^\alpha$	fractional derivative of a function $f(t)$
i	imaginary unit
$E, G, E_\alpha, E_\beta, E_\gamma$	elasticity parameters
α, β, γ	parameters of fractional differentiation
τ, τ_R, τ_C	time constants

1. Introduction

THE FRACTIONAL CALCULUS as a tool to formulate constitutive models is a fairly modern field of scientific research which fits the theory of linear viscoelasticity. Its fundamental idea is based on real-order derivatives of stress and strain which occur in the constitutive equations. If we formulate, for example, a so-called *fractional damping element* of the type $\sigma = E\tau^\alpha d^\alpha \varepsilon / dt^\alpha$ with the order $0 \leq \alpha \leq 1$ of fractional differentiation, we can show that it interpolates between two limit cases: linear Hookean elasticity corresponds to $\alpha = 0$ and linear Newtonian viscosity to $\alpha = 1$ (see e.g. LION [11]). In the intermediate region, i.e. for $0 < \alpha < 1$, it leads to strongly nonlinear stress-strain characteristics (power functions) and to a relaxation or creep behaviour of the power law type; it can be shown that the dynamic moduli and compliances are also of this type. More complicated models containing fractional derivatives lead to creep and relaxation functions, for example of the Mittag Leffler type (cf. KOELLER [10], MAINARDI and BONETTI [15], NONNENMACHER [17] or LION [11]) and are frequently applied to represent the short and long term behaviour of viscoelastic materials (cf. BAGLEY and TORVIK [1], METZELER *et al.* [16], LION [12] or HAUPT *et al.* [7]). In the case of many viscoelastic materials one can observe a very fast rate of relaxation at the very beginning of the process (short term behaviour) and a super-slow rate after some time (long term behaviour). These phenomena can easily be represented using fractional calculus because one needs only a very few number of material parameters. On the basis of relaxation functions of the exponential type one would need a very large number of terms and thus of material constants. The authors BAGLEY and TORVIC [1] have shown that the dynamic behaviour of a corning glass can be represented over seven log cycles in the frequency domain by means of a fractional model containing only four material constants. A model of similar type was applied by HAUPT *et al.* [7] to describe the dynamic properties of polyethylene and by HARTMANN *et al.* [6] to represent the creep behaviour of concrete. Models of the fractional type have also been implemented into finite element codes (cf. SCHMIDT *et al.* [19]).

The fundamental theory of viscoelasticity based on the fractional calculus was originally applied by CAPUTO and MAINARDI [3]. HEYMANS and BAUWENS [9] worked out relations between self similar rheological models and fractional differential equations. In addition to this, they proposed fractional differential equations whose structure is motivated more or less by rheological models. An interesting question concerns the thermodynamical properties of linear viscoelastic models containing fractional derivatives. A first attempt towards this direction was undertaken by BAGLEY and TORVIK [2] who derived restrictions on the material parameters to satisfy the non-negativity of dynamic moduli under sinusoidal loadings. Unfortunately, the free energy function is not specified in their paper.

For comparison, FRIEDRICH [4] has investigated the temporal decay behaviour of the relaxation function belonging to a fractional differential equation under thermodynamical aspects.

The present work starts with some fundamentals of fractional calculus and the classical theory of linear viscoelasticity. Then we study the thermodynamical properties of a frequently used generalisation of the standard linear solid. We demonstrate that this type of generalisation, named type A, can violate the natural laws of thermodynamics if the material constants are not chosen correctly. Motivated by this result, we propose a systematic approach to formulate fractional generalisations of the standard linear solid, named type B and C. It is based on rheological networks of fractional damping elements. These models lead, for example, to non-negative relaxation spectra and dynamic moduli for any values of their parameters and arbitrary loading processes.

2. Fundamentals

In order to prepare the following studies, let us summarise some basic properties of the dynamic moduli in the sense of the classical theory of linear viscoelasticity. For more details and a deeper understanding concerning the mathematical theory of linear viscoelasticity we refer the interested reader to the textbooks of TSCHOEGL [20] and GROSS [5].

Let us assume that we have a linear viscoelastic material which is loaded by a uniaxial harmonic deformation process $\varepsilon(t)$ of the type

$$(2.1) \quad \varepsilon(t) = \varepsilon_0 \sin(\omega t).$$

As we know, the stationary stress response $\sigma(t)$ is also a sinusoidal function and can be written as

$$(2.2) \quad \sigma(t) = \varepsilon_0 (G'(\omega) \sin(\omega t) + G''(\omega) \cos(\omega t)).$$

The parameter ω is the loading (angular) frequency and ε_0 the deformation amplitude. The frequency-dependent *storage* modulus $G'(\omega)$ describes that part of the stationary stress response which is in phase with the deformation process. The function $G''(\omega)$ is the *dissipation modulus* representing that part of stress which is in phase with the deformation rate.

2.1. Thermodynamic compatibility in terms of the relaxation spectrum

In continuum mechanics it is a common practise to express the second law of thermodynamics in the form of the *Clausius Duhem inequality*. For details we

refer the reader to HAUPT [8]. In its uniaxial and isothermal form the Clausius Duhem inequality reads

$$(2.3) \quad \delta = -\rho\dot{\psi} + \sigma\dot{\varepsilon} \geq 0,$$

where σ is the stress, ε the strain, δ the rate of dissipation and $\rho\psi$ the specific free energy per unit volume. This inequality expresses the fact that the temporal change in the free energy has to be equal or smaller than the supplied stress power. In the case of linear viscoelastic constitutive models possessing a so-called *relaxation spectrum* $H(\nu)$, it can be motivated that the isothermal free energy, i.e. the mechanical energy which is stored in the material, has the functional form

$$(2.4) \quad \rho\psi(t) = \frac{1}{2} \int_0^\infty H(\nu) \left(\int_0^t e^{-\nu(t-s)} \dot{\varepsilon}(s) ds \right)^2 d\nu,$$

where the variable ν is the relaxation frequency or the reciprocal relaxation time (cf. LION [11]). Now we show that a model of linear viscoelasticity whose free energy is given by (2.4) with

$$(2.5) \quad H(\nu) \geq 0$$

is compatible with the second law of thermodynamics. To this end we calculate the material time rate of (2.4), insert the result into the Clausius-Duhem inequality (2.3) and rearrange terms:

$$(2.6) \quad \delta = \left[\sigma - \int_0^\infty H(\nu) \left(\int_0^t e^{-\nu(t-s)} \dot{\varepsilon}(s) ds \right) d\nu \right] \dot{\varepsilon}(t) \\ + \int_0^\infty \nu H(\nu) \left(\int_0^t e^{-\nu(t-s)} \dot{\varepsilon}(s) ds \right)^2 d\nu \geq 0.$$

In order to satisfy this inequality for any value of the strain rate at the current time t , the first term in brackets has to vanish. Interchanging the sequence of integration in the first term leads then to the following relations for the stress and the rate of dissipation:

$$(2.7) \quad \sigma(t) = \int_0^t \left(\int_0^\infty H(\nu) e^{-\nu(t-s)} d\nu \right) \dot{\varepsilon}(s) ds$$

$$(2.7) \quad \text{and} \quad \delta(t) = \int_0^\infty \nu H(\nu) \left(\int_0^t e^{-\nu(t-s)} \dot{\epsilon}(s) ds \right)^2 d\nu \geq 0.$$

[cont.]

Taking a look at (2.7)₂ we see that the rate of dissipation $\delta(t)$ is non-negative if the relaxation spectrum $H(\nu)$ is non-negative; the same statement is valid for the free energy (2.4). If we have any constitutive equation of linear viscoelasticity and we can show that its relaxation spectrum is non-negative, the model is compatible with the second law of thermodynamics. Thus $H(\nu) \geq 0$ is a sufficient condition for thermodynamic compatibility.

2.2. Thermodynamic compatibility in terms of the dynamic moduli

Let us consider the general stress-strain relation given by (2.7)₁ and a harmonic deformation process in the form of (2.1). Based on standard calculations we can demonstrate that the asymptotic stress response has the form of (2.2) where the storage and dissipation moduli read

$$(2.8) \quad G'(\omega) = \int_0^\infty H(\nu) \frac{\omega^2}{\nu^2 + \omega^2} d\nu \quad \text{and} \quad G''(\omega) = \int_0^\infty H(\nu) \frac{\omega\nu}{\nu^2 + \omega^2} d\nu.$$

In order to express the free energy $\psi(t)$ and the specific dissipation $\delta(t)$ in terms of the dynamic moduli G' and G'' we take the sinusoidal deformation $\epsilon(t) = \epsilon_0 \sin(\omega t)$ into account, evaluate the formulae (2.4) and (2.7)₂ and consider large times t , so that the initial transients are vanished. As an intermediate result we obtain the relations

$$(2.9) \quad \rho\psi(t) = \frac{\epsilon_0^2}{2} \int_0^\infty H(\nu) \left(\frac{\omega^2}{\nu^2 + \omega^2} \sin(\omega t) + \frac{\omega\nu}{\nu^2 + \omega^2} \cos(\omega t) \right)^2 d\nu,$$

and

$$(2.10) \quad \delta(t) = \epsilon_0^2 \int_0^\infty \nu H(\nu) \left(\frac{\omega^2}{\nu^2 + \omega^2} \sin(\omega t) + \frac{\omega\nu}{\nu^2 + \omega^2} \cos(\omega t) \right)^2 d\nu$$

leading finally to

$$(2.11) \quad \rho\psi = \frac{\epsilon_0^2}{4} \left(\int_0^\infty \frac{H(\nu)\omega^2}{\nu^2 + \omega^2} d\nu - \int_0^\infty \frac{H(\nu)(\omega^4 - \omega^2\nu^2)}{(\nu^2 + \omega^2)^2} d\nu \cos(2\omega t) + 2 \int_0^\infty \frac{H(\nu)\nu\omega^3}{(\nu^2 + \omega^2)^2} d\nu \sin(2\omega t) \right)$$

and

$$(2.12) \quad \delta = \frac{\varepsilon_0^2 \omega}{2} \left(\int_0^\infty \frac{H(\nu) \omega \nu}{\nu^2 + \omega^2} d\nu + \omega \int_0^\infty \frac{H(\nu) (\nu^3 - \nu \omega^2)}{(\nu^2 + \omega^2)^2} d\nu \cos(2\omega t) \right. \\ \left. + 2\omega \int_0^\infty \frac{H(\nu) \omega \nu^2}{(\nu^2 + \omega^2)^2} d\nu \sin(2\omega t) \right)$$

Comparing these integrals with the dynamic moduli (2.8), we can show that the final form of the free energy and the specific dissipation under sinusoidal deformations can be written as

$$(2.13) \quad \rho\psi(t) = \frac{\varepsilon_0^2}{4} \left(G'(\omega) - \left(G'(\omega) - \omega \frac{dG'}{d\omega} \right) \cos(2\omega t) \right. \\ \left. + \left(G''(\omega) - \omega \frac{dG''}{d\omega} \right) \sin(2\omega t) \right)$$

and

$$(2.14) \quad \delta(t) = \frac{\varepsilon_0^2}{2} \left(G''(\omega) + \omega \frac{dG''}{d\omega} \cos(2\omega t) + \omega \frac{dG'}{d\omega} \sin(2\omega t) \right).$$

These expressions correspond to those which were obtained earlier by TSCHOEGL [20] but on the basis of different mathematical techniques. As we see, the current values of the free energy $\psi(t)$ and the rate of dissipation $\delta(t)$ depend in a complicated manner on the storage and dissipation moduli as well as on their first and second derivatives with respect to the frequency ω . Both expressions consist of a positive mean value which is superimposed by harmonic oscillations. Since both mechanisms energy storage and dissipation occur in tension and compression as well, it is obvious that ψ and δ depend on the double frequency 2ω .

2.3. Consequences for the storage and dissipation moduli

Let us estimate the amount of energy w_d dissipated per loading cycle. To this end we integrate the specific dissipation (2.14) over one cycle and obtain

$$(2.15) \quad w_d = \int_t^{t+2\pi/\omega} \delta(s) ds = \pi \varepsilon_0^2 G''(\omega).$$

Since w_d or equivalently, the hysteresis area under cyclic stress strain curves are non-negative, the dissipation modulus $G''(\omega)$ has to be non-negative for any frequency ω :

$$(2.16) \quad G''(\omega) \geq 0.$$

Otherwise the model violates the natural laws of thermodynamics. In addition one can show that the expression $w_d = \pi \varepsilon_0^2 G''(\omega)$ is also valid in the case of non-linear constitutive models whose stationary stress response is given by a complete Fourier series (cf. LION [13]). Calculating the temporal mean value of the free energy (2.13) over one cycle, we obtain

$$(2.17) \quad \rho\psi_{\text{mean}} = \frac{\varepsilon_0^2}{4} G'(\omega),$$

implying $G'(\omega) \geq 0$ and confirming the interpretation of the storage modulus G' assumed by FERRY [21]. Taking an additional look at (2.13) we see that the free energy reduces to

$$(2.18) \quad \rho\psi(t_k) = \frac{\varepsilon_0^2}{4} \omega \frac{dG'}{d\omega}$$

for $\omega t_k = k\pi$ and $k = 0, 1, 2, \dots$ implying $dG'/d\omega \geq 0$, i.e. the storage modulus is an increasing function. Additional consequences can be derived by analysing (2.13) and (2.14) in more detail but this is not the aim of the present work.

2.4. Introduction to fractional calculus

The mathematical theory of the fractional calculus is explained in detail in the textbook of OLDHAM and SPANIER [18] but we also refer the reader to the original paper of CAPUTO and MAINARDI [3]. For our purpose we apply the *Riemann Liouville definition* and define the operator of fractional differentiation of real order $\alpha \geq 0$ as

$$(2.19) \quad \frac{d^\alpha f}{dt^\alpha} = \frac{d^m}{dt^m} \left(\frac{1}{\Gamma(m-\alpha)} \int_0^t s^{m-\alpha-1} f(t-s) ds \right),$$

where the natural number m is chosen so that $m-\alpha > 0$ and $m-\alpha-1 \leq 0$. The quantity $\Gamma(x)$ is the Eulerian Gamma function satisfying the functional relation $x\Gamma(x) = \Gamma(x+1)$. Carrying out the differentiation d^m/dt^m of (2.19) we find the equivalent representation

$$(2.20) \quad \frac{d^\alpha f}{dt^\alpha} = \sum_{k=0}^{m-1} \frac{t^{(k-\alpha)} f^{(k)}(0)}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t s^{m-\alpha-1} \frac{d^m}{dt^m} f(t-s) ds.$$

As we see, the sum incorporating the initial conditions of the function $f(t)$ vanishes for large values of t ; if $f(t)$ satisfies homogeneous initial conditions, it is even zero. To calculate, for example, the fractional derivative of order $0 \leq \alpha < 1$

we set $m = 1$ so that the sum is given by only one term. If $f(t)$ satisfies the initial condition $f(0) = 0$, this term is zero and we obtain

$$(2.21) \quad \frac{d^\alpha f}{dt^\alpha} = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{1}{(t - s)^\alpha} f'(s) ds \quad \text{with } 0 \leq \alpha < 1.$$

In the case of $1 \leq \alpha < 2$ we have to set $m = 2$ in (2.20) and to prescribe an additional initial condition. In other cases one has to proceed in a similar way.

Since we shall calculate the dynamic moduli corresponding to fractional differential equations between stresses and strains, we need the fractional derivative of the complex exponential function

$$(2.22) \quad x(t) = e^{i\omega t}$$

in the stationary case, i.e. for large times t where the influence of the initial conditions is vanished. Application of (2.20) then leads to the expression

$$(2.23) \quad \frac{d^\alpha}{dt^\alpha} e^{i\omega t} = \frac{(i\omega)^m e^{i\omega t}}{\Gamma(m - \alpha)} \int_0^t s^{m-\alpha-1} e^{-i\omega s} ds = \frac{(i\omega)^m e^{i\omega t}}{\Gamma(m - \alpha)} \int_0^\infty s^{m-\alpha-1} e^{-i\omega s} ds$$

where the upper limit of integration can be replaced by ∞ under stationary conditions. Introducing the transformation $i\omega s = u$ we obtain the intermediate result

$$(2.24) \quad \frac{d^\alpha}{dt^\alpha} e^{i\omega t} = \frac{(i\omega)^\alpha e^{i\omega t}}{\Gamma(m - \alpha)} \int_0^\infty u^{(m-\alpha)-1} e^{-u} du,$$

where the integral in the limits between $u = 0$ and $u = \infty$ equals the Eulerian Gamma function $\Gamma(m - \alpha)$. Thus we found the simple relation

$$(2.25) \quad \frac{d^\alpha}{dt^\alpha} e^{i\omega t} = (i\omega)^\alpha e^{i\omega t},$$

which holds under stationary conditions for any real number $\alpha \geq 0$.

3. Standard linear solid

Let us first define the so-called *standard linear solid* which is visualised in Fig. 1.

The model is given by a linear Maxwell element (modulus E and viscosity η) which is in parallel to a Hookean spring with the modulus G . The stress σ splits into σ_1 and σ_2 , the strain ε into ε_1 and ε_{in} , and the constitutive relations read

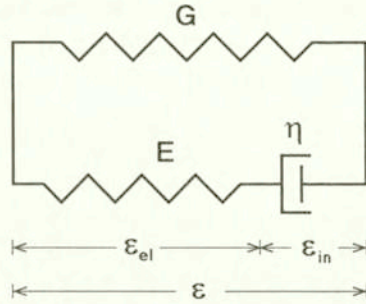


FIG. 1. Standard linear solid.

$$(3.1) \quad \sigma = \sigma_1 + \sigma_2,$$

$$(3.2) \quad \varepsilon = \varepsilon_{el} + \varepsilon_{in},$$

$$(3.3) \quad \sigma_1 = G\varepsilon,$$

$$(3.4) \quad \sigma_2 = E\varepsilon_{el},$$

$$(3.5) \quad \sigma_2 = \eta\dot{\varepsilon}_{in}.$$

After some calculations and eliminating the internal variables we obtain the following differential equation of the first order between the stress and the strain:

$$(3.6) \quad \tau_R \dot{\sigma} + \sigma = G(\tau_C \dot{\varepsilon} + \varepsilon) \quad \text{with} \quad \tau_R = \frac{\eta}{E} \quad \text{and} \quad \tau_C = \tau_R \frac{E + G}{G}.$$

For the purpose of calculating the *complex dynamic modulus* $G^* = G' + iG''$ we prescribe a harmonic deformation in the form of a complex exponential function

$$(3.7) \quad \varepsilon(t) = \varepsilon_0 e^{i\omega t},$$

where ε_0 is the strain amplitude. For the stationary stress response we assume

$$(3.8) \quad \sigma(t) = \sigma_0(i\omega) e^{i\omega t},$$

where $\sigma_0(i\omega)$ is the frequency-dependent stress amplitude. Inserting (3.7) and (3.8) into (3.6) and application of (2.25) leads to

$$(3.9) \quad \sigma_0 = G^*(i\omega)\varepsilon_0 \quad \text{with} \quad G^* = G \frac{1 + (i\omega\tau_C)}{1 + (i\omega\tau_R)}.$$

Computing the real and imaginary parts of G^* we obtain the storage and dissipation moduli

$$(3.10) \quad G' = G \frac{1 + \omega^2 \tau_R \tau_C}{1 + (\omega \tau_R)^2} \quad \text{and} \quad G'' = G \frac{\omega(\tau_C - \tau_R)}{1 + (\omega \tau_R)^2}.$$

Taking a look at (3.10)₂ we recognise that the dissipation modulus G'' is non-negative for arbitrary frequencies ω . Considering (3.6) we see that the requirement $\tau_C \geq \tau_R$ is automatically satisfied for the standard linear solid.

4. Fractional generalisation (Type A)

To obtain a so-called fractional generalisation of the standard linear solid (3.6) there are different methods. From the point of view of the author, a quite formal generalisation, named as type A, can be obtained if one replaces the first order time derivatives of stress and strain in (3.6) by arbitrary non-integer derivatives $0 \leq \alpha < 1$ and $0 \leq \beta < 1$ (cf. BAGLEY and TORVIK [2]):

$$(4.1) \quad \tau_R^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G \left(\tau_C^\beta \frac{d^\beta \varepsilon}{dt^\beta} + \varepsilon \right).$$

The powers of the time constants τ_R^α and τ_C^β have been introduced for dimensional reasons and in comparison with (3.6), the model now contains 5 non-negative material parameters. For the purpose of computing the dynamic modulus G^* we apply a harmonic deformation in the form of (3.7) and assume stationary conditions so that the stress is given by (3.8). Inserting these assumptions into (4.1) and application of (2.25) leads finally to the modulus

$$(4.2) \quad G^* = G \frac{1 + (i\omega\tau_C)^\beta}{1 + (i\omega\tau_R)^\alpha}.$$

To calculate the real and imaginary parts G' and G'' of G^* , we need the following representation of the α -th power of the imaginary unit $i = \sqrt{-1}$:

$$(4.3) \quad i^\alpha = e^{i\alpha\pi/2} = \cos(\alpha\pi/2) + i \sin(\alpha\pi/2).$$

Taking this formula in combination with the modulus (4.2) into account leads to

$$(4.4) \quad G' = G \frac{1 + (\omega\tau_C)^\beta \cos(\beta\pi/2) + (\omega\tau_R)^\alpha \cos(\alpha\pi/2) + (\omega\tau_R)^\alpha (\omega\tau_C)^\beta \cos((\beta - \alpha)\frac{\pi}{2})}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)},$$

$$(4.5) \quad G'' = G \frac{(\omega\tau_C)^\beta \sin(\beta\pi/2) - (\omega\tau_R)^\alpha \sin(\alpha\pi/2) + (\omega\tau_R)^\alpha (\omega\tau_C)^\beta \sin((\beta - \alpha)\frac{\pi}{2})}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)}$$

for the storage and dissipation moduli. As we see, the storage modulus G' is non-negative for any frequency $\omega \geq 0$ but the numerator of dissipation modulus

$G''(\omega)$ contains one term with a minus sign. In addition we recognise that the sign of the third term changes in dependence on the difference between the fractional orders α and β of the stress and strain differentiation.

4.1. Investigation of dissipation modulus and relaxation spectrum

To analyse the frequency-dependence of the dissipation modulus (4.5) let us start with the case of $\beta > \alpha$, so that the sign of $\sin((\beta - \alpha)\pi/2)$ is positive. Then we rewrite the modulus G'' as

$$(4.6) \quad G'' = G\omega^\alpha \frac{\omega^{\beta-\alpha}\tau_C^\beta \sin(\beta\pi/2) - \tau_R^\alpha \sin(\alpha\pi/2) + \tau_C^\beta \tau_R^\alpha \omega^\beta \sin(\beta - \alpha)\pi/2}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)}.$$

Since we assumed both $\beta > 0$ and $\beta - \alpha > 0$, the power functions ω^β and $\omega^{\beta-\alpha}$ tend to zero in the limit $\omega \rightarrow 0$. Thus for sufficiently small frequencies, the *dissipation modulus becomes negative* provided that the other material parameters are different from zero, i.e. $\alpha > 0$, $G > 0$ and $\tau_R > 0$. In the case of $\alpha = 0$ or $\tau_R = 0$ the dissipation modulus remains non-negative for any value of ω . The same statement holds in the case of $G\tau_C^\beta > 0$ and $G = 0$ which can easily be realised by introducing a new parameter for the product $G\tau_C^\beta$ in the fractional differential Eq. (4.1). Taking a look at Eq. (4.1), the cases of $\alpha = 0$ or $\tau_R = 0$ are nearly identical but the requirements $G\tau_C^\beta > 0$ and $G = 0$ would change the type of the model.

For the purpose of analysing the case of $\alpha > \beta$ we rewrite the dissipation modulus (4.5) as

$$(4.7) \quad G'' = G\omega^\beta \frac{\tau_C^\beta \sin(\beta\pi/2) - \omega^{\alpha-\beta}\tau_R^\alpha \sin(\alpha\pi/2) - \tau_C^\beta \tau_R^\alpha \omega^\alpha \sin((\alpha - \beta)\pi/2)}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)}.$$

Since $\alpha > 0$ and $\alpha - \beta > 0$ is assumed, the terms ω^α and $\omega^{\alpha-\beta}$ tend to ∞ in the limit $\omega \rightarrow \infty$. As a consequence, the dissipation modulus G'' tends to $-\infty$ for sufficiently large frequencies. Looking at Eq. (2.16) this corresponds to an unphysical effect.

In the third case of $\alpha = \beta$ the dissipation modulus G'' (cf. Eq. (4.5)) can be simplified to the expression

$$(4.8) \quad G'' = G \frac{\omega^\alpha (\tau_C^\alpha - \tau_R^\alpha) \sin(\alpha\pi/2)}{1 + (\omega\tau_R)^{2\alpha} + 2(\omega\tau_R)^\alpha \cos(\alpha\pi/2)},$$

which is non-negative for any frequency if the condition $\tau_C^\alpha - \tau_R^\alpha \geq 0$, or equivalently $\tau_C \geq \tau_R$, is satisfied. If we compare (4.8) with the dissipation modulus of the standard linear solid (3.10)₂ we observe similar mathematical forms.

Practically the same properties can be found if we analyse the relaxation spectrum $H(\nu)$ of the model defined by Eq. (4.1). The common method to compute the spectrum on the basis of the dynamic modulus uses the *inverse Stieltjes transformation* (cf. GROSS [5] or TSCHOEGL [20]):

$$(4.9) \quad H(\nu) = \frac{1}{2\pi i} \lim_{\gamma \rightarrow 0} \left(\frac{G^*(-\nu - i\gamma)}{-\nu - i\gamma} - \frac{G^*(-\nu + i\gamma)}{-\nu + i\gamma} \right).$$

To calculate the spectrum on the basis of Eq. (4.9) we rewrite the complex number $-\nu \pm i\gamma$ in terms of the complex exponential function and carry out the limit transition:

$$(4.10) \quad \lim_{\gamma \rightarrow 0} -\nu \pm i\gamma = \lim_{\gamma \rightarrow 0} \sqrt{\nu^2 + \gamma^2} e^{\pm i(\pi - \arctan(\gamma/\nu))} = \nu e^{\pm i\pi}.$$

Inserting this formula into Eq. (4.9), considering the complex modulus Eq. (4.2) and elementary calculations lead to the following relaxation spectrum:

$$(4.11) \quad H(\nu) = G \frac{(\nu\tau_C)^\beta \sin(\beta\pi) - (\nu\tau_R)^\alpha \sin(\alpha\pi) + (\nu\tau_R)^\alpha (\nu\tau_C)^\beta \sin((\beta - \alpha)\pi)}{\pi\nu(1 + (\nu\tau_R)^{2\alpha} + 2(\nu\tau_R)^\alpha \cos(\alpha\pi))}.$$

Since the mathematical structure of Eq. (4.11) is similar to that of the dissipation modulus (4.5), the same changes in sign of the numerator occur, but now as a function of the relaxation frequency ν . Since the analysis would run along the same lines as before, we do not repeat it here. Let us take a look at the general expression for the rate of dissipation (2.7)₂. Since the spectrum $H(\nu)$ can be negative in certain ranges of the relaxation frequency, there may exist deformation processes which lead to a negative rate of dissipation, excluding the case of $\alpha = \beta$ with $\tau_C^\alpha - \tau_R^\alpha \geq 0$. The result that only the case $\alpha = \beta$ is physically meaningful was proposed by a different line of argumentation by BAGLEY and TORVIK [2].

This analysis has shown that the formal generalisation (type A) of the standard linear solid leads in general to *thermodynamically inconsistent models* which can have a negative dissipation modulus for certain processes. Only in the special case where the parameters of fractional differentiation of stress and strain are equal ($\alpha = \beta$), the model leads to a non-negative dissipation modulus and a non-negative spectrum. This type of a fractional model was originally applied by CAPUTO and MAINARDI [3] but no thermodynamic analysis has been carried out.

5. Fractional generalisation (Type B)

In order to specify a more physically based generalisation of the standard linear solid which is compatible with the natural laws of thermodynamics for any

values of its parameters, we first consider a so-called *fractional damping element*. It can be understood as an additional rheological element and is compatible with the Clausius Duhem inequality (2.3) (cf. LION [11]). Its free energy can easily be specified and the proof of the thermodynamic compatibility is based on the non-negativity of the relaxation spectrum. A fractional damping element is defined by the linear functional

$$(5.1) \quad \sigma(t) = E_\alpha \tau^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} \quad \text{or} \quad \sigma(t) = \frac{E_\alpha \tau^\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \dot{\varepsilon}(s) ds$$

with $0 \leq \alpha < 1$,

where the constant $\tau = 1s$ is introduced for dimensional reasons. Its relaxation spectrum $H(\nu)$ can easily be calculated (cf. TSCHOEGL [20] or LION [14]),

$$(5.2) \quad H(\nu) = \frac{E_\alpha \tau^\alpha}{\Gamma(1-\alpha)\Gamma(\alpha)\nu^{1-\alpha}},$$

and is positive for $E_\alpha > 0$ implying that the rate of dissipation (2.7)₂ is non-negative for any deformation process; thus the thermodynamical compatibility is shown. As a consequence of Eqs. (5.2) and (2.7)₁, the functional form Eq. (5.1)₂ of the fractional differential Eq. (5.1)₁ can be rewritten as

$$(5.3) \quad \sigma(t) = \frac{E_\alpha \tau^\alpha}{\Gamma(1-\alpha)} \int_0^t \left(\int_0^\infty \frac{1}{\Gamma(\alpha)\nu^{1-\alpha}} e^{-\nu(t-s)} d\nu \right) \dot{\varepsilon}(s) ds.$$

This equation can physically be interpreted and expresses that a fractional damping element in the form of Eq. (5.1) corresponds to a superposition of an infinite number of continuously distributed Maxwell elements in parallel.

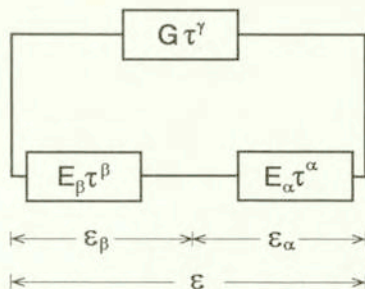


FIG. 2. Fractional linear solid, type B generalisation.
<http://rcin.org.pl>

Motivated by this discussion we replace the rheological elements of the standard linear solid in Fig. 1 by three different fractional damping elements in the form of Eq. (5.1). This idea is visualised in Fig. 2 and named the type B generalisation.

The stress σ is decomposed into the sum of two internal stresses, named σ_1 and σ_2 in Eq. (5.4) but this is not shown in the figure. In the lower branch we have a splitting of the strain ε into the sum of two internal strains ε_α and ε_β corresponding to two fractional elements in series. As we see, the model contains 6 material parameters, namely the non-negative constants E_α , E_β and G as well as three parameters of fractional differentiation $0 \leq \alpha, \beta, \gamma < 1$ and the constitutive equations read

$$(5.4) \quad \sigma = \sigma_1 + \sigma_2,$$

$$(5.5) \quad \varepsilon = \varepsilon_\alpha + \varepsilon_\beta,$$

$$(5.6) \quad \sigma_1 = G\tau^\gamma \frac{d^\gamma \varepsilon}{dt^\gamma},$$

$$(5.7) \quad \sigma_2 = E_\alpha \tau^\alpha \frac{d^\alpha \varepsilon_\alpha}{dt^\alpha},$$

$$(5.8) \quad \sigma_2 = E_\beta \tau^\beta \frac{d^\beta \varepsilon_\beta}{dt^\beta}.$$

Comparing these relations with those of the standard linear solid (3.1) – (3.5) we notice a similar structure. The fundamental difference is that the order of differentiation in (3.3) – (3.5) is given by the integers 0, 0 and 1, whereas in (5.6) – (5.8) it is given by the real numbers γ , α and β .

To eliminate the internal strains and stresses ε_α , ε_β , σ_1 and σ_2 , we first replace the strain ε_α in Eq. (5.7) using (5.5) and differentiate the result fractionally with the order β :

$$(5.9) \quad \frac{d^\beta \sigma_2}{dt^\beta} = E_\alpha \tau^\alpha \left(\frac{d^{\alpha+\beta} \varepsilon}{dt^{\alpha+\beta}} - \frac{d^{\alpha+\beta} \varepsilon_\beta}{dt^{\alpha+\beta}} \right).$$

Then we differentiate Eq. (5.8) with the order α , rearrange terms and obtain the intermediate result

$$(5.10) \quad \frac{d^{\alpha+\beta} \varepsilon_\beta}{dt^{\alpha+\beta}} = \frac{1}{E_\beta \tau^\beta} \frac{d^\alpha \sigma_2}{dt^\alpha},$$

which can be used to eliminate the internal strain ε_β in Eq. (5.9). The final result is

$$(5.11) \quad \tau^\beta \frac{d^\beta \sigma_2}{dt^\beta} = E_\alpha \tau^{\alpha+\beta} \frac{d^{\alpha+\beta} \varepsilon}{dt^{\alpha+\beta}} - \frac{E_\alpha \tau^\alpha}{E_\beta} \frac{d^\alpha \sigma_2}{dt^\alpha}.$$

To eliminate the internal stress in Eq. (5.11) we express σ_2 by means of Eqs. (5.4) and (5.6) and find

$$(5.12) \quad \tau^\beta \frac{d^\beta}{dt^\beta} \left(\sigma - G\tau^\gamma \frac{d^\gamma \varepsilon}{dt^\gamma} \right) = E_\alpha \tau^{\alpha+\beta} \frac{d^{\alpha+\beta} \varepsilon}{dt^{\alpha+\beta}} - \frac{E_\alpha \tau^\alpha}{E_\beta} \frac{d^\alpha}{dt^\alpha} \left(\sigma - G\tau^\gamma \frac{d^\gamma \varepsilon}{dt^\gamma} \right),$$

leading to the final expression in the form of

$$(5.13) \quad \tau^\beta \frac{d^\beta \sigma}{dt^\beta} + \frac{E_\alpha}{E_\beta} \tau^\alpha \frac{d^\alpha \sigma}{dt^\alpha} = G\tau^{\gamma+\beta} \frac{d^{\gamma+\beta} \varepsilon}{dt^{\gamma+\beta}} + E_\alpha \tau^{\alpha+\beta} \frac{d^{\alpha+\beta} \varepsilon}{dt^{\alpha+\beta}} + \frac{E_\alpha}{E_\beta} G\tau^{\gamma+\alpha} \frac{d^{\gamma+\alpha} \varepsilon}{dt^{\gamma+\alpha}},$$

after rearranging terms. In comparison with the type A generalisation specified by Eq. (4.1) we recognise an additional term on the right-hand side of Eq. (5.13); all terms are fractionally differentiated with a different order. If we set, for example, one of the parameters α or β to zero, we recognise that in Eq. (5.13) no strain derivative of the order 0 occurs.

For the purpose of solving the fractional differential equation (5.13) one can prescribe, for example, the deformation process ε . Since the orders α and β of fractional stress differentiation are between 0 and 1, it is sufficient to prescribe only one initial condition, namely $\sigma(0) = 0$. Taking this into account, the fractional differential operator in the form of Eq. (2.21) can be applied to differentiate the stress. The strain differentiation is more complicated because, in dependence on the values of the sums $\gamma + \beta$, $\alpha + \beta$ and $\gamma + \alpha$, the general form Eq. (2.20) of the differential operator has to be taken into account. If we have, for example, $1 \leq \gamma + \beta < 2$, we set $m = 2$ in Eq. (2.20) and have to consider the initial value of the strain rate. Assuming the initial strain $\varepsilon(0)$ to be zero we obtain

$$(5.14) \quad \frac{d^{\gamma+\beta} \varepsilon}{dt^{\gamma+\beta}} = \frac{t^{1-\gamma-\beta} \dot{\varepsilon}(0)}{\Gamma(2-\gamma-\beta)} \int_0^t s^{1-\beta-\gamma} \ddot{\varepsilon}(t-s) ds.$$

Since the exponent $1 - \gamma - \beta$ is negative, the first term incorporating the initial strain rate vanishes for large times or under stationary conditions.

5.2. Investigation of dissipation modulus and relaxation spectrum

Under harmonic loads and the assumption of stationary conditions, the analysis in the frequency domain is much easier. In this case we assume the representations Eqs. (3.7) and (3.8) for the strain and the stress, insert them into (5.13)

and apply (2.25). Then the complex modulus reads

$$(5.15) \quad G^* = G(i\omega\tau)^\gamma + \frac{E_\alpha E_\beta (i\omega\tau)^{\alpha+\beta}}{E_\beta (i\omega\tau)^\beta + E_\alpha (i\omega\tau)^\alpha}.$$

Splitting (5.15) into real and imaginary parts leads to the formulae

$$(5.16) \quad G' = G(\omega\tau)^\gamma \cos(\gamma\pi/2) + \frac{E_\alpha^2 E_\beta (\omega\tau)^{2\alpha+\beta} \cos(\beta\pi/2) + E_\alpha E_\beta^2 (\omega\tau)^{\alpha+2\beta} \cos(\alpha\pi/2)}{E_\alpha^2 (\omega\tau)^{2\alpha} + E_\beta^2 (\omega\tau)^{2\beta} + 2E_\alpha E_\beta (\omega\tau)^{\alpha+\beta} \cos((\alpha-\beta)\pi/2)},$$

$$(5.17) \quad G'' = G(\omega\tau)^\gamma \sin(\gamma\pi/2) + \frac{E_\alpha^2 E_\beta (\omega\tau)^{2\alpha+\beta} \sin(\beta\pi/2) + E_\alpha E_\beta^2 (\omega\tau)^{\alpha+2\beta} \sin(\alpha\pi/2)}{E_\alpha^2 (\omega\tau)^{2\alpha} + E_\beta^2 (\omega\tau)^{2\beta} + 2E_\alpha E_\beta (\omega\tau)^{\alpha+\beta} \cos((\alpha-\beta)\pi/2)},$$

for the storage and dissipation moduli. Taking a look at Eqs. (5.16) or (5.17) we recognise that both functions are non-negative for any value of the frequency ω and any value of the material constants compatible with $E_\alpha, E_\beta, G \geq 0$ and $0 \leq \alpha, \beta, \gamma < 1$.

Calculating the relaxation spectrum $H(\nu)$ on the basis of the inverse Stieltjes transformation Eqs. (4.9) and (4.10), a series of calculations leads to the final expression

$$(5.18) \quad H(\nu) = \frac{1}{\pi\nu} \left(G(\nu\tau)^\gamma \sin(\gamma\pi) + \frac{E_\alpha^2 E_\beta (\nu\tau)^{2\alpha+\beta} \sin(\beta\pi) + E_\alpha E_\beta^2 (\nu\tau)^{\alpha+2\beta} \sin(\alpha\pi)}{E_\alpha^2 (\nu\tau)^{2\alpha} + E_\beta^2 (\nu\tau)^{2\beta} + 2E_\alpha E_\beta (\nu\tau)^{\alpha+\beta} \cos((\alpha-\beta)\pi)} \right),$$

which is non-negative for any value of material constants $0 \leq E_\alpha, E_\beta, G$ and $0 \leq \alpha, \beta, \gamma < 1$. As a fundamental result we recognise that the fractional model of type B is compatible with the second law of thermodynamics (cf. Sec. 2.1).

5.3. Correlation between the thermodynamical consistent form of type A and the model of type B

The thermomechanical consistent form of the first generalisation, named type A, corresponds to the case of $\alpha = \beta$ in the fractional differential equation (4.1). This model can very easily and without any further investigation be derived on the basis of the type B generalisation Eq. (5.13). The only thing to do is to set $\beta = 0$ and $\gamma = 0$. The result can be interpreted as a spring with modulus G in

parallel with a fractional Maxwell element consisting of a spring with modulus E_β in series with a fractional damping element. Let us confront both equations:

$$(5.19) \quad \tau_R^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G \left(\tau_C^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon \right), \quad \frac{E_\alpha}{E_\beta} \tau^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = G \left(\frac{G + E_\beta}{G} \frac{E_\alpha}{E_\beta} \tau^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon \right).$$

As we have shown, the requirement on the material constants for the thermodynamical consistency of the type A generalisation Eq. (5.19)₁ is $\tau_C^\alpha - \tau_R^\alpha \geq 0$. If we take a look at Eq. (5.19)₂, we see that this requirement is automatically satisfied if the generalisation is based on fractional damping elements.

6. Fractional generalisation (Type C)

To discuss a further generalisation based on the fractional damping element, let us take a look at Fig. 3, where the type C generalisation is shown. The rheological model corresponds to the Kelvin-Voigt form of the three-parameter solid: a linear spring is in series with a Kelvin element.

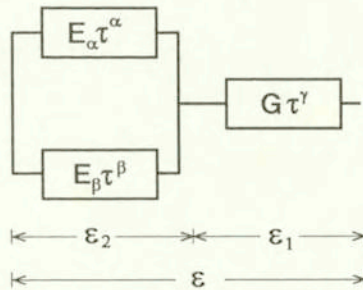


FIG. 3. Fractional linear solid, type C generalisation.

The corresponding constitutive equations read

$$(6.1) \quad \varepsilon = \varepsilon_1 + \varepsilon_2,$$

$$(6.2) \quad \sigma = \sigma_\alpha + \sigma_\beta,$$

$$(6.3) \quad \sigma = G \tau^\gamma \frac{d^\gamma \varepsilon_1}{dt^\gamma},$$

$$(6.4) \quad \sigma_\alpha = E_\alpha \tau^\alpha \frac{d^\alpha \varepsilon_2}{dt^\alpha},$$

$$(6.5) \quad \sigma_\beta = E_\beta \tau^\beta \frac{d^\beta \varepsilon_2}{dt^\beta},$$

and the fractional differential equation in the form of (6.6), where the internal variables are eliminated, can be obtained in a similar way as described above,

$$(6.6) \quad \tau^\gamma \frac{d^\gamma \sigma}{dt^\gamma} + \frac{E_\beta}{G} \tau^\beta \frac{d^\beta \sigma}{dt^\beta} + \frac{E_\alpha}{G} \tau^\alpha \frac{d^\alpha \sigma}{dt^\alpha} = E_\alpha \tau^{\alpha+\gamma} \frac{d^{\alpha+\gamma} \varepsilon}{dt^{\alpha+\gamma}} + E_\beta \tau^{\beta+\gamma} \frac{d^{\beta+\gamma} \varepsilon}{dt^{\beta+\gamma}}.$$

In comparison with (5.13), this equation contains three terms depending on the stress σ and two depending on the strain ε . The dynamic modulus G^* is calculated under the assumption of stationary conditions and harmonic stress and strain processes and reads

$$(6.7) \quad G^* = \frac{GE_\alpha(i\omega\tau)^{\alpha+\gamma} + GE_\beta(i\omega\tau)^{\beta+\gamma}}{G(i\omega\tau)^\gamma + E_\beta(i\omega\tau)^\beta + E_\alpha(i\omega\tau)^\alpha}.$$

Splitting the modulus G^* into its real and imaginary parts G' and G'' and calculating the relaxation spectrum on the basis of the inverse Stieltjes transformation specified by Eqs. (4.9) and (4.10), leads to

$$(6.8) \quad G'(\omega) = \frac{g'(\omega)}{N(\omega)}, \quad G''(\omega) = \frac{g''(\omega)}{B(\omega)}, \quad H(\nu) = \frac{h(\nu)}{M(\nu)},$$

with

$$(6.9) \quad g' = G^2(\omega\tau)^{2\gamma} \left(E_\alpha(\omega\tau)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + E_\beta(\omega\tau)^\beta \cos\left(\frac{\beta\pi}{2}\right) \right) \\ + G(\omega\tau)^\gamma \cos\left(\frac{\gamma\pi}{2}\right) \left(E_\alpha^2(\omega\tau)^{2\alpha} + E_\beta^2(\omega\tau)^{2\beta} \right. \\ \left. + 2E_\alpha E_\beta(\omega\tau)^{\alpha+\beta} \cos\left(\frac{(\alpha-\beta)\pi}{2}\right) \right),$$

$$(6.10) \quad g'' = G^2(\omega\tau)^{2\gamma} \left(E_\alpha(\omega\tau)^\alpha \sin\left(\frac{\alpha\pi}{2}\right) + E_\beta(\omega\tau)^\beta \sin\left(\frac{\beta\pi}{2}\right) \right) \\ + G(\omega\tau)^\gamma \sin\left(\frac{\gamma\pi}{2}\right) \left(E_\alpha^2(\omega\tau)^{2\alpha} + E_\beta^2(\omega\tau)^{2\beta} \right. \\ \left. + 2E_\alpha E_\beta(\omega\tau)^{\alpha+\beta} \cos\left(\frac{(\alpha-\beta)\pi}{2}\right) \right),$$

$$(6.11) \quad N = E_\alpha^2(\omega\tau)^{2\alpha} + E_\beta^2(\omega\tau)^{2\beta} + G^2(\omega\tau)^{2\gamma} + 2E_\alpha G(\omega\tau)^{\alpha+\gamma} \cos\left(\frac{(\alpha-\gamma)\pi}{2}\right) \\ + 2E_\beta G(\omega\tau)^{\beta+\gamma} \cos\left(\frac{(\beta-\gamma)\pi}{2}\right) + 2E_\alpha E_\beta(\omega\tau)^{\alpha+\gamma} \cos\left(\frac{(\alpha-\gamma)\pi}{2}\right),$$

$$\begin{aligned}
 (6.12) \quad h = & G^2(\nu\tau)^{2\gamma}(E_\alpha(\nu\tau)^\alpha \sin(\alpha\pi) + E_\beta(\nu\tau)^\beta \sin(\beta\pi)) \\
 & + G(\nu\tau)^\gamma \sin(\gamma\pi)(E_\alpha^2(\nu\tau)^{2\alpha} + E_\beta^2(\nu\tau)^{2\beta}) \\
 & + 2E_\alpha E_\beta(\nu\tau)^{\alpha+\beta} \cos((\alpha - \beta)\pi),
 \end{aligned}$$

$$\begin{aligned}
 (6.13) \quad M = & E_\alpha^2(\nu\tau)^{2\alpha} + E_\beta^2(\nu\tau)^{2\beta} + G^2(\nu\tau)^{2\gamma} + 2E_\alpha G(\nu\tau)^{\alpha+\gamma} \cos((\alpha - \gamma)\pi) \\
 & + 2E_\beta G(\nu\tau)^{\beta+\gamma} \cos((\beta - \gamma)\pi) + 2E_\alpha E_\beta(\nu\tau)^{\alpha+\gamma} \cos((\alpha - \gamma)\pi).
 \end{aligned}$$

Taking a look at the relations (6.8) – (6.13) we see that both the dynamic moduli $G'(\omega)$ and $G''(\omega)$ and the relaxation spectrum $H(\nu)$ are non-negative functions of their arguments for any values of material parameters satisfying $E_\alpha, E_\beta, G \geq 0$ and $0 \leq \alpha, \beta, \gamma < 1$.

6.1. Correlation between the thermodynamical consistent form of type A and the model of type C

The thermomechanical consistent formulation of the type A generalisation, i.e. the case $\alpha = \beta$, can also be derived on the basis of the type C generalisation (6.6). If we set $\beta = 0$ and $\gamma = 0$ we obtain the fractional differential equation in the form of (6.14)₂:

$$\begin{aligned}
 (6.14) \quad \tau_R^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma = & G \left(\tau_C^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon \right), \quad \frac{E_\alpha}{G + E_\beta} \tau^\alpha \frac{d^\alpha \sigma}{dt^\alpha} + \sigma \\
 = & \frac{GE_\beta}{G + E_\beta} \left(\frac{E_\alpha}{E_\beta} \tau^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} + \varepsilon \right).
 \end{aligned}$$

As shown above, the requirement on the material parameters to satisfy the thermodynamical consistency of (6.14)₁ is $\tau_C^\alpha - \tau_R^\alpha \geq 0$. Comparing (6.14)₁ and (6.14)₂ we notice that this condition is automatically satisfied if the fractional generalisation of the standard linear solid is based on the rheological model visualised in Fig. 3.

7. Discussion

In this paper we demonstrate that the quite formal generalisation of linear differential equations by replacing the first order time derivatives by fractional derivatives leads in general to thermodynamically inconsistent constitutive models. We show that the relaxation spectrum and the dissipation modulus can become negative, so that the natural laws of thermodynamics are violated. In order to avoid these problems we propose a more physically-based method to

formulate constitutive relations of the fractional type: to this end an additional rheological element, a so-called *fractional damping element*, is introduced which is compatible with the natural laws of thermodynamics. The idea of the proposed method is to replace the Hookean springs and Newtonian dashpots in a given rheological network by rheological elements of the fractional type. For two special examples we show that the relaxation spectra and the dynamic moduli are non-negative for arbitrary values of the material parameters and the independent process variables. It is obvious that the proposed method of generalisation leads not to the most general form of a fractional constitutive model but it leads to a thermodynamically consistent model. We are sure that this method can also be applied to formulate more complicated models of the fractional type and to formulate three-dimensional stress/strain relations. In the isotropic case one has only to replace the uniaxial stress and strain variables by the stress and strain deviators and to formulate corresponding relations for the hydrostatic pressure and the volume deformation.

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