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Modeling of elastic slab with periodic breaks

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An elastic slab having periodic breaks across its cross-section is loaded by space-harmonic forces applied to its surfaces. Due to the breaks, the slab displacement response to the load includes a series of space-harmonics. A method is proposed for evaluation of the slab harmonic response within a presumed spectral domain.

1. Introduction

Periodic structures are frequently encountered in mechanical constructions: as a modern example recall a space-shuttle frame with periodically attached ceramic tiles of comparable stiffness. Other examples are: 1) an elastic body (a halfspace, for instance) with surface-breaking or subsurface cracks, 2) similarly looking is the composite ultrasonic transducer with deep periodic cuts that lower its acoustic impedance, 3) a plate with ribs, composite structure with periodic fillings between bonds, etc. In all cases the period of the structure (the period of elastic slabs that can be distinguished as attached to a solid frame) can be comparable to a wavelength of the applied traction, so that any simplifications like homogenization or equivalent discrete loading is not appropriate for serious analysis. Also note that in all the above cases, there is a uniform frame that keeps the slabs in order.

The solid frames in the above examples, an elastic halfspace or an elastic plate, can be conveniently characterized by a planar harmonic Green's function that is the relation between the surface displacement and the surface traction for any spatial frequency of these surface wave-fields, say $u = GT$ in standard notation. If we know the similar description for the layer of periodic slabs mentioned above, say $u_s = HT_s$, then the boundary-value problem for static deformation or vibration of the structure can be formulated like $u = u_s$, $T = T_s$, on the contact plane. Evaluation of H is the ultimate goal of the analysis; simple one-dimensional structure is considered in this paper (Fig. 1) but there are no substantial obstacles in generalization of the presented method.

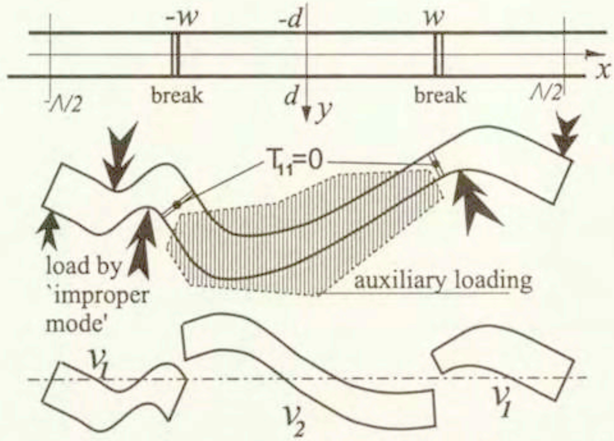

FIG. 1. The analyzed elastic slab with periodic breaks, and the slab deformed under the load of 'improper mode' in the domain $(-\Lambda/2, -w) + (w, \Lambda/2)$ (with auxiliary load in the shadowed area that allows the first part of the slab to relax at breaks). This, and the similarly evaluated deformation in the other domain, yield together what is shown in the bottom figure. Note the displacement discontinuity at breaks.

The vibration (or static deformation) of such structures poses a rather difficult problem that, naturally, can be solved using purely numerical methods. Analytical approach, being valuable due to its ability to bring better understanding of the analyzed structure, requires models of its components; here – a model of a broken slab. It is evident that the broken slab works as an entity only when attached to a certain solid frame (otherwise its periodic components would spill in chaos). This must be taken into account by allowing suitable functional space for deformation and stress fields in the system.

A natural choice is a domain of harmonic functions; moreover, the harmonic analysis is a primary tool in the theory of periodic structures [1]. Here, the mechanical field is expanded in a Bloch series like this

$$\Psi(x, y) = \sum_n \Psi^{(n)}(y) e^{-j(r+nK)x}, \tag{1.1}$$

where $K = 2\pi/\Lambda, \Lambda$ is the period of breaks distributed along the x-axis of the $2d$-thick elastic slab (Fig. 1, here the field is considered to be independent of z, and z-component of displacement is neglected for simplicity; also note that there are two breaks per period); r is a spectral variable, and $0 < r < K$ to avoid ambiguity (the inverse Fourier transform becomes an integral over r, any aperiodic field can be represented by the above series [2]).

The planar harmonic Green's function (actually a matrix $\mathbf{H}$ dependent on r) characterizing the analyzed slab is sufficient for most applications. It is involved in the relation

$$[\mathbf{u}^{(n)}] = [\mathbf{H}^{(nm)}][\mathbf{T}^{(m)}], \tag{1.2}$$

where $\mathbf{u}$ is the particle displacement vector on either of the slab surfaces ($y = \mp d$); $\mathbf{T}$ is the corresponding traction (surface load) on these surfaces, and n, m are the harmonic numbers in the Bloch series (1.1). It is convenient to define $\mathbf{u}^{(n)}$ and $\mathbf{T}^{(m)}$ as 4-dimensional vectors describing fields on both slab surfaces, $y = -d$ d, explicitly: $\mathbf{u} = [u_1^-, u_2^-, u_1^+, u_2^+]^T$, and similarly $\mathbf{T} = [T_{21}^-, T_{22}^-, T_{21}^+, T_{22}^+]^T$, with superscripts $-, +$ referring to $y = \mp d$.

For a homogeneous slab without breaks, all matrices $\mathbf{H}^{(nm)} = 0$ except for $n = m$, and $\mathbf{H}^{(nn)}$ dependent on spectral variable $p = r + nK \in (-\infty, \infty)$, can be easily derived analytically (Appendix) from the known equations of motion of isotropic homogeneous elastic layer [3 – 5]; this is an ordinary boundary-value problem of mechanics [6] discussed briefly in the next section below. The slab with breaks however, includes all matrices $\mathbf{H}^{(mn)}$ because of its periodic inhomogeneity. This is the aim of this paper to propose a method for their evaluation. The method is described on a certain simple example.

2. Fields in homogeneous slab

Deformation of isotropic homogeneous elastic slab is governed by a system of second order differential equations

$$\mathcal{L}\mathbf{u} = 0, \tag{2.1}$$

which, assuming x-dependence in harmonic form $\exp(-jpx)$ (and also $\exp j\omega t$ in the case of time-dependent vibrations), leads to the system

$$\begin{aligned} &\mu \frac{d^2}{dy^2}u_x - p^2(\lambda + 2\mu)u_x - jp(\lambda + \mu)\frac{d}{dy}u_y = 0, \\ &(\lambda + 2\mu)\frac{d^2}{dy^2}u_y - p^2\mu u_y - jp(\lambda + \mu)\frac{d}{dy}u_x = 0, \end{aligned}$$

in the example for isotropic elastic body characterized by Lamé constants μ, λ, under a possible static surface load that is to be accounted for in boundary conditions, but without any internal forces [6].

The system (2.1) has four partial solutions $\phi_k(y)$; conditions $\phi_k(y = 0) = 1$ normalize these solutions. The field inside the slab is

$$\begin{aligned} u_i(x, y) &= \textstyle\sum_k \tilde{U}_{i(k)}\phi_k(y)F_k e^{-jpx}, \\ T_{ij}(x, y) &= \textstyle\sum_k \tilde{T}_{ij(k)}\phi_k(y)F_k e^{-jpx}, \end{aligned} \tag{2.2}$$

where coefficients F_k can be evaluated from the boundary conditions of the given boundary-value problem,

$$T_{2i}(y = \mp d) = \sum_k \tilde{T}_{2i(k)}\phi_k(y = \mp d)F_k, \tag{2.3}$$

for instance (neglecting harmonic dependence on x), which substituted into Eqs. (2.2) yield Eq. (1.2) presented earlier, with substitution $p = r + nK$.

The field at $y = 0$ is

$$\begin{aligned} u_i(x) &= \textstyle\sum_k \bar{U}_{i(k)}F_k e^{-jpx}, \\ T_{11}(x) &= \textstyle\sum_k \bar{T}_{(k)}F_k e^{-jpx}, \end{aligned} \tag{2.4}$$

for instance, due to $\phi_k(0) = 1$.

3. Slab with breaks – deformation modes

For simplicity of the presented example, only the stress component T_{11} is required to vanish at breaks and only at the center of the slab cross-section (at $y = 0$). There are two required conditions, at $x = -w$ and $x = w$ ($w < \Lambda/2$, Fig. 1):

$$T_{11}|_{x=-w} = 0, \text{ and } T_{11}|_{x=w} = 0. \tag{3.1}$$

The involved stress depends on the applied Bloch expansion components,

$$T_{11}(x = -w, y = 0) = \sum_n \bar{\mathbf{T}}^{(n)}\mathbf{F}^{(n)}e^{j(r+nK)w}, \tag{3.2}$$

$$T_{11}(x = -w, y = 0) = \sum_n \bar{\mathbf{T}}^{(n)}\mathbf{F}^{(n)}e^{-j(r+nK)w}, \tag{3.3}$$

where each $\mathbf{F}^{(n)}$ is a 4-dimensional column vector $[F_k]$, and similarly $\bar{\mathbf{T}}^{(n)}$. The summation over harmonics is limited to finite numbers, say $n \in (-N, N]$; this defines the representation space for the mechanical field in the slab. Applying large N, one gets better representation at the cost of computation time.

Now, it is necessary to evaluate a vector $\mathbf{F}$ (comprising all $\mathbf{F}^{(n)}$) that minimizes the divergence of the solution from the conditions (3.2); explicitly, a minimum is sought of

$$\mathcal{E} = \mathbf{F}'\mathbf{A}'\mathbf{A}\mathbf{F}, \quad \mathbf{A}^{(n)} = \begin{bmatrix} \bar{\mathbf{T}}^{(n)}e^{j(r+nK)w} \\ \bar{\mathbf{T}}^{(n)}e^{-j(r+nK)w} \end{bmatrix}, \tag{3.4}$$

where $\mathbf{F}'$ and $\mathbf{A}'$ are Hermitian conjugate to column vectors $[\mathbf{F}^{(n)}]$ and $[\mathbf{A}^{(n)}]$, correspondingly, and $\bar{\mathbf{T}}^{(n)}$ depends on all $\bar{T}_{(k)}$ of Eqs. (2.4).

It is evident that the null-space $\mathbf{O}$ of $\mathbf{E} = \mathbf{A}'\mathbf{A}$ solves the problem. In this particular case, the size of the null-space is $2N - 2$, that is shorter by 2 (2 is the number of conditions at breaks) from the whole space dimension $2N$. The *MATLAB* [7] is a convenient tool for computation of $\mathbf{O}$.

Each vector $\mathbf{F}$ from the null-space $\mathbf{O}$ satisfies conditions (3.1) as accurately as possible within the assumed representation space, and thus they should be accounted for in the displacement and stress fields of both the homogeneous (for which all the above equations were derived), and the broken slabs as well. This is very important because they constitute the majority of possible vectors in the considered space. We call these solution the 'proper modes' of deformation of homogeneous slab because they simultaneously satisfy the conditions for the broken slab.

Using Eqs. (2.2), it is possible to evaluate all harmonic components of displacement and traction on the slab surfaces $y = \mp d$

$$[\mathbf{u}^{(n)}] = \tilde{\mathbf{U}}\mathbf{O}\boldsymbol{\alpha}, \ [\mathbf{T}^{(n)}] = \tilde{\mathbf{T}}\mathbf{O}\boldsymbol{\alpha}, \tag{3.5}$$

where $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{T}}$ are somewhat rearranged matrices $[\tilde{U}_{i,(k)}(r+nK)\phi(\mp d)]$ and $[\tilde{T}_{i,(k)}(r+nK)\phi(\mp d)]$ that appeared earlier in Eqs. (2.2), and $\boldsymbol{\alpha}$ is a column vector of 'modal amplitudes' (note that the same amplitudes appear in both the above equations).

In fact, the amplitudes $\boldsymbol{\alpha}$ are the least important as we seek the slab characterization in the form of Eq. (1.2), where only the Bloch harmonics of $\mathbf{u}$ and $\mathbf{T}$ are involved. There is no way to eliminate $\boldsymbol{\alpha}$ from Eqs. (3.4) however, because this system of equations is incomplete. There are too few unknowns $\boldsymbol{\alpha}$, because the dimension of $\mathbf{O}$ is smaller by 2 (in this example) from the number of involved harmonics ($2N$).

4. Improper modes

The null-space is only by 2, in our example, smaller than the whole space defined by the matrix $\mathbf{E}$, thus there are only 2 'improper modes' $\mathbf{Z}$ which are eigenvectors of $\mathbf{E}$ not belonging to $\mathbf{O}$. In general, these 'modes' do not satisfy conditions (3.1). The surface traction associated with these modes can be evaluated from the second of Eqs. (3.4), replacing $\mathbf{O}$ by $\mathbf{Z}$ and $\boldsymbol{\alpha}$ by $\boldsymbol{\beta}$,

$$[\mathbf{T}^{(n)}] = \sum_n \tilde{\mathbf{T}}\mathbf{Z}\boldsymbol{\beta} = \mathbf{T}_z, \tag{4.1}$$

in the spectral representation. The first equation of (3.4) is useless here because it concerns the unbroken homogeneous slab where T_{11} does not vanish at $x = \mp w$. It is evident that the broken slab will deform differently under the load $\mathbf{T}_z$. Due to the breaks, the displacement may suffer a jump at the breaking points. It is the task of this section to evaluate the broken slab response (in its surface displacements) to the 'modal' loading represented by Eq. (4.1).

The following trick is applied to evaluate this response. There are two breaks per period, at $x = \mp w$. Let us define a periodic 'window function' first that will be used to discriminate the domains between and outside breaks:

$$f(x) = \begin{cases} 1, -\Lambda/2 < x < -1, \\ 0, -w < x < w, \\ 1, w < x < \Lambda/2, \end{cases} \tag{4.2}$$

it is different from zero outside the breaks ($1 - f$ has a support between breaks), and periodic in the remaining domain of x. Due to the breaks, the slab displacements under the load

$$\mathbf{t}_1^{(k)}(x) = f(x)\mathcal{F}^{-1}\{\mathbf{T}_z^{(k)}\}, \ \mathbf{T}_z^{(k)} = \tilde{\mathbf{T}}\mathbf{Z}_k \tag{4.3}$$

is rather constrained to the domain of $\mathbf{t}^{(k)}$, that is to the support of f (neglecting certain residual internal tractions at breaks that can build up in the applied approximation), and independent of the load $\mathcal{F}^{-1}\{\mathbf{T}_z\}(1-f)$ in the other domain. In the above equation, k is the number that counts the considered mode $\mathbf{Z}_k$, Eq. (4.1), and $\mathcal{F}^{-1}$ means the inverse Fourier transform (actually the fast Fourier transform, FFT, is used in computations).

Let us evaluate the slab response in the support of f first. Applying the equations for a homogeneous unbroken slab, the spectral representation to $\mathbf{t}_1^{(k)}$ is evaluated first. The resulting $T_{11}(x = \mp d, y = 0)$ from Eqs. (2.4) indicates how much the evaluated solution (2.1 – 4) differs from the conditions at breaks. Considering still the homogeneous slab, an auxiliary load is applied in the other domain

$$\mathbf{t} = \sum_k \gamma_k \mathcal{F}^{-1}\{\mathbf{T}_z^{(k)}\}(1 - f) \tag{4.4}$$

being a combination of 'improper modes' ($k = 1, 2$ in our example) to make the conditions at breaks (7) satisfied. There are two conditions, and two constants γ_k, which can be evaluated to satisfy these conditions. The physical interpretation of this step relies on helping, by means of $\mathbf{t}$, the slab in the other domain to deform properly in order to allow the slab in the first domain to deform freely at breaks $x = \mp w$, that is to help realizing the conditions $T_{11} = 0$ there. In other words, we apply (by means of $\mathbf{t}$) additional T_{11} at breaks to compensate the force exerted by the other part of the slab that would not allow it to deform freely there.

Under both combined loads, $\mathbf{t}_1 + \mathbf{t}$, the slab deformation $\mathbf{v}_1^{(k)}$ can be evaluated from Eqs. (2.2) (this requires evaluation of the corresponding F_k first, from the second equation). We are interested only in the displacements in the domain of the applied load $\mathbf{t}_1^{(k)}$. This is $\mathbf{v}_1 f$: the *broken* slab response to the load $\mathbf{t}_1$, with vanishing stress at breaks (within the applied approximation); both the analyzed stress and displacements belonging to the same domain of x. Here, the displacement is evaluated with accuracy to the additive 'proper' modes which yield zero stress at breaks; the ambiguity to be removed by means of the energy conservation law. Some computed results are shown in Fig. 2.

Next, the other domain is analyzed in the same manner. It suffices to remark here that, replacing f by $1-f$, the analysis repeats the above presented one. The result is $\mathbf{v}_2^{(k)}$ – a response to the load $\mathbf{t}_2^{(k)}$ in the domain $(-w, w)$. Finally

$$\mathbf{v}^{(k)}(x) = \mathbf{v}_1^{(k)} f + \mathbf{v}_2^{(k)}(1-f), \text{ and } \mathbf{T}(x) = \mathbf{t}_1^{(k)} + \mathbf{t}_2^{(k)}, \tag{4.5}$$

define the corresponding functions over the whole domain of x, moreover $\mathbf{T}(x)$ is exactly $\mathbf{T}_z$ from Eq. (4.1) in the spatial domain. The Fourier transform of $\mathbf{v}$ yields the needed spectral representation $\mathbf{u}^{(n)}$. Further on, for loads being a combination of 'improper modes' $\tilde{\mathbf{T}}\mathbf{Z}_k\beta_k$, the n-th Bloch component of the displacement vector is

$$[\mathbf{u}^{(n)}] = [\mathcal{F}\{\mathbf{v}^{(k)}\}]\boldsymbol{\beta}, \quad [\mathbf{T}^{(n)}] = \tilde{\mathbf{T}}\mathbf{Z}\boldsymbol{\beta}, \tag{4.6}$$

the second equation being repeated after (4.1) for convenience.

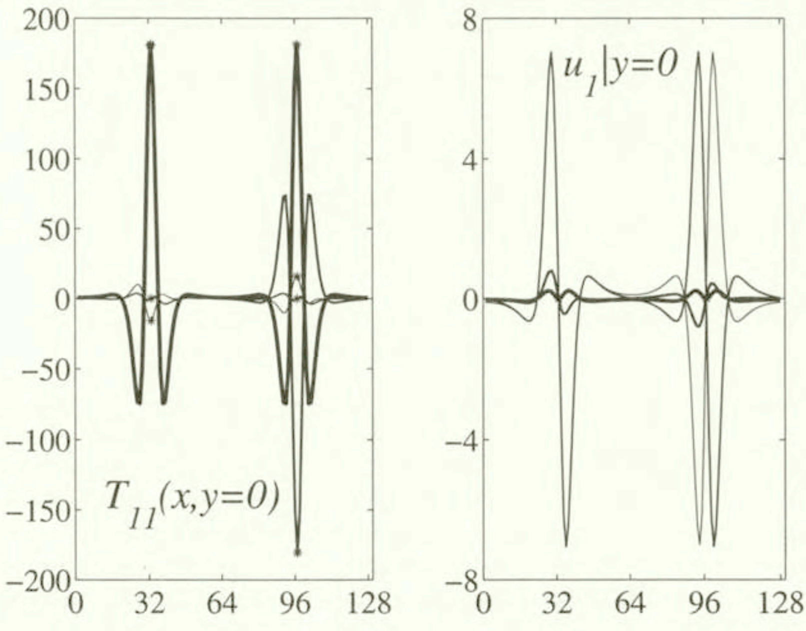

FIG. 2a.

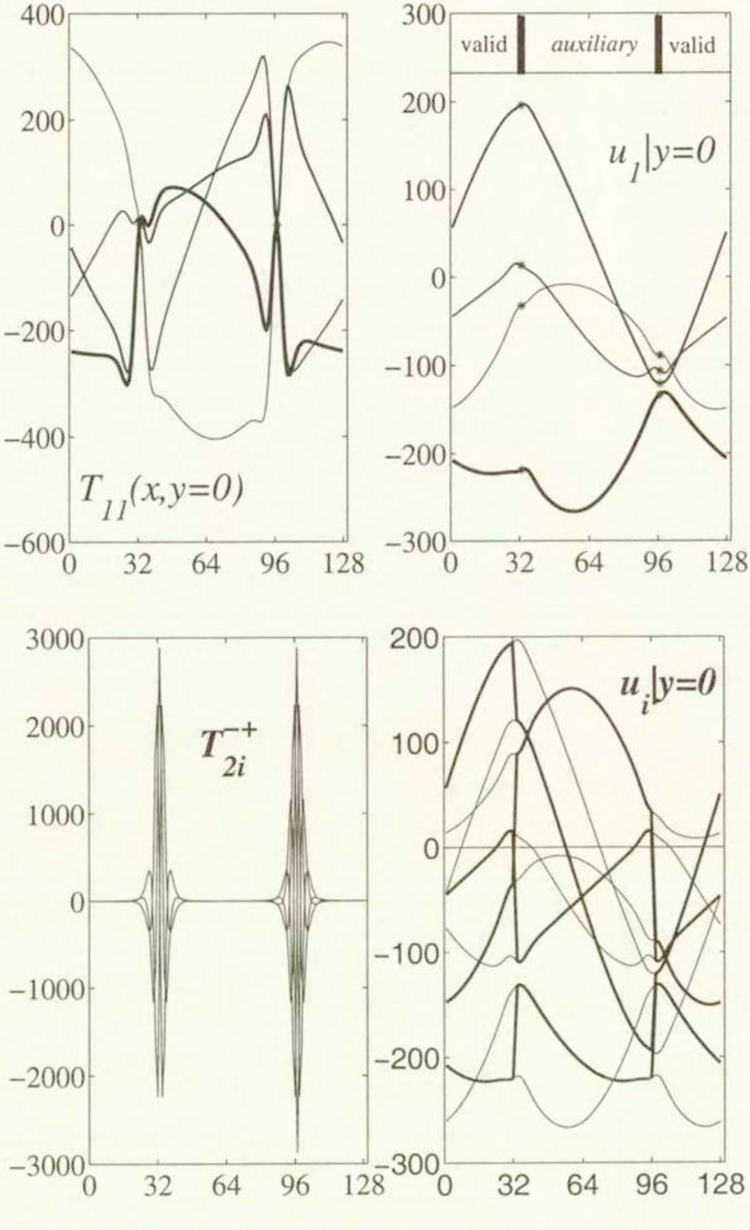

FIG. 2. Real (thick lines) and imaginary parts (thin lines) of $T_{11}(x)$ (left figure) for two 'improper modes.' Note that T_{11} has a maximum rather than zero at breaks; the corresponding homogeneous slab deformation is shown at right. Center: T_{11} and u_1 at $y = 0$ resulting from the loading from (4.3-4) with γ_k evaluated in order to set T_{11} to zero at breaks. Bottom: real and imaginary parts of slab surface traction $T_{21}^{\mp}$ corresponding to 'improper modes' (left figure) and the evaluated broken slab response, u_i at $y = 0$ (both complex parts are shown). Note the displacement discontinuity at breaks (thick lines); thin lines represent $v_{1,2}$ from Eq.(4.5), but evaluated at $y = 0$.

Now, we can join both Eq.(3.4) and the above Eq.(4.6) to obtain equal numbers of harmonics ($2N$) and unknown coefficients ($\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ combined). They can be eliminated yielding the dependence between $\mathbf{u}^{(n)}$ and $\mathbf{T}^{(m)}$ directly.

5. Discussion

A generalization of the above method in order to satisfy $T_{1i} = 0$, $i = 1, 2$ at many points in the slab cross-section, is straightforward. It requires only the evaluation of the corresponding matrix $\mathbf{A}$ resulting in a new $\mathbf{E}$. It will have smaller null-space $\mathbf{O}$, thus more computations will be needed to evaluate numerous 'improper modes' and the corresponding responses $\mathbf{v}$ of larger number.

Naturally, other conditions can be formulated at breaks, for example using integration of $|T_{1i}|^2$ over the slab cross-section. This results only in a minor modification to $\mathbf{E}$ and the rest of the analysis remains as presented above. In this case however, there may be null-space in no rigorous meaning. But it always exist within the computational accuracy and this suffices for the analysis. Indeed, the solution differs from the conditions at the break by $\|\lambda\mathbf{F}\|$, that is small for small eigenvalues $|\lambda|$ of $\mathbf{E}$.

In the above example, only the traction modal load has been analyzed. Analogously, the modal displacement can be set at the slab surfaces and the traction response sought for. Both results help to find the slab characterization in the form of Eq. (1.2).

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Appendix

Planar harmonic Green's function $\mathbf{H}$ for homogeneous elastic slab of a material characterized by Lamé constants λ, μ (in standard notation) and mass density ρ, vibrating with frequency ω (this is zero in statics), possesses the following symmetry:

$$\text{(A.1)} \qquad \begin{bmatrix} \circ & \triangleleft & \triangleright & \cdot \\ -\triangleleft & \bullet & \cdot & * \\ -\triangleright & \cdot & -\circ & \triangleleft \\ \cdot & -* & -\triangleleft & -\bullet \end{bmatrix}.$$

Its matrix elements are:

$$
\begin{aligned}
H_{11} &= js_t k_t^2[(1-L^2T^2)w + (x^2-z)(T^2-L^2)]/D = -H_{33},\\
H_{12} &= jp[(1+L^2T^2)w(x-2s_l s_t) + (z-x^2)(L^2+T^2)(x+2s_l s_t)\\
&\qquad + 4LTx(2xs_l s_t - z)]/D = -H_{21} = -H_{43} = -H_{34},\\
H_{13} &= -j2s_t k_t^2[x^2T(1-L^2) + zL(1-T^2)]/D = -H_{31},\\
H_{14} &= j4ps_l s_t k_t^2 x[L(1+T^2) - T(1+L^2)]/D = H_{23} = H_{32} = H_{41},\\
H_{22} &= \;!js_t k_t^2[(1-L^2T^2)w + (x^2-z)(L^2-T^2)]/D = -H_{44},\\
H_{24} &= -j2s_l k_t^2[L(1-T^2)x^2 + T(1-L^2)z]/D = -H_{42},\\
k_t^2 &= \rho\omega^2/\mu,\ D = \mu[(1-L^2)(1-T^2)w^2 + 4x^2z(L-T)^2],\\
k_l^2 &= \rho\omega^2/(\lambda+2\mu),\quad s_{l,t} = \sqrt{k_{l,t}^2 - p^2},\quad p < k_{l,t},\\
&\qquad \text{otherwise } = -j\sqrt{p^2 - k_{l,t}^2},\\
x &= s_t^2 - p^2,\quad z = 4p^2 s_l s_t,\quad w = x^2 + z,\quad T = \exp(-j2s_t d),\\
L &= \exp(-j2s_l d).
\end{aligned}
\tag{A.2}
$$

Applying $d \to \infty$ splits the above matrix into independent two of dimensions 2×2, being Green's matrix functions of the upper, $y > 0$, or lower, $y < 0$, elastic halfspaces. It suffices to take the limits $T \to 0$ and $L \to 0$ simultaneously.

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