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Onset of convection in a sparsely packed porous layer with throughflow

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THE ONSET of Rayleigh-*Bénard* convection in a sparsely packed porous layer with vertical throughflow is investigated using Brinkman's modification of the Darcy flow model with fluid viscosity different from effective viscosity. The critical Rayleigh numbers are obtained for free-free, rigid-rigid and rigid-free boundaries which are insulated to temperature perturbations. It is noted that an increase in the value of viscosity ratio is to delay the onset of convection. Further, it is observed that the throughflow can be used either to suppress or augment convection, depending on the nature of boundaries and also on the values of physical parameters.

1. Introduction

THE POROUS-*BÉNARD* problem has been widely studied in recent decades owing to its numerous fundamental and industrial applications. Some examples of interest can be found in the thermal insulation engineering, water movements in geothermal reservoirs, underground spreading of chemical waste, geothermal engineering and enhanced recovery of petroleum reservoirs. The growing volume of works devoted to this topic is amply documented in the reviews by NIELD [1], TIEN and VAFAI [2], KAKAC *et al.* [3], KAVIANY [4] and NIELD and BEJAN [5].

Several applications, such as fixed-bed catalytic reactors and in situ coal gasification, involve the non-isothermal flow of fluids through porous media, which is called throughflow. The effect of throughflow in a porous medium was first studied by WOODING [6], who treated the case in which the basic-state temperature field is dominated by the convective effects on the throughflow. Later SUTTON [7] presented a linear stability analysis for small throughflow with rigid and conducting boundaries at both top and bottom and insulating walls at the sides. HOMSY and SHERWOOD [8] extended the analysis of [7] to larger throughflow by considering an infinite horizontal domain. All these investigators concluded that the throughflow always stabilizes the convection. JONES and PERSICHETTI [9] investigated the convective instability in a porous medium with throughflow and assumed that the boundaries are conducting and either permeable or impermeable. They found that a small amount of throughflow can provide a destabilizing effect in at least one situation, when the throughflow is from a more restrictive boundary (the dynamically rigid boundary) towards a less restrictive boundary (the dynamically free boundary). They thought that this behaviour may be due to numerical inaccuracy and did not offer any reason for the same. NIELD [10] and SHIVAKUMARA [11] indicated that the destabilization did occur for the above type of situation and a physical explanation for the same was offered. All these studies assumed Darcy flow behaviour in which inertia and viscous effects are neglected.

However, it is well known that in many cases involving porous media with high permeability, the viscous effects due to frictional drag at the boundary and the inertia effects are significant, particularly at high Peclet numbers. These effects are studied recently by SHIVAKUMARA [12] through the use of the Brinkman extended Darcy model. The analysis is limited to the case of the effective viscosity equal to fluid viscosity. But the recent work of GIVLER and ALTOBELLI [13] suggests that this assumption does not result in a good agreement with experiments for high porosity porous media. Therefore, a theoretical solution that is general enough to yield accurate results for porous media having high permeability is of fundamental and practical interest. This is the object of the present paper.

In the present study, the linear stability characteristics of a sparsely packed porous layer with simultaneous temperature gradient and a vertical throughflow is examined using Brinkman's modification of the Darcy model. The effective viscosity is taken to be different from fluid viscosity. The boundaries are assumed to be either rigid/free and insulated to temperature perturbations. The resulting eigenvalue problem is solved in Sec. 3 using regular perturbation technique with wavenumber **a** as a perturbation parameter. The results obtained are discussed in Sec. 4.

2. Mathematical formulation

We consider a fluid-saturated horizontal porous layer of depth d with a constant throughflow of magnitude w_0 in the vertical direction. The physical configuration is shown in Fig. 1. Cartesian axes are chosen with the z-axis directed vertically upwards in the gravitational field. The lower boundary $z = 0$ and the upper boundary $z = d$ are maintained at uniform temperatures T_0 and T_1 $(T_1 < T_0)$, respectively.

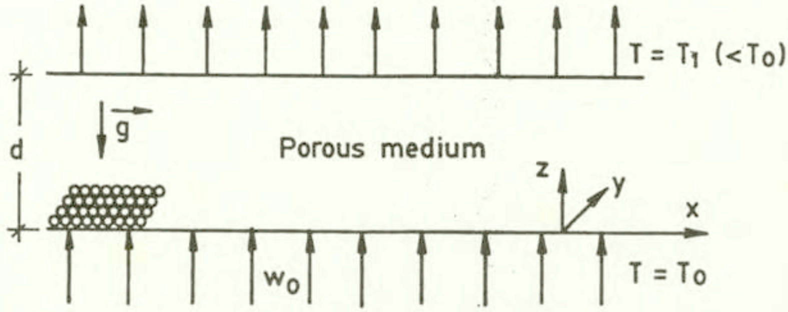

FIG. 1. Physical configuration.

The governing equations are:

$$\nabla \cdot \mathbf{q} = 0\,, \tag{2.1}$$

$$\rho_0 \left[\frac{1}{\phi}\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\phi^2}\,(\mathbf{q}\cdot\nabla)\mathbf{q} \right] = -\nabla p + \rho \mathbf{g} - \frac{\mu}{k}\,\mathbf{q} + \tilde{\mu}\,\nabla^2 \mathbf{q}\,, \tag{2.2}$$

$$A\frac{\partial T}{\partial t} + (\mathbf{q}\cdot\nabla)T = \kappa \nabla^2 T\,, \tag{2.3}$$

$$\rho = \rho_0\,[1 - \alpha\,(T - T_0)]\;. \tag{2.4}$$

Here $\mathbf{q} = (u, v, w)$ is the velocity vector, T the temperature, p the pressure, $A = (\rho c_p)_m/(\rho c_p)_f$ the ratio of specific heats, κ the effective thermal diffusivity, μ the fluid viscosity, $\tilde{\mu}$ the effective viscosity, ρ the density, α the volumetric expansion coefficient, $\mathbf{g}$ the gravitational acceleration, k the permeability, ϕ the porosity and c_p the heat capacity at constant pressure. The subscripts m and f refer to the fluid-solid mixture and the fluid respectively.

The basic steady state solution is given by

$$\mathbf{q}_b = w_0\hat{k},\ \theta_b(z) = T_b - T_0 = (T_1 - T_0)\,\frac{(1 - e^{w_0 z/\kappa})}{(1 - e^{w_0 d/\kappa})}\,, \tag{2.5}$$

where $\hat{k}$ is the unit vector in the increasing z – direction and the subscript b denotes the basic state. Note that the basic temperature distribution deviates from linear to nonlinear in z due to throughflow which has a significant influence on the stability of the system.

We now perturb the steady state basic solution as follows:

$$\mathbf{q} = w_0\hat{k} + \mathbf{q}', \quad T - T_0 = \theta_b(z) + \theta', \quad p = p_b(z) + p', \tag{2.6}$$

where $\mathbf{q}'$, θ' and p' are infinitesimal perturbations.

Substituting Eq. (2.6) into Eqs. (2.1) to (2.4) and neglecting nonlinear terms, we get the following equations (after dropping the primes):

$$\nabla \cdot \mathbf{q} = 0\,, \tag{2.7}$$

$$\rho_0 \left[\frac{1}{\phi}\frac{\partial \mathbf{q}}{\partial t} + \frac{w_0}{\phi^2}\,\frac{\partial \mathbf{q}}{\partial z} \right] = -\nabla p + \alpha \rho_0 g \theta\, \hat{k} - \frac{\mu}{k}\mathbf{q} + \tilde{\mu}\,\nabla^2 \mathbf{q}\,, \tag{2.8}$$

$$A\frac{\partial \theta}{\partial t} + w_0 \frac{\partial \theta}{\partial z} + \frac{\partial \theta_b}{\partial z} w = \kappa \nabla^2 \theta\,. \tag{2.9}$$

We eliminate the pressure and the horizontal velocity components from the governing Eqs. (2.7) to (2.8) by standard manipulations. Then we non-dimensionalize the equations using the notations

$$z^* = z/d,\ t^* = (\kappa/d^2\phi)t, \quad \mathbf{q}^* = (d/\kappa)\mathbf{q}, \quad \theta^* = \theta/(T_0 - T_1)$$

and drop the asterisks for simplicity to obtain the following equations:

$$\frac{1}{\mathrm{Pr}}\left[\frac{\partial}{\partial t} + Q\frac{\partial}{\partial z}\right] \nabla^2 w - (\Lambda \nabla^2 - \sigma^2)\nabla^2 w = R\nabla_1^2 \theta\,, \tag{2.10}$$

$$\frac{A}{\phi}\frac{\partial \theta}{\partial t} + Q\frac{\partial \theta}{\partial z} + f(z) w = \nabla^2 \theta\,. \tag{2.11}$$

Here $R = \alpha g(T_0 - T_1)d^3/\nu\kappa$ is the Rayleigh number, $\mathrm{Pr} = \nu/\kappa\phi^2$ is the modified Prandtl number, $Q = w_0 d/\kappa$ is the throughflow-dependent Peclet number, $\sigma^2 = d^2/k$ is the inverse of Darcy number, $\Lambda = \tilde{\mu}/\mu$ is the ratio of viscosities, $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator, $\nabla^2 = \nabla_1^2 + \partial^2/\partial z^2$ is the Laplacian operator and $\partial\theta_b(z)/\partial z = f(z) = -Qe^{Qz}/(e^Q - 1)$ is the nonlinear steady state basic temperature gradient. Since the instability appears in non-oscillatory form (see [8]), we drop the time derivatives in Eqs. (2.10) and (2.11) and seek a steady cellular solution in the form $(w, \theta) = [W(z), \Theta(z)]\, \exp\{i(lx + my)\}$, to get

$$(D^2 - a^2)\left[\Lambda(D^2 - a^2) - \sigma^2 - MD\right] W = Ra^2 \Theta\,, \tag{2.12}$$

$$(D^2 - a^2)\Theta - QD\Theta = f(z)W. \tag{2.13}$$

Here $D = d/dz$, $a^2 = l^2 + m^2$ is the square of the overall horizontal wavenumber and $M = Q/\mathrm{Pr}$. These equations are to be solved using appropriate boundary conditions depending on the nature of boundaries. We take the boundaries to be either rigid (however permeable) or free from tangential stress and insulating to temperature perturbations (see CHANDRASEKHAR [14] and NIELD [15]). Accordingly, the boundary conditions take the following form:

On the rigid boundary

$$W = DW = 0 = D\Theta \tag{2.14}$$

and on the stress-free boundary

$$W = D^2W = 0 = D\Theta. \tag{2.15}$$

Equations (2.12) and (2.13) together with the boundary conditions (2.14) and / or (2.15) constitutes a two-point boundary value problem with R as an eigenvalue.

3. Method of solution

Since the boundaries are insulated to temperature perturbations, the solution for the eigenvalue problem can be obtained in a closed form for arbitrary Q because, in this case, R attains its minimum value (R=Rc) at an arbitrarily small wavenumber [15]. Accordingly, we look for solutions of Eqs. (2.12) and (2.13) in the form

$$(W, \Theta) = (W_0, \Theta_0) + a^2(W_1, \Theta_1) + \cdots . \tag{3.1}$$

Substituting Eq. (3.1) into Eqs. (2.12) and (2.13) and equating the coefficients of the same powers in a^2, we get

$$\Lambda D^4W_0 - MD^3W_0 - \sigma^2D^2W_0 = 0\,, \tag{3.2}$$

$$D^2\Theta_0 - QD\Theta_0 = f(z)W_0\,, \tag{3.3}$$

$$\Lambda D^4W_1 + (-2\Lambda D^2 + \sigma^2 + MD)W_0 - MD^3W_1 - \sigma^2D^2W_1 = R\Theta_0\,, \tag{3.4}$$

$$D^2\Theta_1 - QD\Theta_1 = \Theta_0 + f(z)W_1. \tag{3.5}$$

These equations have to be solved subject to the boundary conditions

$$W_i = DW_i = D\Theta_i = 0 \quad (i = 0, 1) \qquad \text{at} \quad z = 0, 1 \tag{3.6}$$

in the case of rigid-rigid boundaries, and

$$W_i = D^2W_i = D\Theta_i = 0 \qquad (i = 0, 1) \qquad \text{at} \quad z = 0, 1 \tag{3.7}$$

in the case of stress-free boundaries.

For any combination of boundaries (i.e. rigid-rigid, free-free and rigid-free), the solution of Eqs. (3.2) and (3.3) is $W_0 = 0$ and $\Theta_0 = 1$. Then Eqs. (3.4) and (3.5) become

$$\Lambda D^4W_1 - MD^3W_1 - \sigma^2D^2W_1 = R\,, \tag{3.8}$$

$$D^2\Theta_1 - QD\Theta_1 = 1 + f(z)W_1. \tag{3.9}$$

The general solution of Eq. (3.8) is found to be

$$W_1 = C_0 + C_1 z + C_2 e^{\alpha z} + C_3 e^{\beta z} - Rz^2/2\sigma^2 , \tag{3.10}$$

where $\alpha, \beta = \left[M \pm \sqrt{M^2 + 4\Lambda\sigma^2}\right]/2\Lambda$ and C_i ($i = 0$ to 3) are constants of integration. Applying the solvability condition [15] to Eqs. (3.8) and (3.9) [15, 16] (i.e. the zeroth order solutions are orthogonal to the first order solutions), it follows that

$$< RW_0 >= 0 \text{ and } < [1 + f(z)W_1]\Theta_0 >= 0, \text{ where } < \cdots >= \int_0^1 (\cdots)dz .$$

In these, the first condition is satisfied trivially and the second requires that

$$< 1 + f(z)W_1 >= 0 . \tag{3.11}$$

To determine R, which in fact is a critical Rayleigh number Rc, we should find the constants C_i ($i = 0$ to 3) in Eq. (3.10) which depend on the nature of boundaries. The critical Rayleigh numbers obtained for different boundary combinations are detailed below.

3.1. Both boundaries free

In this case the required boundary conditions are

$$W_1 = D^2 W_1 = D\Theta_1 = 0 \qquad \text{at} \quad z = 0, 1. \tag{3.12}$$

The constants appearing in Eq. (3.10) are now determined using the boundary conditions, and they are found to be

$$\begin{aligned} C_0 &= R\left[\alpha^2(e^\alpha - 1) - \beta^2(e^\beta - 1)\right]/\Delta_1\sigma^2 , \\ C_1 &= R\left[\alpha^2\beta^2(e^\beta - e^\alpha) + 2(\alpha^2 - \beta^2)(1 - e^\alpha)(1 - e^\beta)\right]/2\Delta_1\sigma^2 , \\ C_2 &= R\beta^2(e^\beta - 1)/\Delta_1\sigma^2 , \\ C_3 &= -R\alpha^2(e^\alpha - 1)/\Delta_1\sigma^2 , \end{aligned}$$

where $\Delta_1 = \alpha^2\beta^2(e^\beta - e^\alpha)$.

Making use of these values of C_i ($i = 0$ to 3) in Eq. (3.11), we get an expression for Rc, which can be written, after some algebraic simplifications, in the form

$$Rc = \frac{4Q^2\sigma^2(\gamma^2 - \delta^2)^2 \,(\cosh q_2 - \cosh q_1)}{[b_1 \cosh q_1 + b_2 \cosh q_2 + b_3 \cosh q_3 + b_4 \sinh q_1 + b_5 \sinh q_2 + g_1]} , \tag{3.13}$$

where

$$b_1 = \frac{4Q^2(\gamma^2-\delta^2)(\gamma-\delta)}{Q+\gamma+\delta} + 4(\gamma^2-\delta^2)^2,$$

$$b_2 = -\frac{4Q^2(\gamma^2-\delta^2)(\gamma+\delta)}{Q+\gamma-\delta} - 4(\gamma^2-\delta^2)^2,$$

$$b_3 = 16\gamma\delta Q^2 - 8\delta Q^3\frac{\{\gamma^2+\delta^2+2\gamma(Q+\gamma)\}}{(Q+\gamma)^2-\delta^2},$$

$$b_4 = -2Q\{(\gamma^2-\delta^2)^2 - 8\gamma\delta\},$$

$$b_5 = 2Q\{(\gamma^2-\delta^2)^2 + 8\gamma\delta\},$$

$$g_1 = -16Q\gamma\delta\,(\sinh q_3 + \sinh q_4),$$

with $\delta = \sqrt{(M^2+4\Lambda\sigma^2)}/2\Lambda$, $\gamma = M/2\Lambda$, $q_{1,2} = Q/2 \pm \delta$ and $q_{3,4} = Q/2 \pm \gamma$.

It is observed that Rc is an even function of Q and the direction of throughflow (i.e. Q>0 or Q<0) does not change the value of Rc.

As $\sigma \to \infty$ (Darcy case), Eq. (3.13) gives

$$Rc \sim \frac{2Q^2\sigma^2}{[Q\coth(Q/2) - 2]}. \tag{3.14}$$

Equation (3.14) is an even function of Q but independent of Pr and it coincides with that of NIELD [10]. Further as $Q \to 0$ (i.e. in the absence of throughflow), Eq. (3.13) gives

$$Rc \sim \frac{12\Lambda\tilde{\sigma}^5}{[\tilde{\sigma}^3 - 12\tilde{\sigma} + 24\tanh(\tilde{\sigma}/2)]}, \tag{3.15}$$

where $\tilde{\sigma} = \sigma/\sqrt{\Lambda}$. From this equation it follows that, with $\Lambda = 1$

$$Rc \sim 120 \quad \text{as } \sigma \to 0 \tag{3.16}$$

and

$$Rc \sim 12\sigma^2 \quad \text{as } \sigma \to \infty \tag{3.17}$$

which are the known exact values.

3.2. Both boundaries rigid

The boundary conditions are

$$W_1 = DW_1 = D\Theta_1 = 0 \qquad \text{at} \quad z = 0, 1. \tag{3.18}$$

Then from Eq. (3.10), on using Eq. (3.18), we get

$$C_0 = R\left[\alpha(1+e^{\alpha}) - \beta(1+e^{\beta}) + 2(e^{\beta}-e^{\alpha})\right]/\Delta_2\sigma^2,$$
$$C_1 = R\left[2\alpha(e^{\beta}-1) - 2\beta(e^{\alpha}-1) - \alpha\beta(e^{\beta}-e^{\alpha})\right]/\Delta_2\sigma^2,$$
$$C_2 = R\left[\beta(1+e^{\beta}) - 2(e^{\beta}-1)\right]/\Delta_2\sigma^2,$$
$$C_3 = R\left[2(e^{\alpha}-1) - \alpha(1+e^{\alpha})\right]/\Delta_2\sigma^2,$$

where $\Delta_2 = 2(\beta-\alpha)\left(1-e^{\alpha}-e^{\beta}+e^{\alpha\beta}\right) + 2\alpha\beta(e^{\alpha}-e^{\beta})$.

Substituting these values of C_i ($i = 0$ to 3) in Eq. (3.11) and performing some integration, we obtain an expression for Rc, which can be written in the form

$$(3.19)\qquad Rc = \frac{2Q^2\sigma^2[-2\delta(\sinh q_4 + \sinh q_3 - \sinh q_1 - \sinh q_2) + f_1]}{(d_1\cosh q_1 + d_2\cosh q_2 + d_3\cosh q_3 + d_4\sinh q_1 + f_2)},$$

where

$$d_1 = \frac{2Q(\gamma+\delta)^2}{Q+\gamma+\delta} - (\gamma^2-\delta^2),$$
$$d_2 = -\frac{2Q(\gamma-\delta)^2}{Q+\gamma-\delta} + (\gamma^2-\delta^2),$$
$$d_3 = \frac{4Q^3\delta}{(Q+\gamma)^2-\delta^2} - 4Q\delta,$$
$$d_4 = \frac{Q(\gamma+\delta)^2(\gamma-\delta)}{Q+\gamma+\delta} - 4\delta,$$
$$d_5 = -\frac{Q(\gamma-\delta)^2(\gamma+\delta)}{Q+\gamma-\delta} - 4\delta,$$
$$d_6 = \frac{Q^2(\gamma+\delta)(Q+2\delta)}{Q+\gamma+\delta} + 4\delta,$$
$$d_7 = -\frac{Q^3(\gamma+\delta)}{Q+\gamma-\delta} + 4\delta,$$
$$f_1 = (\gamma^2-\delta^2)(\cosh q_1 - \cosh q_2),$$
$$f_2 = d_5\sinh q_2 + d_6\sinh q_3 + d_7\sinh q_4.$$

We note that Rc is an even function of Q, as in the previous case, because of symmetric boundary conditions and hence the direction of throughflow does not have any influence on the stability of the system.

As $\sigma \to \infty$, Eq. (3.19) gives

$$Rc \sim \frac{2Q^2\sigma^2}{[Q\coth(Q/2) - 2]}, \tag{3.20}$$

which is the same as Eq. (3.14). Further as $Q \to 0$, Eq. (3.19) gives

$$Rc \sim \frac{12\Lambda\tilde{\sigma}^4(2 + \tilde{\sigma}\sinh\tilde{\sigma} - 2\cosh\tilde{\sigma})}{[4(6 - \tilde{\sigma}^2) + \tilde{\sigma}(24 + \tilde{\sigma}^2)\sinh\tilde{\sigma} - 8(\tilde{\sigma}^2 + 3)\cosh\tilde{\sigma}]}. \tag{3.21}$$

Hence (with $\Lambda = 1$)

$$Rc \sim 720 \quad \text{as } \sigma \to 0 \tag{3.22}$$

and

$$Rc \sim 12\sigma^2 + 72\sigma \quad \text{as} \quad \sigma \to \infty. \tag{3.23}$$

The first term on the right-hand side of Eq. (3.23) is the value given by the Darcy equation and the second term is the Brinkman boundary layer correction.

3.3. Lower boundary rigid and upper boundary free

In this case the boundary conditions are given by

$$W_1 = DW_1 = 0 = D\Theta_1 \qquad \text{at } z = 0, \tag{3.24}$$

$$W_1 = D^2W_1 = 0 = D\Theta_1 \qquad \text{at } z = 1. \tag{3.25}$$

Hence

$$\begin{aligned}
C_0 &= R\left[\alpha^2 e^\alpha - \beta^2 e^\beta + 2(e^\beta - \beta) - 2(e^\alpha - \alpha)\right]/2\Delta_3\sigma^2, \\
C_1 &= R\left[\alpha^2\beta e^\alpha - \alpha\beta^2 e^\beta + 2\alpha(e^\beta - 1) - 2\beta(e^\alpha - 1)\right]/2\Delta_3\sigma^2, \\
C_2 &= R\left[\beta^2 e^\beta - 2(e^\beta - \beta - 1)\right]/2\Delta_3\sigma^2, \\
C_3 &= R\left[2(e^\alpha - \alpha - 1) - \alpha^2 e^\alpha\right]/2\Delta_3\sigma^2,
\end{aligned}$$

where $\Delta_3 = \beta^2 e^\beta(e^\alpha - \alpha - 1) - \alpha^2 e^\alpha(e^\beta - \beta - 1)$.

Making use of these values in Eq. (3.11), Rc is found in the form

$$Rc = \frac{2Q^2\sigma^2(e^Q - 1)\Delta_3}{\left[e_1 e^{Q+\gamma+\delta} + e_2 e^{Q+\gamma} + e_3 e^{Q+\delta} + e_4 e^Q + e_5 e^\gamma + e_6 e^\delta + e_7 e^{(\gamma+\delta)} + e_8\right]}, \tag{3.26}$$

where

$$
\begin{aligned}
e_1 &= \frac{Q^3(\delta^2-2)}{Q+\gamma} + (\gamma^2-\delta^2)(Q^2+2Q+2)\,,\\
e_2 &= Q\gamma^2(\delta+2) + 2Q\delta - 2(1+\delta)\gamma^2 - \frac{2Q^2\gamma(1+\delta)}{Q+\gamma}\,,\\
e_3 &= \frac{2Q^2\delta(\gamma+1)}{Q+\delta} - Q\delta^2(\gamma+2) + 2Q\gamma + 2(1+\gamma)\delta^2\,,\\
e_4 &= -2Q(\gamma+\delta)\,,\\
e_5 &= \frac{Q\delta^2(\gamma^2-2)}{Q+\delta} + 2(1+\delta)\gamma^2\,,\\
e_6 &= -\frac{Q\gamma^2(\delta^2-2)}{Q+\gamma} - 2(1+\gamma)\delta^2\,,\\
e_7 &= 2(\delta^2-\gamma^2)\,,\\
e_8 &= \frac{2Q^3(\gamma-\delta)(\gamma+\delta+Q+1)}{(Q+\gamma)(Q+\delta)} - 2Q(Q+1)(\gamma-\delta).
\end{aligned}
$$

A glance at Eq. (3.26) reveals that Rc is not an even function of Q. Hence the direction of throughflow alters the value of Rc.

As $\sigma \to \infty$, Eq. (3.26) gives

$$
Rc \sim \frac{2Q^2\sigma^2}{[Q\coth(Q/2)-2]}, \tag{3.27}
$$

which is the same as Eq. (3.14). Thus we observe that for a densely packed porous medium, Rc is independent of the types of boundaries. In other words, the Darcy model fails to take care of the boundary and inertia effects.

Further as $Q \to 0$, Eq. (3.26) gives

$$
Rc \sim \frac{12\Lambda\tilde{\sigma}^5(\sinh\tilde{\sigma} - \tilde{\sigma}\cosh\tilde{\sigma})}{[4\tilde{\sigma}(\tilde{\sigma}^2-6)\sinh\tilde{\sigma} + (24-\tilde{\sigma}^4)\cosh\tilde{\sigma} + 12(\tilde{\sigma}^2-2)]}. \tag{3.28}
$$

From this equation it follows that (with $\Lambda = 1$)

$$
Rc \sim 320 \quad \text{as } \sigma \to 0 \tag{3.29}
$$

and

$$
Rc \sim 12\sigma^2 + 36\sigma \quad \text{as } \sigma \to \infty. \tag{3.30}
$$

The second term on the right-hand side of Eq. (3.30) is the boundary layer correction as seen in the previous case (see Eq. (3.23)).

4. Results and discussion

Onset of convection in a sparsely packed porous layer with throughflow is investigated in understanding the control of convective instability by the adjustment of vertical throughflow. The Brinkman-extended Darcy model with fluid viscosity different from effective viscosity is used to describe the flow in the porous media. The boundaries are assumed to be either rigid-permeable or free of tangential stresses and insulated to temperature perturbations. Analytical expression for critical Rayleigh number Rc, for free-free, rigid-rigid and rigid-free boundaries have been obtained and are evaluated numerically for different values of Q, Λ and Pr. The results are presented graphically in Figs. 2 – 9.

Figure 2 shows the variations of Rc as a function of $|Q|$ for both free-free and rigid-rigid (i.e. symmetric) boundary conditions. These plots are for $\sigma^2 = 10, 100$, $\Lambda = .1, 1, 10$ and $\mathrm{Pr} = 7$. Note that increase in the values of σ^2 and Λ increases Rc as expected and makes the system more stable. The direction of throughflow is not altering the stability of the system and increase in the value of $|Q|$ is to increase Rc. This is because the effect of throughflow is to confine significant thermal gradients to a thermal boundary layer at the boundary toward which the throughflow is directed. The effective length scale is thus smaller than the thickness of the porous layer d and so the Rayleigh number, which is proportional to the cube of the porous layer thickness, will be much smaller than the actual value of Rc. Therefore higher values of Rc are needed for the onset of convection with an increase in $|Q|$.

The presence of constant vertical throughflow in the basic state brings in the effect of inertia through the appearance of Prandtl number, which complicates the situation. The effect of Prandtl number on the stability of the system is shown distinctly in Figs. 3 and 4 for free-free and rigid-rigid boundaries, respectively. These figures are for $|Q| = 2$, $\Lambda = 0.1, 1, 10$ and $\sigma^2 = 10, 100$. For $\sigma^2 = 10$, the values of Rc are shown on the left-hand side of the figure while for $\sigma^2 = 100$ they are shown on the right-hand side. From these figures it is evident that Rc decreases initially with an increase in the value of Pr and passes through a minimum, depending on the value of Λ, before attaining an asymptotic value with further increase of Pr. Also, an increase in the value of Λ is to decrease the range of values of Pr upto which the system becomes unstable (i.e. Rc decreases). The variation in Rc with Pr for free-free boundaries is found to be not so significant as compared to the rigid-rigid boundaries.

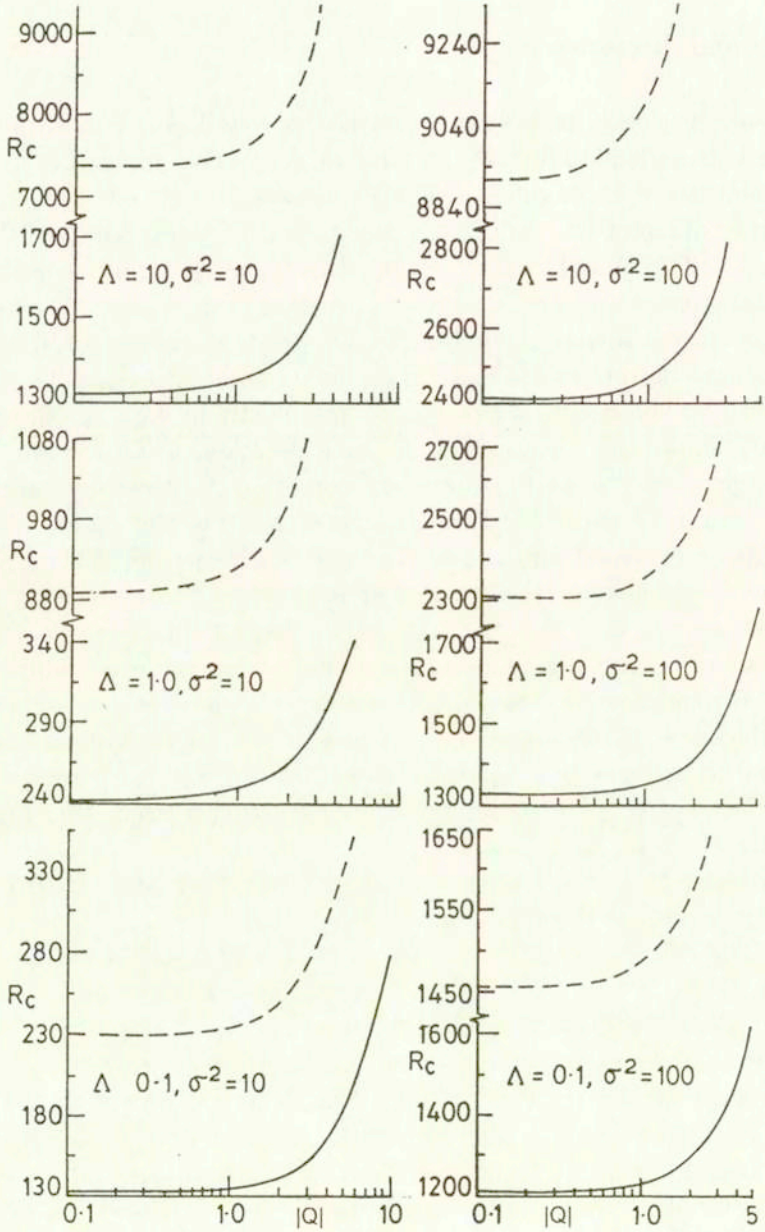

FIG. 2. Critical rayleigh number R_c vs. $|Q|$ for different values of σ^2 and Λ for free-free (——) and rigid-rigid (- - - -) boundaries when $\Pr = 7 \cdot 0$.

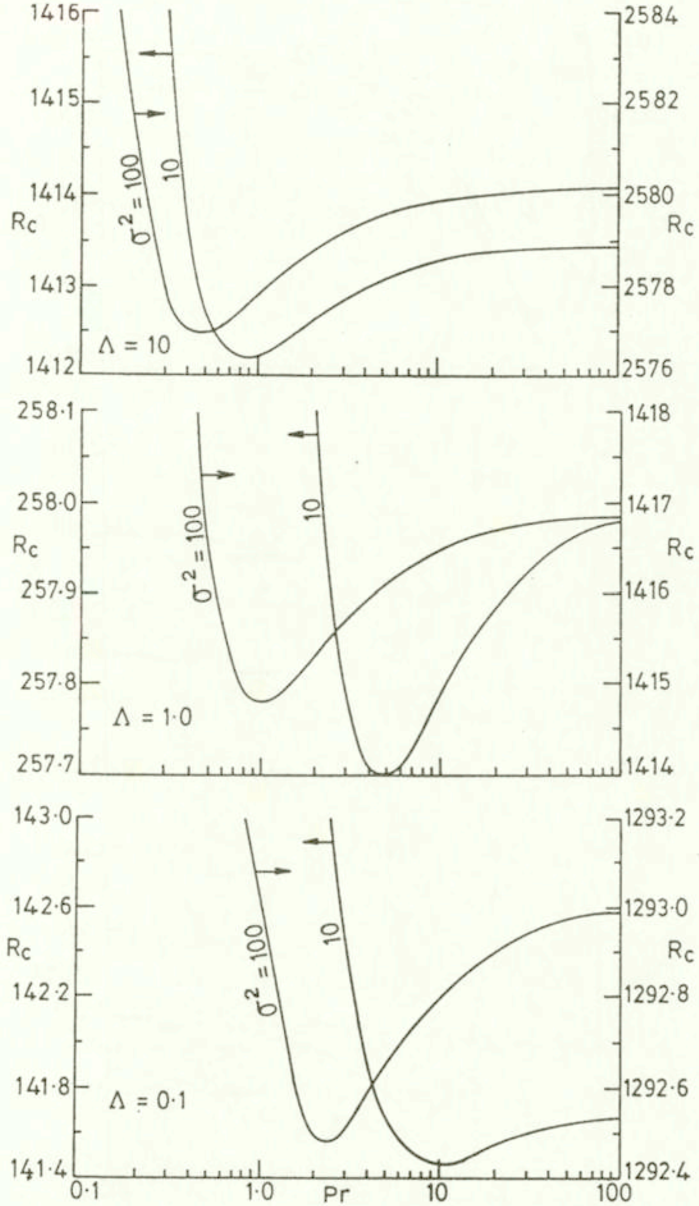

FIG. 3. Critical rayleigh number vs. Pr for different values of Λ and σ^2 for free-free boundaries when $|Q| = 2 \cdot 0$.

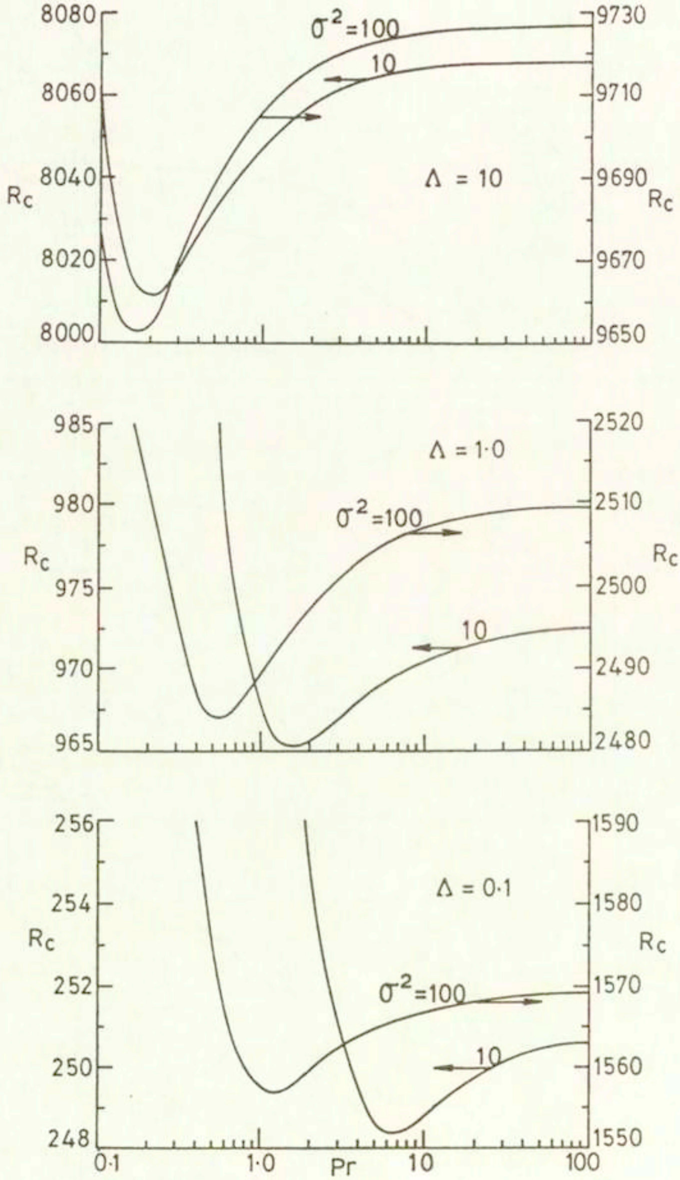

FIG. 4. Critical rayleigh number vs. Pr for different values of Λ and σ^2 for rigid-rigid boundaries when $|Q| = 2 \cdot 0$.

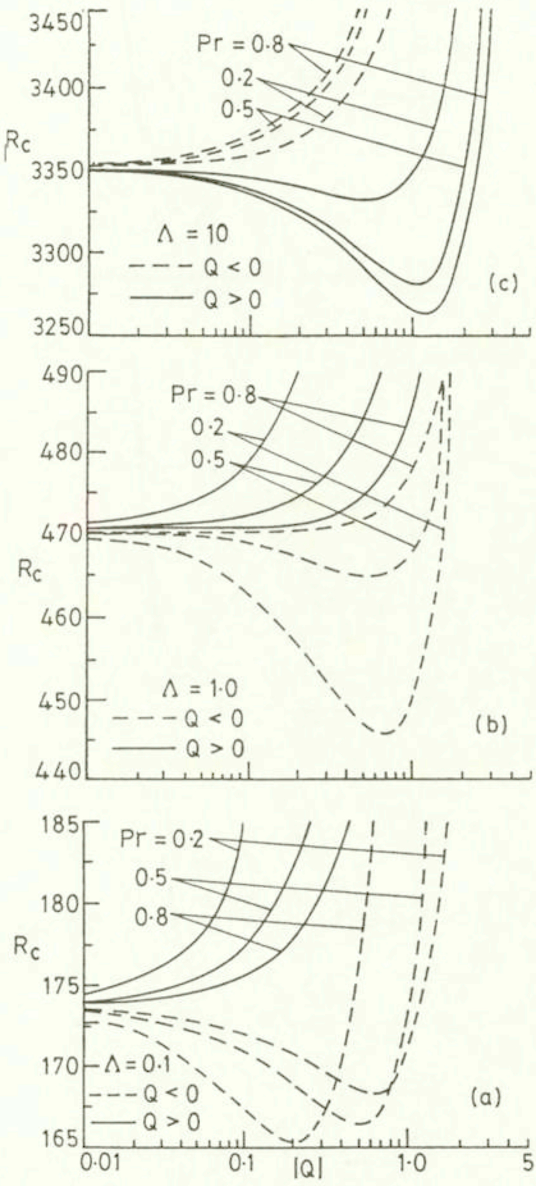

FIG. 5. Critical rayleigh number vs. $|Q|$ for different values of Pr and Λ for rigid-free boundaries when $\sigma^2 = 10$.

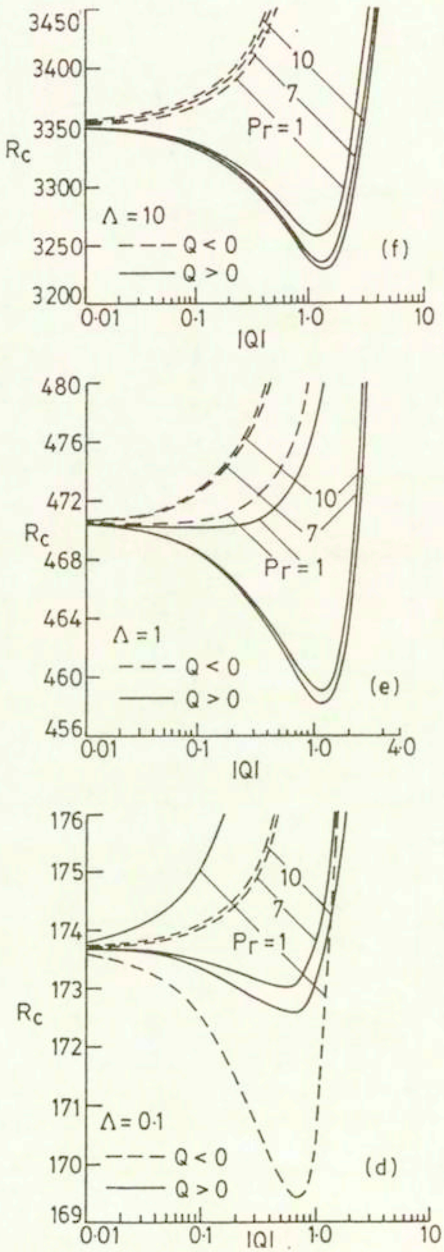

FIG. 5 (continued)

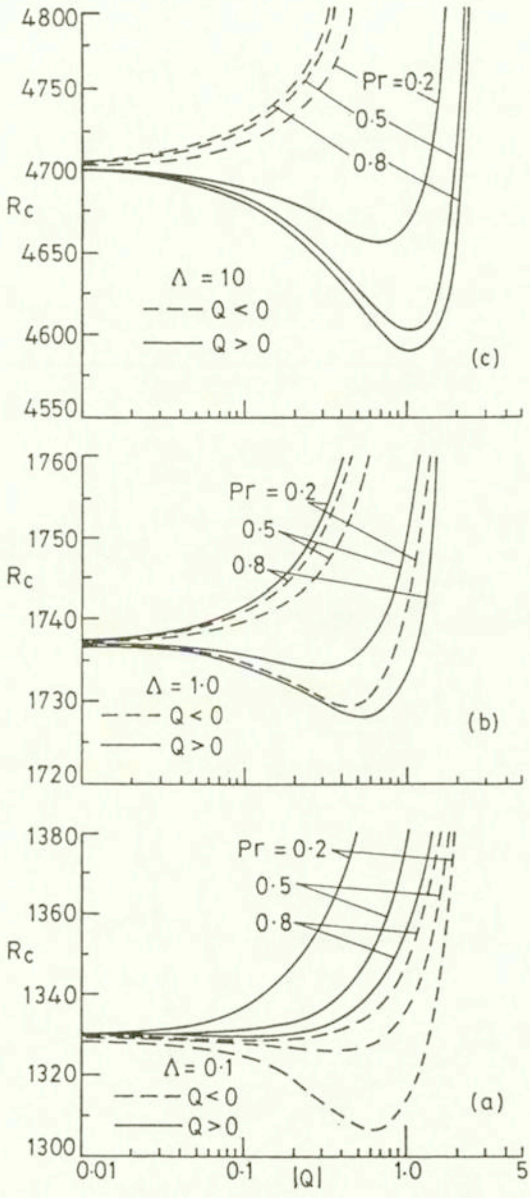

FIG. 6. Critical rayleigh number vs. $|Q|$ for different values of Λ and Pr for rigid-free boundaries when $\sigma^2 = 100$.

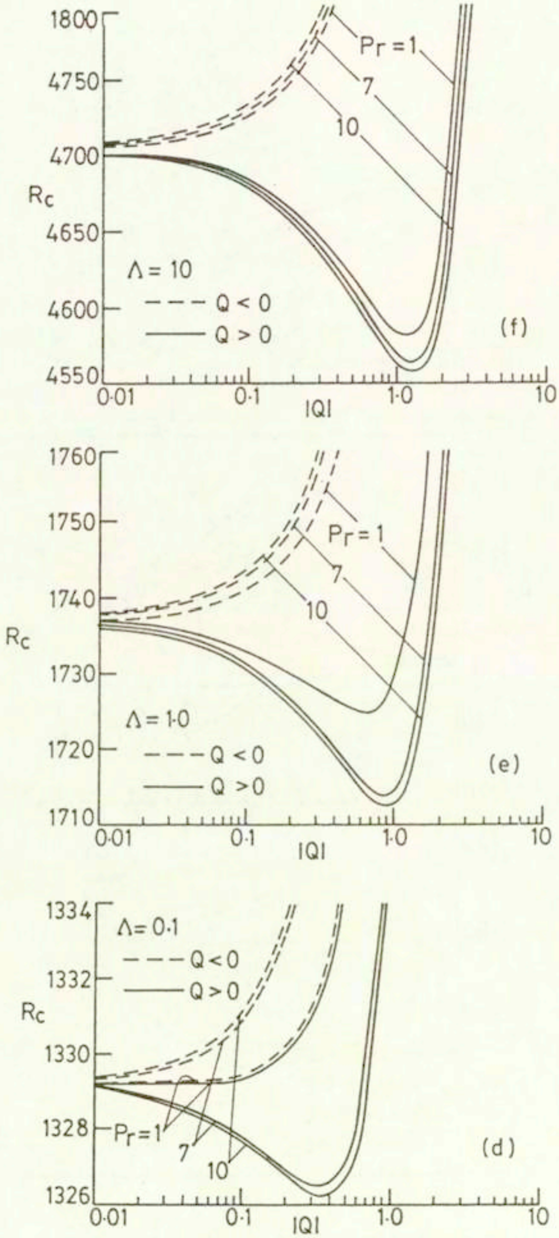

FIG. 6 (continued)

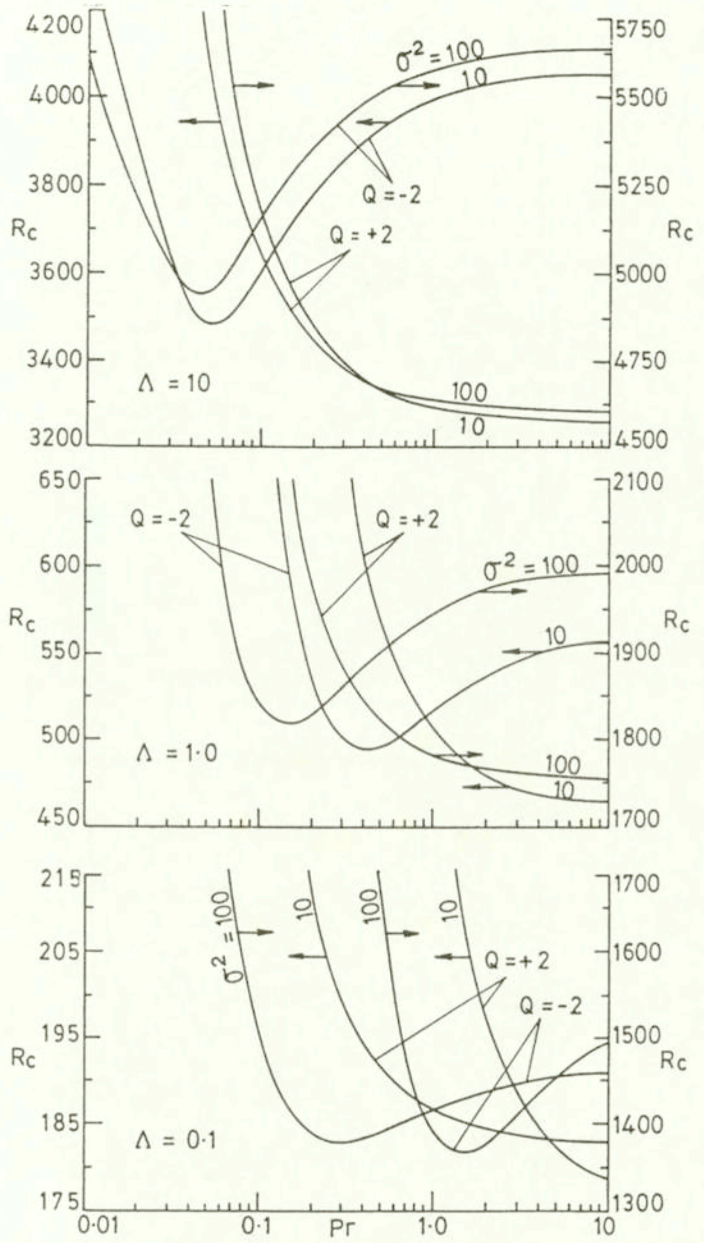

FIG. 7. Critical rayleigh number vs. Pr for different values of Λ and σ^2 for rigid-free boundaries when $Q = \pm 2$.

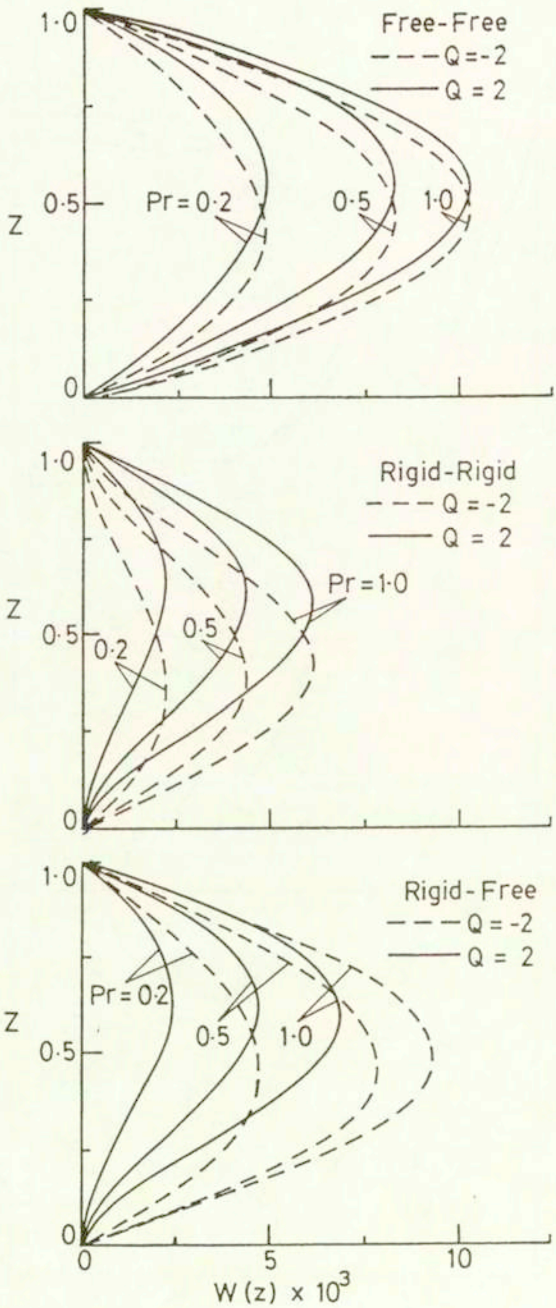

FIG. 8. Velocity eigenfunction for different values of Pr for different boundaries when $\Lambda = 0 \cdot 1$ and $\sigma^2 = 10$.

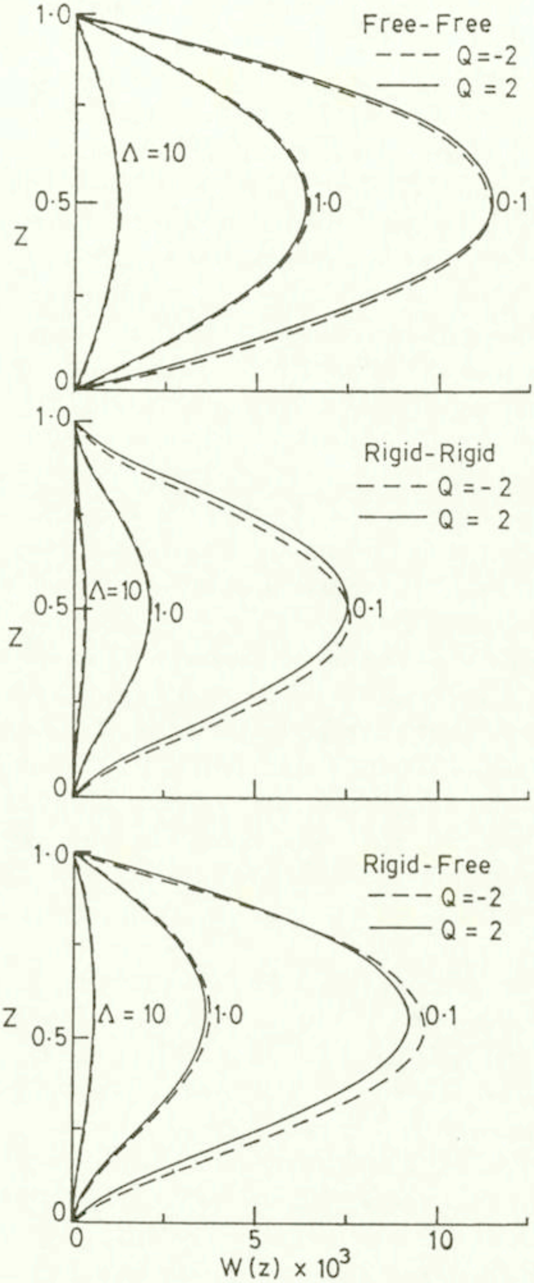

FIG. 9. Velocity eigenfunction for different values of Λ for different boundaries when $\Pr = 7 \cdot 0$ and $\sigma^2 = 10$.

When the boundaries are of asymmetric type (i.e. rigid-free) the situation observed is totally different, in contrast to the symmetric boundaries, and the same is depicted in Figs. 5 and 6 for $\sigma^2 = 10$ and 100 respectively, for different values of Pr and Λ. Figures 5(a, b, c) and 6(a, b, c) show the results for Pr < 1, while Figs. 5(d, e, f) and 6(d, e, f) are for Pr > 1. These figures show that the direction of throughflow alters the stability of the system and also the existence of a critical Prandtl number below which the downward flow destabilizes the system and above which upward flow destabilizes the system. Note that the destabilization manifests itself as a minimum in the $Rc - |Q|$ curve up to certain values of Q depending on the values of σ^2, Λ , Pr and as well as the direction of throughflow. For antigravity throughflow ($Q > 0$), it is found that an increase in the value of Pr decreases the value of Rc and thus makes the system unstable. For gravity aligned throughflow ($Q < 0$) an opposite kind of behaviour is noticed, in general, with some exceptions when $\sigma^2 = 10$ and $\Lambda = 0.1$ and 1 (see Figs. 5a and 5b). Increase in the value of Λ tends to make the system more stable and also to increase the range of values of Q up to which the system gets destabilized. For $\Lambda = 0.1$ and 10, it is found that the destabilization is greater for Pr < 1 and Pr > 1 respectively for the values of σ^2 considered, where as for $\Lambda = 1$ the destabilization is greater for Pr < 1 when $\sigma^2 = 10$ and for $\sigma^2 = 100$ the same is found to be true for Pr > 1. The destabilization may be due to the distortion of the steady state basic temperature distribution by the vertical throughflow. A measure of this is given by the term $< f(z)W\Theta >$ and this can be interpreted as a rate of transfer of energy into the disturbance by interaction of the perturbation convective motion with basic temperature gradient. The maximum temperature occurs at a place where the perturbed vertical velocity is large and this leads to an increase in energy supply for destabilization. The destabilization may also be due to other mechanisms, i.e. the momentum transport and the thermal energy transport.

Figure 7 shows Rc as a function of Pr for rigid-free boundaries. The results exhibited are for two values of $\sigma^2 = 10$ (shown on the left-hand side of the figure) and 100 (shown on the right-hand side of the figure) for $Q = \pm 2$ and $\Lambda = 0.1$, 1, 10. It may be noted that the Prandtl number plays a dual role in deciding on the stability of the system, depending on the direction of throughflow. For antigravity throughflow Rc decreases monotonically with Pr, while for gravity aligned throughflow it decreases initially with Pr and increases again with further increase of Pr before attaining an asymptotic value.

The velocity eigenfunction $W(z)$ for different boundary combinations are illustrated in Figs. 8 and 9 for different values of Pr and Λ in the case $\sigma^2 = 10$ and $Q = \pm 2$. As can be seen, increase in the value of Pr (see Fig. 8) and decrease in the value of Λ (see Fig. 9) increases the convection in the porous layer.

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