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A note on the existence and global stability of deformations of compressible nonlinearly elastic solids

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THE EXISTENCE AND GLOBAL STABILITY of certain deformations of isotropic compressible hyperelastic solids is investigated for the case when the body forces are zero and boundary conditions of place are satisfied. Certain global stability results are obtained and, using a phase-plane analysis, the existence of deformations describing the bending of rectangular blocks into annular cylindrical sectors is established.

1. Introduction

THE MAIN PROBLEM of nonlinear elastostatics is the determination of solutions to equilibrium equations which satisfy appropriate boundary conditions and which are stable, in the sense that they minimise the total energy relative to an appropriate class of variations. Exact solutions to relevant boundary value problems are, however, rarely available and when approximate solutions are sought, it is essential that the existence and stability of solutions are established beforehand. As discussed recently in [1], there are two ways of answering questions of this nature. One way is to establish the existence of minimisers for the total energy within an appropriate function space which are subsequently shown to satisfy the corresponding Euler-Lagrange equation; another way is to establish the existence of solutions to the boundary value problem for the Euler-Lagrange equation, which are subsequently shown to be minimizers to the total energy.

In this paper we are concerned with the existence and global stability (i.e. stability relative to variations of class C^1 and of arbitrary magnitude) of equilibrium solutions for isotropic compressible elastic solids and, in this context, we draw attention to the existence and uniqueness result established in [2] for deformations that describe the straightening of annular cylindrical sectors and to the global stability results obtained in [3] – [6]. Here, assuming (as in [2] – [6]) that the body forces are zero and that the deformations satisfy boundary conditions of place, we obtain certain generalisations of the stability results in [5, 6] (which, in turn, are generalisations of the global stability results obtained in [2, 3] respectively) and using an approach similar to that adopted in [2], find conditions which ensure the existence and uniqueness of deformations that describe the bending of rectangular blocks into annular cylindrical sectors. We show that in certain instances it is possible to combine these results to conclude that certain deformations exist and are globally stable.

2. Preliminaries

We consider deformations of isotropic compressible hyperelastic solids which, in a reference configuration, occupy a domain D of the Euclidean space R^3. Such deformations are smooth and invertible transformations $\mathbf{f} : D \to \hat{D}$, where $\hat{D}$ denotes another domain of R^3. The deformations $\mathbf{f}$ are required to satisfy

$$\det\mathbf{F} > 0, \qquad \mathbf{F} \equiv \nabla \mathbf{f}, \tag{2.1}$$

where det and ∇ stand for determinant and gradient, respectively. The principal stretches, denoted here by λ_i, $i = 1, 2, 3$, are the eigenvalues of $\left(\mathbf{F}\mathbf{F}^T\right)^{1/2}$ where $\mathbf{F}^T$ denotes the transpose of $\mathbf{F}$.

The materials under consideration are characterised by strain-energy functions

$$W = W(\mathbf{F}) = \hat{W}(\lambda_1, \lambda_2, \lambda_3) \tag{2.2}$$

whose domains are restricted to deformations that obey (2.1), which are symmetric in λ_1, λ_2, λ_3, and which satisfy

$$\hat{W}(\lambda_1, \lambda_2, \lambda_3) \geq 0, \qquad \hat{W}_i(1,1,1) = 0, \qquad \hat{W}_i \equiv \frac{\partial \hat{W}}{\partial \lambda_i}, \qquad i = 1, 2, 3, \tag{2.3}$$

equality in $(2.3)_1$ being possible if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 1$. If $W = W(\mathbf{F})$ is of class C^2, then it is convex if and only if [7]

$$\begin{gathered} \sum_{i,j=1}^{3} \hat{W}_{ij}\xi_i\xi_j \geq 0, \quad \forall(\xi_1, \xi_2, \xi_3) \in R^3, \quad \hat{W}_{ij} \equiv \frac{\partial^2 \hat{W}}{\partial \lambda_i \partial \lambda_j}, \quad i, j, = 1, 2, 3, \\ \hat{W}_m + \hat{W}_n \geq 0, \qquad m, n = 1, 2, 3, \quad m \neq n. \end{gathered} \tag{2.4}$$

As pointed out by many authors (see [8], Sec. 52, for instance), the condition (2.4) is not acceptable in nonlinear elasticity.

The stress-deformation relation is

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \tag{2.5}$$

where $\mathbf{S}$ denotes the first Piola-Kirchhoff stress tensor [8, Sec. 43A] and the equilibrium condition reads

$$\operatorname{Div}\mathbf{S}(\mathbf{F}) = \mathbf{0} \quad \text{on } D, \tag{2.6}$$

when the body forces are zero. According to the principle of virtual work ([9], Sec. 2.6), an equilibrium solution $\mathbf{f}$ satisfies

$$\int_D \operatorname{tr}\left[\mathbf{S}(\nabla\mathbf{v})^T\right] dD = 0 \tag{2.7}$$

for all $\mathbf{v} \in V \equiv \{\mathbf{v} \in C^1(D) : \mathbf{v}|_{\partial D} = 0, \det(\mathbf{F} + \nabla\mathbf{v}) > 0\}$, where tr is the trace operator, $C^1(D)$ is the set of continuously differentiable functions on D and ∂D is the boundary of D.

In the absence of body forces, an equilibrium solution $\mathbf{f}$ (that satisfies boundary conditions of place) is said to be globally stable if

$$\int_D W(\mathbf{F} + \nabla\mathbf{v})dD \geq \int_D W(\mathbf{F})dD, \qquad \mathbf{F} = \nabla\mathbf{f}, \tag{2.8}$$

for every variation $\mathbf{v} \in V$. When D is a bounded open set, a strain-energy function W which satisfies the condition (2.8) at a particular deformation gradient $\mathbf{F}$ is said to be quasiconvex at $\mathbf{F}$ (see [9], Sec. 5, for instance).

3. Stability results

In this section we confine our attention to a sub-class of the class of materials characterised by (2.2) and (2.3), namely the materials whose strain-energy function can be written in the form

$$\hat{W} = H\left[\hat{P}(\lambda_1, \lambda_2, \lambda_3)\right] + h(J), \tag{3.1}$$

where $P(\mathbf{F}) \equiv \hat{P}(\lambda_1, \lambda_2, \lambda_3)$ is a scalar-valued function (assumed to be symmetric in $\lambda_1, \lambda_2, \lambda_3$) and where, for convenience, we employ the notation $J \equiv \lambda_1\lambda_2\lambda_3$. Our first result is similar to that established in [3] and [5].

PROPOSITION 1. Let $h(J) = aJ + b$, where a and b are constants, let $P = P(\mathbf{F})$ be a convex function, and let $H'' \geq 0$. Then, for boundary conditions of place and zero body forces, an equilibrium solution $\mathbf{f}^*$ is globally stable provided that

$$H'\left[P(\mathbf{F}^*)\right] \geq 0, \qquad \mathbf{F}^* \equiv \nabla\mathbf{f}^*. \tag{3.2}$$

P r o o f. By the well-known identities $\partial J/\partial \mathbf{F} \equiv J\mathbf{F}^{-T}$ and $\mathrm{Div}(J\mathbf{F}^{-T}) \equiv \mathbf{0}$ (see [5] and [10]), it is clear that we have $\mathrm{Div}(\partial h/\partial \mathbf{F}) = \mathbf{0}$ and thus, for the materials under consideration, the equilibrium condition (2.6) reduces to

$$\mathrm{Div}\left\{H'\left[P(\mathbf{F})\right]\frac{\partial P}{\partial \mathbf{F}}\right\} = \mathbf{0}. \tag{3.3}$$

By assumption, $\mathbf{f}^*$ is an equilibrium solution and therefore, in view of (2.7), (3.3) leads to

$$\int\limits_D H'\left[P(\mathbf{F}^*)\right]\mathrm{tr}\left[\frac{\partial P}{\partial \mathbf{F}}(\mathbf{F}^*)(\nabla \mathbf{v})^T\right] dD = 0, \qquad \forall \mathbf{v} \in V. \tag{3.4}$$

Our convexity assumptions also imply

$$P(\mathbf{F}^* + \nabla \mathbf{v}) - P(\mathbf{F}^*) \geq \mathrm{tr}\left[\frac{\partial P}{\partial \mathbf{F}}(\mathbf{F}^*)(\nabla \mathbf{v})^T\right], \qquad \forall \mathbf{v} \in V, \tag{3.5}$$

and

$$H[P(\mathbf{F}^* + \nabla \mathbf{v})] - H[P(\mathbf{F}^*)] \geq H'[P(\mathbf{F}^*)][P(\mathbf{F}^* + \nabla \mathbf{v}) - P(\mathbf{F}^*)], \qquad \forall \mathbf{v} \in V, \tag{3.6}$$

which, together with (3.2) and (3.4), yield

$$\int\limits_D \{H[P(\mathbf{F}^* + \nabla \mathbf{v})] - H[P(\mathbf{F}^*)]\} dD \geq 0. \tag{3.7}$$

Since

$$\int\limits_D \{h\left[J(\mathbf{F}^* + \nabla \mathbf{v})\right] - h\left[J(\mathbf{F}^*)\right]\} dD = 0, \qquad \forall \mathbf{v} \in V, \tag{3.8}$$

we find (see (3.1)) that

$$\int\limits_D \left[W(\mathbf{F}^* + \nabla \mathbf{v}) - W(\mathbf{F}^*)\right] dD \geq 0, \qquad \forall \mathbf{v} \in V, \tag{3.9}$$

which shows that $\mathbf{f}^*$ is globally stable.

For illustration we restrict ourselves to the case of plane strain and consider a class of materials characterised by a strain-energy function of the form

$$\hat{W} = \frac{2\mu}{\gamma}\left[H(\lambda_1^\gamma + \lambda_2^\gamma) - \lambda_1\lambda_2\right] \tag{3.10}$$

where $\mu > 0$ and $\gamma > 1$ are constants and where H is a function that satisfies $H(2) = 1$, $H'(2) = 1/\gamma$, $H'' > 0$. We note that if $\gamma = 1$, (3.10) represents the class of harmonic materials introduced in [11]. An example of material that belongs to the class (3.10) is provided by

$$\hat{W} = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2\right) + \frac{\lambda+\mu}{8}\left(\lambda_1^2 + \lambda_2^2 - 2\right)^2 - \mu\lambda_1\lambda_2, \tag{3.11}$$

where μ and λ are the infinitesimal Lamé moduli that satisfy $\mu > 0$, $\mu + \lambda > 0$.

Using the specialisation of conditions (2.4) to plane strain (see [7]), we find that

$$P(\mathbf{F}) = \hat{P}(\lambda_1, \lambda_2) = \lambda_1^\gamma + \lambda_2^\gamma, \qquad \gamma > 1, \tag{3.12}$$

is convex. Also, according to the existence result established in [2], equilibrium solutions of the form

$$x = f(R), \qquad y = \alpha\Theta \qquad \alpha = \text{const}, \qquad \alpha > 0, \tag{3.13}$$

where (x, y) are Cartesian coordinates in $\hat{D}$ and (R, Θ) are polar coordinates in D, exist for materials (3.10), boundary conditions of place, and zero body forces, provided that

$$(\gamma - 1)H' + \gamma\lambda_1^\gamma H'' > 0. \tag{3.14}$$

We note that the condition (3.14) is equivalent (for material (3.10)) to the tension-extension condition ([8], Sec. 51)

$$\hat{W}_{11} > 0 \tag{3.15}$$

and that the deformation (3.13) represents the straightening of annular cylindrical sectors. Clearly, if the condition $H' > 0$ is satisfied (for materials (3.11) this is equivalent to $\lambda < 0$), by the above proposition (3.13) are globally stable solutions (relative to the class of materials (3.10)) for any of the boundary conditions of place. If the latter condition is violated however, the stability condition (3.2) may be satisfied only if the boundary conditions are restricted in a suitable manner (see [5], for instance).

Our next Proposition 2 generalises (in a certain sense) the global stability results obtained in [4] and [6].

PROPOSITION 2. Let $h'' \geq 0$ ($h'' \not\equiv 0$), $H'' \geq 0$ and let $P = P(\mathbf{F})$ be a convex function. Then, for boundary conditions of place and zero body forces, an equilibrium solution $\mathbf{f}^*$ that obeys the condition

$$\det(\nabla\mathbf{f}^*) = \text{const}, \tag{3.16}$$

is globally stable provided that the condition (3.2) is satisfied.

P r o o f. Since, by (3.16) and the identities quoted at the beginning of the proof of Proposition 1, we have

$$\operatorname{Div}\left[\frac{\partial h}{\partial \mathbf{F}}(\det \mathbf{F}^*)\right] = \mathbf{0} \tag{3.17}$$

we find, as before, that (3.2) implies (3.7) and, by the convexity of h, it also follows that

$$\int_D \{h[J(\mathbf{F}^* + \nabla \mathbf{v})] - h[J(\mathbf{F}^*)]\}\, dD \geq h'[J(\mathbf{F}^*)] \cdot \int_D [J(\mathbf{F}^* + \nabla \mathbf{v}) - J(\mathbf{F}^*)]\, dD = 0, \qquad \forall \mathbf{v} \in V. \tag{3.18}$$

The condition (3.9) therefore follows from (3.7) and (3.18).

4. An existence result

In this section we consider the plane-strain bending of a rectangular block into an annular cylindrical sector. This deformation may be given in the form

$$r = f(X), \qquad \theta = \alpha Y, \qquad \alpha = \text{const}, \qquad \alpha > 0, \tag{4.1}$$

where (r, θ) are polar coordinates in $\hat{D}$ and (X, Y) are Cartesian coordinates in D, and the equilibrium condition is (see [12], Sec. 5.2)

$$\begin{gathered} \frac{d}{dX}\hat{W}_1[f'(X), \alpha f(X)] = \alpha \hat{W}_2[f'(X), \alpha f(X)], \\ X \in (-A, A), \qquad A = \text{ const}, \quad A > 0. \end{gathered} \tag{4.2}$$

In order to satisfy the boundary conditions of place we prescribe α and require

$$f(-A) = f_1, \qquad f(A) = f_2, \qquad f_2 > f_1 > 0. \tag{4.3}$$

We seek solutions to the boundary-value problem described above, which satisfy

$$f, f' > 0 \qquad \text{on} \qquad X \in [-A, A], \tag{4.4}$$

and to this end we assume that $\hat{W}$ is of class C^3 and satisfies the tension-extension condition (3.15).

The region of physical interest (see (4.3) and (4.4)) is given by

$$\Omega \equiv \{(f, g) : f_1 \leq f \leq f_2, g > 0\}, \tag{4.5}$$

where g represents the possible values of f', and we define

$$\varphi(f, g) \equiv g\hat{W}_1(g, \alpha f) - \hat{W}(g, \alpha f), \qquad (f, g) \in \Omega. \tag{4.6}$$

Clearly any C^2 solution of (4.2) satisfies

$$\varphi(f,g)=\varphi(f(X),f'(X))=c, \tag{4.7}$$

for some constant c, and conversely, by the condition (see (3.15))

$$\frac{\partial\varphi}{\partial g}>0, \qquad (f,g)\in\Omega, \tag{4.8}$$

and the implicit function theorem, any solution of (4.7) which is in Ω must be of class C^2 and satisfy (4.2).

Solutions of (4.2) also correspond to solutions of the two-dimensional system

$$\frac{d}{dX}f=g, \qquad \frac{d}{dX}g=\frac{\alpha\left(\hat{W}_2(g,\alpha f)-g\hat{W}_{12}(g,\alpha f)\right)}{\hat{W}_{11}(g,\alpha f)} \tag{4.9}$$

and thus we may regard X as a time parameter of an autonomous planar system defined in Ω. The level curves $\varphi=c$ therefore give the trajectories of (4.9). Since the vector field (4.9) is C^1, it is locally Lipshitz and, given any point (f_0,g_0) in Ω and some value X_0 of X, by standard results (see e.g. [13]) there exists a unique maximal solution $(f(X),g(X))$ of (4.9) with $f(X_0)=f_0$, $g(X_0)=g_0$, which must be at least C^2. We can choose such a solution to start at $f=f_1$, at time $X=-A$, as required.

The condition (4.8) implies that each level curve intersects each line $f=$ constant at most once. Thus a trajectory can pass from $f=f_1$ to $f=f_2$ if and only if there is a value of c for which the level curve $\varphi=c$ intersects each line $f=\text{constant}$ between $f=f_1$ and $f=f_2$. Introducing the definitions

$$\begin{aligned} q_{\min}(f) &\equiv \lim_{g\to 0}\varphi(f,g)\in[-\infty,\infty),\\ q_{\max}(f) &\equiv \lim_{g\to\infty}\varphi(f,g)\in(-\infty,\infty], \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} q_0 &\equiv \max_{f\in[f_1,f_2]} q_{\min}(f)\in[-\infty,\infty],\\ q_1 &\equiv \min_{f\in[f_1,f_2]} q_{\max}(f)\in[-\infty,\infty], \end{aligned} \tag{4.11}$$

we find that c must belong to the interval (q_0,q_1) and thus we must certainly have $q_0<q_1$ for a solution of (4.2) that satisfies (4.3).

Let $c\in(q_0,q_1)$ and denote the solution curve $(f,g_c(f))$, where $g_c>0$, the curve which satisfies $\varphi=c$. As discussed above, each curve defines a unique trajectory $(f(X), g(X))$ such that $f(-A) = f_1$. It will also satisfy $f(A) = f_2$ if and only if the time taken to cross from $f = f_1$ to $f = f_2$ is $2A$, that is

$$\int_{f_1}^{f_2} \frac{1}{f'(X)} df = \int_{f_1}^{f_2} \frac{1}{g_c} df = 2A. \tag{4.12}$$

Since $\varphi(f, g_c(f)) = c$ we have (by (4.8)) that, for fixed f,

$$\frac{d}{dc} \int_{f_1}^{f_2} \frac{1}{g_c} df = -\int_{f_1}^{f_2} \frac{g_c'}{g_c^2} df = -\int_{f_1}^{f_2} \frac{1}{\varphi_g g_c^2} df < 0, \tag{4.13}$$

which shows that there is at most one solution which satisfies the boundary conditions.

The maximum and minimum values of the integral in (4.12) are given when c approaches q_0 and q_1 respectively. Defining

$$I_k \equiv \lim_{c \to q_k} \int_{f_1}^{f_2} \frac{1}{g_c} df, \qquad k = 0, 1, \tag{4.14}$$

we have demonstrated the following proposition.

PROPOSITION 3. If the $T - E$ condition (3.15) is satisfied, then there is a unique solution of class C^2 to Eq. (4.2), which satisfies Eqs. (4.3) and (4.4), if and only if $q_0 < q_1$ and

$$I_1 < 2A < I_0. \tag{4.15}$$

REMARK. Interpreting the following integrals in the obvious sense when the integrand is infinite (see [2]), we have

$$I_k = \int_{f_1}^{f_2} \frac{1}{g_{q_k}} df, \qquad k = 0, 1. \tag{4.16}$$

Thus if $q_{\max}(f)$ is constant, then the level curves tend uniformly to infinity as c increases, and so $I_1 = 0$. Similarly, if $q_{\min}(f)$ is constant, then $I_0 = \infty$. It then follows that if condition (3.15) is satisfied, and if $q_{\max}(f)$ and $q_{\min}(f)$ are both constant, for each given value A, there is a unique solution to the bending problem for all possible boundary conditions of place.

An example of material which satisfies the condition (3.15) and for which $q_{\max}(f)$ and $q_{\min}(f)$ are both constant is provided by

$$\hat{W} = \mu \left[\frac{1}{2} \lambda_1^3 \lambda_2 + \frac{1}{2} \lambda_1 \lambda_2^3 - 2\lambda_1 \lambda_2 + 1 \right], \qquad \mu = \text{const}, \quad \mu > 0. \tag{4.17}$$

From (4.7) we find in this case that

$$f(X) = (C_1 X + C_2)^{3/4} \tag{4.18}$$

where C_1, C_2 are constants of integration. It is easily seen in this instance that, from the boundary conditions (4.3), the arbitrary constants can be determined uniquely for any choice of f_1 and f_2 (with $f_2 > f_1 > 0$).

For materials (3.10) (where we now allow $\gamma \geq 1$ and assume that (3.14) holds) however, we find that (see (4.10))

$$q_{\min}(f) = -\frac{2\mu}{\gamma} H(\alpha^{\gamma} f^{\gamma}), \tag{4.19}$$

which shows that, for a prescribed value of A, the condition (4.15) restricts the choice of the boundary conditions available. As discussed in the previous section, this choice may be further restricted by the stability requirement (3.2). For a harmonic material of a special kind both these types of restrictions are considered in [14].

References

1. S. S. ANTMAN, *Nonlinear problems of elasticity*, Springer-Verlag, New York 1995.
2. M. ARON, C. CHRISTOPHER and Y. WANG, *On the straightening of compressible, nonlinearly elastic, annular cylindrical sectors*, Mathematics and Mechanics of Solids, **3**, 131, 1998.
3. D. J. STEIGMANN and A. C. PIPKIN, *Stability of harmonic materials in plane strain*, Quarterly of Applied Mathematics, **46**, 559, 1988.
4. J. SIVALOGANATHAN, *The generalized Hamilton-Jacobi inequality and the stability of equilibria in nonlinear elasticity*, Archive for Rational Mechanics and Analysis, **107**, 347, 1989.
5. M. ARON, *A note on the stability of certain deformations of compressible nonlinearly elastic media*, Zeitschrift für Angewandte Mathematik und Physik, **49**, 558, 1998.
6. M. ARON and S. AIZICOVICI, *On a class of deformations of compressible, isotropic, nonlinearly elastic solids*, J. of Elasticity, **49**, 175, 1997.
7. P. ROSAKIS, *Characterization of convex isotropic functions*, J. of Elasticity, **49**, 257, 1998.
8. C. TRUESDELL and W. NOLL, *The non-linear field theory of mechanics* [In:] Handbuch der Physik, Volume III/3, Springer-Verlag, New York 1965.
9. P. G. CIARLET, *Mathematical Elasticity Volume 1: Three-Dimensional Elasticity*, North-Holland, Amsterdam 1988.
10. R. M. BOWEN, *Introduction to continuum mechanics for engineers*, Plenum Press, New York 1989.
11. F. JOHN, *Plane strain problems for a perfectly elastic material of harmonic type*, Communications in Pure and Applied Mathematics, **13**, 239, 1960.
12. R. W. OGDEN, *Non-linear elastic deformations*, Ellis Horwood, Chichester 1984.
13. H. K. WILSON, *Ordinary differential equations*, Addison-Wesley, London 1971.

14. M. ARON and Y. WANG, *Remarks concerning the flexure of a compressible nonlinearly elastic rectangular block*, J. of Elasticity, **40**, 99, 1995.

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