Arch. Mech., **53**, 2, pp. 91–104, Warszawa 2001

On resonances of nonlinear elastic waves in a cubic crystal

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USING THE METHOD of weakly nonlinear geometric optics, we obtain asymptotic transport evolution equations for high-frequency, small amplitude nonlinear elastic waves in a cubic crystal. Both geometrical and physical nonlinearities are included in our model. We expand strain energy up to the third order terms with respect to the strain matrix components. The nonlinear resonant asymptotic equations obtained are of integro-differential type. The coefficients of these equations are called resonant interaction coefficients (RIC). They determine whether and between which waves the nonlinear resonant interactions occur. We have calculated all the RIC in the explicit analytical form for three different crystalline directions of a one-dimensional wave motion. Comparison of the results shows that the direction of propagation influences the resonant interactions in an essential way. Moreover, our analytical formulas for RIC can be used to determine the material constants of a crystal.

1. Introduction

THE NONLINEAR RESONANCE of two waves, contrary to the classical linear superposition, consists in producing a new wave with a fixed wave number and frequency being the combination of the componential wave numbers and frequencies. The generation of the second harmonics is a classical example. Recently, the analysis of nonlinear resonant interactions of waves attracts many mathematicians and physicists. In the last decade, a new asymptotic method called *weakly nonlinear geometric optics*, $WNGO$ in short, was mathematically rigorously developed to analyze nonlinear resonances [1 – 3].

In this paper we are interested in resonant interactions of nonlinear elastic waves in anisotropic media. For simplicity, we focus on a cubic crystal being the simplest nonlinear, anisotropic, elastic medium. To analyze the problem of resonant interactions we derive the equations of motion as the first order quasilinear hyperbolic system of partial differential equations (PDE). Both the geometrical and physical nonlinearities are included in our model.

Employing $WNGO$ we reduced our complicated system of PDE to a relatively simpler set of transport evolution equations with integro-differential terms that describe the nonlinear resonances of the interacting waves. These transport equations can be solved, in general, numerically only. The solution gives the information how the shapes of particular waves evolve in time and space. Even before solving the set of our asymptotic equations, the knowledge of the analytical form of all the nonvanishing RIC provides useful physical information: which waves interact, how strong the nonlinear resonance is and what new waves are produced.

We start with a general formulation of nonlinear dynamical elasticity equations in three space dimensions. After expanding the strain energy up to the third order terms in an arbitrary anisotropic homogeneous medium, we specify the energy form explicitly for a cubic crystal. Then, for simplicity of our presentation, we restrict ourselves to the one space dimension and show the analytical formulas characterizing our hyperbolic system of PDE for three selected directions of wave propagation. Finally, we calculate analytically all the RIC, which we then briefly analyze.

2. Nonlinear elasticity equations

2.1. General form in three-dimensional space

We consider the equations of nonlinear elasticity which, in Lagrangian coordinates, take the following form:

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= \nabla(\mathbf{F}\,\mathbf{T}), \\ \frac{\partial \mathbf{F}}{\partial t} &= \nabla \mathbf{v}, \end{aligned} \tag{2.1}$$

where we introduce: $\mathbf{v} = \dfrac{\partial \mathbf{u}}{\partial t}$ – velocity, $\mathbf{u}$ – displacement, $\nabla \mathbf{u}$ is the displacement gradient matrix with respect to the space variable $\mathbf{x} = [x_1, x_2, x_3] \in \mathbb{R}^3$, t – time, ρ_0 – density and $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$ is the deformation gradient.

In a nonlinear hyperelastic medium the stress tensor $\mathbf{T}$ is characterized by the relation:

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{E}}, \tag{2.2}$$

where the strain energy $W = W(\mathbf{E})$ is an analytic matrix-valued function that we later expand up to the third order terms with respect to the strain matrix $\mathbf{E}$ components, and

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}). \tag{2.3}$$

Using the definition of $\mathbf{F}$ we can express the strain tensor $\mathbf{E}$ as the function of the gradient of displacement:

$$\mathbf{E} = \mathbf{E}(\nabla\mathbf{u}) = \frac{1}{2}\left\{\nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T\,\nabla\mathbf{u}\right\}. \tag{2.4}$$

It shows that we include *geometrical nonlinearity* in our formulation, apart from the so-called *physical nonlinearity* expressed in $W(\mathbf{E})$ (cf. (2.5) below).

2.2. Nonlinear anisotropic medium

We assume the expansion of the energy W in the following physically nonlinear form:

$$W(\mathbf{E}) = \frac{1}{2}\sum_{i,j,k,l}^{3} c_{ij,kl}E_{ij}E_{kl} + \frac{1}{6}\sum_{i,j,k,l,m,n}^{3} c_{ij,kl,mn}E_{ij}E_{kl}E_{mn}. \tag{2.5}$$

Given the explicit form of the energy $W = W(\mathbf{E})$ as a function of the strain $\mathbf{E}$, we compute the stress $\mathbf{T} = \{T_{ij}\}_{i,j=1}^{3}$ by formally differentiating W with respect to $\mathbf{E}$:

$$T_{ij} = \sum_{k,l}^{3} c_{ij,kl}E_{kl} + \frac{1}{2}\sum_{k,l,m,n}^{3} c_{ij,kl,mn}E_{kl}E_{mn}. \tag{2.6}$$

Then employing the formula (2.4), we obtain the energy $\widetilde{W}$ as the function of the gradient of displacement: $\widetilde{W}(\nabla\mathbf{u}) := W(\mathbf{E}(\nabla\mathbf{u}))$.

From now on, to shorten the notation, we use the standard abbreviated indices c_{ij}, c_{ijk}, $i,j,k = 1,...,3$ according to the known rule:

$$\begin{array}{lll} 1,1 \to 1 & 2,2 \to 2 & 3,3 \to 3 \\ 2,3 \to 4 & 3,2 \to 4 & 1,3 \to 5 \\ 3,1 \to 5 & 1,2 \to 6 & 2,1 \to 6 \end{array} \tag{2.7}$$

2.3. Cubic crystal

After BIRCH [4], the explicit form of the strain energy function $W = W(\mathbf{E})$ in the simplest cubic crystal is the following:

$$
\begin{aligned}
(2.8)\qquad W = {} & \frac{1}{2}\hat{c}_{11}(E_{11}^2 + E_{22}^2 + E_{33}^2) + \hat{c}_{12}(E_{11}E_{22} + E_{22}E_{33} + E_{11}E_{33}) \\
& + \hat{c}_{44}(E_{12}^2 + E_{21}^2 + E_{23}^2 + E_{32}^2 + E_{31}^2 + E_{13}^2) + \hat{c}_{111}(E_{11}^3 + E_{22}^3 + E_{33}^3) \\
& + \hat{c}_{112}\{E_{11}^2(E_{22} + E_{33}) + E_{22}^2(E_{11} + E_{33}) + E_{33}^2(E_{11} + E_{22})\} \\
& + \frac{1}{2}\hat{c}_{144}\{E_{11}(E_{23}^2 + E_{32}^2) + E_{22}(E_{13}^2 + E_{31}^2) + E_{33}(E_{12}^2 + E_{21}^2)\} \\
& + \frac{1}{2}\hat{c}_{166}\{(E_{11} + E_{22})(E_{12}^2 + E_{21}^2) + (E_{22} + E_{33})(E_{23}^2 + E_{32}^2) \\
& + (E_{11} + E_{33})(E_{13}^2 + E_{31}^2)\} \\
& + \hat{c}_{123}E_{11}E_{22}E_{33} + \hat{c}_{456}(E_{12}E_{23}E_{31} + E_{21}E_{32}E_{13}),
\end{aligned}
$$

provided that the wave propagation direction is $\mathbf{z}_1 = [1,0,0]$. The coefficients $\hat{c}_{ij}$, $\hat{c}_{ijk}$ in (2.8) are related to c_{ij}, c_{ijk} in (2.5), (2.7) as follows: $\hat{c}_{ij} = c_{ij}$,

$$
\begin{aligned}
\hat{c}_{111} &= \frac{1}{6}c_{111}, & \hat{c}_{144} &= 2c_{144}, & \hat{c}_{123} &= c_{123}, \\
\hat{c}_{112} &= \frac{1}{2}c_{112}, & \hat{c}_{166} &= 2c_{166}, & \hat{c}_{456} &= 4c_{456}.
\end{aligned}
$$

Given any other direction $\mathbf{z}$ of wave propagation, we need to transform the formula (2.8) for the energy $W = W(\mathbf{E})$ according to the rule:

$$
W(\mathbf{E}) \longrightarrow W(\mathbf{Q}_z\mathbf{E}\mathbf{Q}_z^T) \tag{2.9}
$$

where $\mathbf{Q}_z$ is the unitary matrix of the rotation that transforms the vector $\mathbf{z}_1 = [1,0,0]$ into the vector $\mathbf{z}$ and where the superscript T denotes matrix transposition.

In this paper we also consider the two other canonical directions of propagation: $\mathbf{z}_2 = [1,1,0]$ and $\mathbf{z}_3 = [1,1,1]$. The rotation matrix corresponding to the direction $\mathbf{z}_2$ is

$$
\mathbf{Q}_{z_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \tag{2.10}
$$

and to the direction $\mathbf{z}_3$ is

$$
\mathbf{Q}_{z_3} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-\sqrt{2}}{\sqrt{3}} & 0 \end{bmatrix}. \tag{2.11}
$$

2.4. First order system in one dimensional-space

In the one-dimensional case ($\mathbf{x} = [x, 0, 0]$), making use of the formalism from Sec. 2.1, and under natural assumptions guaranteeing hyperbolicity, our nonlinear elasticity equations (2.1) lead to the *quasilinear hyperbolic system* of the form:

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}(\mathbf{w})\,\frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}, \tag{2.12}$$

with the matrix $\mathbf{A}(\mathbf{w})$:

$$\mathbf{A}(\mathbf{w}) = -\begin{bmatrix} \mathbf{0} & \dfrac{1}{\rho_0}\mathbf{B}(\mathbf{w}) \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{2.13}$$

where $\mathbf{I}$ is the 3×3 identity matrix and

$$\mathbf{w} = [\,\mathbf{v}(x,t), \mathbf{u}_{,x}(x,t)\,] = [\,w_1, w_2, w_3, w_4, w_5, w_6\,].$$

The 3×3 matrix $\mathbf{B} = \{B_{ij}\}_{i,j=1}^{3}$ in (2.13) is derived from the equation of motion in (2.1) and is given by the general formula:

$$B_{ij} = \frac{\partial^2 \widetilde{W}}{\partial w_{i+3}\partial w_{j+3}} \qquad \text{with} \qquad \widetilde{W}(\mathbf{u}_{,x}) := W(\mathbf{E}(\mathbf{u}_{,x})). \tag{2.14}$$

The matrix $\mathbf{B}$ can be further specified after expanding the energy $W = W(\mathbf{E})$ up to the *third order* terms with respect to the strain matrix $\mathbf{E}$ (cf. Sec. 2.2).

In the final form of $\mathbf{B}$ obtained by using (2.14), we neglect higher than the first order terms in $\mathbf{w}$. The form of $\mathbf{B}$ depends on the direction $\mathbf{z}$ of the wave propagation in the cubic crystal, as the energy function W does (cf. Sec. 2.3, (2.8), (2.9)). The analytical form of the matrix $\mathbf{B}$ will be explicitly given later for three directions: $\mathbf{z}_1 = [1,0,0]$, $\mathbf{z}_2 = [1,1,0]$ and $\mathbf{z}_3 = [1,1,1]$.

3. WNGO approach

We are interested in the Cauchy problem for the quasilinear hyperbolic system (2.12) with periodic initial conditions

$$\begin{aligned} &\frac{\partial \mathbf{w}^\epsilon}{\partial t} + \mathbf{A}(\mathbf{w}^\epsilon)\,\frac{\partial \mathbf{w}^\epsilon}{\partial x} = \mathbf{0}, \\ &\mathbf{w}^\epsilon(x,0) = \mathbf{w}_0 + \epsilon\,\mathbf{w}_1(x, x/\epsilon), \end{aligned} \tag{3.1}$$

where ϵ is a small parameter and $\mathbf{w}_0$ is a constant state solution to the PDE system (2.12).

This problem can be solved by using asymptotic methods. The heuristic expansions, without proving their asymptotic correctness, were applied by physicists to study interactions of nonlinear waves already in the fifties. The modern, mathematically rigorous approach, called weakly nonlinear geometric optics (*WNGO*), was initiated by CHOQUET-BRUHAT [5], followed by HUNTER and KELLER [6]. In the eighties and nineties, many papers devoted to *WNGO* were published, e.g. [1, 2], several of them containing the rigorous proofs of the validity of *WNGO* [3, 7].

The *WNGO* method has been already applied to study resonances in different physical situations: gasdynamics [1], magnetohydrodynamics [8], isotropic elasticity [9], magnetoelasticity [10]. We have analyzed the resonances in nonlinear electrodynamics [11].

3.1. Asymptotics

A weakly *nonlinear geometric optics* asymptotic solution to the system (3.1) is sought for in the form:

$$\mathbf{w}^{\epsilon}(x,t) = \mathbf{w}_0 + \epsilon \sum_j a_j \left(x, t, \frac{x - \lambda_j t}{\epsilon} \right) \mathbf{r}_j + \mathcal{O}(\epsilon^2) \tag{3.2}$$

with the unknown amplitudes a_j. We assume here that a_j are periodic with zero mean.

Here λ_j is the eigenvalue of $\mathbf{A}(\mathbf{w}_0)$, $j = 1, \ldots, 6$, and $\mathbf{r}_j$ is the right eigenvector corresponding to λ_j:

$$(\mathbf{A}(\mathbf{w}_0) - \lambda_j \mathbf{I})\mathbf{r}_j = \mathbf{0}, \qquad \mathbf{l}_j\,(\mathbf{A}(\mathbf{w}_0) - \lambda_j \mathbf{I}) = \mathbf{0}, \qquad \mathbf{l}_i \cdot \mathbf{r}_j = \delta_{ij}$$

while $\mathbf{l}_j$ is the left eigenvector of $\mathbf{A}(\mathbf{w}_0)$ and δ_{ij} is the Kronecker delta.

Inserting (3.2) into (3.1), expanding $\mathbf{A}$ around $\mathbf{w}_0$, into Taylor's series, using multiple scale analysis and employing the solvability condition, we obtain the *transport evolution equations* for the amplitudes a_j of resonantly interacting waves.

In general, for strictly hyperbolic systems, these are integro-differential equations of the form [1]:

$$\frac{\partial a_j}{\partial t} + \lambda_j \frac{\partial a_j}{\partial x} + \Gamma^j_{jj} a_j \frac{\partial a_j}{\partial \eta} + \sum_{p,q}{}' \, \Gamma^j_{pq} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a'_p \, a_q \, ds \; = 0 \tag{3.3}$$

where

$$a_q = a_q(x, t, \eta + s(\lambda_j - \lambda_q)), \qquad a'_p = \frac{\partial a_p}{\partial \eta}(x, t, \eta + s(\lambda_j - \lambda_p)),$$

and $\sum'$ indicates summation avoiding repeated indices, while $\eta \equiv (x - \lambda_j t)\,\epsilon^{-1}$.

The *nonlinear resonance* takes place when at least one of the integro-differential terms with Γ^j_{pq} in (3.3), $j, p, q = 1, 2, \ldots, 6$, does not vanish.

3.2. Resonant interaction coefficients

The fundamental feature of a nonlinear resonance of waves is the generation of a new wave with a fixed wave number and a frequency being the combination of the componential wave numbers and frequencies [1, 3]. It is of great practical importance to investigate whether and when such nonlinear resonances take place.

The strength of the j-th wave produced through the nonlinear resonant interaction of p-th and q-th waves is represented by the coefficient Γ^j_{pq} in (3.3). It is called the *resonant interaction coefficient (RIC)*. It can be put into relatively simple form:

$$\Gamma^j_{pq}(\mathbf{w}_0) = \mathbf{l}_j \cdot (\nabla_{\mathbf{w}}\, \mathbf{A})\big|_{\mathbf{w}=\mathbf{w}_0} \mathbf{r}_p \cdot \mathbf{r}_q \tag{3.4}$$

convenient for further manipulations with the help of the symbolic calculation program *Mathematica*.

The formula (3.4) for *RIC* involves right and left eigenvectors of the matrix $\mathbf{A}$ of our PDE set, and the derivatives of $\mathbf{A}$ with respect to the unknown vector $\mathbf{w}$, all evaluated at the constant state $\mathbf{w}_0$ around which we expand our asymptotics. In fact, the $(\nabla_{\mathbf{w}}\, \mathbf{A})$ is the Hessian of the matrix $\mathbf{A}$. Thus, together with the eigenvectors of $\mathbf{A}$ in the analytical form, the formula (3.4) can be quite lengthy and tedious. Nevertheless, it is in a suitable form for symbolic calculation programs like *Mathematica* or *Maple*.

Specially designed procedures have been developed in the *Mathematica* language. These procedures allow us to calculate analytically and efficiently simplify all the *RIC* of the system (3.3) for any given constant state $\mathbf{w}_0$.

4. Nonlinear resonances in a cubic crystal

The explicit form of *all* the *RIC* have been computed for system (2.12) of nonlinear elasticity equations for a cubic crystal, at the zero constant state ($\mathbf{w}_0 = \mathbf{0}$) for three different directions of wave propagation: $\mathbf{z}_1 = [1, 0, 0]$, $\mathbf{z}_2 = [1, 1, 0]$ and $\mathbf{z}_3 = [1, 1, 1]$.

In each case, apart from the analytical formulas for *RIC*, we present the particular analytical form of the matrix $\mathbf{B}$ (cf. (2.13)) and the eigenvalues of the matrix $\mathbf{A}(\mathbf{w}_0)$. All the formulas are presented as functions of the elastic material constants c_{ij}, c_{ijk}.

Moreover, in Fig. 1 we show a graphical representation of nonlinearly interacting waves. This representation can be interpreted as follows. Let us recall that a nonvanishing Γ^j_{pq} in (3.3) represents the strength of the j-th wave produced through the nonlinear resonant interaction of p-th and q-th waves. For each new j-th wave, $j = 1, ..., 6$, we show a 6×6 matrix of respectively shaded squares that represent the p-th and q-th waves, by interaction of which the new j-th wave is produced. The letters in the squares and the shades of the squares correspond to the analytical formulas for the respective RIC. They, in turn, are referred to the graphics by the letters in squares in the formulas.

Our graphical representation helps to determine all the nonvanishing RIC and to find the relations between them (e.g. which of them are equal) without presenting lengthy redundant equations. Such a visualization allows us to compare easily nonlinear resonant interaction of waves for different directions of propagation or for different constant states.

Let us notice that the structure of the eigensystem of our matrix $\mathbf{A}$ in (2.12) is preserved, irrespectively of the direction of the wave propagation $\mathbf{z}_j$. There are always three pairs of eigenvalues of opposite sign:

$$\lambda_1 = -\lambda_2, \qquad \lambda_3 = -\lambda_4, \qquad \lambda_5 = -\lambda_6, \tag{4.1}$$

with the degeneracy $\lambda_3 = \lambda_5$ in the cases 1 and 3. In each case, the calculated eigenvectors of $\mathbf{A}(\mathbf{w}_0)$ can be expressed in terms of the eigenvalues (4.1) and they take the following form:

$$\begin{aligned}
\mathbf{r}_1 &= [\lambda_1, 0, 0, 1, 0, 0], & \mathbf{l}_1 &= \frac{1}{2}[\lambda_1^{-1}, 0, 0, 1, 0, 0],\\
\mathbf{r}_2 &= [\lambda_2, 0, 0, 1, 0, 0], & \mathbf{l}_2 &= \frac{1}{2}[\lambda_2^{-1}, 0, 0, 1, 0, 0],\\
\mathbf{r}_3 &= [0, \lambda_3, 0, 0, 1, 0], & \mathbf{l}_3 &= \frac{1}{2}[0, \lambda_3^{-1}, 0, 0, 1, 0],\\
\mathbf{r}_4 &= [0, \lambda_4, 0, 0, 1, 0], & \mathbf{l}_4 &= \frac{1}{2}[0, \lambda_4^{-1}, 0, 0, 1, 0],\\
\mathbf{r}_5 &= [0, 0, \lambda_5, 0, 0, 1], & \mathbf{l}_5 &= \frac{1}{2}[0, 0, \lambda_5^{-1}, 0, 0, 1],\\
\mathbf{r}_6 &= [0, 0, \lambda_6, 0, 0, 1], & \mathbf{l}_6 &= \frac{1}{2}[0, 0, \lambda_6^{-1}, 0, 0, 1].
\end{aligned} \tag{4.2}$$

Formula (4.2) shows that in all the three cases analyzed we have a complete set of linearly independent eigenvectors satisfying $\mathbf{l}_i \cdot \mathbf{r}_j = \delta_{ij}$, in spite of the fact that our system (2.12) is not strictly hyperbolic at $\mathbf{w}_0$ in the cases 1 and 3. The completeness of the eigenvectors allows us to calculate RIC even in these degenerate cases.

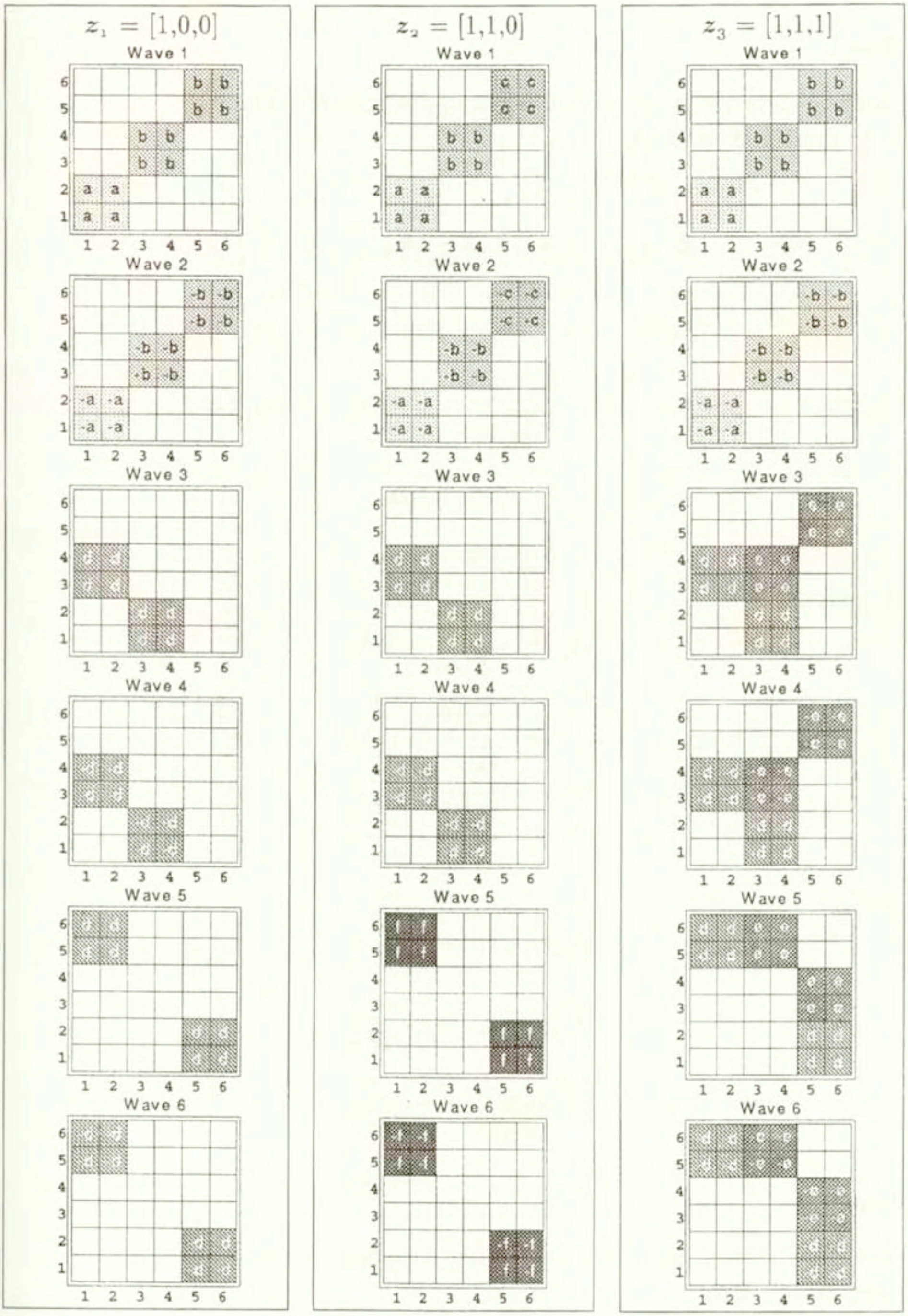

FIG. 1. Graphical representation of nonlinear resonances in a cubic crystal for different directions of wave propagation (see page 8).

4.1. Case 1

For the direction $\mathbf{z}_1 = [1, 0, 0]$ of wave propagation we have:
The matrix $\mathbf{B}$ in (2.13):

$$\mathbf{B} = \begin{bmatrix} \alpha + \beta w_4 & \gamma w_5 & \gamma w_6 \\ \gamma w_5 & \delta + \gamma w_4 & 0 \\ \gamma w_6 & 0 & \delta + \gamma w_4 \end{bmatrix}$$

with

$$\alpha = c_{11},$$

$$\beta = 3c_{11} + c_{111},$$

$$\gamma = c_{11} + c_{166},$$

$$\delta = c_{44}.$$

Eigenvalues:

$$\lambda_1 = -\sqrt{\frac{\alpha}{\rho_0}} = -\lambda_2,$$

$$\lambda_3 = -\sqrt{\frac{\delta}{\rho_0}} = -\lambda_4,$$

$$\lambda_5 = -\sqrt{\frac{\delta}{\rho_0}} = -\lambda_6.$$

RIC:

$$\boxed{\mathrm{a}} \leftrightarrow \Gamma^1_{11} = \frac{-\beta}{2\sqrt{\rho_0\,\alpha}},$$

$$\boxed{\mathrm{b}} \leftrightarrow \Gamma^1_{33} = \frac{-\gamma}{2\sqrt{\rho_0\,\alpha}},$$

$$\boxed{\mathrm{d}} \leftrightarrow \Gamma^3_{13} = \frac{-\gamma}{2\sqrt{\rho_0\,\delta}}.$$

4.2. Case 2

For the direction $\mathbf{z}_2 = [1, 1, 0]$ of wave propagation we have:
The matrix $\mathbf{B}$ in (2.13):

$$\mathbf{B} = \begin{bmatrix} \alpha + \beta w_4 & \gamma w_5 & \gamma^* w_6 \\ \gamma w_5 & \delta + \gamma w_4 & 0 \\ \gamma^* w_6 & 0 & \delta^* + \gamma^* w_4 \end{bmatrix},$$

with

$$\alpha = \frac{1}{2}(c_{11} + c_{12} + 2c_{44}),$$
$$\beta = \frac{3}{2}(c_{11} + c_{12} + 2c_{44}) + \frac{1}{4}(c_{111} + 3c_{112} + 12c_{166}),$$
$$\gamma = \frac{1}{2}(c_{11} + c_{12} + 2c_{44}) + \frac{1}{4}(c_{111} - c_{112}),$$
$$\gamma^* = \frac{1}{2}(c_{11} + c_{12} + 2c_{44} + c_{144} + c_{166} + 2c_{456}),$$
$$\delta = \frac{1}{2}(c_{11} - c_{12}),$$
$$\delta^* = c_{44}.$$

Eigenvalues:

$$\lambda_1 = -\sqrt{\frac{\alpha}{\rho_0}} = -\lambda_2,$$
$$\lambda_3 = -\sqrt{\frac{\delta}{\rho_0}} = -\lambda_4,$$
$$\lambda_5 = -\sqrt{\frac{\delta^*}{\rho_0}} = -\lambda_6.$$

RIC:

$$\boxed{\text{a}} \leftrightarrow \Gamma^1_{11} = \frac{-\beta}{2\sqrt{\rho_0\,\alpha}},$$
$$\boxed{\text{b}} \leftrightarrow \Gamma^1_{33} = \frac{-\gamma}{2\sqrt{\rho_0\,\alpha}},$$
$$\boxed{\text{c}} \leftrightarrow \Gamma^1_{55} = \frac{-\gamma^*}{2\sqrt{\rho_0\,\alpha}},$$
$$\boxed{\text{d}} \leftrightarrow \Gamma^3_{13} = \frac{-\gamma}{2\sqrt{\rho_0\,\delta}},$$
$$\boxed{\text{f}} \leftrightarrow \Gamma^5_{15} = \frac{-\gamma^*}{2\sqrt{\rho_0\,\delta^*}}.$$

4.3. Case 3

For the direction $\mathbf{z}_3 = [1, 1, 1]$ of wave propagation we have:

The matrix $\mathbf{B}$ in (2.13):

$$\mathbf{B} = \begin{bmatrix} \alpha + \beta w_4 & \gamma w_5 & \gamma w_6 \\ \gamma w_5 & \delta + \gamma w_4 + \varepsilon w_5 & \varepsilon w_6 \\ \gamma w_6 & \varepsilon w_6 & \delta + \gamma w_4 + \varepsilon w_5 \end{bmatrix}$$

with

$$\begin{aligned}
\alpha &= \frac{1}{3}(c_{11} + 2c_{12} + 4c_{44}),\\
\beta &= c_{11} + 2c_{12} + 4c_{44}\\
&\quad + \frac{1}{9}(c_{111} + 6c_{112} + 12c_{144} + 24c_{166} + 2c_{123} + 16c_{456}),\\
\gamma &= \frac{1}{3}(c_{11} + 2c_{12} + 4c_{44}) + \frac{1}{9}(c_{111} - 3c_{144} + 6c_{166} - c_{123} - 2c_{456}),\\
\delta &= \frac{1}{3}(c_{11} - c_{12} + c_{44}),\\
\varepsilon &= \frac{\sqrt{2}}{18}(c_{111} - 3c_{112} + 3c_{144} - 3c_{166} + 2c_{123} - 2c_{456}).
\end{aligned}$$

Eigenvalues:

$$\lambda_1 = -\sqrt{\frac{\alpha}{\rho_0}} = -\lambda_2,$$

$$\lambda_3 = -\sqrt{\frac{\delta}{\rho_0}} = -\lambda_4,$$

$$\lambda_5 = -\sqrt{\frac{\delta}{\rho_0}} = -\lambda_6.$$

RIC:

$$\boxed{\mathrm{a}} \leftrightarrow \Gamma^1_{11} = \frac{-\beta}{2\sqrt{\rho_0\,\alpha}},$$

$$\boxed{\mathrm{b}} \leftrightarrow \Gamma^1_{33} = \frac{-\gamma}{2\sqrt{\rho_0\,\alpha}},$$

$$\boxed{\mathrm{d}} \leftrightarrow \Gamma^3_{13} = \frac{-\gamma}{2\sqrt{\rho_0\,\delta}},$$

$$\boxed{\mathrm{e}} \leftrightarrow \Gamma^3_{33} = \frac{-\varepsilon}{2\sqrt{\rho_0\,\delta}}.$$

5. Conclusions

The coefficients of the transport evolution equations (3.3) of weakly nonlinear elastic waves in a homogeneous cubic crystal have been calculated explicitly in a general *analytical* form. These coefficients, called RIC, represent resonant interactions of nonlinear waves and are expressed in terms of material constants of the medium. The RIC formulas have been analyzed for various crystalline directions of one-dimensional wave propagation.

The knowledge of the complete set of RIC for a given direction of propagation allows one to determine precisely which waves interact, how strong the nonlinear resonance is and what new waves are produced.

The analysis of the derived formulas shows that the most important factors influencing the resonant interactions are: the direction of wave propagation and the character of the nonlinearity of a crystal.

Closer examination of the Fig. 1 leads to the following observations:

• For all the three cases analyzed, the following formula holds true:

$$\Gamma^{j}_{pq} = -\Gamma^{j+1}_{pq}$$

for all p, q and $j = 1,3,5$. Let us also recall that $\lambda_j = -\lambda_{j+1}$. The physical interpretation is that resonant waves are produced always in pairs of waves propagating in opposite directions.

• The waves numbered 1 and 2 can only be produced by the interaction of waves in the same pair, namely: (1,2), (3,4) or (5,6).

• In the cases 1 and 2 there are the same nonvanishing RIC – meaning that the same resonances may occur in both cases; only the magnitudes of the nonlinear interactions differ.

• In the case 3 (propagation along the cubic diagonal) more nonzero RIC occur than in the other cases. All the additional RIC are expressed in terms of the third order material constants (cf. the definition of ε). Therefore, the *physical nonlinearity* is crucial here.

• All the nonvanishing RIC except those ε-dependent are not zero, even without the physical nonlinearity included in the energy formula expansion. Thus, nonlinear interactions manifest themselves already in the models with *geometrical nonlinearity* only. However, higher-order, *physical* nonlinearities influence substantially the *magnitude* of the resonances (cf. formulas for β and γ).

• Analytical formulas for RIC can be useful for determining elastic constants of a crystal in suitably designed measurements.

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Received April 26, 2000; revised version October 27, 2000.