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Laminar dispersed two-phase flows at low concentration III Pseudo-turbulence

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Two preceding papers (Parts I and II) present a generalised system of equations to describe non-turbulent flows of inclusion-fluid mixtures. The two first-order equations for each phase correspond to equations of the standard two-fluid models. The closure problem, consisting in the derivation of constitutive laws for unknown quantities (mainly interaction terms between phases) is not encountered in our approach; these quantities are provided by two infinite sequences of higher order equations controlling more and more conditioned "disturbance fields". The difficulty is then to truncate in a consistent way these sequences using a dilutness assumption. Agitation terms are also unknowns, both in first and higher orders equations. These terms, which result from various types of micro-motions of both phases (the so-called pseudo-turbulence phenomena), have not been related to disturbance fields at the end of the two preceding papers; they just came out as unclosed correlation functions. This paper gives each correlation function a specific expression; some of them must be approximated due to dilutness in order to be effectively computed.

1. Introduction

Very early in the derivation of averaged two-phase flows models, Buyevich [3, 4] was one of a few authors to discuss agitation terms which influence the bulk momentum transfer of both the inclusions and the continuous phase, as soon as inertia effects are important. Even if they appear as formally similar unknown correlations, they represent six (at least) different types of phenomena. It is worth to start a preliminary discussion of the physical origin of these types which renews that proposed by Buyevich and Shchelchkova [5]. Recall that two-phase flows considered in our analysis must be dilute since otherwise the collisions between inclusions become an essential and quite distinct mechanism of agitation.

The first case to adress corresponds to turbulent motions of the carrying phase modified by inclusions; turbulence would even exist in the absence of inclusions. Particle-turbulence interactions have been experimentally investigated in a number of situations. Among these, let us quote bubbly flows in basic turbulence fields studied by LANCE and collaborators [6 – 9], and bubbly flows in ducts [10, 11]. Interactions with solid particles have been considered notably by TSUJI *et al.* [12] and by FAETH and coworkers [13, 14]. Even under turbulent conditions, the role of the particle-induced turbulence has been emphasized, but it is experimentally quite difficult to distinguish this contribution from that of the shear-induced turbulence. On the other hand, DNS has thrown some light on the turbulence modulation by small rigid inclusions of variable inertia [15, 16, 17]. To illustrate the complex coupling between phases occuring even at low concentrations, let us mention the occurence of a preferential accumulation of inert particles in regions of low vorticity or high strain rate, a phenomena first identified from simulations [18, 19, 20]. The case of bubbles has also been adressed [21]. More practical and more versatile models are also extracted from all these fundamental studies [22].

The second possibility is that correlation functions may result from the turbulent motion of the carrying phase solely induced by inclusions. The very presence of inclusions may trigger the hydrodynamic unstability of the fluid flow which would be otherwise laminar. This case seems to be beyond our present analysis capabilities (cf. Sec. 3 – Part I).

Surface tension is unable to maintain spherical bubbles and drops against various forces (gravity, inertia, viscosity...) beyond some size. Then, inclusion shapes result from interactions with the ambient fluid, causing generally highly complex free boundary value problems. Interfaces are subjected to various instabilities generating pulsations in both phases. Inclusion trajectories themselves can be spiraling, pulsating, zig-zagging (CLIFT *et al.* [23]) due to shape oscillations or simply modifications (i.e. sphere to ellipsoid) associated to vortex shedding. The ensuing determination of agitation terms due to this third mechanism in the bulk equations is still in infancy, but progress can be expected in the near future from DNS undertaken on bubbly flows [24].

Very small suspended particles (colloidal particles) may be affected by translational (or rotational if they are non-spherical) Brownian motion (RUSSEL *et al.* [25]). To describe this constant state of random motion, a particle velocity autocorrelation function is first calculated from the Langevin equation and then the time change of the variance in position follows; finally, the diffusion coefficients useful for a "population balance" approach (first kinetic equation) can be estimated. An exhaustive study of the influence of Brownian motion is now available. This mechanism of agitation (restricted here to inclusions) as well as the three preceding ones (affecting both phases) are excluded from our analysis. We are left with the two last ones which will be adressed now.

Any local technique for continuous phase velocity measurements in two-phase flows, records inevitably pulsations caused by the local distortions in the fluid flow streamlines caused by the submerged inclusions. They occur in the potential fields at the fore-part, possibly in the viscous boundary layers and mainly in the wake regions. To understand the origin and to have the first estimate of the corresponding fluid velocity variance tensors, it is possible to refer to a deterministic approach. Then, one considers a given dispersed two-phase flow characterized by a configuration of particles having an idealised order: for instance, the particles can be equal-sized spheres arranged in periodic arrays (see the pioneering work of HASIMOTO [26]); at zero or low Reynolds numbers, the corresponding hydrodynamic fields can be solved exactly. The method of homogenisation, which is basically a two-scale method, falls also into this category, when at least it is effectively implemented (MIKSIS and TING, [27]). The result is a macroscopic system of equations for the hydrodynamical variables and the coefficients in this system are given by solving a microscopic problem for a cell to close all unknown terms, and among them, the agitation tensor. This is in essence the approach of NIGMATULIN [28] which proposed an expression for such a tensor based on a potential flow cellular scheme. To our knowledge, there are very few attemps in the literature to calculate variance tensors in various flows conditions (complete range of volume fraction of solid spheres, spherical bubbles, high Reynolds number flows...) having a "frozen configuration", though the corresponding models exist.

In turns out that configurations of particles seldom (if ever) maintain a regular order (CARTELLIER *et al.* [29]); they are very unstable for increasing dispersed phase concentration and decreasing continuous phase viscosity. The time evolution of any configuration of particles which interact together via the suspending fluid, constitutes a highly nonlinear multibody process. The interparticle interaction is accomplished through random pressure and velocity fields in the ambient fluid; this induces lateral and longitudinal pulsations of the particles with respect to a preferred direction and, second, to pulsations in the fluid itself. This so-called "pseudo-turbulence" of both phases which is anisotropic, causes the initiation of additional stresses at the bulk level. This is the last mechanism of agitation we describe. The relevant literature will be evoked in Sec. 4.4.

We will see in our paper that we can pass on in a continuous way from the fourth mechanism description to the fifth one as the concentration Θ increases. To point out the whole process, we restrict our interest to the same type of idealised dispersed two-phase flows which have been selected in the preceding papers [1, 2] (referred to as Parts I and II); e.g. these flows carry spherical inclusions having a radius a small compared to the length scale L of the averaged flows. The concentration Θ or averaged dispersed phase volume fraction is defined by $\Theta = N(a/\mathrm{L})^3$, where N is the total number (assumed to be very large) of inclusions in the studied system, is supposed to be small. Finally, recall that at the end of the Sec. 4 of Part II, all equations of the Lundgren Hierarchy and of the B.B.G.K.Y. hierarchy and, besides, boundary conditions between these two sets of equations at any order but the first one, have been transformed into equations for averaged disturbance flows. The only unclosed terms are agitation terms of various types.

The first three sections adress the velocity variance tensors (agitation or pseudo-turbulent tensors) involving dispersed or continuous phase fluctuations. These tensors differ by their order since they involve fluctuations conditioned by one, two, three inclusions. To get the clue to express any of these tensors in terms of the disturbance fields defined in Part II, we show first in Sec. 2 that any tensor can be expanded in a finite sequence of terms gradually bringing out the contribution of groups of inclusions more and more numerous with respect to the conditioning number characterizing the considered tensor. These expansions open the way to an effective scheme of calculations and two approximation procedures have been devised to simplify them. In Sec. 3, a first approximation based on a preliminary scale analysis exploits diluteness ($\Theta \ll 1$). This assumption precisely permits to extract the leading order terms from the contribution of each group. An extra approximation in Sec. 4 exploits a weak deviation from homogeneity by using a multipole expansion: then only the lowest order multipoles are retained via an asymptotic expansion in terms of $\beta = a/\mathrm{L}$; at the end of this section, proposals available in the literature, concerning some velocity variance tensors we have derived, are briefly compared. In the following two sections, we consider cross-correlations combining properties relative to both phases. In Sec. 5, new terms which correlate the continuous phase presence at one location with the inclusion (s) velocity(ies) at a neighbouring point(s) are expressed in terms of the disturbance fields, while in Sec. 6, fluid-particles velocity covariance tensors (composite agitation tensors) are treated as the above single-phase tensors are. We conclude in Sec. 7 by giving a perspective of this work which will come in a future paper.

2. Expansions of pseudo-turbulent tensors

Pseudo-turbulent tensors appear in momentum equations relative to both phases at any order.

2.1. The dispersed phase pseudo-turbulent tensors

To expand a variance tensor of any order in a finite sequence of contributions relative to groups having an increasing number of test inclusions, we develop a procedure which is based on a property of fine-grained densities introduced in Part I. Given an arbitrary random scalar process $\mathrm{y}(s,t)$, introducing the spiky field of realisations $\delta[\mathrm{y} - \mathrm{y}(s,t)]$ has offered in Part I the possibility to obtain the standard pdf $\mathcal{P}(\mathrm{y})$ by taking its ensemble average i.e. $\mathcal{P}(\mathrm{y}) = E[\delta\{\mathrm{y} - \mathrm{y}(s,t)\}]$. But if δ is a functional of the random process $\mathrm{y}(s,t)$, it is also, for a given realization, a simple pdf of y having the normalization property $\int \delta\{\mathrm{y} - \mathrm{y}(s,t)\}dy = 1$ by definition of the delta function. Extensions of this property for a vectorial random process $\mathrm{y}(s,t)$ are obvious. For instance, the first order tensor, $\mathbb{A}^1_{\mathrm{uu}}(\mathbf{x})$ defined by (4.24 – Part I) can be cast in the following form:

$$
\begin{aligned}
(2.1) \qquad \mathbb{A}^1_{\mathrm{uu}}(\mathbf{x}) &= \sum_{i=1}^{N} E[\varphi_i \mathbf{u}'_i \mathbf{u}'_i] = NE[\varphi_1 \mathbf{u}'_1 \mathbf{u}'_1] \\
&= NE\left[\int \varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}'_1\mathbf{u}'_1(\mathbf{x})d\mathbf{x}^\circ\right] = N\int E[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}'_1\mathbf{u}'_1(\mathbf{x})]d\mathbf{x}^\circ,
\end{aligned}
$$

where the integration is performed, for each $\mathbf{x}$, over the sole domain accessible to the single inclusion centred at $\mathbf{x}^\circ$, knowing that another one is located at $\mathbf{x}$, (non-overlapping condition). Note that throughout this paper, the ensemble average operator $E[\ ]$ applied to integrals involving fine-grained quantities (as φ_1, the first order density) is transformed according to the rules established in Sec. 3 – Part I which result from the definition of $E[\ ]$. By means of the definition of the first fluctuation field (i.e. $\varphi_1\mathbf{u}_1 = \varphi_1\overline{\mathbf{u}}^1 + \varphi_1\mathbf{u}'_1$, see Sec. 4.4 – Part I), the integrand of the r.h.s. may in turn be broken down into three parts:

$$
\begin{aligned}
(2.2) \qquad NE[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}'_1\mathbf{u}'_1(\mathbf{x})] &= NE[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})(\mathbf{u}_1 - \overline{\mathbf{u}}^1)(\mathbf{u}_1 - \overline{\mathbf{u}}^1)] \\
&= NE[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1\mathbf{u}_1] + NE[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})]\overline{\mathbf{u}}^1\overline{\mathbf{u}}^1 \\
&\quad - 2\overline{\mathbf{u}}^1 E[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1]
\end{aligned}
$$

and further can be related to a second-order unsymmetrical (with respect to $\mathbf{x}$ and $\mathbf{x}^\circ$) tensor $\mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ)$, conditional upon the presence at $\mathbf{x}^\circ$ of another inclusion, by introducing the second fluctuation field defined by $\varphi_1\varphi_2\mathbf{u}_1 = \varphi_1\varphi_2\overline{\mathbf{u}}^2 + \varphi_1\varphi_2\mathbf{u}''_1$ (see Sec. 4.4 – Part I):

$$
\begin{aligned}
(2.3) \qquad NE[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}'_1\mathbf{u}'_1(\mathbf{x})] &= \frac{1}{(N-1)}\Big\{\phi^{(2)}\overline{\mathbf{u}}^2\overline{\mathbf{u}}^2 + N(N-1) \\
&\quad E[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}''_1\mathbf{u}''_1(\mathbf{x})] + \phi^{(2)}\overline{\mathbf{u}}^1\overline{\mathbf{u}}^1 - 2\phi^{(2)}(\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{u}}^1\overline{\mathbf{u}}^2\Big\} \\
&= \frac{1}{(N-1)}\Big\{\phi^{(2)}\left[\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^1(\mathbf{x})\right]\left[\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^1(\mathbf{x})\right] + \mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ)\Big\}.
\end{aligned}
$$

Inserting this result into (2.1) we get:

$$
\begin{aligned}
(2.4) \qquad \mathbb{A}^1_{\mathrm{uu}}(\mathbf{x}) = NE[\varphi_1\mathbf{u}'_1\mathbf{u}'_1] &= \frac{1}{(N-1)}\int d\mathbf{x}^\circ\Big\{\phi^{(2)}[\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^1(\mathbf{x})] \\
&\quad [\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^1(\mathbf{x})] + \mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ)\Big\}.
\end{aligned}
$$

Likewise, the second order tensor, $\mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ)$ can be related to an integral for each $\mathbf{x}$ and $\mathbf{x}^\circ$ of $E[\varphi_3(\mathbf{x}^{\circ\circ})\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1''\mathbf{u}_2''(\mathbf{x})]$ over the domain accessible to the single inclusion centred at $\mathbf{x}^{\circ\circ}$ taking into account the non-overlapping condition:

$$(2.5)\qquad \mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ) = N(N-1)E[\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1''\mathbf{u}_1''(\mathbf{x})]$$

$$= N(N-1)E\left[\int \varphi_3(\mathbf{x}^{\circ\circ})\varphi_2(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1''\mathbf{u}_1''(\mathbf{x})d\mathbf{x}^{\circ\circ}\right]$$

$$= \frac{1}{(N-2)}\int d\mathbf{x}^{\circ\circ}\{\phi^{(3)}[\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ)]$$

$$[\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ)] + \mathbb{A}^3_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\}$$

Clearly we have:

$$(2.6)\qquad \mathbb{A}^3_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) = \frac{1}{(N-3)}\int d\mathbf{x}^{\circ\circ\circ}\{\phi^{(4)}[\overline{\mathbf{u}}^4(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ})$$

$$-\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})][\overline{\mathbf{u}}^4(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})]$$

$$+\mathbb{A}^4_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ})\}.$$

It is straightforward to derive a recurrent relation in which $\mathbb{A}^j_{\mathrm{uu}}$ is related to $\mathbb{A}^{j+1}_{\mathrm{uu}}$ where j varies from 1 to $N-1$, and where the various domains of integration must take into account a more and more stringent non-overlapping condition. Thus, one property of the above relation is that it can provide in the same shot $\mathbb{A}^1_{\mathrm{uu}}$, $\mathbb{A}^2_{\mathrm{uu}}$ as well as any higher tensor in terms of a sequence involving all the subsequent tensors.

Of course, a similar recurrent relation holds for mixed agitation tensors of the first kind, which are conditional upon the presence around $\mathbf{x}$ (say $\mathbf{x}^\circ$, $\mathbf{x}^{\circ\circ}$, $\mathbf{x}^{\circ\circ\circ}$,...) of various inclusions; e.g. $\mathbb{A}^1_{\mathrm{u}\omega}(\mathbf{x})$ is expressible in terms of $\mathbb{A}^2_{\omega}(\mathbf{x}|\mathbf{x}^\circ)$ which is itself related to $\mathbb{A}^3_{\omega}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})$ and so on...

Equations ((4.29) and (4.30) – Part I) require also symmetrical (with respect to $\mathbf{x}$ and $\mathbf{x}^\circ$) agitation tensors as $\mathbb{A}^2_{\mathrm{uu}^\circ}$. We can give it an expression in terms of the third order agitation tensor relative to two positions by decomposing properly $\mathbf{u}''$ and $\mathbf{u}^{\circ\prime\prime}$ thanks to (Sec. 4.4 – Part I) evaluated respectively at $\mathbf{x}$ and at $\mathbf{x}^\circ$:

$$(2.7)\qquad \mathbb{A}^2_{\mathrm{uu}^\circ}(\mathbf{x},\mathbf{x}^\circ) = \frac{1}{(N-2)}\int d\mathbf{x}^{\circ\circ}\{\phi^{(3)}[\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ)]$$

$$\cdot[\overline{\mathbf{u}^\circ}^3(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}^\circ}^2(\mathbf{x}^\circ|\mathbf{x})] + \mathbb{A}^3_{\mathrm{uu}^\circ}\}.$$

The second kind of mixed agitation tensor $\mathbb{A}^2_{\omega u^\circ}$ becomes similarly:

$$(2.8)\qquad \mathbb{A}^2_{\omega u^\circ}(\mathbf{x},\mathbf{x}^\circ) = \frac{1}{(N-2)}\int d\mathbf{x}^{\circ\circ}\{\phi^{(3)}[\overline{\boldsymbol{\varpi}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\boldsymbol{\varpi}}^2(\mathbf{x}|\mathbf{x}^\circ)] \cdot[\overline{\mathbf{u}^\circ}^3(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})) - \overline{\mathbf{u}^\circ}^2(\mathbf{x}^\circ|\mathbf{x})] + \mathbb{A}^3_{\omega u^\circ}\}.$$

We have thus shown that any kind of inclusion velocity (linear and rotational) variance tensors could be expressed as finite sums of integrals implying higher and higher pseudo-turbulent tensors. The physical interpretation of the various terms comprised in, say the first order tensor $\mathbb{A}^1_{uu}(\mathbf{x})$, is the following. Each extra term in the infinite sequence gradually brings out the influence of more and more important groups of inclusions upon the first-order agitation tensor. This influence is expressed via increasingly complex space integrals showing definitely its non-local character.

2.2. The continuous phase pseudo-turbulent tensors

There is no basic difference between the ways to expand the velocity variance tensors whether they arise from velocity fluctuations in the dispersed phase or from velocity fluctuations in the continuous phase. For instance, by introducing the presence of an inclusion at $\mathbf{x}^\circ$, the first tensor $\mathbb{A}^1_{vv}$, which involves fluctuations in the continuous phase defined by $X^c\mathbf{v}^c = X^c\overline{\mathbf{v}^c}^1 + X^c\mathbf{v}^{c'}$ (see Sec. 5.4 – Part I), can be cast, in a quite general way, in the following form:

$$(2.9)\qquad \mathbb{A}^1_{vv}(\mathbf{x}) = E[X^c\mathbf{v}^{c'}\mathbf{v}^{c'}](\mathbf{x}) = E\left[\int \varphi_1(\mathbf{x}^\circ)X^c_1\mathbf{v}^{c'}\mathbf{v}^{c'}(\mathbf{x})d\mathbf{x}^\circ\right] = \int E[\varphi_1(\mathbf{x}^\circ)X^c_1\mathbf{v}^{c'}\mathbf{v}^{c'}(\mathbf{x})]d\mathbf{x}^\circ,$$

where the integration is performed, for each $\mathbf{x}$, so that the first inclusion centred at $\mathbf{x}^\circ$ does not overlap $\mathbf{x}$ which must be occupied by the continuous phase; following the same line of reasoning as that giving (2.2) and (2.3), the integrand of the r.h.s. of (2.9) may be related to the second order unsymmetrical (with respect to $\mathbf{x}$ and $\mathbf{x}^\circ$) tensor $\mathbb{A}^2_{vv}(\mathbf{x}|\mathbf{x}^\circ)$, conditional upon the presence at $\mathbf{x}^\circ$ of an inclusion:

$$(2.10)\qquad E[\varphi_1X^c_1\mathbf{v}^{c'}\mathbf{v}^{c'}] = E[\varphi_1X^c_1(\overline{\mathbf{v}^c}^2 - \overline{\mathbf{v}^c}^1)(\overline{\mathbf{v}^c}^2 - \overline{\mathbf{v}^c}^1)] + E[\varphi_1X^c_1\mathbf{v}^{c''}\mathbf{v}^{c''}] = \frac{1}{N}[\phi^{(1)}(\mathbf{x}^\circ)\alpha^{c2}(\mathbf{x}|\mathbf{x}^\circ)(\overline{\mathbf{v}^c}^2 - \overline{\mathbf{v}^c}^1)(\overline{\mathbf{v}^c}^2 - \overline{\mathbf{v}^c}^1)(\mathbf{x}|\mathbf{x}^\circ) + \mathbb{A}^2_{vv}(\mathbf{x}|\mathbf{x}^\circ),$$

where the fluctuation field $\mathbf{v}^{c''}$ defined in Sec. 5.4. – Part I fulfils $\varphi_1X^c_1\mathbf{v}^c(\mathbf{x}^\circ) =$

$\varphi_1 X_1^c \overline{\mathbf{v}^c}^2(\mathbf{x}^\circ|\mathbf{x}) + \varphi_1 X_1^c \mathbf{v}^{c''}(\mathbf{x}^\circ|\mathbf{x})$. The second order tensor $\mathbb{A}^2_{\mathrm{vv}}(\mathbf{x}|\mathbf{x}^\circ)$ in turn becomes:

$$\mathbb{A}^2_{\mathrm{vv}} = NE[\varphi_1 X^c \mathbf{v}^{c''}\mathbf{v}^{c''}] = N\int E[\varphi_1(\mathbf{x}^\circ)\varphi_2(\mathbf{x}^{\circ\circ})X^c_{1,2}\mathbf{v}^{c''}\mathbf{v}^{c''}(\mathbf{x})]d\mathbf{x}^{\circ\circ}, \tag{2.11}$$

where the integration domain must satisfy two conditions at $\mathbf{x}$ and at $\mathbf{x}^\circ$. The integrand in the r.h.s. of (2.11) can be in turn broken down into two parts as in (2.10) and so on... We realise that a finite (up to N) recurrent relation can be derived. The first terms of this relation are:

$$\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \frac{1}{N}\int d\mathbf{x}^\circ\{\phi^{(1)}\alpha^{c2}[\overline{\mathbf{v}^c}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{v}^c}^1(\mathbf{x})]$$

$$\cdot\,[\overline{\mathbf{v}^c}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{v}^c}^1(\mathbf{x})] + \mathbb{A}^2_{\mathrm{vv}}(\mathbf{x}|\mathbf{x}^\circ)\},$$

$$\mathbb{A}^2_{\mathrm{vv}}(\mathbf{x}|\mathbf{x}^\circ) = \frac{1}{(N-1)}\int d\mathbf{x}^{\circ\circ}\{\phi^{(2)}\alpha^{c3}[\overline{\mathbf{v}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\mathbf{v}^c}^2(\mathbf{x}|\mathbf{x}^\circ)]$$

$$\cdot\,[\overline{\mathbf{v}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\mathbf{v}^c}^2(\mathbf{x}|\mathbf{x}^\circ)] + \mathbb{A}^3_{\mathrm{vv}}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\}, \tag{2.12}$$

$$\mathbb{A}^3_{\mathrm{vv}}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) = \frac{1}{(N-2)}\int d\mathbf{x}^{\circ\circ\circ}\{\phi^{(3)}\alpha^{c4}[\overline{\mathbf{v}^c}^4(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ})$$

$$-\overline{\mathbf{v}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})][\overline{\mathbf{v}^c}^4(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{v}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})]$$

$$+\mathbb{A}^4_{\mathrm{vv}}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ})\}.$$

The expressions (3.5) will be used to analyse their order of magnitude as it will be shown in the next section.

The closure problem is not solved yet for any of the above variance tensors. To see this point, consider $\mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ)$ which can be expressed in terms of variables of the whole hierarchy as $\overline{\mathbf{u}}^1(\mathbf{x})$, $\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ)$, $\phi^{(2)}(\mathbf{x},\ \mathbf{x}^\circ)$, $\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})$, $\phi^{(3)}(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})$... or the corresponding disturbance velocities. What would be necessary is to account for a few terms of this sum; more precisely, these terms must be preponderant according to some small parameter and involve variables which appear in equations of order less or equal to two. This aim is reached in the next section where it will be seen that the influence of each group of test inclusions must be broken down in subclasses effects.

3. Approximation of pseudo-turbulent tensors

3.1. Breaking down the contribution of each group

To break down the influence of each group of inclusions into subclasses effects, it suffices to express conditional velocities differences appearing in the recurrence relations, in terms of disturbance fields. For instance, consider the one-position agitation tensors, relative to the dispersed phase. Our treatement is based on the definition of disturbance fields given in Part II which are repeated for convenience: the first disturbance is $\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ) = \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^1(\mathbf{x})$, the second order disturbance is given by $\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) = \overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^{\circ\circ}) + \overline{\mathbf{u}}^1(\mathbf{x})$ and the third order one obeys $\mathbf{u}^{***}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) = \overline{\mathbf{u}}^4(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{u}}^3((\mathbf{x}|\mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) + \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) + \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^{\circ\circ}) + \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{u}}^1(\mathbf{x})$. Combining these definitions leads to the following identities:

$$
\begin{aligned}
\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) - \overline{\mathbf{u}}^1(\mathbf{x}) &= \mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ), \\
\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) &= \mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) + \mathbf{u}^*(\mathbf{x}|\mathbf{x}^{\circ\circ}), \\
\overline{\mathbf{u}}^4(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) &= \mathbf{u}^{***}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) \\
&\quad + \mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ\circ}) + \mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) + \mathbf{u}^*(\mathbf{x}|\mathbf{x}^{\circ\circ\circ}). \\
&\ldots
\end{aligned}
\tag{3.1}
$$

Inserting the above first relation into (2.4) leads to:

$$
\mathbb{A}^1_{\mathrm{uu}}(\mathbf{x}) = \frac{1}{(N-1)} \int d\mathbf{x}^\circ [\phi^{(2)} \mathbf{u}^* \mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ) + \mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ)]. \tag{3.2}
$$

The second order tensor in the r.h.s. becomes, using (2.5) and the second relation (3.1):

$$
\begin{aligned}
\frac{1}{(N-1)} \int d\mathbf{x}^\circ \mathbb{A}^2_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ) &= \frac{1}{(N-1)} \int d\mathbf{x}^\circ [\phi^{(2)} \mathbf{u}^* \mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)] \\
&+ \frac{1}{(N-1)(N-2)} \int d\mathbf{x}^\circ \int d\mathbf{x}^{\circ\circ} \{\phi^{(3)} [\mathbf{u}^{**} \mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) \\
&+ 2\mathbf{u}^*(\mathbf{x}|\mathbf{x}^{\circ\circ}) \mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})] + \mathbb{A}^3_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})\}.
\end{aligned}
\tag{3.3}
$$

Using (2.6) and the third relation (3.1), the third order tensor in the r.h.s. of (3.3) can be expressed, in its turn, as:

$$
\frac{1}{(N-1)(N-2)} \int d\mathbf{x}^\circ \int d\mathbf{x}^{\circ\circ} \mathbb{A}^3_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}) \tag{3.4}
$$

(3.4) [cont.]
$$= \frac{1}{(N-1)} \int d\mathbf{x}^\circ [\phi^{(2)} \mathbf{u}^* \mathbf{u}^* (\mathbf{x}|\mathbf{x}^\circ)]$$

$$+ \frac{1}{(N-1)(N-2)} \int d\mathbf{x}^\circ \int d\mathbf{x}^{\circ\circ} \{\phi^{(3)} [2\mathbf{u}^{**}\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})$$

$$+ 4\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})]\}$$

$$+ \frac{1}{(N-1)(N-2)(N-3)} \int d\mathbf{x}^\circ \int d\mathbf{x}^{\circ\circ} \int d\mathbf{x}^{\circ\circ\circ} \{\phi^{(4)} [\mathbf{u}^{***}\mathbf{u}^{***}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})$$

$$+ 2\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ\circ})\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ}) + 2\mathbf{u}^{***}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})\mathbf{u}^*(\mathbf{x}|\mathbf{x}^{\circ\circ\circ})$$

$$+ 2\mathbf{u}^{***}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})\mathbf{u}^{**}(bfx|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ\circ})$$

$$+ 2\mathbf{u}^{***}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})] + \mathbb{A}^4_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})\}.$$

Such a process can be continued by expanding $\mathbb{A}^4_{\mathrm{uu}}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ}, \mathbf{x}^{\circ\circ\circ})$ and so on, as far as $\mathbb{A}^{N-1}_{\mathrm{uu}}$ is reached. Collecting in each expression identical terms, we obtain:

$$\mathbb{A}^1_{\mathrm{uu}}(\mathbf{x}) = \int d\mathbf{x}^\circ [\phi^{(2)}(\mathbf{x}^\circ, \mathbf{x})\mathbf{u}^*\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)] \tag{3.5}$$

$$+\frac{1}{2} \int d\mathbf{x}^\circ \int d\mathbf{x}^{\circ\circ} \{\phi^{(3)}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})[\mathbf{u}^{**}\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})$$

$$+ 2\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})]\} + \ldots$$

where the $2\sum_{i=1}^{N-2} i = (N-1)(N-2)$ has been used. Thus we have succeded in breaking down the influence of each group of inclusions into one-inclusion, two-inclusions, ... effects. There's nothing surprising about that; for example, contributions of two, three... test inclusions contain cases where a single neighbouring inclusion at $\mathbf{x}^\circ$ is close to the first one at $\mathbf{x}$. These cases feed what will be the leading order of an asymptotic sequence in terms of Θ which will be derived from all these series in the next section.

Note that we can also deduce expressions for $\mathbb{A}^2_{\mathrm{u^\circ u^\circ}}(\mathbf{x}^\circ|\mathbf{x})$ and $\mathbb{A}^2_{\mathrm{u^\circ u}}(\mathbf{x}^\circ, \mathbf{x})$ which are required in (4.16 – Part II):

$$\mathbb{A}^2_{\mathrm{u^\circ u^\circ}}(\mathbf{x}^\circ|\mathbf{x}) = \int d\mathbf{x}^{\circ\circ} \phi^{(3)}(\mathbf{x}, \mathbf{x}^\circ, \mathbf{x}^{\circ\circ})\mathbf{u}^*\mathbf{u}^*(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ}) \tag{3.6}$$

$$+ \int d\mathbf{x}^{\circ\circ} \{\phi^{(3)}(\mathbf{x}, \mathbf{x}^\circ, \mathbf{x}^{\circ\circ})[\mathbf{u}^{**}\mathbf{u}^{**}(\mathbf{x}^\circ|\mathbf{x}, \mathbf{x}^{\circ\circ})$$

$$+ 2\mathbf{u}^*(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ})\mathbf{u}^{**}(\mathbf{x}^\circ|\mathbf{x}, \mathbf{x}^{\circ\circ})]\} + \ldots$$

$$(3.7)\qquad \mathbb{A}^2_{\mathrm{u^\circ u}}(\mathbf{x}^\circ,\mathbf{x}) = \int d\mathbf{x}^{\circ\circ}\phi^{(3)}(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\mathbf{u}^*(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ})\mathbf{u}^*(\mathbf{x}|\mathbf{x}^{\circ\circ})$$
$$+\int d\mathbf{x}^{\circ\circ}\{\phi^{(3)}(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})[\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\mathbf{u}^{**}(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})$$
$$+\int d\mathbf{x}^{\circ\circ}\{\phi^{(3)}(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})[\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\mathbf{u}^{**}(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})$$
$$+\mathbf{u}^*(\mathbf{x}|\mathbf{x}^{\circ\circ})\mathbf{u}^{**}(\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})+\mathbf{u}^*(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ})\mathbf{u}^{**}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})]\}+\ldots$$

An expression for $\mathbb{A}^1_{\omega\mathrm{u}}(\mathbf{x})$ appearing in ((4.11) – Part II) as well as expressions for $\mathbb{A}^2_{\omega^\circ\mathrm{u}^\circ}(\mathbf{x}^\circ|\mathbf{x})$ and $\mathbb{A}^2_{\omega^\circ\mathrm{u}}(\mathbf{x}^\circ,\mathbf{x})$ in ((4.17) – Part II) are required. It is straightforward to derive them from (3.5), (3.6) and (3.7), respectively.

A treatment of any kind of tensor seen in Sec. 2.1 as well as agitation tensors relative to the continuous phase parallels exactly the above analysis. We find for example:

$$(3.8)\qquad \mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \int d\mathbf{x}^\circ[\phi^{(1)}(\mathbf{x}^\circ)\alpha^{c2}\mathbf{v}^*\mathbf{v}^*(\mathbf{x}|\mathbf{x}^\circ)]$$
$$+\frac{1}{2}\int d\mathbf{x}^\circ\int d\mathbf{x}^{\circ\circ}\{\phi^{(2)}(\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\alpha^{c3}[\mathbf{v}^{**}\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})$$
$$+2\mathbf{v}^*(\mathbf{x}|\mathbf{x}^\circ)\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})]\},$$

$$\ldots$$

$$(3.9)\qquad \mathbb{A}^2_{\mathrm{v^\circ v^\circ}}(\mathbf{x}^\circ|\mathbf{x}) = \int d\mathbf{x}^{\circ\circ}\phi^{(2)}(\mathbf{x},\mathbf{x}^{\circ\circ})\alpha^{c3}\mathbf{v}^*\mathbf{v}^*(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ})$$
$$+\int d\mathbf{x}^{\circ\circ}\{\phi^{(2)}(\mathbf{x},\mathbf{x}^{\circ\circ})\alpha^{c3}[\mathbf{v}^{**}\mathbf{v}^{**}(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})$$
$$+2\mathbf{v}^*(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ})\mathbf{v}^{**}(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})]\}.$$

$$\ldots$$

We first observe that $\mathbb{A}^1_{\mathrm{vv}}$, $\mathbb{A}^2_{\mathrm{vv}}\ldots$ have not the same dimension as the corresponding order dispersed phase agitation tensors $\mathbb{A}^1_{\mathrm{uu}}$, $\mathbb{A}^2_{\mathrm{uu}}$ ... since, at a given order, they do not involve the same p.d.f. in their integrand. On the other hand, contrary to the disturbance interfacial force densities which only imply the next order variables, we also observe that all higher order disturbance velocities are requested. Now, it remains to show that all the expansions we have devised can be considered as expansions in terms of the averaged concentration Θ. Only after that, the closure problem will be claimed to be solved.

3.2. Expansion in terms of Θ

To set an example of our procedure, we will consider the above one-point agitation tensor (3.8), relative to the continuous phase. To put this tensor in a dimensionless form, some scales will be specified in a very general way:

(i) The length scale of the first order averaged fields is L as it has been pointed out in Sec. 2.1. – Part II; it is given by the specification of some boundary or initial conditions. The length scale of all the disturbance averaged fields is expected to be a function of a single inclusion dynamics. In the first approximation, it will be set equal to the inclusion radius a. These two scales hold for both phases.

(ii) There are representative scales for the various densities in physical space, ϕ_1, ϕ_2, ...; they are simply $1/\mathrm{L}^3$, $1/\mathrm{L}^6$... The reason of this normalisation is that $\phi_1 \mathrm{L}^3$, $\phi_2 \mathrm{L}^6$,... have an order of magnitude of 1 since they are probabilities.

(iii) The first order averaged disturbance field, $\mathbf{v}^*$, has a scale denoted V_2, the possible values of which will be given in a future paper. The same scale can be maintained for higher-order disturbance fields. In relation to our present concern, theses values are irrelevant. Furthermore, it will be shown that disturbance fields of higher order have at most the same scale V_2.

Takin into account all these assumptions which are not restrictive, we obtain:

$$\begin{aligned}(3.10)\qquad \mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = {} & \Theta V_2^2 \int d\mathbf{x}^\circ [\phi_1 \alpha^{c2} \mathbf{v}^* \mathbf{v}^* (\mathbf{x}|\mathbf{x}^\circ)] \\ & + \Theta^2 V_2^2 \int d\mathbf{x}^\circ \int d\mathbf{x}^{\circ\circ} \{\phi_2 \alpha^{c3} [\mathbf{v}^{**}\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})/2 \\ & + \mathbf{v}^*(\mathbf{x}|\mathbf{x}^\circ)\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}^\circ, \mathbf{x}^{\circ\circ})]\} + O[\Theta^3] + O[1/N],\end{aligned}$$

where all the variables in the r.h.s. are dimensionless. Asymptotic sequences can also be derived for any agitation tensor by using the same (or very close) nondimensionalisation process.

However, it is expected that integrals at the r.h.s. may be not convergent in many cases, especially when disturbance flows are creeping and have long-range hydrodynamic interactions (CAFLISCH and LUKE [30]). This means that the above scaling and especially, the length scale of the disturbance averaged fields, is by far too naive and ought to be reconsidered. The role of the above expansion which appears, to the order $O(1/N)$, as an asymptotic sequence in terms of Θ, is to start a necessary truncation procedure based on dilutness. In this way asymptotic expansions of any unknowns in terms of Θ are sought; investigating the source of nonconverging integrals will amount to determining the regions of nonuniformities for the above expansions. These critical problems will be adressed in a future paper.

4. Expansions in multipoles

Pseudo-turbulent (or agitation) tensors for both phases and interfacial force densities for the continuous phase share common features: they involve some hydrodynamic fields, velocity fluctuations for the former, interfacial stresses for the latter, which result from multipole contributions at a given observation point $\mathbf{x}$, brought by inclusions placed at various positions in the neighborhood of $\mathbf{x}$. To be calculated more easily, all these quantities need to be treated by a multipole expansion method. As explained in Sec. 5.2. – Part II, this amounts to muster the causes of multipole contributions at one location. However, there is a difference between force densities and agitation tensors: hydrodynamic fields are picked off at the interface for the former while they are selected in the bulk for the latter. In the following, we will restrict to expand terms in tensors which involve two-inclusions interactions.

4.1. Slow and fast independant spatial variables

Pseudo-turbulent tensors for both phases have been expressed in Sec. 3.1. in terms of velocity disturbance flows. These fields are solutions of problems which explicitly depend on both fast-varying and slow-varying independent spatial variables. All consequences of this property will be envisaged in a future paper. Here, only some elementary considerations necessary to estimate the order of magnitude of various terms in the obtained multipoles expansions are introduced. They are based on the change of variable:

$$\mathbf{x} = \mathbf{x} \quad \text{and} \quad \mathbf{r} = \mathbf{x}^\circ - \mathbf{x}. \tag{4.1}$$

consider a typical first order disturbance, say $\mathbf{u}^*$; we will be led in the future to let

$$\hat{\mathbf{u}}(\mathbf{r}, \mathbf{x}) = \mathbf{u}^*(\mathbf{x} + \mathbf{r}|\mathbf{x}) = \mathbf{u}^*(\mathbf{x}^\circ|\mathbf{x})$$

be the new unknown. The entire two-inclusions problem can be expressed in scaling the first coordinate $\mathbf{r}$ with a and the second coordinate $\mathbf{x}$ with L. All these new small-scale functions are defined throughout an (usually) unbounded region denoted $\mathcal{V}^d_{\mathbf{x},\mathbf{r}}$ which replace $\mathcal{V}^d_{\mathbf{x},\mathbf{x}^\circ}$. Likewise, a similar change of variables for the second-order disturbance fields:

$$\mathbf{x} = \mathbf{x}, \quad \mathbf{r} = \mathbf{x}^\circ - \mathbf{x} \quad \text{and} \quad \mathbf{r}^\circ = \mathbf{x}^{\circ\circ} - \mathbf{x}^\circ \tag{4.2}$$

gives rise to $\mathbf{u}^{\hat{}\hat{}}(\mathbf{r}, \mathbf{r}^\circ, \mathbf{x})$.

All the concerned velocity disturbance field equations of the continuous phase can be made dimensionless in the same way as above as $\mathbf{v}^{\circ *}$ which becomes:

$$\mathbf{v}^{\hat{}}(\mathbf{r}, \mathbf{x}) = \mathbf{v}^{\circ *}(\mathbf{x} + \mathbf{r}|\mathbf{x}) = \mathbf{v}^{\circ *}(\mathbf{x}^\circ|\mathbf{x}).$$

4.2. The pseudo-turbulent tensors in the dispersed phase

The leading term of the agitation tensor $\mathbb{A}^1_{\rm uu}(\mathbf{x})$ which is defined in (3.5) can be transformed by (4.1):

$$\text{(4.3)} \quad \mathbb{A}^1_{\rm uu}(\mathbf{x}) = \int d\mathbf{x}^\circ[\phi^{(2)}\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)] = \int d\mathbf{r}[\phi^{(2)}(\mathbf{x},\mathbf{x}+\mathbf{r})\mathbf{u}^*\mathbf{u}^*(\mathbf{x}|\mathbf{x}+\mathbf{r}].$$

The integrand is denoted by:

$$\text{(4.4)} \qquad f(\mathbf{r},\mathbf{x}) = \phi^{(2)}(\mathbf{x},\mathbf{x}+\mathbf{r})\mathbf{u}^*\mathbf{u}^*(\mathbf{x}|\mathbf{x}+\mathbf{r}).$$

Expanding **f** with the respect to the second argument around $\mathbf{x}-\mathbf{r}$ according to the formula ((2.5) – Part II) gives:

$$\text{(4.5)} \qquad f(\mathbf{r},\mathbf{x}) = \sum_{\rm m=0}^{\infty} \frac{(-1)^{\rm m}}{\rm m!}\mathbf{r}^{\rm m} \boxed{\rm m} \frac{\partial^{\rm m}}{\partial \mathbf{x}^{\rm m}} f(\mathbf{r},\mathbf{x}-\mathbf{r}).$$

The resulting expansion for the leading term of the agitation tensor can be written:

$$\text{(4.6)} \qquad \int d\mathbf{x}^\circ[\phi^{(2)}\mathbf{u}^*\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)] = (N-1)\sum_{\rm m=0}^{\infty} \frac{\partial^{\rm m}}{\partial \mathbf{x}^{\rm m}} \boxed{\rm m}\, \phi^{(1)}(\mathbf{x})\mathbb{N}^1_{\rm uu,m}(\mathbf{x}),$$

where the $\rm m^{th}$ agitation multipole of the first-order for the dispersed phase is:

$$\text{(4.7)} \qquad \mathbb{N}^1_{\rm uu,m}(\mathbf{x}) = \frac{1}{\rm m!}\int d\mathbf{r}\mathbf{r}^{\rm m}\chi_2\mathbf{u}^*\mathbf{u}^*(\mathbf{x}+\mathbf{r}|\mathbf{x}).$$

In this multipole $\mathbf{r}^{\rm m}$ denotes an m-fold tensor product of $\mathbf{r}$. It describes the agitation due to the first-order disturbance flow of the dispersed phase over a test inclusion centred at $\mathbf{x}$. It is a tensor of rank $\rm m+2$, symmetric in its first m indices.

In the first-order dispersed-phase linear momentum equation ((4.10) – Part II), the above agitation tensor appears under a divergence term which becomes, using (4.6):

$$\text{(4.8)} \qquad \rho^{\rm d}(\phi^{(1)})^{-1}\frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^1_{\rm uu} = \Theta[\rho^{\rm d}U_2^2/L]\left\{\phi_1^{-1}\frac{\partial}{\partial \mathbf{x}} \cdot \phi_1\mathbb{N}^1_{\rm uu,0} + \beta\phi_1^{-1}\frac{\partial}{\partial \mathbf{x}} \cdot \left[\frac{\partial}{\partial \mathbf{x}} \cdot \phi_1\mathbb{N}^1_{\rm uu,1}\right] + O(\beta^2)\right\} + O(1/N),$$

where all the variables of the r.h.s. have been made dimensionless; note that U_2 is the uknown representative linear velocity scale for $\mathbf{u}^*$. It has been shown that the agitation tensor contribution is of $O(\Theta)$ compared to the classical bulk flow convective term (see Eq. (4.10) – Part II).

The agitation tensor $\mathbb{A}^1_{\omega\mathrm{u}}(\mathbf{x})$ can obtain an expression similar to (4.6) in which U_2/a is chosen as the unknown representative angular velocity scale for ω^*. This tensor appears in ((4.11) – Part II) under a divergence term and a formula similar to (4.8) can be derived.

4.3. The pseudo-turbulent tensors in the continuous phase

The two leading terms of the agitation tensor $\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x})$ which is defined in (3.8) can be transformed by the change of variables (4.1):

$$\begin{aligned}\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \int d\mathbf{r}\left[\phi^{(1)}(\mathbf{x}+\mathbf{r})\alpha^{c2}\mathbf{v}^*\mathbf{v}^*(\mathbf{x}|\mathbf{x}+\mathbf{r}\right] \\ + \frac{1}{2}\int d\mathbf{x}^{\circ\circ}\int d\mathbf{r}\left\{\phi^{(2)}(\mathbf{x}+\mathbf{r},\mathbf{x}^{\circ\circ})\alpha^{c3}[\mathbf{v}^{**}\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}+\mathbf{r},\mathbf{x}^{\circ\circ})\right. \\ \left. + \ 2\mathbf{v}^*(\mathbf{x}|\mathbf{x}+\mathbf{r})\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}+\mathbf{r},\mathbf{x}^{\circ\circ})]\right\} + \ldots\end{aligned} \tag{4.9}$$

Expanding the two integrands with respect to the variable $\mathbf{x}$ around $\mathbf{x}-\mathbf{r}$ accot-ding to the formula ((2.5) – Part II) provides:

$$\begin{aligned}\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \sum_{\mathrm{m}=0}^{\infty}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}^{\mathrm{m}}}\boxed{\mathrm{m}}\,\phi^{(1)}(\mathbf{x})\mathbb{N}^1_{\mathrm{vv,m}}(\mathbf{x}) \\ + 1/2\sum_{\mathrm{m}=0}^{\infty}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}^{\mathrm{m}}}\int d\mathbf{x}^{\circ\circ}\phi^{(2)}(\mathbf{x},\mathbf{x}^{\circ\circ})\mathbb{N}^2_{\mathrm{vv,m}}(\mathbf{x},\mathbf{x}^{\circ\circ}) + \ldots\end{aligned} \tag{4.10}$$

where the m^{th} agitation multipole of the first- and second-order for the continuous phase have been defined by:

$$\mathbb{N}^1_{\mathrm{vv,m}}(\mathbf{x}) = \frac{1}{\mathrm{m}!}\int d\mathbf{r}\mathbf{r}^{\mathrm{m}}\alpha^{c2}\mathbf{v}^*\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x}) \tag{4.11}$$

$$\begin{aligned}\mathbb{N}^2_{\mathrm{vv,m}}(\mathbf{x},\mathbf{x}^{\circ\circ}) = \frac{1}{\mathrm{m}!}\int d\mathbf{r}\{\mathbf{r}^{\mathrm{m}}\alpha^{c3}[\mathbf{v}^{**}\mathbf{v}^{**}(\mathbf{x}+\mathbf{r}|\mathbf{x},\mathbf{x}^{\circ\circ}) \\ + 2\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})\mathbf{v}^{**}(\mathbf{x}+\mathbf{r}|\mathbf{x},\mathbf{x}^{\circ\circ})]\}.\end{aligned} \tag{4.12}$$

The m^{th} agitation multipole of order 1 and 2 describe the agitation due to the first and the second disturbance flows of the continuous phase over a test inclusion centred at $\mathbf{x}$. It is a tensor of rank $\mathrm{m}+2$, symmetric in its first m indices. Observe that $\mathbb{N}^2_{\mathrm{vv,m}}(\mathbf{x},\mathbf{x}^{\circ\circ})$ is not symmetrical.

In the first-order continuous-phase momentum Eq. (4.1) – Part II, the above agitation tensor appears under a divergence term which becomes, using (4.10):

$$(4.13)\qquad (\alpha^{c1})^{-1}\frac{\partial}{\partial\mathbf{x}}\cdot\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \frac{\Theta V_2^2}{L\alpha^{c1}}\left\{\frac{\partial}{\partial\mathbf{x}}\cdot\phi_1\mathbb{N}^1_{\mathrm{vv},0}(\mathbf{x}) + \beta\frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{x}}\phi_1\mathbb{N}^1_{\mathrm{vv},1}(\mathbf{x}) + O(\beta^2)\right\} + \frac{\Theta^2V_2^2}{2L\alpha^{c1}}\left\{\frac{\partial}{\partial\mathbf{x}}\cdot\left[\int d\mathbf{x}^{\circ\circ}\phi_2\mathbb{N}^2_{\mathrm{vv},0}(\mathbf{x},\mathbf{x}^{\circ\circ})\right] + \beta\frac{\partial}{\partial\mathbf{x}}\cdot\left[\frac{\partial}{\partial\mathbf{x}}\int d\mathbf{x}^{\circ\circ}\phi_2\mathbb{N}^2_{\mathrm{vv},1}(\mathbf{x},\mathbf{x}^{\circ\circ})\right] + O(\beta^2)\right\} + O(1/N),$$

where all the variables in the r.h.s. are dimensionless and where (4.2) have been used. Note that V_2 is the uknown representative linear velocity scale for $\mathbf{v}^*$ and $\mathbf{v}^{**}$.

The same arguments as those leading to (4.10) show that the agitation tensor for the field with one fixed inclusion which is defined in (3.9), can be expanded according to:

$$(4.14)\qquad \mathbb{A}^2_{\mathrm{v}^\circ\mathrm{v}^\circ}(\mathbf{x}^\circ|\mathbf{x}) = \sum_{\mathrm{m}=0}^{\infty}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}^{\circ\mathrm{m}}}\boxed{\mathrm{m}}\,\phi^{(2)}(\mathbf{x},\mathbf{x}^\circ)\mathbb{N}^1_{\mathrm{vv,m}}(\mathbf{x}^\circ) + \sum_{\mathrm{m}=0}^{\infty}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}^{\circ\mathrm{m}}}\boxed{\mathrm{m}}\,\phi^{(2)}(\mathbf{x},\mathbf{x}^\circ)\mathbb{N}^2_{\mathrm{vv,m}}(\mathbf{x}^\circ,\mathbf{x}) + \ldots$$

In the second-order continuous-phase momentum equation ((4.3) – Part II), the above agitation tensor appears under a divergence term combined with $(\alpha^{c1})^{-1}\dfrac{\partial}{\partial\mathbf{x}^\circ}\cdot\mathbb{A}^1_{\mathrm{v}^\circ\mathrm{v}^\circ}$. The leading order of this combination is:

$$(4.15)\qquad \frac{\partial}{\partial\mathbf{x}^\circ}\cdot\mathbb{A}^1_{\mathrm{v}^\circ\mathrm{v}^\circ}/\alpha^{c1} - \frac{\partial}{\partial\mathbf{x}^\circ}\cdot\mathbb{A}^2_{\mathrm{v}^\circ\mathrm{v}^\circ}/\alpha^{c2}\phi^{(1)} = -N\frac{\partial}{\partial\mathbf{x}^\circ}\cdot\chi^*(\mathbf{x}^\circ|\mathbf{x})\int d\mathbf{r}^\circ[\mathbf{v}^*\mathbf{v}^*(\mathbf{x}^\circ+\mathbf{r}^\circ|\mathbf{x}^\circ)].$$

4.4. Comparison with previous work

The first-order agitation tensors $\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x})$ and $\mathbb{A}^1_{\mathrm{uu}}(\mathbf{x})$ are the only quantities for which comparison with previous work can be envisaged. Other pseudo-turbulent quantities appearing in this paper are specific to our approach. Even so restricted, a comparison is not an easy task for several reasons. Various authors apply different averaging techniques, e.g. they take volume average of the two phases separately; they take volume or ensemble average of the whole mixture as well;

some of them even mix two types of averaging. Concerning our method, only phasic ensemble averaging is used. Secondly, they consider at the outset very specific types of carrier flows: mostly incompressible potential or creeping flows; their way to process the equations takes advantage of the specific properties they have selected. Moreover, in the limit of creeping flows, many studies which use kinetic theory concepts, focus on the first-order averaged field and do not even consider the agitation tensors [31, 32]. Our approach is marked off by being based on general Navier-Stokes equations. Finally they provide explicit closure relations, generally based on specified local distribution of inclusions (periodic arrays or uniform distributions); although the above expressions of the agitation tensors $\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x})$ and $\mathbb{A}^1_{\mathrm{uu}}(\mathbf{x})$ are presented in the form of computable quantities, we have up to now not furnished such explicit results. To get them, higher order disturbance equations (e.g. $\mathbf{v}^*$ and $\mathbf{u}^*$ equations) have to be simplified according to a given particular approximation of disturbance flows and then truncated due to dilution; this will only be achieved in the next paper (Part IV). However, we can anticipate somehow and obtain some straightforward results concerning $\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x})$ which can precisely be found in the literature.

Concerning $\mathbb{A}^1_{\mathrm{uu}}(\mathbf{x})$, the available studies mentioning this quantity are fairly recent [33, 34, 35, 36]. All these authors derived average equations for a suspension of spherical inclusions (possibly massless bubbles) carried by an incompressible irrotational flow. They propose expressions of a quantity they term kinetic (or translational) part of the dispersed-phase stress, which are mainly formal while they compute explicitly the interaction terms (they call it "potential part" of the stress) in both phases momentum equations. Moreover, they introduce a new dispersed phase momentum equation controlling the apparent momentum that can be attributed to an inclusion, i.e. including the impulse of the fluid surrounding each inclusion. In conclusion, we were unable to compare (4.8) with any analytical result.

Let us return to $\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x})$ for which there are plenty of studies. First, we will restrict ourselves to a quasi-homogeneous ($\beta \ll 1$) and dilute ($\Theta \ll 1$) mixture of spherical inclusions non-rotating and non-pulsating. In this case, the leading order of the agitation tensor given in (3.10) is, rewritten in a dimensional form:

$$\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \phi^{(1)}(\mathbf{x}) \int d[\mathbf{v}^*\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})] + ... \tag{4.16}$$

The agitation results from the superimposition of fluctuations due to each inclusion. The above formula has been used by KOCH [37] for moderate inclusion Reynolds numbers, $\mathrm{Re}^{\mathrm{d}} \approx O(1-10)$. For very low particle Reynolds number, within the point-particle approximation, and in the dilute limit, KOCH and SHAQFEH [38] based their analysis upon a similar formula; they also included an extra contribution which formally corresponds to the second term at the r.h.s.

of (3.8). Albeit they did not define averaged disturbance flows as we did, the likeness of our results is even closer if we refer to the expression they derived in the Appendix (see their Eq. (A3)). Here, the result (3.8) has been obtained under very general conditions granted that the scale analysis made in Sec. 3.2. is relevant.

Consider now specific carrier phase flows. A simple way to evaluate the integral in the r.h.s. of (4.16) is to consider the disturbance velocity $\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})$ as that of a pure fluid set in motion by the presence of a single test inclusion moving with the relative velocity $\overline{\mathbf{u}}^1$ (cf. (4.12) – Part II), and to integrate $\mathbf{v}^*\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})$ over the fluid volume outside the test inclusion. This corresponds to the simplest approximation of the one-inclusion problem ((4.3) and (4.7) – Part II). Two models of carrier phase flows can be considered: irrotational flows and Stokes flows.

When $\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})$ is assumed as irrotational, it decays as $1/r^3$, r being the distance from the centre of the test inclusion. Computation of the integral in the r.h.s. of (4.16) is straigthforward:

$$\mathbb{A}^1_{\mathrm{vv}}(\mathbf{x}) = \alpha^{\mathrm{d1}}(\mathbf{x})[k_1\overline{\mathbf{u}}^1\cdot\overline{\mathbf{u}}^1(\mathbf{x})\mathbb{I} + k_2\overline{\mathbf{u}}^1\overline{\mathbf{u}}^1(\mathbf{x})] \tag{4.17}$$

where $k_1 = 3/20$ and $k_2 = 1/20$. BIESHEUVEL and VAN WIJNGAARDEN [39] considered the more general case of compressible bubbles; employing ensemble averaging and introducing at certain stages volume averaging which holds for a statistically homogeneous medium, they found the same coefficients. For ellipsoidal bubbles, LANCE [6] obtain an expression similar to (4.17) with a multiplicative coefficient which, for linear trajectories, increases almost linearly with the eccentricity. Besides, he has experimentally confirmed the validity of (4.17), at least over a limited range of void fraction. The cell model which is used as an ad-hoc approximation of the conditionaly-averaged micro-problem around a given test inclusion has allowed many investigators to produce results having the same structure. GARIPOV [40] even found the same coefficients. NIGMATULIN [28] obtained the above formula with $k_1 = 1/6$ and $k_2 = -1/2$. ARNOLD, DREW and LAHEY [41] extended the concept of cell averaging technique to acommodate gradients in the phase distributions and in the discrete phase velocity. In this limiting case, their results are the same as ours.

The Stokes limit, although extensively treated, is still much debated. Early, CAFLISCH and LUKE [30] have evaluated the integral in (4.16) for a random structure of monodispersed particles in an infinite medium, and pointed out the aforementioned divergent behaviour since $\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})$ decays as the velocity induced by a Stokeslet, i.e. as $1/r$. These foundings have been corroborated by direct simulations [42] but not by experiments (at least not in an unambiguous way). Various arguments have been put forward to solve this issue, and it is worth to briefly recall them in order to illustrate the difficulties encountered when using

(4.16). The first type of arguments is associated with the existence of screening effects which damp the $1/r$ behaviour at infinity. Various screening mechanisms have been proposed. One would be due to a departure from equilibrium of the microstructure of the suspension. For example, considering the interaction of a pair of particles with a third one, KOCH and SHAQFEH [38] predicted a deficit of the pair probability which decreases as $1/r$ at long range, leading to a screening distance of order a/Θ and a variance $v'^2 U_\infty^2 \approx O(1)$, where U_∞ denotes the terminal velocity. So far, the existence of such a microstructure has not been confirmed. On the contrary, the pair density near contact has been found to be significantly higher than the equilibrium one, both in experiments [43, 44], and in simulations (LADD [42]). A second screening effect which was first propounded by KOCH [37] and recently discussed by BRENNER [45], occurs if the inclusions diffusivity becomes strong enough to hamper the momentum transfer in the carrier phase. This "inertial" screening leads to a scaling v'^2/U_∞^2 as $(\Theta/\mathrm{Re}^d)^{2/3}$ whose validity remains to be checked. The third mechanism would result from long-range interactions, either with walls [45] or with a large ensemble of particles (SEGRÉ, HERBOLZHEIMER and CHAIKIN [46]). The presence of walls not only induces a decay of the perturbation velocity faster than $1/r$ beyond some distance, but modifies the concentration distribution along a direction transverse to the sedimentation. The axial agitation v'^2/U_∞^2 may then evolve as Θ for the weak interaction regime, or as $\Theta^{2/3}$ for the strong interaction regime [45]. The latter scaling seems also to apply when correlated regions of large extent (10 to 20 times the mean interparticle distance) exist in the flow [46]. However, the conditions under which such "blobs" occur are still not understood.

At intermediate Re^d, similar questions arise. Accounting for the inertia brings additional complexity, and the only model we are aware of is due to KOCH [37]. For a slightly polydispersed suspension, a buoyancy screening is proposed which is controlled by pair interactions. The resulting microstructure exibits a pair density deficit in the wake of the test bubble up to a distance of the order $a/(\Theta \mathrm{Re}^d)$, and the velocity variance scales $v'^2/U_\infty^2 \approx \Theta \ln(1/\Theta)/\mathrm{Re}^d$ for Re^d close to unity. Although the existence of the pair probability deficit in the near wake has been confirmed experimentally [47], the validity of the above scaling has not been clearly established.

This brief overview shows that at low Reynolds numbers, the sole equations controlling the one-inclusion problem ((4.3) and (4.7) – Part II) may be insufficient to devise consistent approximations and to provide a correct estimate for the carrier phase agitation tensor; if some of the above damping mechanisms of $\mathbf{v}^*(\mathbf{x}+\mathbf{r}|\mathbf{x})$ as r tends to infinity are correct, the one-inclusion problem must be connected somehow to the next multi-inclusion (two or even three) problems. This is probably so at intermediate particle Reynolds numbers (20 – 40) for which recent experiments have shown that the deviation from the pure fluid problem can be surprisingly strong and the pair density structure is highly anisotropic (CARTELLIER and RIVIÈRE, [48]).

5. Cross-correlation terms of the first type

5.1. New averaged dispersed phase velocities

New terms C^*_α and C^{**}_α which correlate the continuous phase presence at one location with the inclusion(s) velocity(ies) at a neighbouring point(s) appear in the conditioned continuity equations of the continuous phase ((4.7) and (4.8) – Part II). They consist in divergence-type sources which involve specific conditionally-averaged velocities which result from $E[X_1^c\varphi_1\mathbf{u}_1]$, $E[X_2^c\varphi_2\mathbf{u}_2]$, $E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_1]$ and $E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_2]$.

$$
\begin{aligned}
E[X_1^c\varphi_1\mathbf{u}_1] &= \alpha^{c2}(\mathbf{x}^\circ|\mathbf{x})\phi_1\overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ),\\
E[X_2^c\varphi_2\mathbf{u}_2] &= \alpha^{c2}(\mathbf{x}|\mathbf{x}^\circ)\phi_1(\mathbf{x}^\circ)\overline{\mathbf{u}^c}^2(\mathbf{x}^\circ|\mathbf{x}),\\
E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_1] &= \alpha^{c3}(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)\phi_2(\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{u}^c}^3(\mathbf{x}|\mathbf{x}^{\circ\circ},\mathbf{x}^\circ),\\
E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_2] &= \alpha^{c3}(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)\phi_2(\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{u}^c}^3(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ},\mathbf{x}).
\end{aligned}
\tag{5.1}
$$

To interpret these new dispersed phase velocities, some simple transformations based on ((2.6) – Part I) are used, as $NE[X_1^c\varphi_1\mathbf{u}_1] = \phi^{(1)}\overline{\mathbf{u}}^1 - NE[X_1^d\varphi_1\mathbf{u}_1]$ and further:

$$
\alpha^{c2}(\mathbf{x}^\circ|\mathbf{x})\overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ) = \overline{\mathbf{u}}^1(\mathbf{x}) - (N-1)\int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^\circ|\le a}\chi_2(\tilde{\mathbf{x}}|\mathbf{x})\overline{\mathbf{u}}^2(\mathbf{x}|\tilde{\mathbf{x}})d\tilde{\mathbf{x}}.
\tag{5.2}
$$

It can be observed that when $\mathbf{x}^\circ$ increases to infinity, $\mathbf{x}$ being fixed, $\overline{\mathbf{u}}^2(\mathbf{x}|\tilde{\mathbf{x}}) \to \overline{\mathbf{u}}^1(\mathbf{x})$ and using ((2.2) – Part II), we conclude that $\overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ) \to \overline{\mathbf{u}}^1(\mathbf{x})$. By commuting $\mathbf{x}^\circ$ and $\mathbf{x}$, a similar expression can be obtained for $\overline{\mathbf{u}^c}^2(\mathbf{x}^\circ|\mathbf{x})$.

Likewise, the relation $N(N-1)E[X^c\varphi_1\varphi_2\mathbf{u}_1] = \phi^{(2)}\overline{\mathbf{u}}^2 - N(N-1)E[X^d\varphi_1\varphi_2\mathbf{u}_1]$ holds and gives:

$$
\begin{aligned}
\alpha^{c3}(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{u}^c}^3(\mathbf{x}|\mathbf{x}^{\circ\circ},\mathbf{x}^\circ) &= \overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ)\\
&\quad -(N-2)\int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ\circ}|\le a}\chi_3(\tilde{\mathbf{x}}|\mathbf{x}^\circ,\mathbf{x})\overline{\mathbf{u}}^3(\mathbf{x}|\tilde{\mathbf{x}},\mathbf{x}^\circ)d\tilde{\mathbf{x}}.
\end{aligned}
\tag{5.3}
$$

By commuting $\mathbf{x}^\circ$ and $\mathbf{x}$, a similar expression can be obtained for $\overline{\mathbf{u}^c}^3(\mathbf{x}^\circ|\mathbf{x}^{\circ\circ},\mathbf{x})$.

Of course, we also have $\overline{\mathbf{u}^{c}}^{3}(\mathbf{x}^{\circ}|\mathbf{x},\mathbf{x}^{\circ\circ}) \to \overline{\mathbf{u}}^{2}(\mathbf{x}^{\circ}|\mathbf{x})$ and $\overline{\mathbf{u}^{c}}^{3}(\mathbf{x}|\mathbf{x}^{\circ},\mathbf{x}^{\circ\circ}) \to \overline{\mathbf{u}}^{2}(\mathbf{x}|\mathbf{x}^{\circ})$ when $\mathbf{x}^{\circ\circ}$ increases to infinity, $\mathbf{x}$ and $\mathbf{x}^{\circ}$ being fixed.

5.2. Continuity equations for the continuous phase

The above dispersed phase velocities, as given by (5.2) and (5.3), will be expressed in terms of disturbance flows, before calculating C^{*}_{α} and C^{**}_{α} in the conditioned continuity equations of the continuous phase. The equation (5.2) can be written:

$$\overline{\mathbf{u}^{c}}^{2}(\mathbf{x}|\mathbf{x}^{\circ}) = \overline{\mathbf{u}}^{1}(\mathbf{x}) - \frac{N-1}{\alpha^{c2}(\mathbf{x}^{\circ}|\mathbf{x})} \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ}|\le a} \chi_{2}(\tilde{\mathbf{x}}|\mathbf{x})\mathbf{u}^{*}(\mathbf{x}|\tilde{\mathbf{x}})d\tilde{\mathbf{x}} \tag{5.4}$$

leading to the following first-order continuity (Eq. (4.7) – Part II):

$$\begin{aligned}\frac{\partial\alpha^{\circ *}}{\partial t} + \frac{\partial}{\partial\mathbf{x}^{\circ}}\cdot\Big[\alpha^{\circ *}(\overline{\mathbf{v}^{c}}^{1} + \mathbf{v}^{\circ *}) - \alpha^{c1}\mathbf{v}^{\circ *}\Big] \\ = -\overline{\mathbf{u}}^{1}\cdot\frac{\partial}{\partial\mathbf{x}}\alpha^{\circ *} - \frac{N-1}{\phi_{1}}\frac{\partial}{\partial\mathbf{x}}\cdot\phi_{1}\int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ}|\le a}\chi_{2}(\tilde{\mathbf{x}}|\mathbf{x})\mathbf{u}^{*}(\mathbf{x}|\tilde{\mathbf{x}})d\tilde{\mathbf{x}}.\end{aligned} \tag{5.5}$$

The equation (5.3) can also be expressed in terms of disturbance flows:

$$\begin{aligned}\overline{\mathbf{u}^{c}}^{3}(\mathbf{x}|\mathbf{x}^{\circ},\mathbf{x}^{\circ\circ}) = \overline{\mathbf{u}}^{2}(\mathbf{x}|\mathbf{x}^{\circ}) \\ - \frac{N-2}{\alpha^{c3}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ},\mathbf{x})}\int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ\circ}|\le a}\chi_{3}(\tilde{\mathbf{x}}|\mathbf{x}^{\circ},\mathbf{x})[\mathbf{u}^{**}(\mathbf{x}|\tilde{\mathbf{x}},\mathbf{x}^{\circ}) + \mathbf{u}^{*}(\mathbf{x}|\tilde{\mathbf{x}})]d\tilde{\mathbf{x}},\end{aligned} \tag{5.6}$$

and a second one is obtained by commuting $\mathbf{x}^{\circ}$ and $\mathbf{x}$. The second-order continuity (Eq. (4.8) – Part II) becomes then:

$$\begin{aligned}\frac{\partial\alpha^{\circ\circ **}}{\partial t} + \frac{\partial}{\partial\mathbf{x}^{\circ\circ}}\cdot\{\alpha^{\circ\circ **}[\overline{\mathbf{v}^{c}}^{1}(\mathbf{x}^{\circ\circ}) + \mathbf{v}^{*}(\mathbf{x}^{\circ\circ}|\mathbf{x}) + \mathbf{v}^{*}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ}) + \mathbf{v}^{\circ\circ **}] \\ - \alpha^{c1}(\mathbf{x}^{\circ\circ})\mathbf{v}^{\circ\circ **}\} + \frac{\partial}{\partial\mathbf{x}^{\circ\circ}}\cdot\{\alpha^{*}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})[\mathbf{v}^{*}(\mathbf{x}^{\circ\circ}|\mathbf{x}) + \mathbf{v}^{\circ\circ **}]\end{aligned} \tag{5.7}$$

(5.7) [cont.]

$$+ \alpha^*(\mathbf{x}^{\circ\circ}|\mathbf{x})[\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ) + \mathbf{v}^{\circ\circ**}]\} = +\overline{\mathbf{u}}^1 \cdot \frac{\partial}{\partial \mathbf{x}}\alpha^{\circ*} + \overline{\mathbf{u}^\circ}^1 \cdot \frac{\partial}{\partial \mathbf{x}^\circ}\alpha^*$$

$$-\overline{\mathbf{u}}^2 \cdot \frac{\partial}{\partial \mathbf{x}}[\alpha^{\circ\circ**} + \alpha^*(\mathbf{x}^{\circ\circ}|\mathbf{x})] - \overline{\mathbf{u}^\circ}^2 \cdot \frac{\partial}{\partial \mathbf{x}^\circ}[\alpha^{\circ\circ**} + \alpha^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)]$$

$$+ \frac{N-1}{\phi_1}\frac{\partial}{\partial \mathbf{x}} \cdot \phi_1 \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^\circ|\le a} \chi_2(\tilde{\mathbf{x}}|\mathbf{x})\mathbf{u}^*(\mathbf{x}|\tilde{\mathbf{x}})d\tilde{\mathbf{x}}$$

$$+ \frac{N-1}{\phi_1^\circ}\frac{\partial}{\partial \mathbf{x}^\circ} \cdot \phi_1^\circ \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\le a} \chi_2(\tilde{\mathbf{x}}|\mathbf{x}^\circ)\mathbf{u}^*(\mathbf{x}^\circ|\tilde{\mathbf{x}})d\tilde{\mathbf{x}}$$

$$- \frac{N-2}{\phi_2}\frac{\partial}{\partial \mathbf{x}} \cdot \phi_2 \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ\circ}|\le a} \chi_3(\tilde{\mathbf{x}}|\mathbf{x},\mathbf{x}^\circ)[\mathbf{u}^{**}(\mathbf{x}|\tilde{\mathbf{x}},\mathbf{x}^\circ) + \mathbf{u}^*(\mathbf{x}|\tilde{\mathbf{x}})]d\tilde{\mathbf{x}}$$

$$- \frac{N-2}{\phi_2}\frac{\partial}{\partial \mathbf{x}^\circ} \cdot \phi_2 \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ\circ}|\le a} \chi_3(\tilde{\mathbf{x}}|\mathbf{x},\mathbf{x}^\circ)[\mathbf{u}^{**}(\mathbf{x}^\circ|\mathbf{x},\tilde{\mathbf{x}}) + \mathbf{u}^*(\mathbf{x}^\circ|\tilde{\mathbf{x}})]d\tilde{\mathbf{x}}.$$

6. Cross-correlation terms of the second type: composite pseudo-turbulent tensors

6.1. Definitions

The second part of the program concerning cross-correlations between properties relative to each phase, deals with C_v^* and C_v^{**}; they appear in the conditioned momentum equations of the continuous phase ((4.3) and (4.5) – Part II). They collect terms which have already been seen in Sec. 5.1 and two new types of cross-correlations as $E[X_1^c\varphi_1\mathbf{u}_1\mathbf{v}^c]$ and $E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_2\mathbf{v}^c]$; these correlate the continuous phase velocity at one point and the inclusions(s) velocity(ies) at a neighbouring point(s). To break down each cross-correlation into a mean flow convection term and an agitation term, we need to define several new fluctuation fields. To begin with, consider:

$$X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1 = X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ) + X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1^{c''}. \tag{6.1}$$

This first fluctuation field is different from those defined in Sec. 4.4 – Part I. Combining this new field and the standard one $\varphi_1(\mathbf{x})X^c\mathbf{v}^{c''}(\mathbf{x}^\circ)$, defined in Sec. 5.4 –

Part I, yields:

$$(6.2)\qquad \sum_{i=1}^{N} E[X_1^c(\mathbf{x}^\circ)\varphi_i(\mathbf{x})\mathbf{u}_i^{c''}\mathbf{v}^{c''}(\mathbf{x}^\circ)] = NE[X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1^{c''}\mathbf{v}^{c''}(\mathbf{x}^\circ)]$$

$$= NE[X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})(\mathbf{u}_1 - \overline{\mathbf{u}^c}^2)(\mathbf{v}^c - \overline{\mathbf{v}^c}^2)]$$

$$= \phi^{(1)}(\mathbf{x})\alpha^{c2}(\mathbf{x}^\circ|\mathbf{x})\overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ)\overline{\mathbf{v}^c}^2(\mathbf{x}^\circ|\mathbf{x})$$

$$-NE[X_1^c\varphi_1\mathbf{u}_1]\overline{\mathbf{v}^c}^2 - N\overline{\mathbf{u}^c}^2 E[X_1^c\varphi_1\mathbf{v}^c] + NE[X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1\mathbf{v}^c]$$

$$= NE[X_1^c(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1\mathbf{v}^c(\mathbf{x}^\circ)] - \phi^{(1)}(\mathbf{x})\alpha^{c2}(\mathbf{x}^\circ|\mathbf{x})\overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ)\overline{\mathbf{v}^c}^2(\mathbf{x}^\circ|\mathbf{x}).$$

Thus it is shown that the first above cross-correlation function $NE[X_1^c\varphi_1\mathbf{u}_1\mathbf{v}^c]$, which appears in C_{v}^*, is the sum of a mean composite convection term and of a composite pseudo-turbulent tensor of second-order which is defined by:

$$(6.3)\qquad \mathbb{A}^2_{\mathrm{uv}^\circ}(\mathbf{x}^\circ,\mathbf{x}) = \sum_{i=1}^{N} E[X_i^c(\mathbf{x}^\circ)\varphi_i(\mathbf{x})\mathbf{u}_i^{c''}\mathbf{v}^{c''}(\mathbf{x}^\circ)].$$

Similar cross-correlation function appear in C_{v}^{**}. They are broken down in the same way and generate pseudo-turbulent tensors as $\mathbb{A}^2_{\mathrm{uv}^{\circ\circ}}(\mathbf{x}^{\circ\circ},\mathbf{x})$ and $\mathbb{A}^2_{\mathrm{u}^\circ\mathrm{v}^{\circ\circ}}(\mathbf{x}^{\circ\circ},\mathbf{x}^\circ)$ whose definitions are obvious.

The next fluctuation field is defined by:

$$(6.4)\qquad X_{1,2}^c(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}^\circ)\mathbf{u}_1 = X_{1,2}^c(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}^\circ)\overline{\mathbf{u}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) + X_{1,2}^c(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}^\circ)\mathbf{u}_1^{c'''}.$$

It is used to break down the cross-correlation $N(N-1)E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_1\mathbf{v}^c]$ of the second type, via the following equalities:

$$(6.5)\qquad \sum_{i}^{N}\sum_{j\neq i} E[X_{i,j}^c(\mathbf{x}^{\circ\circ})\varphi_i(\mathbf{x})\varphi_j(\mathbf{x}^\circ)\mathbf{u}_i^{c'''}\mathbf{v}^{c'''}(\mathbf{x}^{\circ\circ})]$$

$$= N(N-1)E[X_{1,2}^c(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}^\circ)\mathbf{u}_1^{c'''}\mathbf{v}^{c'''}(\mathbf{x}^{\circ\circ})]$$

$$= N(N-1)E[X_{1,2}^c(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})(\mathbf{u}_1 - \overline{\mathbf{u}^c}^3)(\mathbf{v}^c - \overline{\mathbf{v}^c}^3)]$$

$$+ \varphi^{(2)}(\mathbf{x},\mathbf{x}^\circ)\alpha^{c3}(\mathbf{x}^{\circ\circ})|\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{v}^c}^3(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{u}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})$$

$$- \overline{\mathbf{v}^c}^3 N(N-1)E[X_{1,2}^c\varphi_1\varphi_2\mathbf{u}_1] - \overline{\mathbf{u}^c}^3 N(N-1)E[X_{1,2}^c\varphi_1\varphi_2\mathbf{v}^c]$$

$$+ N(N-1)E[X_{1,2}^c(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}^\circ)\mathbf{u}_1\mathbf{v}^c]$$

$$\text{(6.5) [cont.]} \quad = N(N-1)E[X^c_{1,2}(\mathbf{x}^{\circ\circ})\varphi_1(\mathbf{x})\varphi_2(\mathbf{x}^\circ)\mathbf{u}_1\mathbf{v}^c(\mathbf{x}^{\circ\circ})]$$
$$- \phi^{(2)}(\mathbf{x},\mathbf{x}^\circ)\alpha^{c3}(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{v}^c}^3(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)\overline{\mathbf{u}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}),$$

and we are led to define the corresponding composite pseudo-turbulent tensor of third order by:

$$\text{(6.6)} \qquad \mathbb{A}^3_{\mathrm{uv}^{\circ\circ}}(\mathbf{x}^{\circ\circ},\mathbf{x}|\mathbf{x}^\circ) = \sum_i^N \sum_{j\neq i} E[X^c_{i,j}(\mathbf{x}^{\circ\circ})\varphi_i(\mathbf{x})\varphi_j(\mathbf{x}^\circ)\mathbf{u}_i^{c'''}\mathbf{v}^{c'''}(\mathbf{x}^{\circ\circ})].$$

A second one, $\mathbb{A}^3_{\mathrm{u}^\circ\mathrm{v}^{\circ\circ}}(\mathbf{x}^{\circ\circ},\mathbf{x}^\circ|\mathbf{x})$, is obtained by commuting $\mathbf{x}^\circ$ and $\mathbf{x}$; it is used to break down the similar cross-correlation function $N(N-1)E[X^c_{1,2}\varphi_1\varphi_2\mathbf{u}_2\mathbf{v}^c]$.

6.2. Expression of C^*_{v} in the second-order momentum equations for the continuous phase

The calculation of the composite pseudo-turbulent tensors is based on the procedure already used in Secs. 2.1 and 2.2, i.e. it begins by expanding them in terms expressing the contribution of each group, and goes on to break down the contribution of each group. This procedure will not be repeated here in detail. Moreover, we will restrict ourselves to the second order, i.e. to $\mathbb{A}^2_{\mathrm{uv}^\circ}(\mathbf{x}^\circ,\mathbf{x})$. Thus, we will not be able at the end to present an expression for C^{**}_{v}. The fluctuating part of such an expression is very complicated, much more than the corresponding one for C^{**}_α we have present above (see Eq. (5.6)). Moreover, our general strategy of finding solutions in the diluteness limit which will be presented in the next paper does not generally require the fluctuating part of this term: it suffices to know that it is an $O(\Theta)$ contribution.

We obtain first:

$$\text{(6.7)} \qquad \mathbb{A}^2_{\mathrm{uv}^\circ}(\mathbf{x}^\circ,\mathbf{x}) = NE[X^c_1(\mathbf{x}^\circ)\varphi_1(\mathbf{x})\mathbf{u}_1^{c''}\mathbf{v}^{c''}(\mathbf{x}^\circ)]$$
$$= \frac{1}{N-1}\int d\mathbf{x}^{\circ\circ}\{\phi^{(2)}(\mathbf{x},\mathbf{x}^{\circ\circ})\alpha^{c3}(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})[\overline{\mathbf{u}^c}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \overline{\mathbf{u}^c}^2(\mathbf{x}|\mathbf{x}^\circ)]$$
$$\cdot[\overline{\mathbf{v}^c}^3(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ}) - \overline{\mathbf{v}^c}^2(\mathbf{x}^\circ|\mathbf{x})] + \mathbb{A}^3_{\mathrm{uv}^\circ}(\mathbf{x}^\circ,\mathbf{x}|\mathbf{x}^{\circ\circ})\}.$$

Note that similar computations can be used to express the next third-order composite correlation functions i.e. $\mathbb{A}^3_{\mathrm{uv}^{\circ\circ}}(\mathbf{x}^{\circ\circ},\mathbf{x}|\mathbf{x}^\circ)$ and $\mathbb{A}^3_{\mathrm{u}^\circ\mathrm{v}^{\circ\circ}}(\mathbf{x}^{\circ\circ},\mathbf{x}^\circ|\mathbf{x})$. Using (5.4), (5.6), and the definition of the second-order disturbances, the above expression leads to the following result valid up to $O(\Theta)$:

$$\text{(6.8)} \qquad \mathbb{A}^2_{\mathrm{uv}^\circ}(\mathbf{x}^\circ,\mathbf{x}) = \int d\mathbf{x}^{\circ\circ}\{\phi^{(2)}(\mathbf{x},\mathbf{x}^{\circ\circ})\alpha^{c3}(\mathbf{x}^\circ|\mathbf{x}\mathbf{x}^{\circ\circ})\mathbf{u}^*(\mathbf{x}|\mathbf{x}^\circ)$$
$$\cdot[\mathbf{v}^*(\mathbf{x}^\circ|\mathbf{x}) + \mathbf{v}^{**}(\mathbf{x}^\circ|\mathbf{x},\mathbf{x}^{\circ\circ})]\} + \ldots$$

Having at our disposal $\mathbb{A}^2_{\mathrm{uv}^\circ}$ as well as the cross-correlation of the second type (see Sec. 5), the source term C^*_{v} appearing in the conditioned momentum equation ((4.7) – Part II) can be completely expressed in terms of disturbance flows.

$$C^*_{\mathrm{v}}(\mathbf{x}^\circ|\mathbf{x}) = \overline{\mathbf{v}^c}^2(\mathbf{x}^\circ|\mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{x}} E[\varphi_1 X^c_1 \mathbf{u}_1]/\alpha^{c2}\phi_1 - \frac{\partial}{\partial \mathbf{x}} \cdot E[\varphi_1 X^c_1 \mathbf{u}_1 \mathbf{v}^c]/\alpha^{c2}\phi_1 = \overline{\mathbf{v}^c}^2 \cdot \frac{\partial}{\partial \mathbf{x}} \overline{\mathbf{u}^c}^2 - \frac{\partial}{\partial \mathbf{x}} \cdot \overline{\mathbf{u}^c}^2 \overline{\mathbf{v}^c}^2 + [\overline{\mathbf{u}^c}^2 \overline{\mathbf{v}^c}^2 - \overline{\mathbf{v}^c}^2 \overline{\mathbf{u}^c}^2] \cdot \frac{\partial}{\partial \mathbf{x}} \mathrm{Log}(\alpha^{c2}\phi_1) - \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^2_{\mathrm{uv}^\circ}/\alpha^{c2}\phi^{(1)}. \tag{6.9}$$

7. Conclusions

To describe laminar flows carrying spherical inclusions, we succeeded to obtain at the end of Part I, two infinite sets of equations; the revisited BBGKY hierarchy for the dispersed phase and the revisited Lundgren hierarchy (LUNDGREN [49]) for the continuous phase. The first-order or bulk flow equations issued from both hierarchies constitute the basic building blocks of our two-fluid approach. Higher-order equations, which do not appear in usual two-fluid models, have to be retained as a rule up to some order depending on whether the mixture is more or less dilute. It happens that these higher-order equations are very complicated; as matters stand, they cannot easily be simplified nor truncated. The purpose of the second part was to transform these equations into equations controlling new conditional disturbance fields which are much more tractable.

Whatever their number or form may be, the disturbance flow equations have to be treated on an equal footing as the bulk flow equations in the final resulting model. At any time and at any location, they appear as the natural frame to specify the micro-problems, the solution of which provides the missing information to close the bulk flow equations. Among this information, the most delicate piece concerns pseudo-turbulence. This whole article part has been devoted to show that solution of the above micro-problems can effectively be used to estimate various pseudo-turbulent tensors and correlation functions.

Comparison with other similar studies is difficult. First of all, having regard to most of our correlations, it is simply impossible since usual models do not include these quantities: they are specific of our approach and have been ignored so far (i.e. second and higher-order pseudo-turbulent tensors, correlations and cross-correlations of the first and of the second type). The first-order pseudo-turbulent tensor for the continuous phase is an exception to this rule since it has given rise to many studies. Observe first that analogy with single-phase turbulent momentum transport is not allowed and that it is difficult to model this tensor in a standard approach since fluctuations are only generated by the flow around individual inclusions. As a consequence, some micro-problems corresponding to schools of thought evoked in the introduction of Part I come inevitably to the assistance of modellers. Among various proposals, a formula derived by Koch and Shaqfey [38], in a more restricted framework, has been compared to our results; the two leading order terms of their formula correspond to the first two terms of a multipole expansion of the expression we have found for the first-order pseudo-turbulent tensor.

It might be pointed out that first-order pseudo-turbulent tensors we have proposed are not expressed in terms of bulk-flow variables and cannot be directly used in ordinary two-phase models. As the other unknowns terms met in the previous parts, they are explicitly related to the first-order disturbance flow equations. Thus, they are awaiting a closure relationship which results in our approach from solving specific micro-problems described by higher-order disturbance flow equations. The exact form of these extra equations, the way (one-way or two-way) they are coupled with bulk-flow equations depend on each considered physical situation (e.g. low or high inclusion Reynolds number). So does the ultimate number of equations in the final closed model which is given by the afore-mentioned truncation procedure, based on diluteness, which will be developped in the next Part IV.

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