Arch. Mech., **53**, 1, pp. 65–85, 2001

An orthotropic constitutive model for secondary creep of ice

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As polycrystalline ice undergoes creep deformation over long time-periods, it develops a fabric (oriented structure) and associated, strain-induced anisotropy. In the paper, a frame-indifferent orthotropic constitutive model for secondary creep of ice is formulated, in which the strain-rate is expressed in terms of the deviatoric stress, strain, and three structure tensors based on the principal deformation axes. As an illustration, the model is used to determine the evolution of the creep response of ice to continued uniaxial compression and simple shearing.

Key Words: ice, creep, induced anisotropy, orthotropy, constitutive law.

Notations

$\mathbf{B}$	left Cauchy-Green deformation tensor
b_r $(r = 1,2,3)$	principal values of the deformation tensor $\mathbf{B}$
$\mathbf{D}$	strain-rate tensor
E_a	axial enhancement factor
E_s	shear enhancement factor
$\mathbf{e}^{(r)}$ $(r = 1,2,3)$	unit vectors of the principal stretch axes
$\mathbf{F}$	material deformation gradient tensor
h	material response function
$\mathbf{I}$	unit tensor
J_k $(k = 1,\ldots,21)$	invariants of second-order tensors
K	trace of the deformation tensor $\mathbf{B}$
$\mathbf{M}^{(r)}$ $(r = 1,2,3)$	structure tensors
Q, q	material response functions
$\mathbf{R}$	rotation tensor
$\mathbf{S}$	deviatoric Cauchy stress tensor
$\mathbf{V}$	left stretch tensor
$\mathbf{v}$	velocity vector
x_i $(i = 1,2,3)$	spatial rectangular Cartesian co-ordinates
X_i $(i = 1,2,3)$	material rectangular Cartesian co-ordinates
η_0	isotropic ice fluidity
η_a	axial fluidity
η_s	shear fluidity
κ	shear strain
λ_r $(r = 1,2,3)$	principal stretches

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μ_0	isotropic ice viscosity
σ	Cauchy stress tensor
ϕ_j $(j, \ldots, 12)$	material response functions

1. Introduction

ICE CAPS COVER approximately 15 million square kilometres of Earth's land in Antarctica and Greenland and, subject to seasonal variations, about 18 to 23 million square kilometres of Arctic and Antarctic waters. The presence of such huge masses of ice affects the thermodynamics of both the atmosphere and ocean and has a considerable impact on the global climate. In order to properly describe the processes taking place in polar regions, e.g. for predicting the climate changes in the future, it is necessary to understand the mechanical behaviour of ice and, in particular, to formulate adequate constitutive relations that are capable of capturing the observed behaviour of ice, both on large and small scales.

Ice is a complex material. In natural conditions it usually exists at high homologous temperatures (that is close to the melting point on the absolute temperature scale), therefore its behaviour resembles very much the behaviour of many metals and rocks prior to melting. Ice displays a wide range of mechanical responses that include: pure elasticity, nonlinear viscoelasticity (decelerating primary creep, also referred to as transient creep or delayed elasticity), and irreversible secondary ("steady-state") and tertiary (accelerating) creeps. The latter two types of creep, characteristic for the ductile behaviour of ice, occur at relatively low stress levels. At high stresses and strain-rates, the ice changes considerably its behaviour and becomes a very brittle material (more brittle than, for instance, glass).

In this paper we concentrate on secondary creep of ice, as this deformation mechanism dominates the flow of polar ice masses, and is also important in sea ice applications (since usually during this stage of deformation, a floating ice cover sustains maximum stresses, and hence for these stresses engineering structures are to be designed). An important process associated with the irreversible creep of ice is the formation and subsequent evolution of anisotropy in an initially isotropic material when it is subjected to changing stress and deformation. Such a phenomenon, known as induced anisotropy, is of crucial importance in the case of polar ice. As ice cores drilled at different sites in Antarctica and Greenland have shown (GOW and WILLIAMSON [7], RUSSELL-HEAD and BUDD [15], THORSTEINSSON *et al.* [21]), polar ice reveals strong fabrics, shown by significant alignment of individual ice crystal c-axes along some preferential directions, developing as ice descends from the free surface to depth in an ice sheet. The anisotropy of the medium affects considerably the flow of polar ice masses, what has been proved by the results of numerical simulations carried out by MANGENEY *et al.* [10], MANGENEY *et al.* [11], and STAROSZCZYK and MORLAND [19]. In the case of sea ice, due to relatively short time-scales (years compared with thousands of years for polar ice), the process of fabric evolution plays a negligible role. Nonetheless, the constitutive relations developed for anisotropic polar ice can still be used for describing sea ice behaviour, since this type of ice is usually anisotropic (most often transversely isotropic) from the very moment of its origin. The macroscopic anisotropy of ice is due to underlying processes occurring in the material on the micro-scale of individual ice crystals, as the latter are very strongly anisotropic: shear stresses applied in planes normal to the crystal basal plane give strain-rates up to two orders of magnitude higher than the strain-rates resulting from shearing performed in planes parallel to the basal plane (PATERSON [14]). The main microscopic processes involved in the creation and evolution of the anisotropic fabric in ice are: (1) the rotation of crystal c-axes towards principal axes of compression and away from principal axes of extension, and (2) the process of rotation recrystallisation (or polygonisation), in which new ice grains with orientations similar to old grains are created (LLIBOUTRY and DUVAL [9]).

In order to construct macroscopic constitutive equations for polycrystalline ice, three different methods can be applied. The first method is to derive an average response of an ice aggregate from the properties of individual grains and assumptions on crystal interactions (AZUMA [1], VAN der VEEN and WHILLANS [22], CASTELNAU *et al.* [4]). Since this method, which can be called a discrete-grain approach, requires that the behaviour of several hundred grains at a given material point is followed to yield the macroscopic response, the constitutive theories of this type can be hardly implemented in current large-scale ice sheet numerical models.

Therefore, in order to significantly reduce the number of variables involved in the description of ice fabric, a group of micro-macroscopic models have been developed (LLIBOUTRY [8], SVENDSEN and HUTTER [20], MEYSSONNIER and PHILIP [12], GÖDERT and HUTTER [6], GAGLIARDINI and MEYSSONNIER [5]). In these models the polycrystalline aggregate is treated as a continuum, whose directional properties are described in terms of a so-called orientation distribution function (ODF), defining continuous weightings to the grain c-axes orientations in space. Unlike the discrete-grain models, in which the behaviour of each grain has to be considered, in the micro-macroscopic approach the evolution of only a few functions has to be followed at each node of an ice sheet model.

The third method is to assume that the macroscopic response of ice can be described in terms of the fabric induced entirely by macroscopic deformation, and to ignore all microscopic processes taking place at the crystal level. This leads to a phenomenological model formulated by MORLAND and STAROSZCZYK [13] and further extended by STAROSZCZYK and MORLAND [18]. The adopted assumption

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that the induced anisotropy of ice depends only on the current macroscopic strain and does not depend on the deformation history is a significant simplification, since, in general, the fabric evolution is a path-dependent process. Nevertheless, the model allows a good agreement with observations and its predictions correlate well with the results given by the discrete-grain and micro-macroscopic models (STAROSZCZYK and GAGLIARDINI [17]). It is also believed that such an approximation provides the simplest approach to an evolving anisotropic flow law which can be tractable in large-scale ice sheet dynamics, since it requires that only current deformation gradients are calculated in addition to the velocity and stress fields.

The orthotropic constitutive law formulated by STAROSZCZYK and MORLAND [18] expresses the deviatoric stress in terms of the strain-rate, strain, and three structure tensors defined by the outer products of three orthogonal vectors along the current principal stretch axes. In the present work we formulate an inverse orthotropic flow law, in which the strain-rate is expressed in terms of the stress and deformation. Such a form of the flow law is a conventional glaciology form, despite the fact that it is less useful in applications, as it is more convenient to use with the momentum balance equations the stress – strain-rate form of the constitutive relation. However, it is possible that the inverse form will reveal different features which can improve correlations with experimental data.

The proposed law is derived from a general, frame-indifferent, orthotropic tensor representation given by BOEHLER [2]. The general law is subsequently reduced by retaining only those tensor generators which contribute to viscous responses that can be detected by simple shearing performed in different directions on different planes. Apart from three structure tensors, needed to describe the orthotropic symmetries in the material, the model involves two material response functions with dependence on the principal stretches and an invariant measure of total deformation. These functions are constructed by correlating the predicted model response with the observed limit behaviour of ice at large strains. The constitutive theory is then used to illustrate the evolution of the creep response of an initially isotropic sample of ice during indefinite uniaxial compression and simple shearing.

2. General orthotropic constitutive law

Newly formed compacted polar ice is assumed to be macroscopically isotropic due to the random distribution of individual crystals in the material. As the polycrystalline aggregate starts to deform, all crystal glide planes move in such a way that the crystal c-axes (the axes which are orthogonal to the grain basal planes) are rotated towards principal compression axes and away from principal exten-

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sion axes. This movement of glide planes leads to the formation of an orthotropic fabric in the material, with orthotropic symmetry axes coinciding with the initial principal stretch axes. It is supposed that, due to the symmetric distribution of all glide planes about the principal directions of strain, the reflexional symmetries with respect to the three orthogonal principal stretch planes are maintained throughout the whole process of deformation, even though the orientations of the principal stretch axes change as the material creeps. Since the ice crystal basal planes are those planes over which the material can shear most easily, this implies that macroscopic shearing on the principal stretch planes should have ease of shearing, with fluidities (reciprocal viscosities) ordered by the respective magnitudes of normal compressions (the inverse stretches). Furthermore, the relative magnitudes of such fluidities should depend on, at least, the principal stretches. Therefore, the constitutive flow law should include the dependence on at least the principal stretches as arguments of the response functions, or more generally, on the deformation. The most simple approach which captures an evolving orthotropic fabric is then to relate the strain-rate to the Cauchy deviatoric stress and strain. As a deformation measure we adopt here the left Cauchy-Green deformation tensor which, like the Cauchy stress tensor and the strain-rate tensor, is a frame-indifferent quantity and as such can be used in an objective constitutive equation.

Let Ox_i $(i = 1, 2, 3)$ be the spatial rectangular Cartesian co-ordinates, OX_i $(i = 1, 2, 3)$ particle reference co-ordinates, and v_i – the components of the velocity vector $\mathbf{v}$. Then the material deformation gradient $\mathbf{F}$ and strain-rate $\mathbf{D}$ have the components

$$(2.1)\qquad F_{ij} = \frac{\partial x_i}{\partial X_j}, \qquad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

By the polar decomposition theorem (SPENCER [16]), the deformation gradient tensor $\mathbf{F}$ can be expressed in the form

$$(2.2)\qquad \mathbf{F} = \mathbf{V}\mathbf{R},$$

where $\mathbf{R}$ is the rotation tensor and $\mathbf{V}$ is the left stretch tensor. The principal stretches λ_r $(r = 1, 2, 3)$ along the principal stretch axes defined by the unit vectors $\mathbf{e}^{(r)}$ $(r = 1, 2, 3)$ are given by

$$(2.3)\qquad \mathbf{V}\mathbf{e}^{(r)} = \lambda_r \mathbf{e}^{(r)}, \qquad \det(\mathbf{V} - \lambda_r \mathbf{I}) = 0,$$

where $\mathbf{I}$ is the unit tensor. The left Cauchy-Green deformation tensor $\mathbf{B}$ and its principal values b_r, equal to the squares of the principal stretches λ_r, are defined by

$$(2.4)\qquad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \quad \mathbf{B}\mathbf{e}^{(r)} = b_r \mathbf{e}^{(r)}, \quad \det(\mathbf{B} - b_r \mathbf{I}) = 0, \quad b_r = \lambda_r^2.$$

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The three structure tensors, needed to describe the orthotropy of the material, are defined by the outer products of the principal stretch unit vectors by

(2.5) $$\mathbf{M}^{(r)} = \mathbf{e}^{(r)} \otimes \mathbf{e}^{(r)}, \quad (r = 1,2,3), \quad \mathbf{M}^{(1)} + \mathbf{M}^{(2)} + \mathbf{M}^{(3)} = \mathbf{I}.$$

By ice incompressibility, a common glaciology approximation, we have

(2.6) $$\mathrm{div}\mathbf{v} = 0, \quad \det \mathbf{F} = \lambda_1\lambda_2\lambda_3 = \det \mathbf{B} = b_1 b_2 b_3 = 1,$$

with $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $b_1 = b_2 = b_3 = 1$ in an undeformed isotropic state $\mathbf{F} = \mathbf{I}$ or in a rigid rotation motion $\mathbf{F} = \mathbf{R}$. The deviatoric Cauchy stress $\mathbf{S}$ is defined in terms of the Cauchy stress $\boldsymbol{\sigma}$ and the mean pressure p by

(2.7) $$\mathbf{S} = \sigma + p\mathbf{I}, \qquad p = -\frac{1}{3}\mathrm{tr}\sigma, \quad \mathrm{tr}\mathbf{S} = 0,$$

where $\mathrm{tr}\sigma$ denotes the trace of σ. Due to ice incompressibility, p is a workless constraint not given by a constitutive law, but determined by the momentum balance and boundary conditions.

Any constitutive relation should satisfy the principle of frame-indifference, or objectivity, to ensure that material properties are independent of the observer. Here we are concerned with a frame-indifferent law that relates one symmetric tensor (strain-rate $\mathbf{D}$) to other two symmetric tensors (the deviatoric stress $\mathbf{S}$ and the deformation $\mathbf{B}$). For such a constitutive law the general orthotropic representation, given by BOEHLER [2], is

(2.8) $$\mathbf{D} = \sum_{r=1}^{3} [\phi_r \mathbf{M}^{(r)} + \phi_{r+3}(\mathbf{M}^{(r)}\mathbf{S} + \mathbf{S}\mathbf{M}^{(r)}) + \phi_{r+6}(\mathbf{M}^{(r)}\mathbf{B} + \mathbf{B}\mathbf{M}^{(r)})] + \phi_{10}\mathbf{S}^2 + \phi_{11}\mathbf{B}^2 + \phi_{12}(\mathbf{B}\mathbf{S} + \mathbf{S}\mathbf{B}),$$

where the 12 response coefficients ϕ_i $(i = 1, \ldots, 12)$ are the functions of the 19 invariants formed from the tensors $\mathbf{M}^{(r)}$, $\mathbf{S}$ and $\mathbf{B}$:

(2.9) $$\begin{aligned} J_r &= \mathrm{tr}\mathbf{M}^{(r)}\mathbf{S}, \qquad J_{r+3} = \mathrm{tr}\mathbf{M}^{(r)}\mathbf{B}, \quad J_{r+6} = \mathrm{tr}\mathbf{M}^{(r)}\mathbf{S}^2, \\ J_{r+9} &= \mathrm{tr}\mathbf{M}^{(r)}\mathbf{B}^2, \quad J_{r+12} = \mathrm{tr}\mathbf{M}^{(r)}\mathbf{S}\mathbf{B} \quad (r = 1,2,3), \\ J_{16} &= \mathrm{tr}\mathbf{B}\mathbf{S}^2, \qquad J_{17} = \mathrm{tr}\mathbf{B}^2\mathbf{S}, \quad J_{18} = \det \mathbf{S}, \quad J_{19} = \det \mathbf{B}. \end{aligned}$$

Due to the constraints that the strain-rate tensor $\mathbf{D}$ has zero trace and the material is incompressible, only 11 coefficients ϕ_i are independent, and only 18 invariants are nontrivial, as $J_{19} = \det \mathbf{B} = 1$. Since we suppose that in any state the strain-rate $\mathbf{D}$ vanishes when the deviatoric stress $\mathbf{S}$ vanishes, we require that the coefficients ϕ_1, ϕ_2, ϕ_3, ϕ_7, ϕ_8, ϕ_9, ϕ_{11} vanish when $\mathbf{S}$ vanishes; that is, when the invariants J_1, J_2, J_3, J_7, J_8, J_9, J_{13}, J_{14}, J_{15}, J_{16}, J_{17}, J_{18} vanish.

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The general constitutive model defined by equation (2.8), with 11 independent response functions and 18 invariants as their possible arguments, is far beyond the theory that can be correlated with available experimental data. Therefore, the relation (2.8) has to be significantly simplified by reducing the number of the functions ϕ_i and the invariants J_k. This needs to be done in such a way that the main features of the observed creep response of ice are still captured by a reduced model, and all the model coefficients can be determined from a limited number of simple laboratory tests, most commonly uniaxial compression and simple shearing. In order to simplify the general orthotropic relation (2.8) we follow here a method proposed by MORLAND and STAROSZCZYK [13, 18], based on the concept of so-called instantaneous directional viscosities that can be measured in a series of simple shear tests carried out on different co-ordinate planes.

Consider distinct axial stretches $\lambda_1, \lambda_2, \lambda_3$ along the fixed co-ordinate axes x_1, x_2, x_3, corresponding to the deformation

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \quad \lambda_1\lambda_2\lambda_3 = 1, \tag{2.10}$$

where X_1, X_2, X_3 are particle co-ordinates in the initial isotropic reference state. The left stretch tensor $\mathbf{V}$, deformation gradient $\mathbf{F}$, rotation tensor $\mathbf{R}$ and the left Cauchy-Green deformation tensor $\mathbf{B}$ are given by

$$\mathbf{V} = \mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \mathbf{R} = \mathbf{I}, \quad \mathbf{B} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}. \tag{2.11}$$

The principal stretch axes $\mathbf{e}^{(r)}$ coincide now with the co-ordinate axes, therefore the structure tensors $\mathbf{M}^{(r)}$ $(r = 1, 2, 3)$ are the single diagonal element matrices

$$\mathbf{M}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

Now remove the stress and strain-rate, so the fabric defined by the current stretches $\lambda_1, \lambda_2, \lambda_3$ is frozen, and consider instantaneous responses to shearings performed in different directions on different co-ordinate planes. For simple shear in the x_i direction on a glide plane normal to the x_j direction $(i \neq j)$, with no summation implied by a repeated suffix, the new deformation field is defined by

$$x_i = \lambda_i X_i + \kappa_{ij} X_j, \quad x_j = \lambda_j X_j, \quad x_k = \lambda_k X_k, \tag{2.13}$$

where i, j, k are distinct permutations of $1, 2, 3$, and κ_{ij} is a shear strain. For the shearing occurring in the plane Ox_ix_j, all the components of the deviatoric stress

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tensor are zero except the symmetric entries S_{ij}. Such a stress configuration induces a viscous response, described by (2.8), in which the strain-rate tensor has, in general, three nonzero diagonal components and two nonzero off-diagonal symmetric components D_{ij}. Instantaneously, at the frozen values of λ_1, λ_2, λ_3, the tensors $\mathbf{B}$ and $\mathbf{M}^{(r)}$ $(r = 1, 2, 3)$ are given by the diagonal tensors (2.11) and (2.12). The symmetric tensor generators in (2.8) have for $i \neq j$ the following instantaneous (ij) components, equal to the (ji) components:

$$(\mathbf{M}^{(r)}\mathbf{S} + \mathbf{S}\mathbf{M}^{(r)})_{ij} = \begin{cases} S_{ij} \ (r = i \ \text{or} \ r = j) \\ 0 \ (r \neq i \ \text{and} \ r \neq j) \end{cases}, \tag{2.14}$$

$$(\mathbf{M}^{(r)}\mathbf{B} + \mathbf{B}\mathbf{M}^{(r)})_{ij} = 0, \quad (\mathbf{S}^2)_{ij} = 0, \quad (\mathbf{B}^2)_{ij} = 0, \tag{2.15}$$

$$(\mathbf{B}\mathbf{S} + \mathbf{S}\mathbf{B})_{ij} = (b_i + b_j)S_{ij}\,. \tag{2.16}$$

There are also nonzero diagonal components of the instantaneous strain-rate $\mathbf{D}$, other than those defined above, these are however of no interest at this point, since they cannot be detected in the shearing tests.

The (ij) component $(i \neq j)$ of the constitutive law (2.8) is therefore given by the following relation

$$D_{ij} = [\phi_{i+3} + \phi_{j+3} + (b_i + b_j)\phi_{12}]S_{ij}\,, \tag{2.17}$$

defining an instantaneous fluidity η_{ij} (reciprocal viscosity) for shear in the x_i direction on a glide plane normal to the x_j direction by

$$\eta_{ij} = \frac{2D_{ij}}{S_{ij}} = 2[\phi_{i+3} + \phi_{j+3} + (b_i + b_j)\phi_{12}], \tag{2.18}$$

which depends for each (ij) only on the response functions ϕ_{i+3}, ϕ_{j+3} $(i, j = 1, 2, 3)$, and ϕ_{12}; the other terms in the general law (2.8) do not contribute to the directional fluidities. In view of (2.18), the ratios of the instantaneous directional fluidities are defined by

$$\frac{\eta_{13}}{\eta_{23}} = \frac{\phi_4 + \phi_6 + (b_1 + b_3)\phi_{12}}{\phi_5 + \phi_6 + (b_2 + b_3)\phi_{12}}, \quad \frac{\eta_{12}}{\eta_{13}} = \frac{\phi_4 + \phi_5 + (b_1 + b_2)\phi_{12}}{\phi_4 + \phi_6 + (b_1 + b_3)\phi_{12}}. \tag{2.19}$$

If the values of b_1 and b_2 are interchanged in the first ratio, for any b_3, then that ratio must become η_{23}/η_{13} with the original values, and, similarly, interchanging the values of b_2 and b_3 for any b_1 in the second ratio, must yield η_{13}/η_{12} with the original values. Thus ϕ_{12} must not change when b_1, b_2, b_3 are permuted, the values of ϕ_4 and ϕ_5 are interchanged when b_1 and b_2 are interchanged, the values of ϕ_5 and ϕ_6 are interchanged when b_2 and b_3 are interchanged, and those of ϕ_4

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and ϕ_6 when b_1 and b_3 are interchanged. Therefore, the function ϕ_{12} can depend in the frozen fabric only on the combinations of two invariants

$$J_{20} = \sum_{r=1}^{3} J_{r+3} = \mathrm{tr}\mathbf{B}, \quad J_{21} = \sum_{r=1}^{3} J_{r+9} = \mathrm{tr}\mathbf{B}^2, \tag{2.20}$$

while the functions ϕ_4, ϕ_5 and ϕ_6 can have common dependence on J_{20} and J_{21} and common dependence on $J_4 = b_1$, $J_5 = b_2$ and $J_6 = b_3$, respectively.

3. Reduced model

Following STAROSZCZYK and MORLAND [18] we consider only those terms in the general constitutive relation (2.8) which contribute to the instantaneous directional fluidities (2.18), that is we retain only the terms with the fabric response functions ϕ_4, ϕ_5, ϕ_6 and ϕ_{12}. We further assume that the response functions depend on only two invariants of the deformation tensor $\mathbf{B}$, namely $J_{r+3} = b_r$ and $J_{20} = \mathrm{tr}\mathbf{B}$, which constitute a minimum set of invariants the model has to incorporate in order to satisfy the directional fluidity ratios (2.19). Accordingly, we express the response functions by

$$\phi_{r+3}(J_{r+3}, J_{20}, J_{21}) = \frac{\eta_0}{4}\, h(b_r), \quad \phi_{12}(J_{20}, J_{21}) = \frac{\eta_0}{4}\, q(K), \tag{3.1}$$

where $h(b_r)$ and $q(K)$ are single-argument response functions, $K = \mathrm{tr}\mathbf{B} = b_1 + b_2 + b_3 \geq 3$, and $\eta_0 = \eta_0(\mathrm{tr}\mathbf{S}^2, T)$ is the fluidity of isotropic ice, a function of the second invariant of the deviatoric stress $\mathbf{S}$ and temperature T. With the definitions (3.1), the reduced orthotropic constitutive model takes the following form:

$$\begin{aligned}\mathbf{D} = \frac{\eta_0}{4}\Big\{ \sum_{r=1}^{3} h(b_r)[\mathbf{M}^{(r)}\mathbf{S} + \mathbf{S}\mathbf{M}^{(r)} - \frac{2}{3}\,\mathrm{tr}(\mathbf{M}^{(r)}\mathbf{S})\mathbf{I}] \\ + q(K)[\mathbf{B}\mathbf{S} + \mathbf{S}\mathbf{B} - \frac{2}{3}\,\mathrm{tr}(\mathbf{B}\mathbf{S})\mathbf{I}]\Big\},\end{aligned} \tag{3.2}$$

where the terms with isotropic tensors are introduced to recover zero trace, noting that the included scalar $(\mathbf{M}^{(r)}\mathbf{S}) = J_r$, and the scalar $\mathrm{tr}(\mathbf{B}\mathbf{S})$ is the sum of J_{r+12}. We require that when $\mathbf{B} = \mathbf{I}$, that is when $K = 3$, the relation (3.2) reduces to the isotropic fluid flow law $\mathbf{S} = 2\mu_0\mathbf{D}$, where $\mu_0 = 1/\eta_0$ is the viscosity of isotropic ice; thus

$$h(1) + q(3) = 1. \tag{3.3}$$

By Eq. (3.2), the instantaneous directional fluidity (2.18) becomes

$$\eta_{ij} = \frac{\eta_0}{2}[h(b_i) + h(b_j) + (b_i + b_j)q(K)], \tag{3.4}$$

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and since this must remain bounded for any axial stretch b_r increasing indefinitely, we rewrite q and the normalisation (3.3) as

$$q(K) = K^{-1}Q(K), \quad h(1) + \frac{1}{3}Q(3) = 1. \tag{3.5}$$

We now employ the orthotropic constitutive relation (3.2) to simulate the behaviour of ice in simple configurations corresponding to those applied in typical creep tests, and in particular we predict the evolution of axial and shear fluidities during the uniaxial compression and simple shear experiments. In the first test, unconfined compression of an initially isotropic ice sample along, say, the x_3 axis, there are equal lateral stretches $\lambda_1 = \lambda_2 > 1$, and, due to the incompressibility condition (2.6), the axial stretch (a compression) is $\lambda_3 = \lambda_1^{-2} < 1$. The deformation field for this configuration is defined by the relations (2.10), and the corresponding deformation tensor $\mathbf{B}$ and the structure tensors $\mathbf{M}^{(r)}$ $(r = 1, 2, 3)$ are given by (2.11) and (2.12), respectively. The deviatoric stress tensor has only diagonal components

$$\mathbf{S} = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{11} & 0 \\ 0 & 0 & -2S_{11} \end{pmatrix}, \tag{3.6}$$

and the invariants entering (3.2) are

$$\operatorname{tr}(\mathbf{M}^{(1)}\mathbf{S}) = \operatorname{tr}(\mathbf{M}^{(2)}\mathbf{S}) = S_{11}, \quad \operatorname{tr}(\mathbf{M}^{(3)}\mathbf{S}) = -2S_{11}, \tag{3.7}$$

$$\operatorname{tr}(\mathbf{BS}) = 2S_{11}(b_1 - b_1^{-2}),$$

and $K = \operatorname{tr}\mathbf{B} = 2b_1 + b_1^{-2}$. With the above definitions, the constitutive law (3.2) yields the following viscous response of ice in uniaxial compression:

$$\frac{2\mu_0 D_{11}}{S_{11}} = \frac{2\mu_0 D_{33}}{S_{33}} = \frac{1}{3}h(b_1) + \frac{2}{3}h(b_1^{-2}) + \frac{Q(K)}{3K}(b_1 + 2b_1^{-2}) = \frac{\eta_a}{\eta_0}, \tag{3.8}$$

where η_a/η_0 defines the ratio of the fabric induced axial fluidity to isotropic fluidity. BUDD and JACKA [3] have determined experimentally the limit value of this ratio for indefinite axial compression, together with an analogous limit ratio for indefinite shearing in a plane deformation. These limit ratios, commonly described in glaciology as enhancement factors, are used here to determine the limit values of the response functions h and Q. Hence, as $b_1 \to \infty$, then $K \sim 2b_1$, and from (3.8) we obtain

$$\frac{\eta_a}{\eta_0} \to \frac{2}{3}h(0) + \frac{1}{3}h(\infty) + \frac{1}{6}Q(\infty) = E_a\,, \tag{3.9}$$

where E_a is the axial enhancement factor.

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Next consider a simple shear test on ice. For more generality, assume that the material is not isotropic at the beginning of shear deformation, that is it has already developed a fabric in a plane creep, the strength of which is described by the principal stretches $\lambda_3 = \lambda_1^{-1}$, $\lambda_2 = 1$. Now let us start shearing in the plane Ox_1x_3, driven by a shear stress S_{13}. The deformation field is described by

$$x_1 = \lambda_1 X_1 + \kappa X_3, \quad x_2 = X_2, \quad x_3 = \lambda_1^{-1} X_3, \tag{3.10}$$

where κ is a shear strain in the plane Ox_1x_3. The deformation, deviatoric stress, and strain-rate tensors are now given by

$$\mathbf{B} = \begin{pmatrix} \lambda_1^2+\kappa^2 & 0 & \lambda_1^{-1}\kappa \\ 0 & 1 & 0 \\ \lambda_1^{-1}\kappa & 0 & \lambda_1^{-2} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} S_{11} & 0 & S_{13} \\ 0 & S_{22} & 0 \\ S_{13} & 0 & S_{33} \end{pmatrix}, \tag{3.11}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\dot{\gamma} \\ 0 & 0 & 0 \\ \frac{1}{2}\dot{\gamma} & 0 & 0 \end{pmatrix},$$

with $S_{11} + S_{22} + S_{33} = 0$ and $\dot{\gamma} = \lambda_1 \dot{\kappa}$. The principal stretch squares b_i, $(i = 1,2,3)$, the eigenvalues of $\mathbf{B}$, are

$$b_2 = 1, \quad b_3 = b_1^{-1}, \quad 2b_1 = \lambda_1^2 + \lambda_1^{-2} + \kappa^2 + \sqrt{(\lambda_1^2 + \lambda_1^{-2} + \kappa^2)^2 - 4}\ , \tag{3.12}$$

and the associated principal unit vectors $\mathbf{e}^{(r)}$ are defined by

$$\mathbf{e}^{(2)} = (0,1,0), \quad e_2^{(s)} = 0, \quad \lambda_1^{-1}\kappa\, e_1^{(s)} + (\lambda_1^{-2} - b_s)\, e_3^{(s)} = 0, \tag{3.13}$$
$$[e_1^{(s)}]^2 + [e_3^{(s)}]^2 = 1 \quad (s = 1,3).$$

The structure tensors are given by

$$\mathbf{M}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{(s)} = \begin{pmatrix} e_1^{(s)}e_1^{(s)} & 0 & e_1^{(s)}e_3^{(s)} \\ 0 & 0 & 0 \\ e_1^{(s)}e_3^{(s)} & 0 & e_3^{(s)}e_3^{(s)} \end{pmatrix} \quad (s = 1,3), \tag{3.14}$$

and the invariants are

$$\operatorname{tr}(\mathbf{M}^{(2)}\mathbf{S}) = S_{22}, \ \operatorname{tr}(\mathbf{M}^{(s)}\mathbf{S}) = S_{11}e_1^{(s)}\, e_1^{(s)} + S_{22}\, e_3^{(s)}e_3^{(s)} + 2S_{13}\, e_1^{(s)}e_3^{(s)}, \tag{3.15}$$
$$(s = 1,3), \qquad \operatorname{tr}(\mathbf{BS}) = S_{11}(\lambda_1^2 + \kappa^2 - 1) + S_{33}(\lambda_1^{-2} - 1) + 2S_{13}\lambda_1^{-1}\kappa,$$
$$K = \operatorname{tr}\mathbf{B} = b_1 + 1 + b_1^{-1} = \lambda_1^2 + \lambda_1^{-2} + \kappa^2 + 1.$$

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The tensor combinations appearing in (3.2) are defined by

$$\mathbf{M}^{(2)}\mathbf{S} + \mathbf{S}\mathbf{M}^{(2)} - \frac{2}{3}\mathrm{tr}(\mathbf{M}^{(2)}\mathbf{S})\mathbf{I} = \frac{2}{3}\begin{pmatrix} -S_{22} & 0 & 0 \\ 0 & 2S_{22} & 0 \\ 0 & 0 & -S_{22} \end{pmatrix}, \tag{3.16}$$

$$\mathbf{M}^{(s)}\mathbf{S} + \mathbf{S}\mathbf{M}^{(s)} - \frac{2}{3}\mathrm{tr}(\mathbf{M}^{(s)}\mathbf{S})\mathbf{I} = \begin{pmatrix} A_{11}^{(s)} & 0 & A_{13}^{(s)} \\ 0 & A_{22}^{(s)} & 0 \\ A_{13}^{(s)} & 0 & A_{33}^{(s)} \end{pmatrix} \quad (s = 1, 3), \tag{3.17}$$

where

$$A_{11}^{(s)} = \frac{2}{3}\left(2S_{11}e_1^{(s)}e_1^{(s)} - S_{33}e_3^{(s)}e_3^{(s)} + S_{13}e_1^{(s)}e_3^{(s)}\right), \tag{3.18}$$

$$A_{22}^{(s)} = -\frac{2}{3}\left(S_{11}e_1^{(s)}e_1^{(s)} + S_{33}e_3^{(s)}e_3^{(s)} + 2S_{13}e_1^{(s)}e_3^{(s)}\right), \tag{3.19}$$

$$A_{33}^{(s)} = \frac{2}{3}\left(-S_{11}e_1^{(s)}e_1^{(s)} + 2S_{33}e_3^{(s)}e_3^{(s)} + S_{13}e_1^{(s)}e_3^{(s)}\right), \tag{3.20}$$

$$A_{13}^{(s)} = (S_{11} + S_{33})\, e_1^{(s)}e_3^{(s)} + S_{13}, \tag{3.21}$$

and

$$\mathbf{B}\mathbf{S} + \mathbf{S}\mathbf{B} - \frac{2}{3}\mathrm{tr}(\mathbf{B}\mathbf{S})\mathbf{I} = \begin{pmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22} & 0 \\ C_{13} & 0 & C_{33} \end{pmatrix} \quad (s = 1, 3), \tag{3.22}$$

with

$$C_{11} = \frac{2}{3}[S_{11}\,(2\lambda_1^2 + 2\kappa^2 + 1) - S_{33}\,(\lambda_1^{-2} - 1) + S_{13}\,\lambda_1^{-1}\kappa], \tag{3.23}$$

$$C_{22} = -\frac{2}{3}[S_{11}\,(\lambda_1^2 + \kappa^2 + 2) + S_{33}\,(\lambda_1^{-2} + 2) + 2S_{13}\,\lambda_1^{-1}\kappa], \tag{3.24}$$

$$C_{33} = \frac{2}{3}[-S_{11}\,(\lambda_1^2 + \kappa^2 - 1) + S_{33}\,(2\lambda_1^{-2} + 1) + S_{13}\,\lambda_1^{-1}\kappa], \tag{3.25}$$

$$C_{13} = (S_{11} + S_{33})\,\lambda_1^{-1}\kappa + S_{13}\,(\lambda_1^2 + \lambda_1^{-2} + \kappa^2). \tag{3.26}$$

In wiev of (3.16), (3.17) and (3.22), the constitutive law (3.2) gives for the shear strain-rate D_{13} the following relation:

$$\begin{aligned} 2\mu_0 D_{13} = \frac{1}{2}\,(S_{11} + S_{33})&\left[h(b_1)e_1^{(1)}e_3^{(1)} + h(b_1^{-1})e_1^{(3)}e_3^{(3)} + \frac{Q(K)}{K}\,\lambda^{-1}\kappa\right] \\ &+ \frac{1}{2}S_{13}\left[h(b_1) + h(b_1^{-1}) + \frac{Q(K)}{K}(\lambda_1^2 + \lambda_1^{-2} + \kappa^2)\right], \end{aligned} \tag{3.27}$$

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which involves three stress tensor components, S_{11}, S_{33}, and S_{13}. In order to express the shear strain-rate in terms of the shear stress alone, two more equations relating the three stress components are required. These two equations are obtained from (3.2) by determining two axial strain-rate components, say D_{11} and D_{33}, and then setting them to zero, since in the simple shear test all the axial strain-rates are zero due to lateral constraints imposed on a sample. Hence, for the axial components, Eq. (3.2) yields

$$
\begin{aligned}
(3.28)\qquad 2\mu_0 D_{11} = \frac{1}{3} S_{11} \Big[2h(b_1) e_1^{(1)} e_1^{(1)} + h(1) + 2h(b_1^{-1}) e_1^{(3)} e_1^{(3)} \\
+ \frac{Q(K))}{K}(2\lambda_1^2 + 2\kappa^2 + 1) \Big] + \frac{1}{3} S_{33} \Big[-h(b_1) e_3^{(1)} e_3^{(1)} + h(1) - h(b_1^{-1}) e_3^{(3)} e_3^{(3)} \\
- \frac{Q(K)}{K}(\lambda_1^{-2} - 1) \Big] + \frac{1}{3} S_{13} \Big[h(b_1) e_1^{(1)} e_3^{(1)} + h(b_1^{-1}) e_1^{(3)} e_3^{(3)} \\
+ \frac{Q(K)}{K} \lambda^{-1} \kappa \Big] = 0,
\end{aligned}
$$

$$
\begin{aligned}
(3.29)\qquad 2\mu_0 D_{33} = \frac{1}{3} S_{11} \Big[-h(b_1) e_1^{(1)} e_1^{(1)} + h(1) - h(b_1^{-1}) e_1^{(3)} e_1^{(3)} \\
- \frac{Q(K)}{K}(\lambda_1^2 + \kappa^2 - 1) \Big] + \frac{1}{3} S_{33} \Big[2h(b_1) c_3^{(1)} e_3^{(1)} + h(1) + 2h(b_1^{-1}) e_3^{(3)} e_3^{(3)} \\
+ \frac{Q(K)}{K}(2\lambda_1^{-2} + 1) \Big] + \frac{1}{3} S_{13} \Big[h(b_1) e_1^{(1)} e_3^{(1)} + h(b_1^{-1}) e_1^{(3)} e_3^{(3)} \\
+ \frac{Q(K)}{K} \lambda^{-1} \kappa \Big] = 0.
\end{aligned}
$$

Equations (3.28) and (3.29) provide two relations to eliminate S_{11} and S_{33} in terms of S_{13}, so that (3.27) becomes a relation between D_{13} and S_{13}. The latter relation, expressed in the form $2\mu_0 D_{13}/S_{13} = \eta_s/\eta_0$, describes the evolution of the normalised shear fluidity in terms of the shear strain κ. In the limit, as $\kappa \to \infty$ with λ_1 finite, then $b_1 \sim \kappa^2$ and $K \sim \kappa^2$, and, further, $\mathbf{e}^{(1)} \to (1,0,0)$ and $\mathbf{e}^{(3)} \to (0,0,1)$, so Eq. (3.27) implies that

$$
(3.30)\qquad \frac{\eta_s}{\eta_0} \to \frac{1}{2} h(0) + \frac{1}{2} h(\infty) + \frac{1}{2} Q(\infty) = E_s\,,
$$

where E_s is the shear enhancement factor.

The two relations (3.9) and (3.30) express the three limit values of the response functions, $h(0)$, $h(\infty)$ and $Q(\infty)$, in terms of the two enhancement factors

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for compression and shear (the other limit value of the function Q, at $K = 3$, is defined by Eq. (3.5)). In order to determine uniquely the three limit values we derive a third equation by following STAROSZCZYK and MORLAND [18], who derived a set of equalities and inequalities which should be satisfied by instantaneous directional viscosities μ_{ij} (reciprocal directional fluidities η_{ij} defined by Eq. (3.4), $\mu_{ij} = \eta_{ij}^{-1}$, $i, j = 1, 2, 3$, $i \neq j$). Their relations are based on the assumption that the alignment of ice crystal c-axes towards the direction of compression (and away from the direction of extension) depends on the relative magnitudes of the three principal stretches λ_r $(r = 1, 2, 3)$. The smaller a given principal stretch is compared to the other two stretches, the stronger is the alignment of c-axes towards the direction of this stretch and, therefore, the easier is the crystal basal gliding on the plane normal to this principal stretch axis (that is, the smaller is the corresponding shear viscosity). For any ordering of stretches λ_r, say $\lambda_1 \geq \lambda_2 \geq \lambda_3$, there are six distinct sets of relative values of λ_r, and for each of them corresponding relations order μ_{12}, μ_{13} and μ_{23} in the co-ordinate frame of the principal stretch axes λ_r. By using the viscosity relation corresponding to the plane flow, that is when $\lambda_2 = b_2 = 1$, and hence $b_3 = b_1^{-1}$ and $K = b_1 + 1 + b_1^{-1}$, it is possible to relate $Q(K)$ to $h(b_r)$ explicitly; namely by

$$Q(K) = -\frac{K}{b_1 - b_1^{-1}}[h(b_1) - h(b_1^{-1})], \tag{3.31}$$

where

$$2b_1 = K - 1 + \sqrt{(K-1)^2 - 4}\,, \geq 2. \tag{3.32}$$

The limit of (3.31) as $b_1 \to 1$, $K \to 3$, combined with the normalisation $(3.5)_2$, yields

$$h(1) - h'(1) = 1, \tag{3.33}$$

which is a restriction on $h(b_r)$ at $b = 1$. Further, the limit of (3.31) as $b_1 \to \infty$, when $K \sim b_1$, provides the relation

$$h(0) - h(\infty) - Q(\infty) = 0. \tag{3.34}$$

The system of three linear equations (3.9), (3.30) and (3.34) for $h(0)$, $h(\infty)$ and $Q(\infty)$ has a solution

$$h(0) = E_s\,, \quad h(\infty) = 6E_a - 5E_s\,, \quad Q(\infty) = 6(E_s - E_a), \tag{3.35}$$

which, together with the relation (3.33), defines the general properties which the response functions $h(b_r)$ and $Q(K)$ should satisfy in order that the reduced constitutive model (3.2) yields the limit responses observed in uniaxial compression and simple shear tests. More specific properties of the response functions

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should, ideally, be inferred from experimental results covering the whole range of axial and shear deformations which an ice sample undergoes as it develops anisotropy from its initial isotropic state, instead of using only the limit viscosities represented by the enhancement factors E_a and E_s, as has been done here. Unfortunately, such detailed experimental data are not available yet, therefore we adopt simple monotonic response functions satisfying (3.33) and (3.35) to explore the behaviour of ice as predicted by the orthotropic law (3.2).

4. Illustrations

For illustration purposes, the following response function $h(b_r)$ is adopted to investigate the creep behaviour of ice in uniaxial unconfined compression and simple shearing:

$$h(b_r) = h(\infty) - [h(\infty) - h(0)]\exp(-\alpha b_r^m), \quad \alpha > 0, \quad m > 0, \tag{4.1}$$

where m is a free parameter, and α is determined by the restriction (3.33). The other response function, $Q(K)$, is related to $h(b_r)$ by Eq. (3.31). Two kinds of ice are considered: so-called *warm ice* and *cold ice*. The former is the ice which is near the melting point, and the creep behaviour of such ice has been extensively tested at various stress levels by BUDD and JACKA [3]. Their results have shown that warm ice softens very considerably under both the compression and shear, and for large deformations the compression and shear enhancement factors approach the respective values of $E_a \approx 3$ and $E_s \approx 8$. However, it has turned out that in polar ice sheets the creep response of ice to compressive stresses is different from that observed for ice near melting, and its viscosity increases with axial deformation (which means that the enhancement factor for compression is less than unity). Recently, MANGENEY *et al.* [10] suggested the value $E_s \approx 1/3$ for ice near the bottom of the Greenland ice cap, evaluated on the basis of the data provided by THORSTEINSSON *et al.* [21]. The shear enhancement factor for ice at the bottom of the Greenland ice sheet near its divide (centre) has been calculated to be $E_s \approx 2.5$, though it seems that further away from the divide (where shear strains are much larger than at the divide) a higher value is more relevant. Hence, the chosen enhancement factors E_a and E_s for warm and cold ice and the related limit values of the response functions h and Q, defined by (3.4), are:

$$E_a = 3, \qquad E_s = 8 : \qquad h(0) = 8, \qquad h(\infty) = -22, \qquad Q(\infty) = 30, \tag{4.2}$$

$$E_a = 1/3, \qquad E_s = 5 : \qquad h(0) = 5, \qquad h(\infty) = -23, \qquad Q(\infty) = 28. \tag{4.3}$$

Plots of the selected response functions $h(b_r)$ for cold ice (for warm ice they are very similar) are presented in Fig. 1, in which labels indicate the curves

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corresponding to the values of the free parameter m in (4.1). The same labelling is applied in subsequent plots illustrating the creep behaviour of ice.

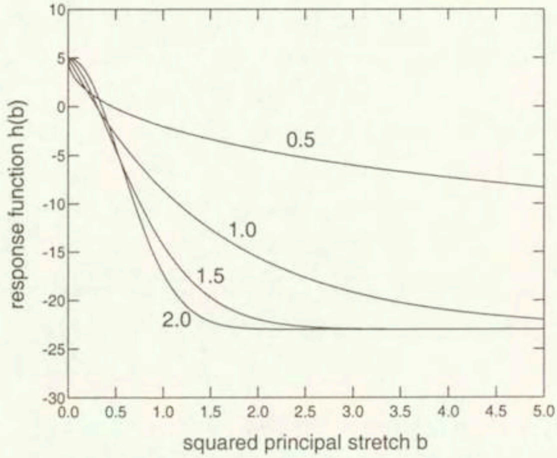

FIG. 1. Adopted forms of the fabric response function $h(b_r)$.

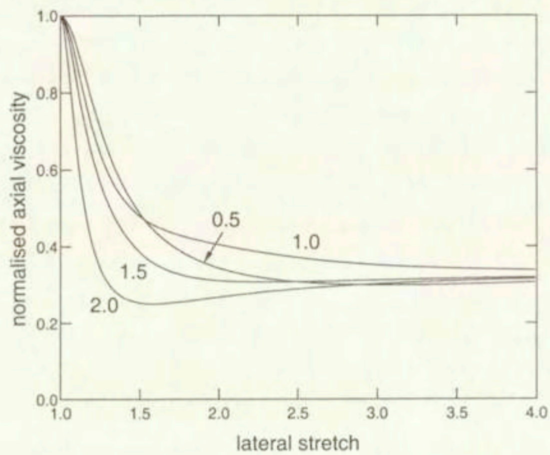

FIG. 2. Evolution of the normalized axial viscosity with increasing stretch λ_1 in uniaxial compression for different response functions $h(b_r)$ (warm ice).

The results of simulations carried out for warm ice are shown in Figs. 2 and 3. The response of ice to uniaxial compression is illustrated in Fig. 2, in which the evolution of the dimensionless axial viscosity $S_{11}/(2\mu_0 D_{11})$, the reciprocal of the axial fluidity described by Eq. (3.8), is shown for different values of the parameter m in the response function $h(b_r)$. Comparing these results with those given by the stress – strain-rate formulation of the constitutive law (STAROSZCZYK and MORLAND [18]), in which analogous response functions have been applied, we

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note that the present model predicts much faster softening of ice (decrease in viscosity with increasing deformation). The results of simple shear simulations are plotted in Fig. 3, illustrating the evolution of shear viscosity $S_{13}/(2\mu_0 D_{13})$, the reciprocal of the shear fluidity calculated from the relations (3.27) – (3.29). Comparison of these results with those obtained from the constitutive model [18] shows again that the inverse constitutive law predicts much faster softening of ice during its shearing, that is the limit shear viscosity defined by the enhancement factor E_s is now much faster approached as the shear deformation, started from an initially isotropic state of ice, proceeds.

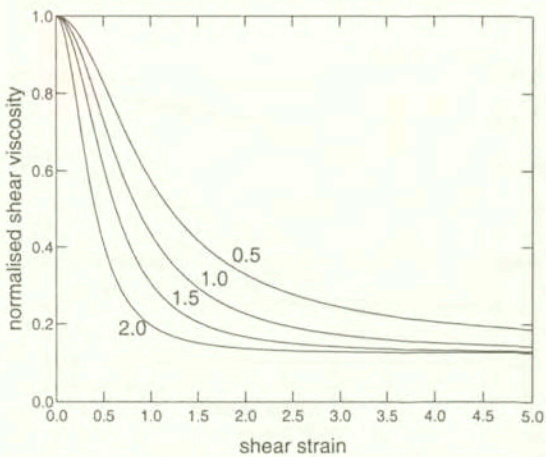

FIG. 3. Evolution of the normalised shear viscosity with increasing strain κ in simple shear started from an isotropic state for different response functions $h(b_r)$ (warm ice).

The creep behaviour of cold ice is illustrated in Figs. 4 to 7. Figure 4 shows the evolution of the axial viscosity for different values of the free parameter m in the function (4.1). It is clearly seen that the value of this parameter, particularly for $m \lesssim 1.5$, considerably affects the predicted response of cold ice to compressive stresses. Such sensitivity of the results to the adopted form of the response function, which is an undesirable feature of the constitutive model significantly restricting its flexibility, has not been observed in the case of the stress – strain-rate formulation [18], as shown by the results of simulations for cold ice presented by STAROSZCZYK and GAGLIARDINI [17]. Figure 5 demonstrates the evolution of the normalised shear viscosity of cold ice with increasing strain κ started from the isotropic state ($\lambda_1 = \lambda_2 = 1$). Contrary to the uniaxial compression, the results for simple shearing change smoothly with varying values of the free parameter m in the response function (4.1). Comparison of the shear viscosities yielded by the inverse model proposed here with the results predicted by the model [18] and presented in [17], shows that, alike the case of warm ice, the limit shear viscosity

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for indefinite shear strain is now approached faster. Additionally, we note that the present constitutive theory leads to the monotonic softening of cold ice with shear strain κ increasing from zero at the isotropic state, while the previous stress - strain-rate formulation [18] predicts initial hardening of ice, with maximum shear viscosities occurring at strains $\kappa \sim 1$, followed then by a progressive decrease in ice viscosity until the limit value, defined by the shear enhancement factor, is reached at large strains (STAROSZCZYK and GAGLIARDINI [17]).

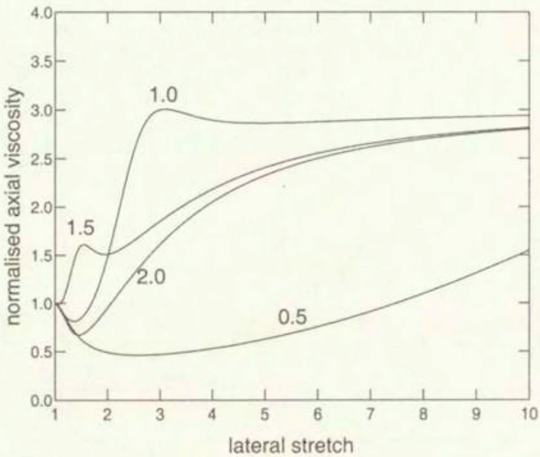

FIG. 4. Evolution of the normalised axial viscosity with increasing stretch λ_1 in uniaxial compression for different response functions $h(b_r)$ (cold ice).

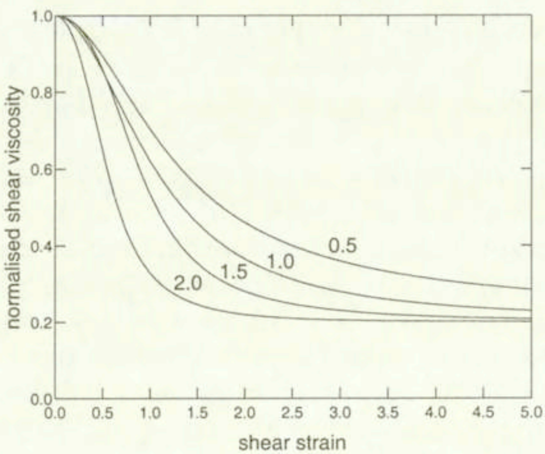

FIG. 5. Evolution of the normalised shear viscosity with increasing strain κ in simple shear started from an isotropic state for different response functions $h(b_r)$ (cold ice).

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Finally, in Figs. 6 and 7 we illustrate the behaviour of cold ice in simple shearing started from anisotropic states induced by an initial plane compression along the x_3 axis, defined by the stretches $\lambda_2 = 1$ and $\lambda_3 \leq 1$. Figure 6 shows, for different values of the free parameter m in the response function $h(b_r)$, the variation of the dimensionless shear viscosity $S_{13}/(2\mu_0 D_{13})$ with the strain κ for ice that has been axially pre-compressed to the stretch $\lambda_3 = \lambda_1^{-1} = 0.5$. Corresponding to the previous plot is Fig. 7, in which, for the function $h(b_r)$ with $m = 1$, the evolution of shear viscosity in simple shearing started from different anisotropic states defined by the axial stretches λ_3 is presented.

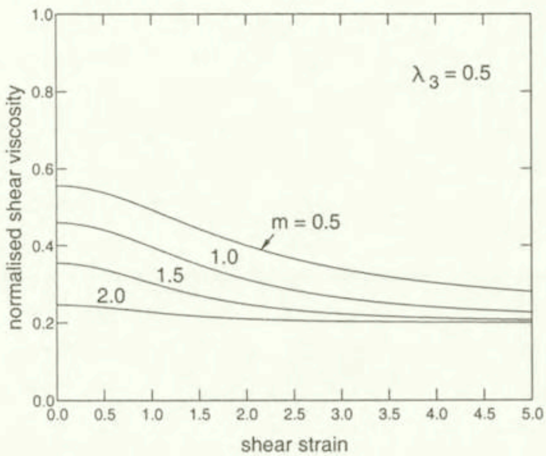

FIG. 6. Evolution of the normalised shear viscosity with increasing strain κ in simple shear started from an anisotropic state defined by $\lambda_3 = 0.5$ for different response functions $h(b_r)$ (cold ice).

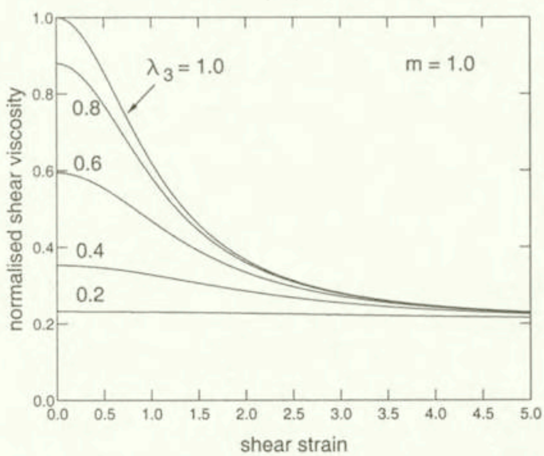

FIG. 7. Evolution of the normalised shear viscosity with increasing strain κ in simple shear started from different anisotropic states defined by λ_3 for the response function $h(b_r)$ with $m = 1.0$ (cold ice).

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5. Conclusions

In the paper an orthotropic constitutive law for viscous flow of ice has been formulated. The law expresses the strain-rate in terms of the deviatoric Cauchy stress, current deformation, and three structure tensors describing the evolving symmetric properties of the material. Since the results of laboratory tests on ice which have been conducted so far are insufficient to correlate the theory with experiment, simple forms of the response functions have been adopted in order to explore the predictions of the proposed theoretical model. The results of numerical simulations for maintained uniaxial compression and simple shear have been compared with the results given by the analogous stress – strain-rate model (STAROSZCZYK and MORLAND [18]). It has been found that the present strain-rate – stress formulation predicts much faster softening of the material during simple shear for both the warm and cold ice, as well as during uniaxial compression of warm ice. The results for the response of cold ice to uniaxial compression have shown that the present model is more sensitive to the specific forms of the response functions, which renders it less flexible than the former stress – strain-rate form of the law [18]. Before more definite conclusions have been drawn, however, the model response functions need to be correlated with the detailed experimental data once they are available, since at the moment these functions have been constructed only on the basis of the limit viscosities measured for indefinite deformations.

The proposed constitutive model can be used to describe the phenomenon of induced anisotropy in other materials, for which the current response is instantaneously viscous. It also seems that it is relatively simple to incorporate in the model other micro-mechanisms occurring in polycrystalline materials, for instance the process of dynamic (migration) recrystallisation, whose inclusion in discrete-grain or micro-macroscopic models is much more complex than in the case of the continuum approach pursued in this work.

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Received September 20, 2000.

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