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On material and geometrical instabilities in finite elasticity and elastoplasticity

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MATERIAL INSTABILITY phenomena arise in homogeneous stress states if nonlinear stress-strain relations are considered. The stability behaviour is investigated by looking at the Gateaux derivative of the first Piola-Kirchhoff stress tensor in the direction of the deformation gradient. This requires to solve a nine-dimensional matrix eigenvalue problem. In the present contribution, it is shown that material instabilities can be clearly differentiated from instabilities of geometrical character. The latter aspect is especially important for the design of new materials, since unstable solution paths under common loading conditions are not desirable. Geometrical instabilities, however, can usually be avoided by choosing appropriate boundary conditions. The derivation in this work leads to a simple stability criterion which allows to describe the stability behaviour of many materials in a very general context.

1. Introduction

MANY CLASSICAL stability investigations in elasticity concern the buckling of rods, plates and shells. A common example is an Euler column exposed to an axial force. At a certain load value, the homogeneous stress state is no longer stable which leads in the presence of imperfections to buckling of the structure into a stable inhomogeneous stress state. The treatment of such stability problems requires to consider large deformations or in other words, the geometrical nonlinearity in the system. For such so-called structural stability behaviour, the nonlinearity of the material law (physical nonlinearity) plays usually a subordinate role and is neglected in most cases. Structural instabilities occur mainly in compressive stress states.

For stability problems in finite elasticity, this is no longer the case. Here, large deformations and *nonlinear* stress-strain relations are considered. It can be expected that this increases the variety of instability phenomena noticeably. In particular, a new class of stability problems comes into play, even if the investigation is restricted completely to homogeneous stress states. Instabilities of such kind are usually due to the physical nonlinearity in the system. The question arises as to whether these so-called material instabilities are of physical origin, or only a result of the material model. Actually, one finds both cases. A physical material stability phenomenon was observed in an experiment documented by TRELOAR [41], where the symmetric deformation state of a biaxially equally loaded sheet became unstable and therefore inaccessible after a certain load level. Instead, the originally square sheet developed a rectangular form.

It is important to emphasize that the appearance of such material instabilities in finite elasticity does not mean that fundamental constitutive requirements such like polyconvexity (see BALL [2], CIARLET [10]) are violated. A good overview of constitutive inequalities is given in MARSDEN and HUGHES [22]. See for more details BAKER and ERICKSEN [1], HILL [14, 17, 18], OGDEN [26], BALL [2] and SIMPSON and SPECTOR [37, 38, 39]. One can show experimentally that these constitutive restrictions are physically reasonable for elastic materials.

This is different in elastoplasticity, where the bifurcation into shear bands is observed experimentally. The phenomenon can be explained mathematically by the loss of ellipticity of the underlying differential equation system. Thus, if one wishes to describe the behaviour of such materials realistically, the material model must be sophisticated enough to include shear banding (localization). Note that polyconvexity includes strong ellipticity such that the latter effect is excluded in finite elasticity. Such phenomena belong to another class of material instabilities which have to be additionally investigated in the context of elastoplasticity.

In many publications dealing with stability problems in finite elasticity, some special examples and material models are considered but only a few general conclusions are derived. Very common examples are for instance a biaxially loaded sheet (see e.g. SHIELD [35], OGDEN [26], KEARSLEY [20], CHEN [8], MÜLLER [24, 25], REESE [28], REESE and WRIGGERS [29]) or a triaxially loaded cube (see e.g. RIVLIN [31], SAWYERS [34], BALL and SCHAEFFER [3], REESE and WRIGGERS [30]). One main goal of the present work is to develop a stability criterion which would be general enough to reproduce the results of previous investigations in a very simple way. This criterion is based on a common aspect of the stability behaviour of elastic materials which has not been fully explored in earlier works. This is partially due to the fact that most investigations are based on the assumption of fixed principal axes which represents a major restriction and makes the complete understanding of the material stability behaviour more difficult. Exceptions are the quite formal derivations of HILL [16, 19], OGDEN [26], Sec. 6.2 and CHEN [9], who also developed general criteria. The results of HILL and OGDEN are similar to those derived in the present paper but based on a different derivation. The emphasis of the present work lies rather on the physical description of non-unique solutions. A further goal is to predict unstable solution paths and to be able to avoid them in practice. Since the present approach is not restricted to purely elastic material behaviour, it is more general than the one of OGDEN [26].

In contrast to other publications, including the ones of HILL [16, 19], OGDEN [26] and CHEN [9], in the present contribution, the classical split of the tangent stiffness into material and geometrical parts is exploited to obtain a useful classification of the instability phenomena into two groups. The second group is again subdivided into three different cases. It will be shown further, that the two main groups represent instabilities of geometrical and material character, respectively.

Although we restrict ourselves completely to homogeneous stress and deformation states, we still observe instabilities of purely geometrical character. Note that these are different from the typical structural instability phenomena (buckling), since they are associated with rigid body rotations as eigenmodes and occur usually in the natural state (undeformed configuration).

Concerning the group of material instabilities, we have to differentiate between purely elastic and elastoplastic material behaviour. In elasticity, material instabilities are characterized by the material tensor losing its positive definiteness. It is important to emphasize that these singularities represent *multiple* bifurcations. Such deformation states become unstable under arbitrary linear combinations of stretch and shear modes.

In elastoplasticity, material instabilities might arise in tension as well as in compression which is due to the fact that they usually appear in the form of shear bands. This aspect will be discussed in detail.

The paper is organized as follows. Based on the balance of linear momentum, the eigenvalue problem for the Gateaux derivative of the first Piola-Kirchhoff stress tensor in the direction of the deformation gradient is formulated. Using frame indifference, the latter fourth order tensor can be split into material and geometrical parts. In Sec. 3, this result is used to classify the instabilities into two main groups. In certain cases, in particular for isotropic elasticity and associated elastoplasticity, we can reformulate the eigenvalue problem for the tangent tensor in such a way that a decoupled structure is obtained. The originally nine-dimensional eigenvalue problem then reduces to one three-dimensional and three two-dimensional sub-problems (Sec. 4). Finally (Sec. 5), the solution of these eigenvalue problems is discussed in a very general context, i.e. neither a special material model nor certain deformation states are specified. In this way, generally applicable stability criteria are derived. In Sec. 6, the use of these criteria is validated by means of several examples.

2. Preliminary remarks

The derivation starts from the balance of linear momentum

$$\operatorname{Div} \mathbf{P} = \mathbf{0}, \tag{2.1}$$

where the absence of volumetric forces (e.g. due to gravitation or inertia) has been exploited and $\mathbf{P}$ denotes the first Piola-Kirchhoff stress tensor. Using the assumption that the strain and stress states are homogeneous, the integration of (2.1) leads to

$$\tilde{\mathbf{g}}(\mathbf{F}) = \tilde{\mathbf{P}}(\mathbf{F}) - \mathbf{P}_L = \mathbf{0}. \tag{2.2}$$

Here, $\mathbf{P}_L$ denotes the first Piola-Kirchhoff stress tensor known either from the tractions $\mathbf{T}_L = \mathbf{P}_L \cdot \mathbf{N}$ prescribed on the boundary $\partial\mathcal{B}_T$ or from the deformation given on $\partial\mathcal{B}_u$. The whole boundary of the reference volume $\mathcal{B}_0$ is given by $\partial\mathcal{B}_0 = \partial\mathcal{B}_T \cup \partial\mathcal{B}_u$. The tensor $\mathbf{F}$ represents the material deformation gradient. In the following, a colon will stand for the scalar product of two tensors, whereas one dot between two tensors characterizes the usual tensor multiplication. For simplicity, we start here with the special case of finite elasticity. Thus, we need to consider here only the dependence on $\mathbf{F}$. In order to account for the inelastic material behaviour, internal variables will be introduced. The extension of the derivation will be discussed in Sec. 4.2.

Due to the restriction to hyperelastic material behaviour, the stress tensor

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} \tag{2.3}$$

can be derived from a potential W, the so-called strain energy function per unit reference volume.

From the implicit function theorem (see Hildebrandt and Graves [13]) we obtain the result that the solution of (2.2) is regular, if the tensor $\dfrac{\partial \mathbf{g}}{\partial \mathbf{F}}$ is invertible, and singular otherwise. In the latter case, one can find a non-zero tensor $\Delta\mathbf{F}$ which fulfills the relation

$$\Delta\mathbf{g} := D\tilde{\mathbf{g}}(\mathbf{F}) : \Delta\mathbf{F} = \frac{d}{d\alpha}\left(\hat{\mathbf{g}}(\mathbf{F} + \alpha\,\Delta\mathbf{F})\right)\bigg|_{\alpha=0} = \frac{\partial \mathbf{g}}{\partial \mathbf{F}} : \Delta\mathbf{F} = \mathbf{0}. \tag{2.4}$$

The tensor $D\tilde{\mathbf{g}}(\mathbf{F}) : \Delta\mathbf{F}$ represents the so-called Gateaux derivative of $\tilde{\mathbf{g}}(\mathbf{F})$ in the direction $\Delta\mathbf{F}$, where $\Delta\mathbf{F}$ is identical with the eigentensor $\tilde{\mathbf{\Phi}}$ of the eigenvalue problem

$$\text{EIG:} \qquad (\mathcal{A} - \omega\,\mathbf{1}^4) : \tilde{\mathbf{\Phi}} = \mathbf{0},$$

if the eigenvalue ω vanishes. See in this context also Beatty [6, 7] and the text-book of Marsden and Hughes [22]. Note that the fourth order tensor

$$\mathcal{A} := \frac{\partial \mathbf{g}}{\partial \mathbf{F}} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} = \frac{\partial^2 \tilde{W}(\mathbf{F})}{\partial \mathbf{F}^2} \tag{2.5}$$

denotes the derivative of the first Piola-Kirchhoff stress tensor **P** with respect to the deformation gradient **F**. It can be stated that for asymmetrical bifurcations, symmetrical bifurcations and limit points, a singular solution always indicates the beginning of an unstable solution path. Thus, to check the stability behaviour on the "primary" solution path, the detection of singularities is sufficient.

Since the constitutive equations are required to be frame-indifferent (see e.g. TRUESDELL and NOLL [42]), the function $\tilde{\mathbf{P}}(\mathbf{F})$ reduces further to

$$\mathbf{P} = \mathbf{F} \cdot \breve{\mathbf{S}}(\mathbf{E}) = \mathbf{F} \cdot \frac{\partial \breve{W}(\mathbf{E})}{\partial \mathbf{E}}, \tag{2.6}$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1})$ represents the Green-Lagrange strain tensor. This fact has not been used to derive $\mathcal{A}$ and is therefore not included in (EIG). Using (2.6), we may rewrite the expression $\Delta \mathbf{g}$ as

$$\Delta \mathrm{g} = \mathcal{A} : \Delta \mathbf{F} = \Delta \mathbf{F} \cdot \breve{\mathbf{S}}(\mathbf{E}) + \mathbf{F} \cdot (D\breve{\mathbf{S}}(\mathbf{E}) : \Delta \mathbf{E}) = \Delta \mathbf{F} \cdot \breve{\mathbf{S}}(\mathbf{E}) + \mathbf{F} \cdot \left(\frac{\partial \mathbf{S}}{\partial \mathbf{E}} : \Delta \mathbf{E} \right), \tag{2.7}$$

where $\Delta \mathbf{E}$ is defined by the Gateaux derivative

$$\Delta \mathbf{E} := D\tilde{\mathbf{E}}(\mathbf{F}) : \Delta \mathbf{F} = \frac{\partial \mathbf{E}}{\partial \mathbf{F}} : \Delta \mathbf{F} = \mathrm{sym}(\mathbf{F}^T \cdot \Delta \mathbf{F}). \tag{2.8}$$

The fourth order tensor

$$\mathcal{L} := \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 \breve{W}(\mathbf{E})}{\partial \mathbf{E}^2} \tag{2.9}$$

is termed a material tensor. If we replace $\Delta \mathbf{F}$ by the eigentensor $\tilde{\mathbf{\Phi}}$ and carry out the scalar multiplication of (2.7) with $\tilde{\mathbf{\Phi}}$ we obtain in case of a singular solution

$$\boxed{\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = \mathrm{sym}(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}}) : \mathcal{L} : \mathrm{sym}(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}}) + \mathbf{S} : (\tilde{\mathbf{\Phi}}^T \cdot \tilde{\mathbf{\Phi}}) = 0.} \tag{2.10}$$

One could derive the result (2.10) also on the basis of the relations (6.2.76) and (6.1.22) stated in OGDEN [26]. A similar statement is found in HILL [19] (Eq. (3.38 a)), where one has to replace the spatial velocity gradient $\frac{\partial v_i}{\partial x_j}$ by $\tilde{\mathbf{\Phi}} \cdot \mathbf{F}^{-1}$. Neither OGDEN nor HILL, however, exploited the information contained in (2.10) any further. This is now done in the following section.

3. Classification of singular solutions

Using index notation, it can be easily shown that the term $\mathbf{S} : (\tilde{\mathbf{\Phi}}^T \cdot \tilde{\mathbf{\Phi}}) = S_{ij}\, \tilde{\Phi}_{ki}\, \tilde{\Phi}_{kj}$ is alternatively represented by means of

$$S_{ij}\,\tilde{\Phi}_{ki}\,\tilde{\Phi}_{kj} = \tilde{\Phi}_{ij}\,\delta_{ik}\,S_{jl}\,\tilde{\Phi}_{kl} := \tilde{\mathbf{\Phi}} : \mathcal{M} : \tilde{\mathbf{\Phi}}, \tag{3.1}$$

where the fourth order tensor $\mathcal{M}$ is computed from

$$\mathcal{M} = \delta_{ik}\,S_{jl}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{3.2}$$

The vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$ represent Cartesian basis vectors and the Einstein summation convention is assumed to hold for lower case indices. The Eq. (2.10) then reads

$$\boxed{\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = \underbrace{\mathrm{sym}\,(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}}) : \mathcal{L} : \mathrm{sym}\,(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}})}_{:=\,\omega_{\mathcal{L}}} + \underbrace{\tilde{\mathbf{\Phi}} : \mathcal{M} : \tilde{\mathbf{\Phi}}}_{:=\,\omega_{\mathcal{M}}} = 0.} \tag{3.3}$$

The scalar factors $\omega_{\mathcal{L}}$ and $\omega_{\mathcal{M}}$ represent the eigenvalues in the eigenvalue problems

$$(\mathcal{L} - \omega_{\mathcal{L}}\,\mathbf{1}^4_{\mathrm{sym}}) : \underbrace{\mathrm{sym}\,(\mathbf{F}^T \cdot \tilde{\mathbf{\Phi}})}_{:=\,\breve{\mathbf{\Phi}}} = \mathbf{0} \quad \text{and} \quad (\mathcal{M} - \omega_{\mathcal{M}}\,\mathbf{1}^4) : \tilde{\mathbf{\Phi}} = \mathbf{0}, \tag{3.4}$$

respectively.

Consider now a given deformation state $\mathbf{F} = \bar{\mathbf{F}}$ and assume that the tensor $\tilde{\mathcal{A}}\,(\bar{\mathbf{F}})$ has at least one vanishing eigenvalue. Let us further introduce the index $\star$ for all eigentensors of $\tilde{\mathcal{A}}\,(\bar{\mathbf{F}})$ which fulfill the equation $\tilde{\mathcal{A}}\,(\bar{\mathbf{F}}) : \tilde{\mathbf{\Phi}} = \mathbf{0}$. Then, for each eigentensor $\tilde{\mathbf{\Phi}}^\star$, (only) one of the following two statements is true.

(1) The eigentensor $\tilde{\mathbf{\Phi}}^\star$ satisfies the equation

$$\breve{\mathbf{\Phi}}^\star := \mathrm{sym}\,(\bar{\mathbf{F}}^T \cdot \tilde{\mathbf{\Phi}}^\star) = \mathbf{0}. \tag{3.5}$$

From (3.3), we obtain the relation

$$\tilde{\mathbf{\Phi}}^\star : \tilde{\mathcal{A}}\,(\bar{\mathbf{F}}) : \tilde{\mathbf{\Phi}}^\star = \tilde{\mathbf{\Phi}}^\star : \tilde{\mathcal{M}}\,(\bar{\mathbf{F}}) : \tilde{\mathbf{\Phi}}^\star = \omega_{\mathcal{M}} = 0. \tag{3.6}$$

Thus, $\tilde{\mathcal{M}}\,(\bar{\mathbf{F}})$ has one zero eigenvalue associated with the eigentensor $\tilde{\mathbf{\Phi}}^\star$.

(2) The eigentensor $\tilde{\mathbf{\Phi}}^\star$ does *not* satisfy (3.5). Let $\bar{\mathbf{E}}$ be defined by $\bar{\mathbf{E}} := \dfrac{1}{2}\,(\bar{\mathbf{F}}^T \cdot \bar{\mathbf{F}} - \mathbf{1})$.

(a) $\omega_{\mathcal{M}} = 0 \;\Rightarrow\; \omega_{\mathcal{L}} = 0$
The eigenvalue of $\breve{\mathcal{L}}\,(\bar{\mathbf{E}})$ associated with the eigentensor $\breve{\mathbf{\Phi}}^\star$ vanishes.

(b) $\omega_{\mathcal{M}} < 0 \;\Rightarrow\; \omega_{\mathcal{L}} > 0$

The eigenvalue of $\breve{\mathcal{L}}(\bar{\mathbf{E}})$ associated with the eigentensor $\breve{\mathbf{\Phi}}^{\star}$ is positive.

(c) $\omega_{\mathcal{M}} > 0 \Rightarrow \omega_{\mathcal{L}} < 0$

The eigenvalue of $\breve{\mathcal{L}}(\bar{\mathbf{E}})$ associated with the eigentensor $\breve{\mathbf{\Phi}}^{\star}$ is negative.

It remains to discuss what these cases mean physically. It is well-known that the split of the tangent matrix

$$A_{ijkl} = F_{im}\, L_{mjnl}\, F_{kn} + M_{ijkl}. \tag{3.7}$$

$(M_{ijkl} = \delta_{ik}\, S_{jl})$ represents a split into a material and a geometrical part. The latter equation is also given by HILL [19] and OGDEN [26] (Sec. 6.1.2).

In *Case 1*, the term $\breve{\mathbf{\Phi}} : \mathcal{L} : \breve{\mathbf{\Phi}}$, i.e. the material part of (3.3), vanishes. A singularity of this type is not related to the choice of the material model and can be considered therefore to be of purely geometrical character. The reversed argument allows the following statement. If $\mathcal{M}$ has a zero eigenvalue and the eigentensor accociated with the vanishing eigenvalue fulfills (3.5), the tensor $\mathcal{A}$ must be singular, too. It will become clear later that this case becomes relevant only in the natural state ($\mathbf{S} = \mathbf{0}$).

Usually, however, *Case 2* is detected, where (apart from 2a) either the first or the second term in (3.3) has a negative sign. The first class is characterized by $\mathcal{L}$ losing its positive definiteness (*Case 2c*). The second type arises, if $\mathcal{M}$ has at least one negative eigenvalue with respect to the eigenform $\tilde{\mathbf{\Phi}}$ (*Case 2b*).

To conclude, *Case 1* represents a geometrical instability, whereas *Case 2c* characterizes in any case a material instability. *Case 2a* would be both, but it does not appear in the context of physically reasonable models. *Case 2b* becomes relevant, if negative stresses dominate. For later use, we exploit the polar decomposition of $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ to rewrite (3.7) as

$$A_{ijkl} = R_{ix}\, \underbrace{(U_{xm}\, L_{mjnl}\, U_{yn} + \delta_{xy}\, S_{jl})}_{E^1_{xjyl}}\, R_{ky} \tag{3.8}$$

leading to

$$\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = (\mathbf{R}^T \cdot \tilde{\mathbf{\Phi}}) : \mathcal{E}^1 : (\mathbf{R}^T \cdot \tilde{\mathbf{\Phi}}). \tag{3.9}$$

In a similar way, the push-forwards $C_{ixky} = F_{im}\, F_{xj}\, F_{km}\, F_{yl}\, L_{mjnl}$ and $\tau_{xy} = F_{xj}\, S_{jl}\, F_{yl}$ ($\boldsymbol{\tau}$ Kirchhoff stress tensor) are used to reformulate (3.7) as

$$A_{ijkl} = (F^{-1})_{jx}\, \underbrace{(C_{ixky} + \delta_{ik}\, \tau_{xy})}_{E^2_{ixky}}\, (F^{-1})_{ly} \tag{3.10}$$

which yields

$$\tilde{\mathbf{\Phi}} : \mathcal{A} : \tilde{\mathbf{\Phi}} = (\tilde{\mathbf{\Phi}} \cdot \mathbf{F}^{-1}) : \mathcal{E}^2 : (\tilde{\mathbf{\Phi}} \cdot \mathbf{F}^{-1}). \tag{3.11}$$

If one of the three tensors, $\mathcal{A}$, $\mathcal{E}^1$ or $\mathcal{E}^2$, is singular, the same holds for the other two. The eigentensors of $\mathcal{E}^1$ and $\mathcal{E}^2$ associated with these singular solutions are, however, different. The eigentensors $\mathbf{\Phi}^1$ of $\mathcal{E}^1$ are obtained from the ones of $\mathcal{A}$ by multiplying $\tilde{\mathbf{\Phi}}$ by $\mathbf{R}^T$ from the left. Analogously, we compute the eigentensors $\mathbf{\Phi}^2$ of $\mathcal{E}^2$ by multiplying $\tilde{\mathbf{\Phi}}$ by $\mathbf{F}^{-1}$ from the right. The L_2-norm $||\mathbf{\Phi}^1||$ is set equal to $||\tilde{\mathbf{\Phi}}||$. On the other hand, we require the relation

$$||\mathbf{\Phi}^2|| = \sqrt{\mathbf{\Phi}^2 : \mathbf{\Phi}^2} = \sqrt{\mathrm{tr}\,(\tilde{\mathbf{\Phi}} \cdot \mathbf{C}^{-1} \cdot \tilde{\mathbf{\Phi}})} \tag{3.12}$$

to hold for $\tilde{\mathbf{\Phi}}^2$. Both tensors, $\mathcal{E}^1$ and $\mathbf{\Phi}^1$ live completely in the reference configuration, whereas $\mathcal{A}$ and $\tilde{\mathbf{\Phi}}$ are two-field tensors. $\mathcal{E}^2$ and $\mathbf{\Phi}^2$ live in the current configuration.

It will be shown later that it is convenient to work with $\mathcal{E}^1$ or $\mathcal{E}^2$ in some cases. The stability investigation can then be carried out in the following way:

- Solution of the eigenvalue problem

  $$\mathsf{EIG}^1: \qquad (\mathcal{E}^1 - \omega\, \mathbf{1}^4) : \mathbf{\Phi}^1 = \mathbf{0}$$

  or

  $$\mathsf{EIG}^2: \qquad (\mathcal{E}^2 - \omega\, \mathbf{1}^4) : \mathbf{\Phi}^2 = \mathbf{0},$$

  respectively.

- In case of a singular solution $\rightarrow$ case differentiation:

Case 1: The symmetric part of $\mathbf{U}^T \cdot \mathbf{\Phi}^1$ or $\mathbf{\Phi}^2$, respectively, is equal to the zero tensor.

Case 2: The symmetric part of $\mathbf{U}^T \cdot \mathbf{\Phi}^1$ or $\mathbf{\Phi}^2$, respectively, is **not** equal to the zero tensor. The sign of

$$\omega_{\mathcal{M}} = \Phi^1_{xj}\, \delta_{xy}\, S_{jl}\, \Phi^1_{yl} = \Phi^2_{ix}\, \delta_{ik}\, \tau_{xy}\, \Phi^2_{ky} \tag{3.13}$$

determines whether we deal with *Case 2a*, *Case 2b* or *Case 2c*.

4. Stability investigation

4.1. Finite elasticity

Up to this point, the derivation is general enough to include anisotropic material behaviour. The goal of the following derivation is to show the use of the stability criterion derived in Sec. 2. For simplicity, we restrict ourselves here to transverse isotropy. Transversely isotropic material behaviour is for instance observed in fiber-reinforced materials, where the material properties in the fiber direction are different from those in the plane perpendicular to the fiber. In this plane the material behaviour is assumed to be isotropic. The considerations carried out in the context of transverse isotropy would hold analogously for general anisotropy.

According to the theoretical works of BOEHLER [4, 5], the strain energy function $\breve{W}(\mathbf{E})$ reduces to an isotropic function of $\mathbf{E}$ and the so-called structural tensor $\mathbf{M} = \mathbf{n} \otimes \mathbf{n}$. The normed vector $\mathbf{n}$ is oriented parallel to the fibers. It is then straightforward to show that the potential W is a function of the three invariants

$$I_1 := \operatorname{tr} \mathbf{E}, \quad I_2 := \frac{1}{2}\left(I_1^2 - \operatorname{tr}(\mathbf{E}^2)\right), \quad I_3 := \det \mathbf{E} \tag{4.1}$$

of $\mathbf{E}$ and the first invariants of $\mathbf{E} \cdot \mathbf{M}$ and $\mathbf{E}^2 \cdot \mathbf{M}$, respectively:

$$I_4 := \operatorname{tr}(\mathbf{E} \cdot \mathbf{M}) = \mathbf{E} : \mathbf{M}, \qquad I_5 := \operatorname{tr}(\mathbf{E}^2 \cdot \mathbf{M}) = \mathbf{E}^2 : \mathbf{M}. \tag{4.2}$$

In general, $\mathbf{E}$ and $\mathbf{M}$ are not coaxial such that the 6 x 6-matrix obtained from writing $\mathcal{L}$ in the Voigt notation would be a full matrix. If, however, coaxiality of $\mathbf{S}$ with $\mathbf{E}$ or $\mathbf{U}$ is assumed, we may write the nine-dimensional matrix representation of $\mathcal{E}^1$ as

$$[\mathcal{E}^1] = \begin{bmatrix} [\mathcal{E}^1]^{\text{stretch}} & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathcal{E}^1]^{\text{shear}(12)} & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathcal{E}^1]^{\text{shear}(23)} & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & [\mathcal{E}^1]^{\text{shear}(31)} \end{bmatrix}, \tag{4.3}$$

where the sub-matrices $[\mathcal{E}^1]^{\text{stretch}}$ and $[\mathcal{E}^1]^{\text{shear}(12)}$ take the form

$$[\mathcal{E}^1]^{\text{stretch}} = \begin{bmatrix} \lambda_1^2 L_{1111} & \lambda_1 \lambda_2 L_{1122} & \lambda_1 \lambda_3 L_{1133} \\ \lambda_2 \lambda_1 L_{1122} & \lambda_2^2 L_{2222} & \lambda_2 \lambda_3 L_{2233} \\ \lambda_3 \lambda_1 L_{1133} & \lambda_3 \lambda_2 L_{2233} & \lambda_3^2 L_{3333} \end{bmatrix} + \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & S_{33} \end{bmatrix} \tag{4.4}$$

and

$$[\mathcal{E}^1]^{\text{shear}(12)} = \begin{bmatrix} \lambda_1^2 L_{1212} & \lambda_1 \lambda_2 L_{1212} \\ \lambda_2 \lambda_1 L_{1212} & \lambda_2^2 L_{1212} \end{bmatrix} + \begin{bmatrix} S_{22} & 0 \\ 0 & S_{11} \end{bmatrix}, \tag{4.5}$$

respectively. The matrices $[\mathcal{E}^1]^{\text{shear}(23)}$ and $[\mathcal{E}^1]^{\text{shear}(31)}$ are constructed analogously. The coefficients refer to the axes defined by $\mathbf{n}_I$ ($I = 1, 2, 3$, $\mathbf{n}_I \cdot \mathbf{n}_J = \delta_{IJ}$) with $\mathbf{n}_1 = \mathbf{n}$. The quantities λ_I ($I = 1, 2, 3$) are given by

$$\lambda_I := U_{II} = \sqrt{2\,E_{II} + 1}, \quad U_{IJ} = 0 \text{ if } I \neq J, \tag{4.6}$$

where U_{II} represent the principal values of the right stretch tensor $\mathbf{U}$.

Due to the special structure of $[\mathcal{E}^1]$, the nine-dimensional eigenvalue problem (EIG^1) can be subdivided into four smaller ones: one for the "stretch matrix" $[\mathcal{E}^1]^{\text{stretch}}$ and three others for the "shear matrices" $[\mathcal{E}^1]^{\text{shear}(12)}$, $[\mathcal{E}^1]^{\text{shear}(23)}$ and $[\mathcal{E}^1]^{\text{shear}(31)}$. This decoupled structure makes an analytical approach possible. Moreover, due to the fact that the latter two matrices are derived from $[\mathcal{E}^1]^{\text{shear}(12)}$ by merely exchanging the indices, we have to solve the two-dimensional eigenvalue problem only once. In isotropic elasticity, the coaxiality of $\mathbf{S}$ and $\mathbf{E}$ is always fulfilled such that the decoupled form of $[\mathcal{E}^1]$ is possible in general.

Many previous works (e.g. BALL and SCHAEFFER [3]) use the assumption that the principal axes do not rotate and can be chosen equal to the fixed Cartesian coordinate axes. In this special case, the left and the right stretch tensor, $\mathbf{V}$ and $\mathbf{U}$, have the same principal axes, and $\mathbf{R}$ is equal to the identity tensor. Then, also the matrix $[\mathcal{A}]$ would be sparse. Such an assumption, however, is here not necessary, since the coaxiality of $\mathbf{S}$ and $\mathbf{E}$ is already sufficient to obtain an analytically tractable form of $\mathcal{E}^1$.

An alternative approach has been given by OGDEN [26], who wrote the expression $\Delta\mathbf{g} = \Delta\tilde{\mathbf{P}}(\mathbf{F}) = \mathbf{0}$ (see the relation (2.4)) in the form

$$\Delta\tilde{\mathbf{P}}(\mathbf{F}) = \Delta(\mathbf{R}\cdot\mathbf{U}\cdot\check{\mathbf{S}}(\mathbf{E})) = \Delta(\mathbf{R}\cdot\check{\mathbf{T}}(\mathbf{U})) = \Delta\mathbf{R}\cdot\mathbf{T} + \mathbf{R}\cdot\Delta\mathbf{T} = \mathbf{0}. \tag{4.7}$$

The quantity $\mathbf{T}$ represents the so-called Biot stress tensor. From the latter equation one obtains the statement

$$\Delta\mathbf{T} = -\mathbf{R}^T\cdot\Delta\mathbf{R}\cdot\mathbf{T}. \tag{4.8}$$

The Biot stress tensor $\mathbf{T}$ is symmetric, if the coaxiality of $\mathbf{S}$ and $\mathbf{U}$ holds.

Since the coaxiality of $\mathbf{S}$ and $\mathbf{U}$ implies the coaxiality of $\boldsymbol{\tau}$ and $\mathbf{V}$, $[\mathcal{E}^2]$ has the same decoupled form. The sub-matrices then read

$$[\mathcal{E}^2]^{\text{stretch}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} \\ C_{1122} & C_{2222} & C_{2233} \\ C_{1133} & C_{2233} & C_{3333} \end{bmatrix} + \begin{bmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix} \tag{4.9}$$

and

$$[\mathcal{E}^2]^{\text{shear}(12)} = \begin{bmatrix} C_{1212} & C_{1212} \\ C_{1212} & C_{1212} \end{bmatrix} + \begin{bmatrix} \tau_{22} & 0 \\ 0 & \tau_{11} \end{bmatrix}. \tag{4.10}$$

4.2. Finite isotropic elastoplasticity

In the case of finite isotropic elastoplasticity with isotropic hardening, we start from the Helmholtz free energy

$$\Psi = W(\mathbf{C}_e) + f(\xi) = W(I_1^{\mathbf{C}_e}, I_2^{\mathbf{C}_e}, I_3^{\mathbf{C}_e}) + f(\xi) \tag{4.11}$$

where $\mathbf{C}_e$ is defined by $\mathbf{C}_e = \mathbf{F}_p^{-T} \cdot \mathbf{C} \cdot \mathbf{F}_p^{-1} = \mathbf{F}_e^T \cdot \mathbf{F}_e$ and $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = 2\,\mathbf{E} + \mathbf{1}$ denotes the right Cauchy-Green tensor. Note that here the multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$ into elastic and plastic parts has been exploited. Since the present model is restricted to isotropy, the function W depends only on the invariants of $\mathbf{C}_e$ which are identical to the invariants of $\mathbf{C} \cdot \mathbf{C}_p^{-1}$ ($\mathbf{C}_p = \mathbf{F}_p^T \cdot \mathbf{F}_p$ "plastic" right Cauchy-Green tensor). Then, besides the accumulated plastic strain ξ, the tensor $\mathbf{C}_p$ plays the role of an internal variable. We further define the yield function $\Phi(\boldsymbol{\tau}, q)$ which is formulated in terms of the invariants of the Kirchhoff stress tensor $\boldsymbol{\tau} = 2\,\dfrac{\partial W}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e$ and the stress-like quantity $q = \dfrac{\partial f(\xi)}{\partial \xi}$. Postulating the evolution equations ($\mathbf{b}_e = \mathbf{F} \cdot \mathbf{C}_p^{-1} \cdot \mathbf{F}^T$)

$$\mathbf{d}_p := \frac{1}{2}\,\mathbf{F}^{-T} \cdot \dot{\mathbf{C}}_p \cdot \mathbf{F}^{-1} = \dot{\gamma}\,\mathbf{b}_e^{-1} \cdot \frac{\partial \Phi}{\partial \boldsymbol{\tau}} \quad \text{and} \quad \dot{\xi} = \dot{\gamma}\,\frac{\partial \Phi}{\partial q} \tag{4.12}$$

and using the Kuhn-Tucker conditions $\dot{\gamma} \le 0$, $\Phi \le 0$ and $\dot{\gamma}\,\Phi = 0$, together with the consistency condition $\dot{\Phi}$ yields the plastic multiplier $\dot{\gamma}$ as

$$\dot{\gamma} = k\,\left(2\,\frac{\partial \Phi}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e\right) : \mathbf{d}, \tag{4.13}$$

where $k = k(\mathbf{b}_e, \xi)$ represents an isotropic function of its two arguments, and the partial derivative of Φ with respect to $\mathbf{b}_e$ is given via

$$\frac{\partial \Phi}{\partial \mathbf{b}_e} = \frac{\partial \Phi}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{b}_e}. \tag{4.14}$$

The deformation rate tensor $\mathbf{d}$ is computed from $\mathbf{d} = \dfrac{1}{2}\,\mathbf{F}^{-T} \cdot \dot{\mathbf{C}} \cdot \mathbf{F}^{-1}$. If we use the fact that the material behaviour is rate-independent, the time derivative $(...)^{\cdot}$ can be replaced by the derivative with respect to some arbitrary parameter s. In the following, we choose the notation

$$\frac{d(...)}{ds} := \Delta(...) \quad \text{and} \quad \mathbf{d}_\Delta := \frac{1}{2}\,\mathbf{F}^{-T} \cdot \Delta\mathbf{C} \cdot \mathbf{F}^{-1}, \qquad \mathbf{d}_{p\,\Delta} := \frac{1}{2}\,\mathbf{F}^{-T} \cdot \Delta\mathbf{C}_p \cdot \mathbf{F}^{-1}. \tag{4.15}$$

The evolution Eq. $(4.12)_1$ can finally be reformulated in the form

$$\mathbf{d}_{p\,\Delta} = k\,\left(\mathbf{b}_e^{-1} \cdot \frac{\partial \Phi}{\partial \boldsymbol{\tau}}\right) \otimes \left(2\,\frac{\partial \Phi}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e\right) : \mathbf{d}_\Delta = \mathcal{G} : \mathbf{d}_\Delta. \tag{4.16}$$

Note that the present model is based on a *hyperelastic* stress relation (see also Lubliner [21] and Simo and Hughes [36]), whereas the model discussed by Hill [16, 19] is formulated in rate form (*hypoelastic* stress relation).

The purpose of the present section is to show that the considerations made in Sec. 2 can be easily extended to a model of isotropic elastoplasticity as given in the form discussed above. In this context, (2.6) is rewritten as

$$\mathbf{P} = \mathbf{F} \cdot \frac{\partial W(I_1^{\mathbf{C}_e}, I_2^{\mathbf{C}_e}, I_3^{\mathbf{C}_e})}{\partial \mathbf{C}} = \mathbf{F} \cdot \mathbf{S}\,(\mathbf{C}, \mathbf{C}_p), \tag{4.17}$$

where $\mathbf{S}$ is represented by means of the function

$$\mathbf{S}\,(\mathbf{C}, \mathbf{C}_p) = \alpha_1\, \mathbf{C}^{-1} + \alpha_2\, \mathbf{C}_p^{-1} + \alpha_3\, \mathbf{C}_p^{-1} \cdot \mathbf{C} \cdot \mathbf{C}_p^{-1}. \tag{4.18}$$

The scalar factors α_i $(i = 1, 2, 3)$ are given as functions of the invariants of $\mathbf{C} \cdot \mathbf{C}_p^{-1}$. It is not difficult to see that the statements of the Secs. 2 and 3 concern the more general case of elastoplasticity, if the increment $\Delta\mathbf{S}$ is computed via

$$\Delta\mathbf{S} = \Bigg(\underbrace{\frac{\partial \mathbf{S}}{\partial \mathbf{E}}}_{\mathcal{L}^{\text{el}}} + 2\, \frac{\partial \mathbf{S}}{\partial \mathbf{C}_p} : \frac{\partial \mathbf{C}_p}{\partial \mathbf{C}}\Bigg) : \Delta\mathbf{E} = \mathcal{L}\,(\mathbf{C}, \mathbf{C}_p, \xi) : \Delta\mathbf{E} \tag{4.19}$$

and the material tensor in (2.9) is replaced by the tensor $\mathcal{L}$ of the latter relation. Obviously, unlike in finite elasticity, $\mathcal{L}$ does not necessarily possess the symmetry property $L_{ijkl} = L_{klij}$. We will derive the fourth-order tensor $\mathcal{C}$ which has the same symmetry properties as $\mathcal{L}$ and is given by

$$\mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T = \mathcal{C} : \mathbf{d}_\Delta. \tag{4.20}$$

Using (4.18), the increment $\mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T$ is determined by

$$\begin{aligned} \mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T = {} & \Delta\alpha_1\, \mathbf{1} + \Delta\alpha_2\, \mathbf{b}_e + \Delta\alpha_3\, \mathbf{b}_e^2 \\ & -2\,\alpha_1\, \mathbf{d}_\Delta + 2\,\alpha_3\, \mathbf{b}_e \cdot \mathbf{d}_\Delta \cdot \mathbf{b}_e - 2\,\alpha_2\, \mathbf{b}_e \cdot \mathbf{d}_{p\,\Delta} \cdot \mathbf{b}_e \\ & -2\,\alpha_3\, (\mathbf{b}_e \cdot \mathbf{d}_{p\,\Delta} \cdot \mathbf{b}_e^2 + \mathbf{b}_e^2 \cdot \mathbf{d}_{p\,\Delta} \cdot \mathbf{b}_e). \end{aligned} \tag{4.21}$$

After a longer derivation, we obtain $\mathcal{C}$ as

$$\begin{aligned} \mathcal{C} = {} & \mathbf{1} \otimes \mathbf{B}_1 - \mathbf{1} \otimes (\mathbf{b}_e \cdot \mathbf{B}_1) : \mathcal{G} + \mathbf{b}_e \otimes \mathbf{B}_2 - \mathbf{b}_e \otimes (\mathbf{b}_e \cdot \mathbf{B}_2) : \mathcal{G} \\ & + \mathbf{b}_e^2 \otimes \mathbf{B}_3 - \mathbf{b}_e^2 \otimes (\mathbf{b}_e \cdot \mathbf{B}_3) : \mathcal{G} - 2\,\alpha_1\, \mathbf{1}^4_{\text{sym}} + 2\,\alpha_3\, \mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e) \\ & -2\,\alpha_2\, \mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e) : \mathcal{G} - 2\,\alpha_3\, (\mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e^2) + \mathcal{X}\,(\mathbf{b}_e^2, \mathbf{b}_e)) : \mathcal{G}, \end{aligned} \tag{4.22}$$

where the equation

$$\Delta\alpha_i = \underbrace{2\, \frac{\partial \alpha_i}{\partial \mathbf{b}_e} \cdot \mathbf{b}_e}_{:=\mathbf{B}_i} : (\mathbf{d}_\Delta - \mathbf{b}_e \cdot \mathbf{d}_{p\,\Delta}), \tag{4.23}$$

has been exploited. The coefficients of $\mathcal{X}(\mathbf{Y},\mathbf{Z})$ with respect to Cartesian coordinates read

$$X_{ijkl} = \frac{1}{2}\,(Y_{ik}\,Z_{lj} + Y_{il}\,Z_{kj}). \tag{4.24}$$

Since the tensors $\mathbf{B}_i$ represent isotropic functions of $\mathbf{b}_e$, the Voigt notation of $\mathcal{C}$ with respect to the principal axes of $\mathbf{b}_e$ takes the sparse form referred to in Sec. 4.1. Clearly, this is not true for $\mathcal{L}$, so that it suggests to carry out the stability investigation by means of the solution of (EIG^2).

The derivation in Sec. 2 was based on the assumption that $\mathcal{E}^2$ has the major symmetry $E^2_{ijkl} = E^2_{klij}$. Obviously, this holds for the second term of E^2_{ijkl}, $\delta_{ik}\,\tau_{jl}$, independently of the material model chosen. The material tensor $\mathcal{C}$, however, as well as $\mathcal{L}$, possesses the major symmetry only in special cases. One of these cases is obtained, if we use the von Mises yield function (σ_Y yield stress)

$$\Phi = ||\operatorname{dev}\boldsymbol{\tau}|| - (\sigma_Y - q) \tag{4.25}$$

and the neo-Hookean elasticity relation (μ shear modulus, Λ Lamé constant, $J_e^2 = \det \mathbf{b}_e$)

$$\boldsymbol{\tau} = \mu\,(\mathbf{b}_e - \mathbf{1}) + \frac{\Lambda}{2}\,(J_e^2 - 1)\,\mathbf{1}. \tag{4.26}$$

The scalar factors α_i and the tensors $\mathbf{B}_i$ ($i = 1,2,3$) take here the forms

$$\alpha_1 = -\mu + \frac{\Lambda}{2}\,(J_e^2 - 1), \quad \alpha_2 = \mu, \quad \alpha_3 = 0 \tag{4.27}$$

and

$$\mathbf{B}_1 = \Lambda\,J_e^2\,\mathbf{1}, \quad \mathbf{B}_2 = \mathbf{B}_3 = \mathbf{0}. \tag{4.28}$$

Using $\operatorname{dev}\boldsymbol{\tau} = \mu \operatorname{dev}\mathbf{b}_e$, the tensor $\mathcal{G}$ reads

$$\mathcal{G} = \tilde{k}\ (\mathbf{b}_e^{-1}\cdot\operatorname{dev}\mathbf{b}_e)\otimes(\operatorname{dev}\mathbf{b}_e\cdot\mathbf{b}_e), \tag{4.29}$$

where

$$\tilde{k} = k\,\frac{4\,\mu}{\operatorname{dev}\mathbf{b}_e : \operatorname{dev}\mathbf{b}_e} \tag{4.30}$$

represents another isotropic function of $\mathbf{b}_e$ and ξ. We finally obtain for $\mathcal{C}$ the expression

$$\begin{aligned}\mathcal{C} = \Lambda\,J_e^2\,\mathbf{1}\otimes\mathbf{1} - \tilde{k}\,\Lambda\,J_e^2\,(\mathbf{1}\otimes\mathbf{b}_e) : (\mathbf{b}_e^{-1}\cdot\operatorname{dev}\mathbf{b}_e)\otimes(\operatorname{dev}\mathbf{b}_e\cdot\mathbf{b}_e)\\ +(2\,\mu - \Lambda\,(J_e^2 - 1))\,\mathbf{1}^4_{\mathrm{sym}} - 2\,\mu\,\mathcal{X}(\mathbf{b}_e,\mathbf{b}_e) : \mathcal{G}.\end{aligned} \tag{4.31}$$

Due to $(\operatorname{dev}\mathbf{b}_e) : \mathbf{1} = 0$, the right-hand term of the first line vanishes. To check the symmetry of the right-hand term of the second line, we use index notation. By means of (4.24) one obtains

$$X_{ijkl}\, G_{klmn} = \frac{\tilde{k}}{2}\left((b_e)_{ik}\,(b_e)_{jl} + (b_e)_{il}\,(b_e)_{jk}\right) \tag{4.32}$$

$$\left((b_e^{-1})_{kx}\,(\operatorname{dev} b_e)_{xl}\,(\operatorname{dev} b_e)_{my}\,(b_e)_{yn}\right) = \tilde{k}\,(b_e)_{ix}\,(\operatorname{dev} b_e)_{xj}\,(\operatorname{dev} b_e)_{my}\,(b_e)_{yn}.$$

In tensor notation, this yields with $\mathbf{b}_e \cdot \operatorname{dev} \mathbf{b}_e = \operatorname{dev} \mathbf{b}_e \cdot \mathbf{b}_e$ the equation

$$\mathcal{X}\,(\mathbf{b}_e, \mathbf{b}_e) : \mathcal{G} = \tilde{k}\,(\mathbf{b}_e \cdot \operatorname{dev} \mathbf{b}_e) \otimes (\mathbf{b}_e \cdot \operatorname{dev} \mathbf{b}_e). \tag{4.33}$$

Thus, $\mathcal{C}$ has major as well as minor symmetries.

The "elastic" part of $\mathcal{C}$, $\mathcal{C}^{\mathrm{el}}$, is derived by carrying out the push-forward of $\mathcal{L}^{\mathrm{el}}$ by means of $\mathbf{F}$. In (4.31) the terms of $\mathcal{C}^{\mathrm{el}}$ can be easily identified as those terms which do *not* depend on $\mathcal{G}$. Taking into account the result (4.33), it is evident that the tensor $\mathcal{C} - \mathcal{C}^{\mathrm{el}}$ consists of dyadic products of the form $\mathbf{A}\,(\mathbf{b}_e, \xi) \otimes \mathbf{B}\,(\mathbf{b}_e, \xi)$. So only $\mathcal{C}^{\mathrm{el}}$ contributes to $[\mathcal{E}^2]^{\mathrm{shear}(12)}$ or, in other words, C_{1212} can be replaced by C^{el}_{1212}. In finite elasticity, L_{1212} is determined by formulating the relation between $\sum_{I=1}^{3} S_{II}\,(E_{11}, E_{22}, E_{33})\,\Delta(\mathbf{n}_I \otimes \mathbf{n}_I)$ and $\sum_{J=1}^{3} E_{JJ}\,\Delta(\mathbf{n}_J \otimes \mathbf{n}_J)$ (see OGDEN [26]). C_{1212} is given by

$$C_{1212} = \lambda_1^2\,\lambda_2^2\,L_{1212} = \lambda_1^2\,\lambda_2^2\,\frac{S_{11} - S_{22}}{\lambda_1^2 - \lambda_2^2} = \frac{\tau_{11}\,\lambda_2^2 - \tau_{22}\,\lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \tag{4.34}$$

if $\lambda_1^2 \neq \lambda_2^2$. In elastoplasticity, we exploit the fact that $\bar{\mathbf{S}} := \mathbf{F}_p \cdot \mathbf{S} \cdot \mathbf{F}_p^T$ represents an isotropic function of $\mathbf{C}_e$. The relation between $\sum_{I=1}^{3} \bar{S}_{II}\,(C_{e\,11}, C_{e\,22}, C_{e\,33})\,\Delta(\bar{\mathbf{n}}_I \otimes \bar{\mathbf{n}}_I)$ and $\dfrac{1}{2}\displaystyle\sum_{J=1}^{3} C_{e\,JJ}\,\Delta(\bar{\mathbf{n}}_J \otimes \bar{\mathbf{n}}_J)$ is then determined analogously to L_{1212} in finite elasticity. Let us call the result $\bar{L}^{\mathrm{el}}_{1212}$. Using further the equations

$$\mathbf{F} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^T = \mathbf{F} \cdot \mathbf{F}_p^{-1} \cdot \Delta\bar{\mathbf{S}} \cdot \mathbf{F}_p^{-T} \cdot \mathbf{F}^T = \mathbf{F}_e \cdot \Delta\bar{\mathbf{S}} \cdot \mathbf{F}_e^T \tag{4.35}$$

and

$$\mathbf{F}^{-T} \cdot \Delta\mathbf{C} \cdot \mathbf{F}^{-1} = \mathbf{F}^{-T} \cdot \mathbf{F}_p^{-T} \cdot \Delta\mathbf{C}_e \cdot \mathbf{F}_p^{-1} \cdot \mathbf{F}^{-1} = \mathbf{F}_e^{-T} \cdot \Delta\mathbf{C}_e \cdot \mathbf{F}_e^{-1} \tag{4.36}$$

which hold for *fixed* $\mathbf{F}_p$, it becomes clear that the coefficient C^{el}_{1212} is given by

$$C^{\mathrm{el}}_{1212} = \lambda_{1\,e}^2\,\lambda_{2\,e}^2\,\bar{L}^{\mathrm{el}}_{1212}. \tag{4.37}$$

We then obtain

$$C^{\mathrm{el}}_{1212} = \lambda_{1\,e}^2\,\lambda_{2\,e}^2\,\bar{L}^{\mathrm{el}}_{1212} = \lambda_{1\,e}^2\,\lambda_{2\,e}^2\,\frac{\bar{S}_{11} - \bar{S}_{22}}{\lambda_{1\,e}^2 - \lambda_{2\,e}^2} = \frac{\tau_{11}\,\lambda_{2\,e}^2 - \tau_{22}\,\lambda_{1\,e}^2}{\lambda_{1\,e}^2 - \lambda_{2\,e}^2}, \tag{4.38}$$

if $\lambda_{1\,e}^2 \neq \lambda_{2\,e}^2$.

To conclude this section, it should be emphasized that the stability investigation based on (EIG^2) seems to be the most convenient, since it is analytically tractable in both cases, finite elasticity and finite elastoplasticity. The results of OGDEN [26] are directly recovered by investigating (EIG^1). After all, it is certainly possible to work with either one of the three eigenvalue problems (EIG), (EIG^1) or (EIG^2). The results are equivalent. But the goal is to derive an analytical stability criterion which is suitably achieved with (EIG^2).

5. Solution of the eigenvalue problem (EIG^2)

In the following, we derive the conditions for which $[\mathcal{E}^2]^{\text{shear}(12)}$ and $[\mathcal{E}^2]^{\text{stretch}}$ are singular. Further, we classify these singular solutions according to the cases listed in Sec. 2.

The analysis presented in this paper is based on the assumption $\partial\mathcal{B}_0 = \partial\mathcal{B}_T$ which means that tractions are prescribed on the whole boundary. Displacement boundary conditions reduce the size of the eigenvalue problem, since not all possible eigenmodes are consistent with the boundary conditions. This usually simplifies the stability investigations but does not lead to new results. Therefore, it is not necessary to consider this case in the present work.

5.1. Solution of the two-dimensional sub-problem

With the shorthand notations $C = C_{1212}$, $\tau_I = \tau_{II}$ and $\Phi_{ij} = \Phi^2_{ij}$, the two-dimensional eigenvalue problem is rewritten as

$$\mathsf{EIG\text{-}SH(12)}: \left(\begin{bmatrix} C & C \\ C & C \end{bmatrix} + \begin{bmatrix} \tau_2 & 0 \\ 0 & \tau_1 \end{bmatrix} - \omega \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The eigenvalues are determined by

$$\omega_{1,2} = X \pm \sqrt{X^2 - \det\,[\mathcal{E}^2]^{\text{shear}(12)}}, \tag{5.1}$$

where X is a shorthand notation for the expression $X = C + \frac{1}{2}(\tau_1 + \tau_2)$. A necessary condition for vanishing of at least one eigenvalue is

$$\det\,[\mathcal{E}^2]^{\text{shear}(12)} = C\,(\tau_1 + \tau_2) + \tau_1\,\tau_2 = 0 \tag{5.2}$$

leading to

$$\omega_1 = X + |X| \quad \text{and} \quad \omega_2 = X - |X|. \tag{5.3}$$

Thus, if X has a positive sign, ω_2 vanishes and ω_1 is equal to $2\,X$, i.e. positive. If X has a negative sign, ω_1 vanishes and ω_2 is equal to $2\,X$, i.e. negative. For $X = 0$, we have a double zero eigenvalue.

In finite elasticity, we have to differentiate between the two situations $\lambda_1 \neq \lambda_2$ (SH-R) and $\lambda_1 = \lambda_2 = \lambda$ (SH-S). In elastoplasticity, we have instead $\lambda_{1\,e} \neq \lambda_{2\,e}$ (SH-R) and $\lambda_{1\,e} = \lambda_{2\,e} = \lambda_e$ (SH-S), where $\lambda_{i\,e}^2$ $(i = 1, 2, 3)$ denote the eigenvalues of $\mathbf{b}_e$.

SH-R:
In the *regular* situation $\lambda_1 \neq \lambda_2$, the coefficient C is determined by

$$C = \frac{\tau_1\,\lambda_2^2 - \tau_2\,\lambda_1^2}{\lambda_1^2 - \lambda_2^2}, \tag{5.4}$$

see e.g. OGDEN [26] for the case of finite elasticity. The same relation holds for isotropic elastoplasticity, if we choose the shorthand notation $\lambda_i = \lambda_{i\,e}$ which is assumed to hold from now on. Inserting the expression for C into (5.2) yields

$$\boxed{\text{C-SH-R} : \lambda_1 \neq \lambda_2 \;\Rightarrow\; \frac{\tau_1^2}{\lambda_1^2} - \frac{\tau_2^2}{\lambda_2^2} = 0.}$$

The condition (C-SH-R) is fulfilled either with $P := \dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_2}{\lambda_2}$ or $P := \dfrac{\tau_1}{\lambda_1} = -\dfrac{\tau_2}{\lambda_2}$. Taking the first possibility (C-SH-R-1) gives

$$\frac{1}{\lambda_1 + \lambda_2}\begin{bmatrix} P\,\lambda_2^2 & -P\,\lambda_1\,\lambda_2 \\ -P\,\lambda_1\,\lambda_2 & P\,\lambda_1^2 \end{bmatrix}\begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{5.5}$$

Thus, we obtain $\Phi_{12} = \dfrac{\lambda_1}{\lambda_2}\,f$, $\Phi_{21} = f$, where f represents an arbitrary factor. Since the symmetric part of $\boldsymbol{\Phi}$ is not zero, we have here *Case 2.*

The second choice (C-SH-R-2) leads to

$$\frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} P\,\lambda_2^2 & P\,\lambda_1\,\lambda_2 \\ P\,\lambda_1\,\lambda_2 & P\,\lambda_1^2 \end{bmatrix}\begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{5.6}$$

Here, the eigenform associated with the vanishing eigenvalue is described by $\Phi_{12} = -\dfrac{\lambda_1}{\lambda_2}\,f$, $\Phi_{21} = f$. This would be *Case 1*, if the principal stretches λ_1 and λ_2 were equal. Since this is not true, we detect again *Case 2.*

SH-S:
In the *special* situation $\lambda_1 = \lambda_2 = \lambda$, the coefficient C is determined by

$$C = \lim\nolimits_{\lambda_1^2 \to \lambda_2^2} \frac{\tau_1\,\lambda_2^2 - \tau_2\,\lambda_1^2}{\lambda_1^2 - \lambda_2^2} = \frac{\partial(\tau_1\,\lambda_2^2)}{\partial\lambda_1^2} - \frac{\partial(\tau_2\,\lambda_1^2)}{\partial\lambda_1^2} = \frac{1}{2}\,(C_{1111}^{\text{el}} - C_{1122}^{\text{el}}). \tag{5.7}$$

Note that in finite elasticity, $\mathcal{C}^{\mathrm{el}}$ is identical with $\mathcal{C}$. In elastoplasticity, this holds only for the shear part. The difference has an important effect on the stability behaviour as will be pointed out in Sec. 5.2.

From (5.2) we obtain

$$\boxed{\mathsf{C\text{-}SH\text{-}S}: \lambda_1 = \lambda_2 \;\Rightarrow\; \tau\left(2C+\tau\right) = 0,}$$

where the relation $\tau = \tau_1 = \tau_2$ has been used. There are two possibilities to fulfill (C-SH-S). The first (C-SH-S-1) is represented by the criterion

$$\text{(5.8)} \qquad \mathsf{CRIT:} \qquad C = -\frac{\tau}{2} \Rightarrow \quad \frac{1}{2}\begin{bmatrix} \tau & -\tau \\ -\tau & \tau \end{bmatrix}\begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The associated eigenform is $\Phi_{12} = \Phi_{21} = f$ (SH-12), a typical shear mode (see Fig. 1), i.e. we have *Case 2*. The condition (CRIT) has an important meaning in the context of the present stability investigation. We will come back to this point later.

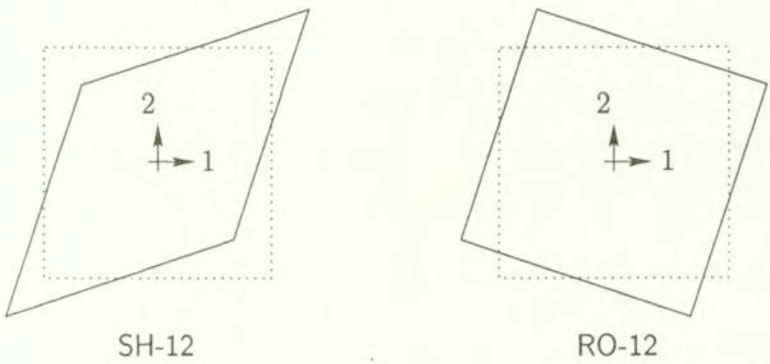

FIG. 1. Shear eigenmode (SH-12) and rotational eigenmode (RO-12).

The second possibility to fulfill (C-SH-S) is $\tau = 0$ (C-SH-S-2) leading to

$$\text{(5.9)} \qquad \begin{bmatrix} C & C \\ C & C \end{bmatrix}\begin{Bmatrix} \Phi_{12} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

The associated eigenform is $\Phi_{12} = -\Phi_{21} = f$ (RO-12), a rotational eigenmode (see Fig. 1). The symmetric part of the eigentensor vanishes, we obtain *Case 1*! See Table 1 for a summary of the latter results.

For physically reasonable material models, τ vanishes only if the external loading is chosen in such a way that the boundaries normal to the 1- and the 2-axis are stress-free. This can be achieved for any kind of elastic material model showing the purely geometrical character of singularities belonging to *Case 1*. When we look at *Case 2*, however, it is important to specify a material model, firstly, in order to differentiate between the possibilities a–c and, secondly, to check whether fulfilling the conditions C-SH-R or C-SH-S is consistent with the physical and mathematical assumptions upon which the derivation of these equations is based.

Table 1. Results of the two-dimensional sub-problem.

- $\lambda_1 \neq \lambda_2$:

(1) $\dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_2}{\lambda_2}$ → *Case 2*

(2) $\dfrac{\tau_1}{\lambda_1} = -\dfrac{\tau_2}{\lambda_2}$ → *Case 2*

- $\lambda_1 = \lambda_2$:

(1) (CRIT) : (SH-12) → *Case 2*

(2) $\tau = 0$: (RO-12) → *Case 1*

5.2. Solution of the three-dimensional sub-problem

Before discussing the results of Sec. 5.1 in the context of a special material model, let us continue with the investigation of the three-dimensional sub-problem

$$\text{EIG-ST}: \quad \left(\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} \\ C_{1122} & C_{2222} & C_{2233} \\ C_{1133} & C_{2233} & C_{3333} \end{bmatrix} + \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix} - \omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \Phi_{11} \\ \Phi_{22} \\ \Phi_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

At this point, we have to differentiate between the elastic and the elastoplastic case. Consider first the special situation that all stretches are equal (ST-SS). In finite elasticity, this is a sufficient condition for the statements $\tau_1 = \tau_2 = \tau_3$, $C_{1111} = C_{2222} = C_{3333}$ and $C_{1122} = C_{1133} = C_{2233}$. In elastoplasticity, also the loading *history* has to be taken into account. Thus, we have to require in addition, that the stretches are equal at *any* time of the loading process. This is tacitly assumed in the following.

We investigate the two sitations $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ (ST-SS) as well as $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 \neq \lambda$ (ST-S).

ST-SS:

Elasticity. In the special situation $\lambda_1 = \lambda_2 = \lambda_2 = \lambda$ ($\Rightarrow \tau_1 = \tau_2 = \tau_3 = \tau$), the determinant of $[\mathcal{E}^2]^{\text{stretch}}$ reads

$$\det [\mathcal{E}^2]^{\text{stretch}} = H^3 + 2\,N^3 - 3\,H\,N^2 = 0, \tag{5.10}$$

where the shorthand notations

$$H = C_{1111} + \tau \quad \text{and} \quad N = C_{1122} \tag{5.11}$$

have been introduced. One evident solution of the latter equation is

$$N = H \quad \Leftrightarrow \quad C_{1111} + \tau = C_{1122}. \tag{5.12}$$

It is identical with (CRIT). Moreover, it is shown easily, that fulfillment of (CRIT) leads in the context of (EIG-ST) to a *two-fold* zero eigenvalue. The associated eigenforms are linear combinations of the stretch modes $\tilde{\Phi}_{11} = f$, $\tilde{\Phi}_{22} = -f$, $\tilde{\Phi}_{33} = 0$ (ST-12) and $\tilde{\Phi}_{11} = 0$, $\tilde{\Phi}_{22} = f$, $\tilde{\Phi}_{33} = -f$ (ST-23). Thus, also the third stretch mode $\tilde{\Phi}_{11} = -f$, $\tilde{\Phi}_{22} = 0$, $\tilde{\Phi}_{33} = f$ (ST-31) is a relevant eigenform. The third zero eigenvalue is obtained for $H = -2\,N$ and correlated with the eigenform $\tilde{\Phi}_{11} = f$, $\tilde{\Phi}_{22} = f$, $\tilde{\Phi}_{33} = f$ (ST-VOL). We obtain *Case 2.* The modes (ST-12) and (ST-VOL) are plotted in Fig. 2.

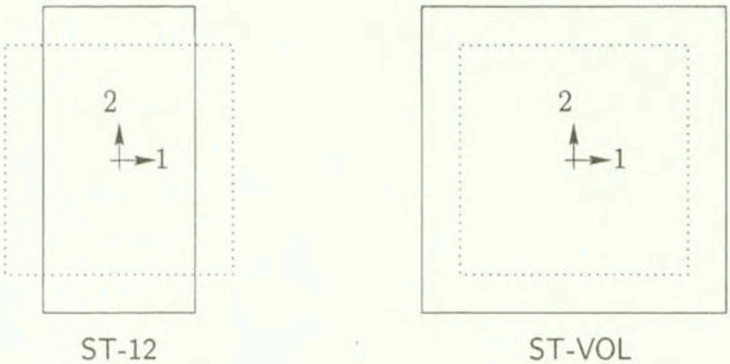

FIG. 2. Stretch modes (ST-12) and (ST-VOL).

Going back to the complete eigenvalue problem (EIG^2), we can make the following statement. If the condition (CRIT) is fulfilled and all three principal stretches are equal, we have a *five-fold* zero eigenvalue corresponding to linear combinations of the three stretch eigenmodes ST-12, ST-23 and ST-31 and the three shear eigenmodes SH-12, SH-23 and SH-31. See Table 1 and 2.

Elastoplasticity. The conditions for $[\mathcal{E}^2]^{\text{stretch}}$ becoming singular are computed in the same way as shown above. But we observe the following important fact. Due to the fact that $C_{iijj} \neq C^{\text{el}}_{iijj}$ $(i,j = 1,2,3)$ holds, the condition (5.12) is **not** identical with (CRIT). The "stretch" and the "shear" singularities do not occur simultaneously!

ST-S:

Elasticity. Let us investigate now the case $\lambda_1 = \lambda_2 = \lambda$ in combination with $\lambda_3 \neq \lambda$. The determinant $\det[\mathcal{E}^2]^{\text{stretch}}$ takes the form

$$\det[\mathcal{E}^2]^{\text{stretch}} = H_3\,H^2 + 2\,N\,N_3^2 - 2\,H\,N_3^2 - H_3\,N^2 = 0. \tag{5.13}$$

In the latter equation, the letters H_3 and N_3 stand for the expressions

$$H_3 = C_{3333} + \tau_3 \quad \text{and} \quad N_3 = C_{1133}. \tag{5.14}$$

Again, one possibility to satisfy $\det[\mathcal{E}^2]^{\text{stretch}} = 0$ is to fulfill (CRIT). The associated eigenmode is (ST-12) and as such related to *Case 2*. The other eigenvalues vanish, if the condition

$$H_3\,(H + N) - 2\,N_3^2 = 0 \tag{5.15}$$

is satisfied. Due to the fact that the form of the latter equation is crucially dependent on the material model, the discussion of (5.15) is postponed to Sec. 5.

In analogy to (ST-SS), fulfillment of (CRIT) leads here to a *two-fold* zero eigenvalue of $\mathcal{E}^2$. The relevant eigenforms are linear combinations of the modes (ST-12) and (SH-12) (see Table 1 and 2).

Elastoplasticity. We have the same situation as in the case of three principal stretches. See the summary in Table 3.

Table 2. Results of the three-dimensional sub-problem (elasticity).

- $\lambda_1 = \lambda_2 \neq \lambda_3$:

(CRIT) : (ST-12) → *Case 2*

- $\lambda_1 = \lambda_2 = \lambda_3$:

(CRIT) : (ST-12), (ST-23), (ST-31) → *Case 2*

Table 3. Results of the two- and the three-dimensional sub-problem (elastoplasticity).

- $\lambda_1 = \lambda_2 \neq \lambda_3$:

$C^{\text{el}}_{1111} - C^{\text{el}}_{1122} + \tau_1 = 0$ (CRIT)	:	(SH-12) → Case 2
$C_{1111} - C_{1122} + \tau_1 = 0$	:	(ST-12) → Case 2
further (distinct stretches)	:	see Table 1

- $\lambda_1 = \lambda_2 = \lambda_3$:

$C^{\text{el}}_{iiii} - C^{\text{el}}_{iijj} + \tau_i = 0$ (CRIT)	:	(SH-12), (SH-23), (SH-31) → Case 2
$C_{iiii} - C_{iijj} + \tau_i = 0$	:	(ST-12), (ST-23), (ST-31) → Case 2

The preceding derivation shows that the local stability investigations in finite elasticity in finite elastoplasticity can be carried out in a very general context. In the following, we will consider the conditions derived above in relation to common material models like the neo-Hookean and the Mooney-Rivlin model. Some of the results have already been achieved in earlier works and thus confirm the correctness of the following derivation.

5.3. Loss of strong ellipticity

In the preceding sections, it is assumed that the singularity of $\mathcal{E}^2$ is caused by the loss of positive definiteness of either $[\mathcal{E}^2]^{\text{stretch}}$, $[\mathcal{E}^2]^{\text{shear(12)}}$, $[\mathcal{E}^2]^{\text{shear(23)}}$ or $[\mathcal{E}^2]^{\text{shear(31)}}$. If we term the corresponding eigenmodes $\{\Phi^{\text{stretch}}\}$, $\{\Phi^{\text{shear(12)}}\}$, $\{\Phi^{\text{shear(23)}}\}$ or $\{\Phi^{\text{shear(31)}}\}$, respectively, the eigenvalues of $\mathcal{E}^2$ can be represented as

$$
\begin{aligned}
\omega = {} & \{\Phi^{\text{stretch}}\}^T [\mathcal{E}^2]^{\text{stretch}} \{\Phi^{\text{stretch}}\}^T \\
& + \{\Phi^{\text{shear(12)}}\}^T [\mathcal{E}^2]^{\text{shear(12)}} \{\Phi^{\text{shear(12)}}\}^T \\
& + \{\Phi^{\text{shear(23)}}\}^T [\mathcal{E}^2]^{\text{shear(23)}} \{\Phi^{\text{shear(23)}}\}^T \\
& + \{\Phi^{\text{shear(31)}}\}^T [\mathcal{E}^2]^{\text{shear(31)}} \{\Phi^{\text{shear(31)}}\}^T.
\end{aligned} \tag{5.16}
$$

Evidently, ω might vanish also, if the four terms on the right-hand side of (5.16) cancel each other. For simplicity, let us restrict ourselves to a two-dimensional investigation ($\lambda_3 = 1$), where the stretch matrix reduces to a 2 x 2-matrix and in addition, only the shear matrix $[\mathcal{E}^2]^{\text{shear(12)}}$ has to be considered. Then we may write

$$
\omega = \omega^{\text{stretch}} + \omega^{\text{shear(12)}}. \tag{5.17}
$$

The vanishing of ω with $\omega^{\text{stretch}} \neq 0$ is possible only if *either* ω^{stretch} *or* ω^{shear12} becomes negative. In elasticity, this is excluded for $\lambda_1 = \lambda_2$ but theoretically possible for $\lambda_1 \neq \lambda_2$. The change of sign of one of the eigenvalue parts requires that one of the matrices, $[\mathcal{E}^2]^{\text{stretch}}$ or $[\mathcal{E}^2]^{\text{shear(12)}}$, loses its positive definiteness. In other words, if the sub-matrices of $[\mathcal{E}^2]$ can be shown to be positive definite for any arbitrary deformation, singularities of the type $\omega^{\text{stretch}} = -\omega^{\text{shear(12)}}$ with $\omega^{\text{stretch}} \neq 0$ are also excluded.

In elastoplasticity the latter type of instability might occur, even if $\lambda_1 = \lambda_2$ holds. This is due to the fact that in the case of two equal stretches, ω^{stretch} and $\omega^{\text{shear(12)}}$ do not vanish simultaneously. The question arises as to which physical meaning such instabilities could have. It is well known that strong ellipticity of the underlying differential equation system is guaranteed if the so-called acoustic tensor $\mathbf{A} := \mathbf{N} \cdot \mathcal{A} \cdot \mathbf{N}$ ($\mathbf{N}$ arbitrary vector) is positive definite (see e.g. Hill [15], Petryk [27] for a detailed discussion). According to Hadamard [12], the singularity of $\mathbf{A}$ indicates a so-called stationary discontinuity. This means that an acceleration wave travels with zero speed through a material continuum. If $\mathbf{A}$ has at least one negative eigenvalue, one speaks of a wave with imaginary speed. One can show further that the loss of ellipticity is sufficient for a bifurcation into a shear band of the orientation $\mathbf{N}$. It is common to use the notion "localization" for this phenomenon, since the deformation is "localized" in some small interior subdomain of the specimen, whereas the boundary remains unperturbed.

The positive definiteness of $\mathbf{A}$ can be expressed also in the form

$$(\boldsymbol{\varphi} \otimes \mathbf{N}) : \mathcal{A} : (\boldsymbol{\varphi} \otimes \mathbf{N}) = (\boldsymbol{\varphi} \otimes \underbrace{\mathbf{N} \cdot \mathbf{F}^{-1}}_{\mathbf{n}}) : \mathcal{E}^2 : (\boldsymbol{\varphi} \otimes \mathbf{N} \cdot \mathbf{F}^{-1}) > 0. \tag{5.18}$$

Thus, we may state the following. If $\mathcal{E}^2$ has a zero eigenvalue associated with the eigentensor $\mathbf{\Phi}^{\text{loc}} = \boldsymbol{\varphi} \otimes \mathbf{n}$, localization takes place. The vectors $\boldsymbol{\varphi}$ and $\mathbf{n}$ will have components in at least two (plane strain) or all three coordinate directions in general. Since $\mathbf{\Phi}^{\text{stretch}} \neq \mathbf{0}$ **and** $\mathbf{\Phi}^{\text{shear}(12)} \neq \mathbf{0}$ holds, this is one of the cases, where $\omega = 0$ with $\omega^{\text{stretch}} = -\omega^{\text{shear}(12)} \neq 0$ is valid. Concerning the classification discussed at the end of Sec. 2, the latter type of singularity belongs to *Case 2*, since the symmetric part of $\boldsymbol{\varphi} \otimes \mathbf{n}$ vanishes only in the physically irrelevant situations $\mathbf{n} = \mathbf{0}$ or $\boldsymbol{\varphi} = \mathbf{0}$. For *Case 2b*, the scalar product $\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}$ must be negative (positive). This means that the stress component orthogonal to the shear band is negative (positive). Both cases are observed experimentally.

6. Examples

6.1. Equitriaxial loading ($\mathbf{P}_L = P_L\,\mathbf{1}$)

Elasticity. In order to remain as simple as possible, we first work with the neo-Hookean model which has been derived by Treloar [41] on the basis of micromechanical considerations. Generalization the original model for compressible material behaviour leads to a strain energy function of the form

$$W = \frac{\mu}{2}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \mu \ln J + \frac{\Lambda}{2}\,(J^2 - 1 - 2\ln J) \tag{6.1}$$

where $J = \det \mathbf{F} = \lambda_1\,\lambda_2\,\lambda_3$ denotes a shorthand notation for the determinant of the deformation gradient. Note that μ and Λ represent elasticity constants, μ being the shear modulus and Λ the Lamé constant.

We consider first the deformation state with equal principal stretches. The eigenvalues of $[\mathcal{E}^2]^{\text{stretch}}$ and $[\mathcal{E}^2]^{\text{shear}(12)}$ can then be evaluated analytically, e.g.

by means of Mathematica. Inserting the material constants $\mu = 1$ and $\Lambda = 10$, one obtains for the "stretch" eigenvalues

$$(6.2) \qquad \omega_1^{\text{stretch}} = 6 + \lambda^2 - 5\,\lambda^6 = \omega_2^{\text{stretch}} \text{ and } \omega_3^{\text{stretch}} = 6 + \lambda^2 + 25\,\lambda^6.$$

The "shear" eigenvalues take the form

$$(6.3) \qquad \omega_1^{\text{shear}(12)} = \omega_1^{\text{stretch}} \text{ and } \omega_2^{\text{shear}(12)} = -6 + \lambda^2 + 5\,\lambda^6.$$

The condition (CRIT) is rewritten as

$$(6.4) \qquad 6 + \lambda^2 - 5\,\lambda^6 = 0.$$

It is now clearly recognizable that fulfilling of (CRIT) leads simultaneously to the vanishing of $\hat{\omega}_1^{\text{stretch}}(\lambda)$, $\hat{\omega}_2^{\text{stretch}}(\lambda)$, $\hat{\omega}_1^{\text{shear}(12)}(\lambda)$, $\hat{\omega}_1^{\text{shear}(23)}(\lambda)$ and $\hat{\omega}_1^{\text{shear}(31)}(\lambda)$. Very interesting is the fact that these functions are even identical. For the present example, we detect the five-fold bifurcation point at $\lambda_{\text{crit}} = 1.061$.

The eigenvalue $\omega_3^{\text{stretch}}$ remains always positive, so that the eigenform (ST-VOL) never becomes relevant. The eigenvalue $\omega_2^{\text{shear}(12)}$ is associated with the rotational eigenmode (RO-12) and consequently changes its sign in the natural state $\tau = 0$ (C-SH-S-2), i.e. for $\lambda = 1$. Analogously, $\omega_2^{\text{shear}(23)}$ and $\omega_2^{\text{shear}(31)}$ change their signs also for $\tau = 0$ but the associated eigenforms are (RO-23) and (RO-31), respectively. To conclude, the deformation state $\lambda_I = \lambda$ is stable only for $1 < \lambda < \lambda_{\text{crit}}$. For all other stretch values, this solution branch is singular ($\lambda = 1$, $\lambda = \lambda_{\text{crit}}$) or unstable ($\lambda < 1$, $\lambda > \lambda_{\text{crit}}$).

It should be emphasized again that the singularity at $\lambda = 1$ occurs independently of the kind of the nonlinearly elastic material model chosen. This is the typical character of singularities of *Case 1*. The singularity for $\lambda = \lambda_{\text{crit}}$, however, can be clearly attributed to *Case 2c*, since $\omega_{\mathcal{M}}$ is positive. Thus, for this stretch value, the material tensors $\mathcal{C}$ or $\mathcal{L}$, respectively, have lost their positive definiteness indicating a material instability in the sense of Sec. 2. Such an instability can be avoided by choosing a different material model, e.g. the one characterized by a constant and positive definite material tensor.

Remark

The fact that the natural state becomes singular, if rigid body rotations are considered as eigenmodes, has already been discussed by Beatty [6, 7] and Fosdick [11]. For the purpose of restricting the class of eigenmodes, the first author introduces the so-called zero moment condition and later uses Korn's inequality in addition. In the context of the present work, additional constraints are not necessary any longer since singularities of *Case 1* can be clearly differentiated from the physically more meaningful *Case 2*. □

In order to compare the results of the present work with the calculations of BALL and SCHAEFFER [3], we derive from (CRIT) the expression

$$\tau = \mu\,(\lambda^2 - 1) + \frac{\Lambda}{2}\,(J^2 - 1) = \Lambda\,(J^2 - 1) - 2\,\mu = -2\,C = C_{1122} - C_{111} \tag{6.5}$$

leading to $\frac{\Lambda}{2}\,(J^2 - 1) = \mu\,(\lambda^2 - 1) + 2\,\mu$ and

$$\tau_{\text{crit}} = 2\,\mu\,\lambda^2. \tag{6.6}$$

In the limit of incompressiblity ($\Lambda/\mu \to \infty$, $J \to 1 \;\Rightarrow\; \lambda \to 1$), one obtains $\tau_{\text{crit}} \to 2\,\mu$, a result which is in agreement e.g. with RIVLIN [33] and BALL and SCHAEFFER [3].

Due to the fact that infinitely many linear combinations of the three stretch modes and the three shear modes become relevant for the singularity determined by (CRIT), also an infinite number of secondary branches run through this bifurcation point. Among these are six branches, where the deformation state can be either described as being plate-like or rod-like (see BALL and SCHAEFFER [3]). In a plate-like deformation state, two of the principal stretches are equal and larger than the third one. The rod-like deformation state is characterized by the opposite situation, i.e. the two equal principal stretches are smaller than the one in the perpendicular direction. One of the rod-like secondary branches ($\lambda_1 = \lambda_2 = \lambda < \lambda_3$) is depicted in Fig. 3.

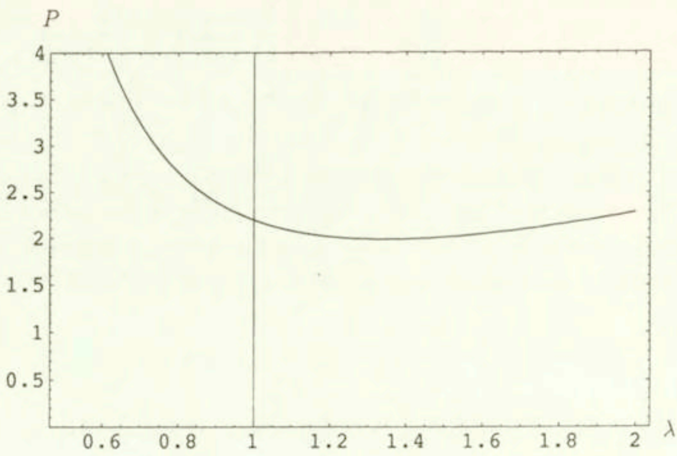

FIG. 3. Stress-stretch curve: $\lambda_1 = \lambda_2 = \lambda < \lambda_3$.

Application of the equality $P_{11} = P_{33}$ yields, with

$$\lambda_3 = \frac{1}{\lambda^3}\,\frac{\mu}{\Lambda}\left(1 \pm \sqrt{1 + (2\,\mu + \Lambda)\,\Lambda\,\lambda^2\,\frac{1}{\mu^2}}\right), \tag{6.7}$$

a relation between λ and λ_3, where the minus sign in (6.7) would apply for a plate-like branch. The determinant of $[\mathcal{E}^2]^{\text{stretch}}$ is plotted in Fig. 4.

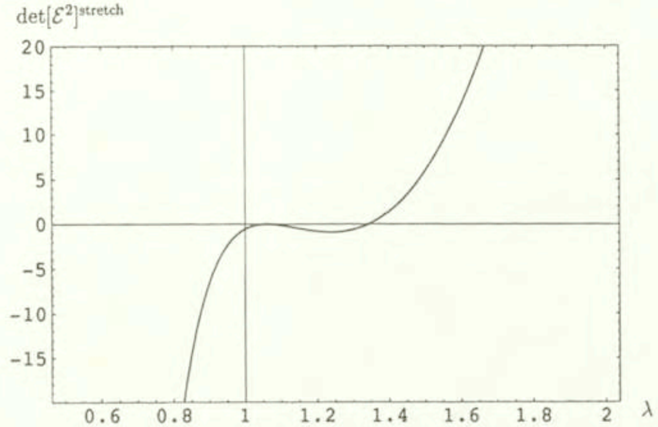

FIG. 4. $\det[\mathcal{E}^2]^{\text{stretch}}$ $(\lambda_1 = \lambda_2 = \lambda < \lambda_3)$.

At $\lambda = \lambda_{\text{crit}}$, we observe a double zero eigenvalue of $[\mathcal{E}^2]^{\text{stretch}}$. This point indicates the bifurcation back to the primary branch or to a plate-like secondary branch. Linear combinations of the stretch eigenforms would lead us to solution branches with three distinct principal stretches. The single zero eigenvalue at $\lambda = 1.343 = \lambda_{\text{lim}}$ is associated with the stress minimum in Fig. 3 and represents a singularity of *Case 2c*. So we do not detect any further bifurcation, since the condition (CRIT) cannot be fulfilled on this branch in the context of the neo-Hookean material model. Note that only for $\lambda > \lambda_{\text{lim}}$, $[\mathcal{E}^2]^{\text{stretch}}$ is positive definite.

The solution of the eigenvalue problem (EIG-SH(12)) yields one positive eigenvalue and one which is only positive for $\lambda > \lambda_{\text{crit}}$. The vanishing of the latter eigenvalue indicates again the bifurcation described by (CRIT). The natural state $\tau = 0$ (C-SH-S-2) is never reached on this secondary branch. Thus, the rotational mode (RO-12) does not become relevant and the eigenvalue associated with it remains positive. The eigenvalue problems (EIG-SH(23)) and (EIG-SH(31)) lead to one zero eigenvalue related to the solution $\dfrac{\tau_2}{\lambda_2} = \dfrac{\tau_3}{\lambda_3}$ or $\dfrac{\tau_3}{\lambda_3} = \dfrac{\tau_1}{\lambda_1}$ (C-SH-R-1), respectively. Due to the fact that $P_{ii} = \dfrac{\tau_i}{\lambda_i}$ holds in this example, (C-SH-R-1) is fulfilled on *all* secondary branches (even when all principal stretches are distinct). Since (C-SH-R-2) cannot be satisfied, the second eigenvalue of these eigenvalue problems is positive.

In order to compare the latter results with the literature, let us carry out the same calculation for the incompressible Mooney-Rivlin material model (MOONEY [23], RIVLIN [31, 32])

$$W = \sum_{R=1}^{2} \left[\frac{\mu_R}{\alpha_R} (\lambda_1^{\alpha_R} + \lambda_2^{\alpha_R} + \lambda_3^{\alpha_R} - 3) \right] + p\,(J-1) = \bar{W} + p\,(J-1) \tag{6.8}$$

where the elasticity constants

$$\mu_1 = \mu,\ \mu_2 = -\gamma\,\mu,\ \alpha_1 = 2,\ \alpha_2 = -2 \tag{6.9}$$

with the parameter γ have been introduced. Using $P_{11} = P_{33}$, the hydrostatic pressure p is derived from

$$p = \frac{1}{\lambda_2\,(\lambda_3 - \lambda_1)} \left(\frac{\partial \bar{W}}{\partial \lambda_3} - \frac{\partial \bar{W}}{\partial \lambda_1} \right). \tag{6.10}$$

With $\mu = 1$, $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \dfrac{1}{\lambda^2}$, the condition (CRIT) reduces to

$$\frac{(1-\lambda^3)\,(2\,\gamma - \lambda + \gamma\,\lambda^3)}{2\,\lambda^2} = 0 \tag{6.11}$$

resulting in the so-called bifurcation condtion

$$\hat{\gamma}\,(\lambda) = \frac{\lambda}{\lambda^3 + 2}. \tag{6.12}$$

The maximum of the function $\hat{\gamma}\,(\lambda)$ is $\gamma_{\max} = \dfrac{1}{3}$ for $\lambda = 1$. Thus, a bifurcation is detected only for $0 < \gamma \le \dfrac{1}{3}$. The same result can be found in BALL and SCHAEFFER [3]. The bifurcation for $\gamma = \dfrac{1}{3}$ occurs at $\lambda = 1$. It then falls together with the bifurcation from the primary path detected also by the term $1 - \lambda^3$ in (6.12). As already discussed before, fulfilling (CRIT) means that $[\mathcal{E}^2]^{\text{stretch}}$ as well as $[\mathcal{E}^2]^{\text{shear}(12)}$ are singular. The relevant eigenmodes are the stretch mode (ST-12) and the shear mode (SH-12). In contrast to the previous investigation based on the neo-Hookean material, (CRIT) indicates for the Mooney-Rivlin material a second bifurcation from the secondary branch into one with distinct principal stretches (if $0 < \gamma < \dfrac{1}{3}$). Evidently, this is a singularity of *Case 2c*.

Elastoplasticity. For the investigation of the elasto-plastic case, we use the Neo-Hooke model in combination with the von Mises yield function and the evolution equations (4.12) presented in Sec. 4.2.

Employing (4.30) and (4.32), we present the material tensor $\mathcal{C}$ in the form

$$\begin{aligned} \mathcal{C} = {} & 2\,\mu\,\mathbf{1}^4_{\text{sym}} + \Lambda\,J_e^2\,\mathbf{1}\otimes\mathbf{1} - \Lambda\,(J_e^2 - 1)\,\mathbf{1}^4_{\text{sym}} \\ & - 2\,\mu\,\tilde{k}\,(\mathbf{b}_e \cdot \operatorname{dev}\mathbf{b}_e) \otimes (\mathbf{b}_e \cdot \operatorname{dev}\mathbf{b}_e). \end{aligned} \tag{6.13}$$

Due to the fact that a purely deviatoric flow rule is assumed, there is no evolution of plastic deformation on the primary branch (all principal stretches equal, $\operatorname{dev}\mathbf{b}_e = \operatorname{dev}\mathbf{b} = \mathbf{0}$), the material behaves elastically.

On the secondary branch with $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$, plastic deformation might evolve, if the yield limit is reached. The evaluation of (CRIT) gives us the same information as above (no further bifurcation for the neo-Hookean case). In addition, we have to investigate the criterion $C_{1111} - C_{1122} + \tau_1 = 0$ which is fulfilled for

$$g(\lambda_e, \lambda_{3\,e}) = -\frac{\Lambda}{2}(J_e^2 - 1) + \mu(\lambda_e^2 + 1) - \frac{2}{9}\mu\tilde{k}\,\lambda_e^4(\lambda_e^2 - \lambda_{e\,3}^2)^2 = 0. \tag{6.14}$$

If the latter function intersects with $f(\lambda_e, \lambda_{e\,3}) = 0$ derived from $P_{11} = P_{33}$, a singular point is detected. Whether such an intersection takes place, depends crucially on the function $\tilde{k}$ which controls the influence of the elastoplastic material behaviour on the stability of some test sample. If such a singular point occurs, it is associated with stretch eigenmodes. One speaks of diffuse failure.

6.2. Equibiaxial loading ($P_{L\,ii} = P$ $(i = 1, 2)$, $P_{L\,33} = 0$, $P_{L\,ij} = 0$ with $i \neq j$)

Elasticity. Here, we start investigating the deformation state $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$. The relationship between λ_3 and λ is derived from the statement $P_{33} = 0$ leading to

$$\lambda_3 = \sqrt{\frac{\Lambda + 2\mu}{\Lambda\lambda^4 + 2\mu}}. \tag{6.15}$$

The three eigenvalues of $[\mathcal{E}^2]^{\text{stretch}}$ are all positive. The condition (CRIT) cannot be fulfilled with the neo-Hookean material model. As expected, one eigenvalue of $[\mathcal{E}^2]^{\text{shear}(12)}$ is equal to one of $[\mathcal{E}^2]^{\text{stretch}}$. The second eigenvalue is zero for the natural state (C-SH-S-2) indicating a singularity of *Case 1* and negative for $\lambda < 1$. The eigenvalues of $[\mathcal{E}^2]^{\text{shear}(23)}$ and $[\mathcal{E}^2]^{\text{shear}(31)}_{2\times 2}$ are positive which is due to the fact that neither $\dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_3}{\lambda_3}$ nor $\dfrac{\tau_1}{\lambda_1} = -\dfrac{\tau_3}{\lambda_3}$ (C-SH-R) can be fulfilled. To conclude, the solution path investigated above is stable for $\lambda > 1$, singular for $\lambda = 1$ and unstable for $\lambda < 1$. But we do not detect any bifurcation for $\lambda \neq 1$, a result which has already been stated in the literature (see e.g. KEARSLEY [20], CHEN [8] and MÜLLER [24, 25]).

Again, in order to compare with previous results let us examine the same example in the context of the incompressible Mooney-Rivlin material model. The hydrostatic pressure

$$p = -\frac{\partial\bar{W}}{\partial\lambda_3}\lambda_3 \tag{6.16}$$

is here derived from $P_{33} = 0$. The condition (CRIT) reduces to the very simple equation

$$\frac{1}{2\,\lambda^4}\,(1 + 3\,\gamma\,\lambda^2 + \lambda^6 - \gamma\,\lambda^8) = 0. \tag{6.17}$$

Evidently, for $\gamma = 0$ this equation can never be fulfilled, whereas for the parameters $\gamma > 0$, always a bifurcation into a path with $\lambda_1 \neq \lambda_2$ takes place (singularity of *Case 2c*). For $\gamma = 0.2$, this is the critical stretch $\lambda_{\text{crit}} = 2.27$. This result is in agreement with MÜLLER [24, 25]. The second eigenvalue of $[\mathcal{E}^2]^{\text{shear}(12)}$ is zero for $\lambda = 1$ (C-SH-S-2) indicating again a singularity of *Case 1*. Note that the secondary branch is singular due to $P_{11} = \dfrac{\tau_1}{\lambda_1} = \dfrac{\tau_2}{\lambda_2} = P_{22}$ (C-SH-R-1). To conclude, for the Mooney-Rivlin material, the primary deformation state is stable only for $1 < \lambda < \lambda_{\text{crit}}$, singular for $\lambda = 1$ or $\lambda = \lambda_{\text{crit}}$ and unstable for $\lambda < 1$ or $\lambda > \lambda_{\text{crit}}$.

Elastoplasticity. In elastoplasticity, the investigation is carried out analogously to the previous case. We obtain again the condition (6.14) which has to be compared for the present loading with $f\,(\lambda_e, \lambda_{e\,3}) = 0$ derived from $P_{33} = 0$.

7. Conclusions

One purpose of the present paper was to show that the material stability behaviour in finite elasticity and elastoplasticity follows a certain logic which has not been fully investigated in previous works. Important is the fact that the basic aspects can be described without specifying a material model. Similar investigations have been mainly carried out in the context of finite elasticity. The closest one is the approach of OGDEN [26] which exploits the coaxiality of the Biot stress tensor $\mathbf{T}$ and the right stretch tensor $\mathbf{U}$. In the present work, we investigate the eigenvalues of the tensor $\mathcal{E}^2$ with the coefficients $E^2_{ijkl} = C_{ijkl} + \delta_{ik}\,\tau_{jl}$. In this way, the stability investigation leads to the more general case of isotropic elastoplasticity, where $[\mathcal{E}^2]$ can be still written in the suitably decoupled form.

The main results of the present work are repeated in the following.

Elasticity:

1) If all stretches are equal, one finds a five-fold zero eigenvalue associated with arbitrary linear combinations of the three stretch modes and the three shear modes. The condition to obtain the zero eigenvalue is $C = -\dfrac{\tau}{2}$ (CRIT).

2) If the stability investigation is based on the assumption that two principal stretches are equal, one detects a double zero eigenvalue for $C = -\dfrac{\tau}{2}$ (CRIT) associated with one stretch mode and one shear mode.

The present examples were based on the so-called Ogden model (see OGDEN [26]). All singularities detected by means of (CRIT) were identified as material instabilities (*Case 2c*).

3) Independently of the material model chosen, we observe singularities of geometrical character (*Case 1*). Apart from the example 3.3 they occur in the natural state and are associated with rigid body rotations as eigenmodes. Obviously, these instabilities can be easily avoided by hindering the rotation of the system.

   It is interesting that *Case 2a* and *Case 2b* do not arise in the context of such common material models like the neo-Hookean and the Mooney-Rivlin material model. This fact confirms the introductory remark that geometrical instabilities (with the exception of *Case 1*) have usually a global character. As such, they do not arise, if the investigation is completely restricted to homogeneous deformation states.

Elastoplasticity:

1) If all stretches are equal, no plastic deformation evolves in the case of a deviatoric flow rule. We obtain elastic material behaviour.
2) In the case of two equal principal stretches, one detects a zero eigenvalue for $C = -\dfrac{\tau}{2}$ (CRIT) which is associated with a shear eigenmode and one for $C_{iiii} - C_{iijj} + \tau_i = 0$ (stretch eigenode). In contrast to elasticitiy, these singularities do not occur simultaneously.
3) As in elasticity, one observes singularities of purely geometrical character (*Case 1*).
4) Localization phenomena are characterized by eigenmodes of the form $\mathbf{\Phi}^{\mathrm{loc}} = \boldsymbol{\varphi} \otimes \mathbf{n}$.

These results enable a very general investigation of the material stability behaviour. Investigations of this kind could become necessary for two reasons. The first case is that instabilities are observed in an experiment. Then, the developed criteria such as e.g. (CRIT) serve to verify whether the developed material model is realistic enough to exhibit these physical effects. The second and usual case is that one wishes to avoid material instabilities. In such a case, the stability conditions derived in this paper are extremely useful to design a material and could be quite easily implemented in any optimization code.

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