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Prediction of mechanical behaviour of inhomogeneous and anisotropic materials using an incremental scheme

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A NEW MULTISITE homogenisation technique is proposed, suitable for the description of the effective properties of composite materials. After a short presentation of the multi-site approach, the traditional self-consistent approximation is developed. The limitations of such an approach are emphasised and a new alternative technique, based on the differential scheme idea, is proposed. The great versatility of this method in predicting the global elastic properties of various inhomogeneous materials is demonstrated. The present examples correspond to extreme situations of composites with voids and that of quasi-rigid (compared to the matrix) reinforcements. Very good agreement between the predicted and measured (or calculated by FEM) equivalent properties are found.

Notations

$\boldsymbol{\sigma}$, $(\boldsymbol{\Sigma})$	Cauchy local (global) stress tensor
$\boldsymbol{\varepsilon}$, $(\mathbf{E})$	local (global) strain tensor
$\mathbf{A}$, $(\mathbf{B})$	strain (stress) concentration tensor
$\mathbf{c}$, $(\mathbf{C})$	local (global) elasticity tensor
$\mathbf{u}$	local displacement field
$\mathbf{G}$, $(\boldsymbol{\Gamma})$	Green's (modified) tensor
$\mathbf{T}^{\mathrm{II}}$	interaction tensor
$\mathbf{I}$	fourth range unit tensor
$\mathbf{S}$	number of steps
$\mathbf{f}$, $\mathbf{f}_i$	global volume fraction, current volume fraction at step i
$\Delta\mathbf{f}$, $\Delta\mathbf{f}_i$	global increment volume fraction, current volume fraction at step i $\left(\Delta f = \frac{f}{S}\right)$

1. Introduction

THE SELF-CONSISTENT SCHEME (SCS) appears to be a powerful tool for prediction of the effective properties of polycrystalline materials, both in elasticity as well as in elasto-plasticity [1, 2, 3]. Some attempts of its direct application to the natural

or manufactured multi-phase composites have demonstrated the inadequacy of such an approach [4, 5, 6]. In fact, the aberrant behaviour is predicted when the reinforcement volume or mass fraction and/or the degree of inhomogeneity between this reinforcement and matrix becomes considerable [5, 6]. This is due to the inaccurate description of interactions between the reinforcement and the surrounding material introduced by the self-consistent approximation. In fact, in this approach to the inclusion modelling, an inhomogeneity is embedded in the effective material, the properties of which can be very different from those of the surrounding medium.

At least three approaches have been proposed to solve this problem. In the MORI-TANAKA model (MTS) [7], the inclusion interacts directly with a uniform matrix, which is subjected to an overall loading equal to the external loading applied to the heterogeneous material. To describe more accurately the interactions of the heterogeneity with the surrounding material, the idea of a "composite" inclusion has been developed. In this approach, the reinforcement is coated with a shell of the matrix material and the resulting composite is embedded in an infinite body with unknown properties of the equivalent homogeneous medium. This model was introduced by CHRISTENSEN and LO [8] for elastic materials and then extended to the nonlinear behaviour by HÉRVÉ and ZAOUI [9]. Recently, CHERKAOUI *et al.* [10] have proposed an alternative but approximate solution of the multi-coated inclusion.

Another approximate method, called the Differential Scheme (DS), was proposed by BRUGGEMAN [11] to predict the effective conductivity of inhomogeneous media. The DS is based on a progressive construction of the composite material. The effective properties of the composite are obtained by gradual addition of infinitesimal quantities of reinforcements. The actual effective properties are determined using the small concentration relations. To find the global properties of the material, a differential tensorial equation has to be solved. The method has been used and improved by many authors. ROSCOE [12] proposed an interpretation of the method based on the assumption that there exist the reinforcements of different size. The construction of the composite begins with insertion of the smallest inclusions and ends with the greatest ones. In this way, at each step, the application of the small concentration procedure is justified. BOUCHER [13] has used the DS to predict the behaviour of porous materials. He obtained a good agreement of the predicted properties with experimental results[1]. Works by MCLAUGHLIN [14], SALGANIK [15], LAWS and DVORAK [16], HASHIN [17] can be cited to illustrate the application of the DS in the field of viscous and cracked media. NORRIS [18] has emphasised that the incremental construction of

[1]It is well known that the SCS does not work at all for such materials when the volume fraction of voids exceeds 25%.

the multi-phase composite is not unique. Consequently, the predicted material properties are not exact. All the applications we have found in the literature are limited to the case of isotropic behaviour of the constituents. The form of these constituents is frequently limited to spherical, cylindrical or penny-shaped inclusions.

The purpose of this paper is to establish a new scheme for prediction of the effective elastic properties of composite materials. The general framework is that of the kinematic integral equation that appears as the formal solution for inhomogeneous materials. This equation will be briefly recalled in the next section where we will show that all existing models can be derived from this expression by adopting appropriate simplifications. In the next section we introduce, following FASSI-FEHRI [19], two tensors to describe the interactions between an inclusion and the surrounding material, and between two inclusions interacting through the matrix. These applications are used to construct one-site and multi-site approximations of the integral equation. The same approach has been applied by FASSI-FEHRI and BERVEILLER [20] for the self-consistent modelling of anisotropic materials with cubic symmetries. El MOUDEN [21] has used similar tensors to generalise the Mori-Tanaka model to the multi-site situation. In his approach the matrix has been supposed to be isotropic. ZATTARIN and LIPINSKI [22] have developed a general multi-site self-consistent scheme (MSCS) for a cluster or periodic arrangement of inclusions. Their results show relatively rapid deviation of the predicted properties from the experimental values for strongly inhomogeneous cases.

The approach developed in this paper is based on the DS idea and was first proposed by VIÉVILLE [23] and VIÉVILLE and LIPINSKI [6] for one-site approximations. We call it Multi-Site Incremental Scheme (MIS). It consists in the finite increment construction of the composite. In each increment, for which the small concentration solution is not valid, the self-consistent procedure is used to calculate the actual properties of the resulting material. The model takes into account the anisotropy of the constituents, the morphological texture of the composite as well as the spatial repartition of the inhomogeneities (topological texture).

The model is used to predict the effective properties of reinforced composites and porous materials. In both cases the calculated properties are close to the experimental data.

2. Preliminary remarks

Consider a representative volume (RV) of a macro-homogeneous and micro-inhomogeneous body in the sense introduced by HILL [1] and MANDEL [30]. We restrict our considerations to the case of linear elastic behaviour of the constitu-

tive phases and the small transformation approximation. The usual Hooke's law is supposed to be valid on the global and local levels. Let $\mathbf{E}$ be the global strain tensor and $\boldsymbol{\Sigma}$ be the overall stress tensor. Our goal is to determine the overall moduli of the body, such that

$$\Sigma_{ij} = C^{eff}_{ijkl} E_{kl}, \tag{2.1}$$

knowing the local Hooke's law at each point r of the RV

$$\sigma_{ij}(r) = c_{ijkl}(r)\varepsilon_{kl}(r), \tag{2.2}$$

where $\boldsymbol{\sigma}(r)$, $\boldsymbol{\varepsilon}(r)$ and $\mathbf{c}(r)$ are respectively the local stress, strain and elastic stiffness tensors. All these second-order tensors are symmetric and the above fourth-order tensors are characterised be the usual elasticity symmetries.

We suppose that all constituents are perfectly bonded together. Under this condition, we can link the macroscopic variables with microscopic ones by the volume averaging procedure over RV:

$$\Sigma_{ij} = \frac{1}{V}\int_V \sigma_{ij}(r)dV, \tag{2.3}$$

$$E_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}(r)dV. \tag{2.4}$$

Following MANDEL [30] and HILL [1], we introduce now a fourth-order concentration tensor $\mathbf{A}$ such that

$$\varepsilon_{ij}(r) = A_{ijkl}(r)E_{kl}. \tag{2.5}$$

Combining (2.2), (2.3), and (2.5) and comparing the resulting expression with (2.1), one can determine the effective properties of the RV

$$C_{ijkl} = \frac{1}{V}\int_V c_{ijmn}(r)A_{mnkl}(r)dV. \tag{2.6}$$

In this paper we use a general kinematic integral equation proposed by DEDERICHS and ZELLER [24]. This equation, what will be shown in the next section, enables to deduce an approximation of $\mathbf{A}$ and to develop new models. In the following we propose a multi-site approximation of this tensor.

3. Multi-Site Self-Consistent Scheme (MSCS)

3.1. Integral equation

Consider a heterogeneous body in equilibrium under the uniform boundary

conditions of the type

$$u_i(r) = E_{ij}x_j, \qquad \forall r \in S,$$

where $r = x_i e_i$ and S is the external surface of the body. When the body forces are neglected, the local equilibrium equation is written as follows:

$$\sigma_{ij,j} = 0,$$

and the displacement vector of any point of the medium can be expressed from the following integral equation proposed first by DEDERICHS and ZELLER [24]

$$u_m(r) = U_m^\circ(r) - \int_V G_{mj,i'}(r,r')\delta c_{ijkl}(r')u_{k,l'}(r')dV, \tag{3.1}$$

where $G_{mj}(r,r')$ is a Green tensor corresponding to the m-th component of the displacement vector at point r due to the j-th component of a unit force applied at r' of a fictitious homogeneous medium with elastic properties defined by $\mathbf{C}^0$. $U_m^\circ(r)$ describes the displacement field, solution of a corresponding auxiliary problem with homogeneous properties, and

$$\delta c_{ijkl}(r) = c_{ijkl}(r) - C_{ijkl}^\circ$$

defines the deviation of the local properties from $\mathbf{C}^0$.

The strain tensor field may now be calculated by differentiation of Eq. (3.1)

$$\varepsilon_{mn}(r) = E_{mn}^\circ - \int_V \Gamma_{mnij}(r,r')\delta c_{ijkl}(r')\varepsilon_{kl}(r')dV \tag{3.2}$$

where a new quantity, called a modified Green tensor, is introduced:

$$\Gamma_{mnij}(r,r') = \frac{1}{2}\left(G_{mj,i'n}(r,r') + G_{nj,i'm}(r,r')\right). \tag{3.3}$$

Expressions (3.1) and (3.2) constitute the formal solution of the heterogeneous body behaviour, and enable to deduce a form of the concentration tensor $\mathbf{A}(r)$. This tensor is obtained by a recurrent procedure (see for instance [3, 19]) which begins with the Voigt approximation of the strain field characterised by

$$\mathbf{A}(r) = \mathbf{A}^0(r) = \mathbf{I}, \tag{3.4}$$

$$\boldsymbol{\varepsilon}(r) = \mathbf{E} = \mathbf{E}^0, \tag{3.5}$$

where $\mathbf{I}$ is a fourth range unit tensor and $\mathbf{A}^0$ satisfies the equation

$$\boldsymbol{\varepsilon}(r) = \mathbf{A}^0(r){:}\mathbf{E}^0. \tag{3.6}$$

Introducing (3.5) into (3.2) one obtains a higher-order approximation. This procedure is repeated, leading to the following expression for the concentration tensor $\mathbf{A}^0(r)$:

$$A(r) = I + \int_V \Gamma(r,r'):\delta\mathbf{c}(r')dV - \int_V \Gamma(r,r'):\delta\mathbf{c}(r'):(\Gamma(r',r''):\delta\mathbf{c}(r'')dV)dV + \ldots .$$

The averaging operation applied to the expression (3.6) gives the link between $\mathbf{E}^0$ and $\mathbf{E}$, and consequently, the final result for the strain concentration tensor $\mathbf{A}^0(r)$ can be written as

$$\mathbf{A}(r) = \mathbf{A}^0(r):\left(\frac{1}{V}\int_V \mathbf{A}^0(r)dV\right)^{-1} . \tag{3.7}$$

In what follows we analyse the problem of a set of N ellipsoidal inclusions interacting through the homogeneous matrix. The solution of this problem constitutes a basis of the multi-site modelling of heterogeneous media, taking into account the effects of the spatial distribution of reinforcements upon the effective macroscopic behaviour of the material.

3.2. Multi-site modelling

Consider an infinite medium with elastic constants $\mathbf{C}^0$, containing N inclusions. Each inclusion I is characterised by its volume V^{I} and the elastic constants $\mathbf{c}^{\mathrm{I}}$. In this case, the field of the elastic properties of the material can be expressed as shown below:

$$\mathbf{c}(r) = \mathbf{C}^0 + (\mathbf{c}^1 - \mathbf{C}^0)\theta^1(r) + \ldots + (\mathbf{c}^I - \mathbf{C}^0)\theta^I(r) + \ldots + (\mathbf{c}^N - \mathbf{C}^0)\theta^N(r), \tag{3.8}$$

where θ^{I} is the characteristic function such that:

$$\theta^I(r) = \begin{cases} 1 & \forall r \in V^I, \\ 0 & \forall r \notin V^I. \end{cases} \tag{3.9}$$

Similar expressions are available for all N inclusions.

Let us introduce new notations

$$\Delta\mathbf{c}^I = \mathbf{c}^I - \mathbf{C}^0.$$

Integral equation (3.2) can now be presented as

$$\boldsymbol{\varepsilon}(r) = \mathbf{E}^0 - \int \boldsymbol{\Gamma}(r,r'): \left(\Delta\mathbf{c}^1\theta^1(r')\right.$$

$$\left. + \ldots + \Delta\mathbf{c}^I\theta^I(r') + \ldots + \Delta\mathbf{c}^N\theta^N(r')\right) : \boldsymbol{\varepsilon}(r')dV$$

which, using the properties of θ (3.9), can be rewritten as

$$\boldsymbol{\varepsilon}(r) = \mathbf{E}^0 - \int\limits_{V_1} \boldsymbol{\Gamma}(r,r'):\Delta\mathbf{c}^1:\boldsymbol{\varepsilon}(r')dV - \ldots - \int\limits_{V_1} \boldsymbol{\Gamma}(r,r'):\Delta\mathbf{c}^1:\boldsymbol{\varepsilon}(r')dV$$

$$- \ldots - \int\limits_{V_N} \boldsymbol{\Gamma}(r,r'):\Delta\mathbf{c}^N:\boldsymbol{\varepsilon}(r')dV.$$

In spite of the important simplification, the above equation still remains very difficult to solve. To simplify the considerations, let us calculate the average strain inside each inclusion

$$\boldsymbol{\varepsilon}^1 = \frac{1}{V_I}\int\limits_{V_I} \boldsymbol{\varepsilon}(r)dV, \tag{3.10}$$

and approximate the real strain field by a function

$$\boldsymbol{\varepsilon}(r) = \boldsymbol{\varepsilon}^1\theta^1(r) + \ldots\boldsymbol{\varepsilon}^I\theta^I(r) + \ldots\boldsymbol{\varepsilon}^N\theta^N(r).$$

Thus, the integral equation becomes:

$$\boldsymbol{\varepsilon}(r) = \mathbf{E}^0 - \left(\int\limits_{V_1} \boldsymbol{\Gamma}(r,r')dV\right):\Delta\mathbf{c}^1:\boldsymbol{\varepsilon}^1 - \ldots - \left(\int\limits_{V_I} \boldsymbol{\Gamma}(r,r')dV\right):\Delta\mathbf{c}^I:\boldsymbol{\varepsilon}^I$$

$$- \ldots - \left(\int\limits_{V_N} \boldsymbol{\Gamma}(r,r')dV\right):\Delta\mathbf{c}^N:\boldsymbol{\varepsilon}^N.$$

Now, one can calculate the average strain tensor inside each inclusion using relation (3.10). This leads to the system of N tensorial equations for N unknown strain tensors $\boldsymbol{\varepsilon}^I$:

$$
\begin{aligned}
&\boldsymbol{\varepsilon}^1 = \mathbf{E}^0 - \mathbf{T}^{11}{:}\Delta\mathbf{c}^1 : \boldsymbol{\varepsilon}^1 - \ldots - \mathbf{T}^{11}{:}\Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I - \ldots - \mathbf{T}^{1N}{:}\Delta\mathbf{c}^N : \boldsymbol{\varepsilon}^N \\
&\vdots \\
&\boldsymbol{\varepsilon}^I = \mathbf{E}^0 - \mathbf{T}^{I1}{:}\Delta\mathbf{c}^1 : \boldsymbol{\varepsilon}^1 - \ldots - \mathbf{T}^{II}{:}\Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I - \ldots - \mathbf{T}^{IN}{:}\Delta\mathbf{c}^N : \boldsymbol{\varepsilon}^N \\
&\vdots \\
&\boldsymbol{\varepsilon}^N = \mathbf{E}^0 - \mathbf{T}^{N1}{:}\Delta\mathbf{c}^1 : \boldsymbol{\varepsilon}^1 - \ldots - \mathbf{T}^{NI}{:}\Delta\mathbf{c}^I : \boldsymbol{\varepsilon}^I - \ldots - \mathbf{T}^{NN}{:}\Delta\mathbf{c}^N : \boldsymbol{\varepsilon}^N
\end{aligned} \tag{3.11}
$$

$\mathbf{T}^{IJ}$ are called interaction tensors. They are defined by the expression

$$
\mathbf{T}^{IJ} = \frac{1}{V_I} \int_{V_I} \int_{V_J} \Gamma(r, r') dV dV \tag{3.12}
$$

and have been studied, for case of two ellipsoidal inclusions embedded in an anisotropic matrix, by BERVEILLER and FASSI-FEHRI [20]. A numerical calculation of these tensors has been performed by LIPINSKI [3], and recently improved by ZATTARIN *et al.* [25].

When the reference homogeneous material $\mathbf{C}^0$ is chosen to be an unknown effective medium $\mathbf{C}^{\text{eff}}$, the self-consistent multi-site approximation is obtained. In this case; $\mathbf{E}^0 = \mathbf{E}$ and the concentration tensors for all inclusions are expressed by:

$$
\mathbf{A}^{\mathrm{I}}_{\mathrm{sc}} = \left(\mathbf{I} + \mathbf{T}^{\mathrm{II}}{:}\Delta\mathbf{c}^{\mathrm{I}}\right)^{-1} : \left(\mathbf{I} - \boldsymbol{\Sigma}_{j \neq I} \mathbf{T}^{IJ}{:}\Delta\mathbf{c}^J{:}\mathbf{A}^J_{\mathrm{sc}}\right). \tag{3.13}
$$

The usual one-site self-consistent method is obtained when all tensors $\mathbf{T}^{IJ}$ with $I \neq J$ are neglected. Of course, the determination of the concentration tensors has to be performed by iterations because the $\mathbf{A}^I_{\mathrm{sc}}$ tensor depends on all unknown $\mathbf{A}^J_{\mathrm{sc}}$ operators. The one-site self-consistent approximation constitutes a very good starting point for this iterative procedure.

3.3. Confrontation with experimental data

The self-consistent, one- or multi-site, models predict the accurate results when they are applied to the multi-phase polycrystalline materials whose elastic or elastic-plastic properties change slowly from one constituent to another [5, 19]. Its direct application to the artificial composites we want to illustrate below what leads to a considerable deviation from the experimental measurements when the relative difference between the matrix and reinforcement characteristics becomes significant. Consider for instance a composite with isotropic epoxy matrix reinforced by the boron fibres, supposed also to be isotropic. Such a composite has been experimentally studied by SABODH [26]. The elastic properties of the constituents are respectively:

– for epoxy : Young's modulus $E = 4.14$ GPa and Poisson's ratio $v = 0.35$;
– for boron : Young's modulus $E = 414$ GPa and Poisson's ratio $v = 0.20$.

The cylindrical fibers have been reproduced numerically by ellipsoidal inclusions with the aspect ratio of 1000. The regular spatial distribution of reinforcements has been introduced corresponding to the periodic square arrangement of parallel reinforcements. The MSCS has been used to predict the elastic properties of the composite. Table 1 shows the evolution of the transversal Young's modulus and the axial shear modulus as functions of the volume fraction of fibres, and a comparison between the experimental and predicted values is given. A reasonably good agreement is obtained for the volume fraction less than 25%. When the concentration of fibres is greater than 40%, the interactions between the inclusion and the surrounding material are strongly overestimated and the predicted moduli become much too high compared to the experimental measurements.

Table 1. Comparison between experimental and predicted moduli for epoxy-boron composite. Experimental data from Sabodh [26].

Volume fraction	Young's modulus [GPa]		Shear modulus [GPa]	
	Experimental	Predicted	Experimental	Predicted
0.25	7.67	7.67	2.93	3.42
0.3	9.15	9.15	3.06	4.03
0.35	11.18	11.18	3.18	4.63
0.40	12.74	13.82	3.79	7.54
0.50	15.36	24.87	5.72	17.89
0.60	20	53.3	8.91	41.1
0.70	33.25	119	14.4	72.36

4. Incremental Scheme (IS)

In order to avoid the divergence of the self-consistent model, VIÉVILLE [23] and VIÉVILLE *et al.* [6] proposed to modify the SCS using the idea of progressive construction of the material developed in the DS approach. Contrary to the DS, construction of the material is made using finite increments of the volume fractions, and for each increment the self-consistent approximation of the homogenisation process is performed instead of a small concentration solution.

4.1. Two-phase material

Let us first analyse a simple case of two-phase material considered by VIÉVILLE [23]. The resulting composite will be characterised by a volume fraction of the second phase $0 < f < 1$. Generally, the phase with the greatest volume fraction is considered as a matrix, and the second phase as a reinforcement. The

composite is built by the S step procedure in such a manner that at step i, the volume fraction of the second phase is

$$f_i = \frac{f}{S} \cdot i = \Delta f \cdot i. \tag{4.1}$$

When the step number S tends to infinity, the usual DS is found. Suppose, to clarify the idea, that at the beginning we have some volume of the matrix V^m of the reinforcement V^r, and that the resulting composite should have a volume V^c such that

$$V^m + V^r = V^c = V.$$

After $(i-1)$ steps the composite corresponds to

$$(i-1)\Delta f \cdot V^r + (1-(i-1)\Delta f) \cdot V^m = V^c_{i-1}, \tag{4.2}$$

where $V^c_{i-1} = V$ is the composite volume for step $i-1$. This volume becomes the matrix for the next step of the building procedure (N.B. this new matrix contains reinforcements).

At step i, to preserve the total volume, we have to cut off some volume of the new matrix to introduce an unknown volume fraction of reinforcement Δf_i such that the following equation will be verified

$$\Delta f_i \cdot V^r + (1-\Delta f_i) \cdot V^c_{i-1} = i\Delta f \cdot V^r + (1-i\Delta f) \cdot V^m. \tag{4.3}$$

Now, introducing (4.2) into (4.3) and comparing factors of V^m or V^r, one can determine the value of Δf_i

$$\Delta f_i = \frac{\Delta f}{1-(i-1)\Delta f}. \tag{4.4}$$

Expression (4.4) shows that the volume fraction of reinforcements continuously increases as a function of the step number i. This corresponds to the ROSCOE'S [29] interpretation of the DS where the filling process begins with the smallest inclusions to end up with the biggest ones. However, as it has been emphasised by HASHIN [17], this interpretation is contradictory to the requirement of infinitesimality of the volume fraction increment.

It is important to point out that the overall properties of the equivalent homogeneous material depend upon the number of steps S. Figure 1 illustrates the evolution of Young's modulus of an equivalent homogeneous material composed of 50% of an isotropic matrix, with shear modulus $G_M = 1.4$ MPa and Poisson's ratio $\nu_M = 0.499$, and 50% of isotropic randomly distributed spheres, with shear

modulus $G_S = 30.2$ GPa and Poisson's ration $v_S = 0.160$. It is easy to determine, for this case, the value of Young's modulus using the DS solution given by BOUCHER [13] in case of spherical inclusions.

One can observe that the degree of heterogeneity of the studied material is very important. Indeed we have:

$$\frac{G_S}{G_M} \cong 21571.$$

On the other hand, Fig. 1 shows that the SCS predicts the value of Young's modulus, given for $S = 1$, which is much higher than that obtained by the DS. Moreover, the One-Site Incremental Scheme (OIS) converges very rapidly to the DS solution.

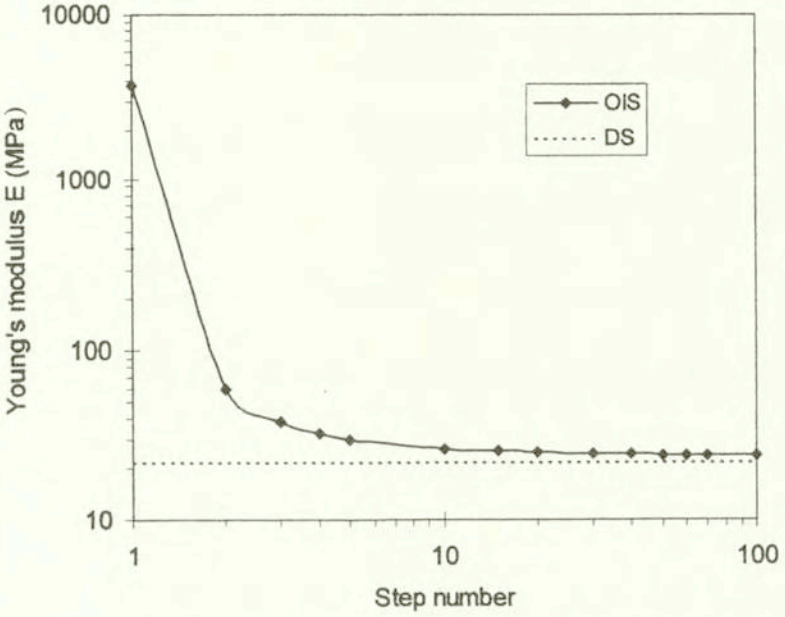

FIG. 1. Influence of the number of steps on the value of Young's modulus. Results obtained using One-Site Incremental Scheme.

4.2. Multi-phase material

Consider now the case of multiphase material (more than two phases) composed of a matrix with N families of reinforcements. To build this material we start with some volume of the matrix V^m, of the reinforcements V^{rJ} (J takes the values from 1 to N) and suppose that the resulting composite should have a volume V^c such that

$$V^m + V^{rJ} = V^c = V.$$

If f^J is the global volume fraction for the family J and S the total number of steps used to build the composite, the incremental fraction for the family J is

$$\Delta f^J = \frac{f^J}{S}. \tag{4.5}$$

After $(i-1)$ steps, the material corresponds to

$$\sum_{J=1}^{N}(i-1)\Delta f^J \cdot V^{rJ} + \left[1 - \sum_{J=1}^{N}(i-1)\Delta f^J \cdot V^{rJ}\right] \cdot V^m = V^c_{i-1} \tag{4.6}$$

where $V^c_{i-1} = V$ is the composite volume after step $i-1$. This volume becomes a matrix for the next step of the building procedure.

At step i, to preserve the total volume, we have to cut off some volume of the new matrix to introduce an unknown volume fraction of reinforcements Δf_i^J such that the following equation should be satisfied:

$$\sum_{J=1}^{N}\Delta f_i^J \cdot V^{rJ} + \left[1 - \sum_{J=1}^{N}\Delta f_i^J\right] \cdot V^c_{i-1} = i\sum_{J=1}^{N}\Delta f^J \cdot V^{rJ} + \left[1 - i\sum_{J=1}^{N}\Delta f^J\right] \cdot V^m. \tag{4.7}$$

Now, introducing (4.6) into (4.7) and comparing factors of V^m or V^r, one can determine the following system of equations for the unknown fractions Δf_i^J:

$$\begin{bmatrix} \frac{1}{(i-1)\Delta f^1} & \cdots & -1 & \cdots & -1 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ -1 & \cdots & \frac{1}{(i-1)\Delta f^J} & \cdots & -1 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ -1 & \cdots & -1 & \cdots & \frac{1}{(i-1)\Delta f^N} \end{bmatrix} \begin{Bmatrix} \Delta f_i^1 \\ \vdots \\ \Delta f_i^J \\ \vdots \\ \Delta f_i^N \end{Bmatrix} = \frac{1}{(i-1)} \begin{Bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{Bmatrix}.$$

The solution of this system takes a form

$$\Delta f_i^J = \frac{\Delta f^J}{1-(i-1)\Delta f^r} \tag{4.8}$$

where

$$\Delta f^r = \sum_{J=1}^{N} \Delta f^J = \frac{\sum_{J=1}^{N} f^J}{S} = \frac{f^r}{S}, \tag{4.9}$$

and f^r is the total volume fraction of reinforcements.

It is important to emphasise that the above solution corresponds to the radial or proportional path in the volume fraction N-dimensional space. As it has been observed by NORRIS [18], this is not a unique possibility to fill the composite. Figure 2 illustrates this idea in 2D fraction space for a three-phase material. The path I corresponds to the Roscoe-Boucher scheme and to the actual proposition, and path II illustrates any arbitrary filling process. It is easy to show, see for instance NORRIS [18], that the resulting equivalent material properties are path-dependent. All the results presented in this paper have been obtained by increasing the volume fractions along path I, expression (4.8). The influence of the filling process corresponding to any other path is not discussed in this paper.

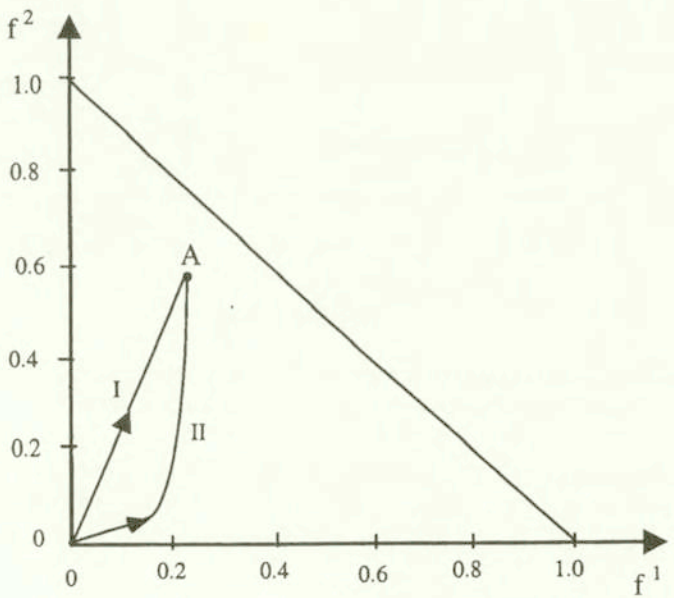

FIG. 2. Path definition in fraction space. I – actual method, II – arbitrary path.

5. Applications

In this section we present some numerical results in order to show the results of the above-described method.

The first application concerns again the epoxy-boron composite of the Sec. 3.3. We compare the transverse Young's modulus predicted by the Multi-Site Self-Consistent Scheme (MSCS) and by the Multi-Site Incremental Scheme (MIS), with experimental values from SABODH [26].

In Fig. 3 we can observe that a relatively good agreement is obtained with MIS even for concentration of fibres. MSCS model, according to Table 1, overestimates (when $f = 0.7$) the Young's modulus by a factor of almost 4.

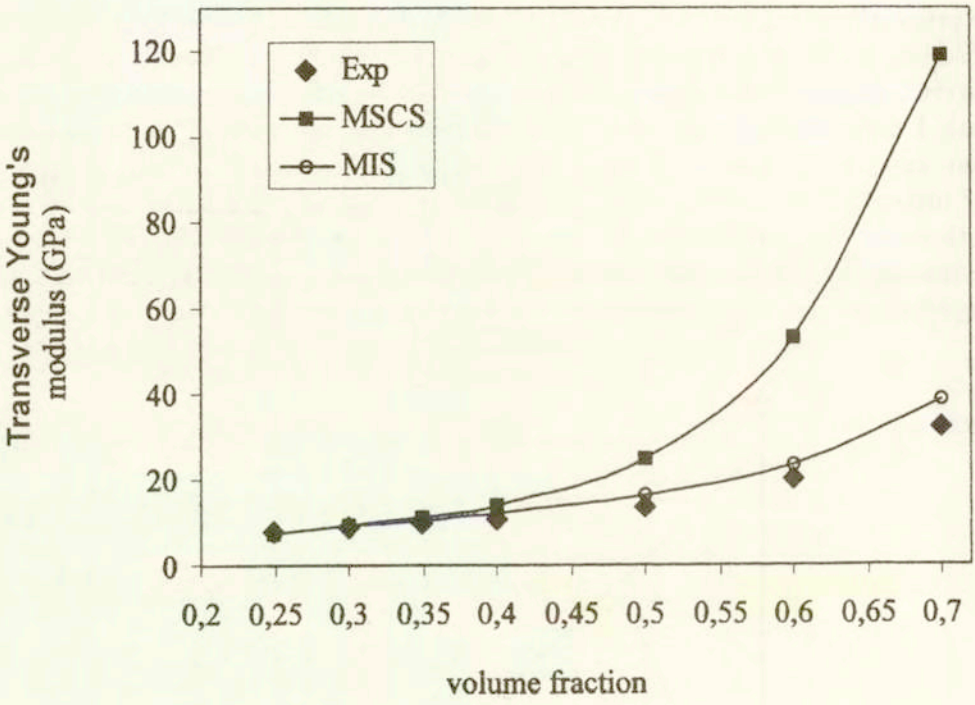

FIG. 3. Comparison of the predicted transverse Young's modulus, using MSCS and MIS with experimental data from SABODH [26].

In the second application we consider a two-phase material consisting of a rigid sphere embedded in an isotropic matrix. The elastic properties are

$$v^M = v^S = 0.3$$

and

$$\frac{E^S}{E^M} = 50,$$

where v^M and v^S are, respectively, Poisson's ratio of the matrix and sphere, E^M and E^S are their Young's moduli. The regular spatial distribution of spheres is introduced corresponding to a periodic cubic arrangement. We used the Multi-Site Self-Consistent Scheme (MSCS) and the Multi-Site Incremental Scheme (MIS) to predict the elastic properties of this composite. The calculated properties are compared with the results obtained by Finite Element Method. The commercial code ANSYS V 5.6 has been used to perform all these calculations. Figure 4 illustrates the mesh of an elementary volume from which the full composite can be constructed by symmetries and periodicity. This mesh is composed on 4600 hexahedral eight-node elements and corresponds to the volume fraction of reinforcements $f = 0.4$. The same figure shows also the obtained axial stress distribution. One can observe that this stress component distribution inside the inclusion is very complex.

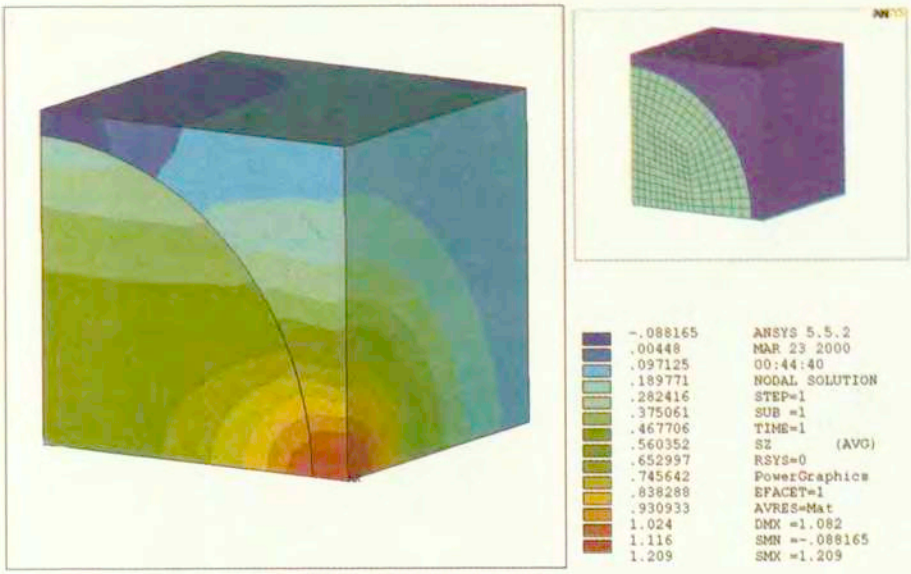

FIG. 4. Finite Element mesh and axial stress distribution.

Figure 5 shows the evolution of the Young modulus as a function of the volume fraction of spheres. A good agreement is obtained between the FEM and the MIS approach, while the MSCS strongly overestimates the value of the Young modulus for the volume fractions greater than 20%. One can observe that, despite the complex stress field inside the inclusion (Fig. 4), the MIS approach predicts very accurately the effective properties of the composite.

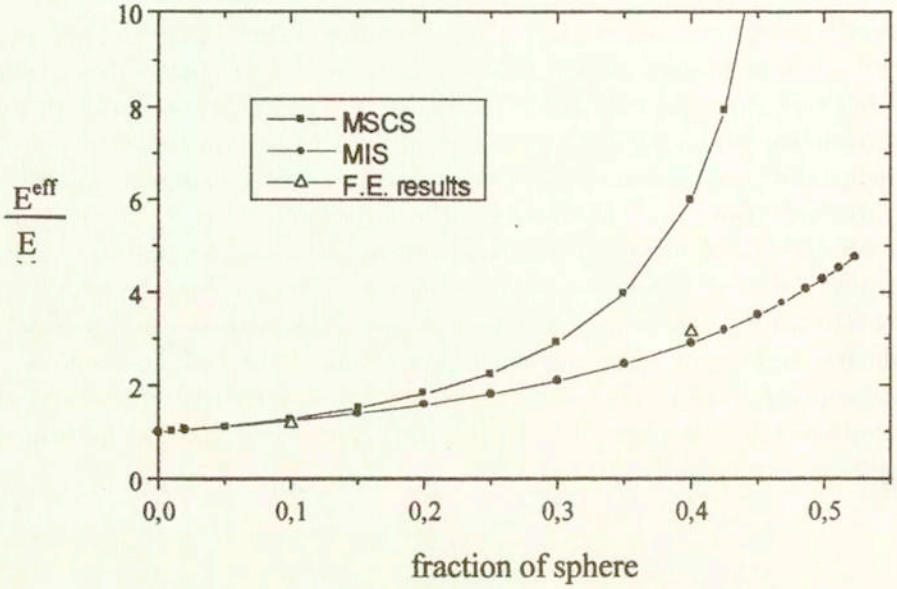

FIG. 5. Evolution of Young's modulus, using MSCS, MIS and FEM approaches, for a composite constituted of rigid spheres embedded in a matrix.

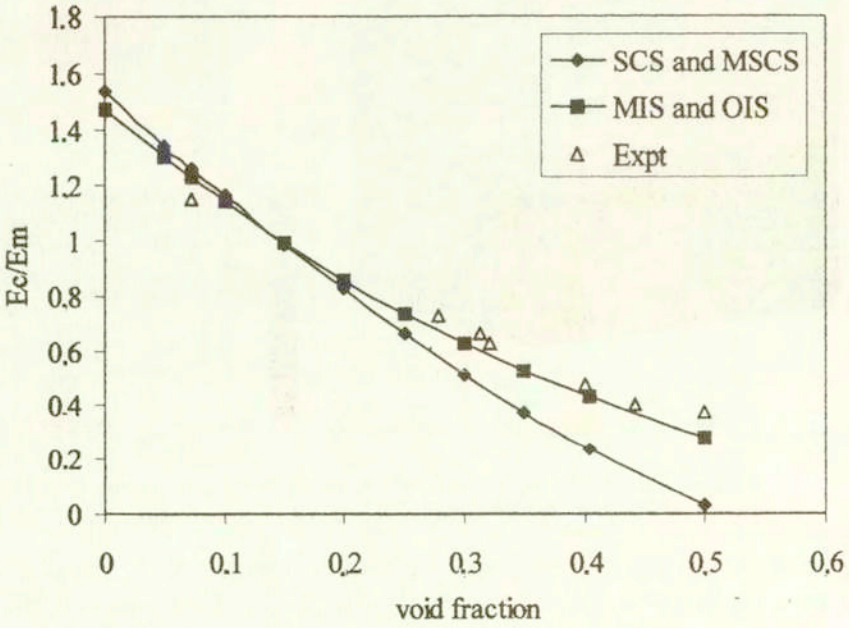

FIG. 6. Evolution (due to the SCS and IS approach) of the Young's modulus for a three-phase composite, and comparison with experimental results.

In the last example, we compare the experimental results obtained by ISHAI and COHEN [27] with the Self-Consistent Scheme (SCS) and the Incremental Scheme (IS) predictions. The material is a three-phase composite consisting of sand and voids in an epoxy matrix. We suppose that sand and voids are in a random arrangement. In this case of perfect disorder, Multi-Site (MSCS or MIS) and One-Site (SCS or OIS) give rise to similar results. HUANG *et al.* [28] showed, using the Cohen data, that the volume fractions of sand f_S and void f_V are interdependent and vary according to:

$$f_S = 0.173(1 - f_V).$$

As Fig. 6 shows, the IS predicts correctly the experimental data over the entire range of void fraction, while the SCS underestimates the results for void fractions greater than $f_V = 0.2$.

6. Conclusions

A new transition scale based model has been proposed for the prediction of effective properties of composite materials. This Multi-Site Incremental Scheme takes into account the anisotropy of the constituents as well as the morphological and the topological textures, what enables the prediction of the effective properties for general cases.

The model has been applied to characterise the elastic behaviour of two- and three-phase composites. The incremental procedure used in this approach permits to avoid the problems of Self-Consistent Scheme distortions for large volume fraction of reinforcement or void.

The comparison of the numerical results with the experimental data known from literature shows a very good agreement between the calculations and measurements. The same conclusion concerns the comparison between the predicted values and Finite Elements results.

The MIS scheme appears as a powerful alternative to the complex and time-consuming Finite Element applications for prediction of the effective properties of elastic heterogeneous materials.

References

1. R. HILL, *A self-consistent mechanics of composite materials*, J. Mech. Phys. Solids, **13**, 213–222, 1965.
2. E. KRONER, *Berechnung der elastischen Konstanten des Vielkristalls aus den Konstanten der Einkristalls*, Physik, 151, 1958.
3. P. LIPINSKI, *Modélisation du comportement des métaux, en transformations élastoplastiques finies, à partir des méthodes de transition d'échelles*, Habilitation, Université de Metz, 1993.

4. F. CORVASCE, P. LIPINSKI, M. BERVEILLER, *The effects of thermal plastic and elastic stress concentrations on the overall behavior of metal matrix composites*, IUTAM Symposium on "Inelastic deformation of composite materias", 389–408, Troy, USA 1990.

5. P. ZATTARIN, P. LIPINSKI, P. VIEVILLE, *Modélisation à sites multiples du comportement anisotrope des composites ordonnés*, 3ème Congrès Marocain de Mécanique, 839–844, 1997.

6. P. VIEVILLE, P. LIPINSKI, *Application du schéma autocohérent par étapes à la modélisation des propriétés viscoélastiques des composites*, 9ème Journées Nationales des Composites, 392–397, Paris, France 1995.

7. T. MORI, K. TANAKA, *Average stress in matrix and average energy of materials with misfitting inclusions*, Acta Metall., **21**, 571, 1973.

8. R. M. CHRISTENSEN, K. H. LO, *Solutions for effective shear properties in three-phase sphere and cylinder models*, J. Mech. Phys. Solids, **27**, 315–330, 1979.

9. E. HERVE, A. ZAOUI, *Modeling the effective behavior of nonlinear matrix-inclusion composites*, Eur. J. Mech., A/Solids, **9**, 505–515, 1990.

10. M. CHERKAOUI, H. SABAR, M. BERVEILLER, *Micromechanical approach of the coated inclusion problem and applications to composite materials*, J. Engng. Mat. and Tech., **116**, 274–278, 1994.

11. D. A. G. BRUGGEMAN, *Berchnung verschiedener physkalisher Konstante von heterogene Substanzen*, Ann. Physik, **24**, 636, 1935.

12. R. ROSCOE, *Isotropic composites with elastic or viscoelastic phases: General bounds for the moduli and solutions for special geometries*, Rheol. Acta, **12**, 404–411, 1973.

13. S. BOUCHER, *Modules effectifs de matériaux composites quasi homogènes et quasi isotropes constitués d'une matrice élastique et d'inclusions élastiques. I – Cas des concentrations infinitésimales en inclusions*, Revue M, **21**, 3, 1975, *II – Cas des concentrations finies en inclusions*, Revue M, **22**, 1, 1976.

14. R. MC LAUGHLIN, *A study of the differential scheme for composite material*, Int. J. Engng. Sci., **15**, 237–244, 1977.

15. R. L. SALGANIK, *Mechanics of bodies with many cracks*, Mekhanika tverdogo tela (Mech. Mats), **8**, 135, 1973.

16. N. LAWS, G. J. DVORAK, Int. J. Solids Struct., **23**, 1269, 1987.

17. Z. HASHIN, *The differential scheme and its applications to cracked materials*, J. Mech. Phys. Solids, **36**, 6, 719–734, 1988.

18. A. N. NORRIS, *A differential scheme for effective moduli of composites*, Mechanics of Materials, **4**, 1–16, 1985.

19. O. FASSI-FEHRI, *Le problème de la paire d'inclusions plastiques et hétérogènes dans une matrice anisotrope – Application à l'étude du comportement des matériaux composites et de la plasticité*, Thèse d'état, Université de Metz, 1985.

20. O. FASSI-FEHRI, A. HIHI, M. BERVEILLER, *Multiple site self consistent scheme*, Int. J. Engng. Sci., 495–502, 1989.

21. M. EL MOUDENE, *Une nouvelle méthode d'homogénéisation des matériaux composites élastiques*, Thèse-Université de Metz, 1995.

22. P. ZATTARIN, P. LIPINSKI, *Modélisation du comportement anisotrope des composites par le schéma autocohérent multisite*, 13ème Congrès Français de Mécanique, 357–360, Poitiers 1997.

23. P. VIEVILLE, *Influence des paramètres architecturaux sur les caractéristiques viscoélastiques du bois à ses différentes échelles d'hétérogénéité*, Thèse-Institut National Polytechnique de Lorraine, 1992.

24. P. H. DEDERICHS, R. ZELLER, *Variational treatment of the elastic constants of disordered materials*, Phys. Stat., **259**, 103–113, 1973.

25. P. ZATTARIN, A. CARMASOL, P. LIPINSKI, *Une nouvelle approche numérique pour calculer les interactions entre deux inclusions dans un milieu anisotrope*, 2ème Congrès Marocain de Mécanique, 845–850, 1995.

26. S. G. SABODH, *Analysis of structural composite material*, M. Dekker in New-York 1973.

27. O. ISHAI, O. J. COHEN, *Elastic properties of filled and porous epoxy composites*, Int. J. Mech. Sci., **9**, 539–546, 1967.

28. Y. HUANG, K. X. HU, X. WEI, A. CHANDRA, *A generalized self-consistent mechanics method for composite materials with multiphase inclusions*, J. Mech. Phys. Solids, **42**, 491–504, 1994.

29. R. ROSCOE, *The viscosity of suspension of rigid spheres*, Brit. J. Appl. Phys., **3**, 267–269, 1952.

30. J. MANDEL, *Une généralisation de la théorie de la plasticité de W. T. Koiter*, Int. J. Solids Struct., **1**, 273–295, 1965.

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