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Brief Notes

On the order of singularity at V-shaped notches in anisotropic bodies

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The self-similar problem of stress singularity at the notch in infinite two-dimensional elastic orthotropic body was considered. The considerations were restricted to the notches symmetrically oriented with respect to the axes of orthotropy. Both, the extension and shear modes were studied. It was confirmed that in the limiting case of zero opening angle (semi-infinite crack), the order of singularity is the same as in the case of isotropic material – $r^{-1/2}$. This is not true in the case of finite opening angles. If the orientation of the notch axis is parallel to the axis of maximal stiffness, the order of singularity is lower than that for the case of perpendicular orientation. In the last case, if the ratio E_T/E_L is small enough, then the order of singularity in tension does not practically decrease with growing opening angle 2α up to $\alpha \approx \pi/4$.

1. Preliminary remarks

The problem of formulation of the fracture criteria at the tips of V-shaped notches in anisotropic materials needs some knowledge on the order of singularity involved [1] (see also [2, 3]). One may expect that values of this parameter essentially depend not only on the opening angle, like in the case of isotropic material, but on the material anisotropy as well. Another important practical problem of computational mechanics of strongly anisotropic materials consists in a proper choice of the parameters of "singular" finite elements for the calculation of the stress distribution in the vicinity of the notch tip. To this end one also needs exact knowledge on the order of singularity. In the foregoing sections we shall briefly sketch the way leading to the family of analytic self-similar solutions in polar co-ordinates as well as to the numeric procedure of choice of these solutions which fulfill the imposed homogenous boundary value conditions at free edges. In the present paper we shall confine our attention to the cases, when the axis of symmetry of the infinite notch coincides with the axis of orthotropy. The solutions for arbitrarily oriented notches can be readily obtained, however their interpretation is not trivial and it will be postponed to the separate paper.

2. Basic relations

For the description of the plane orthotropic problem we shall follow the way proposed in [4]. We assume that the co-ordinate axes $\{x_1, x_2\}$ are chosen along the axes of symmetry of the orthotropic plane elastic medium. In such a case two-dimensional constitutive relations of linear elasticity can be expressed as follows:

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{E_1}\left(\sigma_{11} + \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\sigma_{22}\right),\\
\varepsilon_{22} &= \frac{1}{E_1}\left(\gamma_1^2\gamma_2^2\,\sigma_{22} + \frac{\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2}\sigma_{11}\right),\\
\varepsilon_{12} &= \frac{\gamma_3^2}{E_1}\sigma_{12},
\end{aligned}
\tag{2.1}
$$

where E_1 denotes Young's modulus corresponding to the uniaxial tension in x_1 direction[1], $\gamma_1, \gamma_2, \gamma_3$ are dimensionless constants, fulfilling the following relations:

$$
\gamma_1^2\gamma_2^2 = \frac{E_1}{E_2}, \qquad \gamma_1^2+\gamma_2^2 = 2\left(\frac{E_1}{2\mu} - \nu_{12}\right), \qquad \gamma_3^2 = \frac{E_1}{2\mu}.
$$

The meaning of E_2, μ and ν_{12} is obvious. Without any loss of generality $\gamma_1, \gamma_2, \gamma_3$ can be assumed to be positive. Imposing the conditions of the Poisson ratios and elastic energy positiveness, one can obtain the following constraints which should be imposed on γ_1, γ_2, and γ_3:

$$
\gamma_1^2+\gamma_2^2 < 2\gamma_3^2 < (\gamma_1+\gamma_2)^2\,. \tag{2.2}
$$

This means that two of them, e.g. γ_1 and γ_2, can, in principle, assume any positive values.

Let us introduce Airy stress function $\Phi(x_1, x_2)$, such that

[1] We assume plane stress conditions, where all material constants under consideration are the same as in the three-dimensional case, corresponding values for the plane strain can be readily obtained by assuming $\varepsilon_{33} = 0$.

$$\begin{aligned}\sigma_{11} &= \Phi_{,22},\\ \sigma_{22} &= \Phi_{,11},\\ \sigma_{12} &= -\Phi_{,12},\end{aligned} \tag{2.3}$$

where comma denotes partial derivative in Cartesian co-ordinates. Combining relations (2.3) and (2.1) and substituting the result into the strain compatibility condition

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}, \tag{2.4}$$

one can obtain in a standard way the following orthotropic counterpart of the biharmonic equation describing the isotropic material:

$$\Phi_{,2222} + \left(\gamma_1^2 + \gamma_2^2\right)\Phi_{,1122} + \gamma_1^2\gamma_2^2\Phi_{,1111} = 0 \tag{2.5}$$

(compare (2.9) in [4]). The last equation can be rewritten in the following form:

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x_2^2} + \gamma_1^2\frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial^2}{\partial x_2^2} + \gamma_2^2\frac{\partial^2}{\partial x_1^2}\right)\Phi(x_1, x_2)\\ &\qquad = \left(\frac{\partial}{\partial x_2} + i\gamma_1\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - i\gamma_1\frac{\partial}{\partial x_1}\right)\\ &\qquad \times\left(\frac{\partial}{\partial x_2} + i\gamma_2\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - i\gamma_2\frac{\partial}{\partial x_1}\right)\Phi(x_1, x_2) = 0,\end{aligned} \tag{2.6}$$

where the symbol $\times$ (used here for typographic reasons only) denotes superposition of operators, $i = \sqrt{-1}$. Thus, any differentiable function of any of the following complex variables:

$$\eta = x_1 + i\gamma_1 x_2, \quad \overline{\eta} = x_1 - i\gamma_1 x_2, \quad \xi = x_1 + i\gamma_2 x_2, \quad \overline{\xi} = x_1 - i\gamma_2 x_2, \tag{2.7}$$

satisfies Eq. (2.5).

Looking for singular solution it is reasonable to take into consideration power functions of these variables. Adopting polar co-ordinates one can express e.g. η as follows:

$$\eta = r\left(\cos\varphi + i\gamma_1\sin\varphi\right). \tag{2.8}$$

Note that here variables r and φ do not denote the modulus and argument of η, instead we have

$$|\eta| = r\left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{\frac{1}{2}}, \qquad \mathrm{Arg}(\eta) = \mathrm{Arctan}\left(\gamma_1\tan\varphi\right), \tag{2.9}$$

thus, one can write:

$$\eta^{2-\lambda} = r^{2-\lambda}\left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{1-\lambda/2} \times \{\cos\left[(2-\lambda)\,\mathrm{Arctan}\,(\gamma_1\tan\varphi)\right] + i\sin\left[(2-\lambda)\,\mathrm{Arctan}\,(\gamma_1\tan\varphi)\right]\}. \tag{2.10}$$

For practical calculations certain care must be taken to keep the values

$$\beta_1(\varphi) \equiv \mathrm{Arctan}\,(\gamma_1\tan\varphi), \qquad \beta_2(\varphi) \equiv \mathrm{Arctan}\,(\gamma_2\tan\varphi) \tag{2.11}$$

in the same quarter-plane as φ: $\mathrm{Sgn}(\cos\beta_i) = \mathrm{Sgn}(\cos\varphi)$, $\mathrm{Sgn}(\sin\beta_i) = \mathrm{Sgn}(\sin\varphi)$ for $i = 1, 2$.

For further considerations we shall restrict our attention to real values of λ parameter[2]. For this case we can look for the following Airy stress function:

$$F(r,\varphi) = r^{2-\lambda}\left[\Phi(\varphi) + \Psi(\varphi)\right], \tag{2.12}$$

where

$$\begin{aligned} \Phi(\varphi) &= A\Phi_1(\varphi) + B\Phi_2(\varphi), \\ \Psi(\varphi) &= C\Psi_1(\varphi) + D\Psi_2(\varphi), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \Phi_1(\varphi) &= \left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{1-\lambda/2}\cos\left[(2-\lambda)\beta_1\right], \\ \Phi_2(\varphi) &= \left(\cos^2\varphi + \gamma_2^2\sin^2\varphi\right)^{1-\lambda/2}\cos\left[(2-\lambda)\beta_2\right], \\ \Psi_1(\varphi) &= \left(\cos^2\varphi + \gamma_1^2\sin^2\varphi\right)^{1-\lambda/2}\sin\left[(2-\lambda)\beta_1\right], \\ \Psi_2(\varphi) &= \left(\cos^2\varphi + \gamma_2^2\sin^2\varphi\right)^{1-\lambda/2}\sin\left[(2-\lambda)\beta_2\right], \end{aligned} \tag{2.14}$$

A, B, C, D are arbitrary constants.

3. Solution of the boundary value problem

In the polar co-ordinate system, the following expressions for the stress field components in terms of Airy function derivatives hold true [5]:

[2]The authors hope to return in the future to the discussion on the physical sense of high-amplitude oscillations of stress at the vicinity of singular point like $r^{-\alpha}\cos(\ln(r))$, which would follow from the imaginary part of λ if an expression of type (2.10) were taken as a stress function.

$$
\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial F(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F(r,\varphi)}{\partial \varphi^2},\\
\sigma_{\varphi\varphi} &= \frac{\partial^2 F(r,\varphi)}{\partial r^2},\\
\sigma_{r\varphi} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F(r,\varphi)}{\partial \varphi}\right).
\end{aligned}
\tag{3.1}
$$

As it was already mentioned, in the present paper we shall focus our attention on the restricted class of boundary value problems: V-shaped notches of the opening angle 2α, $0 < \alpha < \pi/2$ symmetric with respect to x_1 axis (compare Fig. 1).

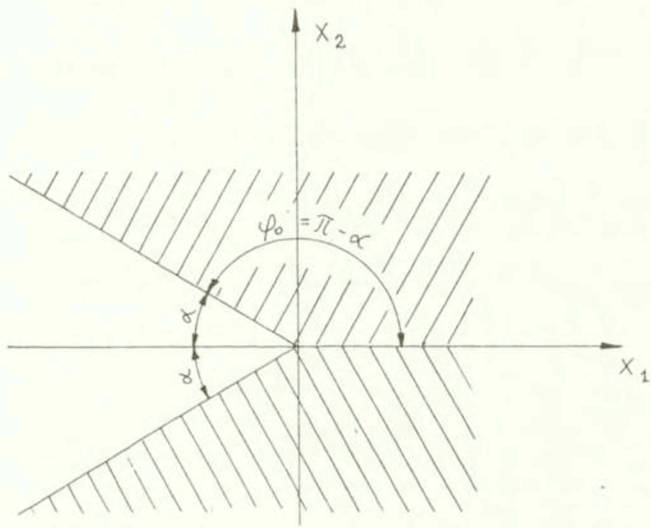

FIG. 1. Orientation of the V-shaped notch with respect to axes of orthotropy.

We assume also the absence of contact forces at the boundaries:

$$
\begin{aligned}
\sigma_{\varphi\varphi}(r, \pm\varphi_0) &= 0,\\
\sigma_{r\varphi}(r, \pm\varphi_0) &= 0,
\end{aligned}
\tag{3.2}
$$

where $\varphi_0 = \pi - \alpha$. Bearing in mind the symmetry properties of the functions $\Phi(\varphi)$ and $\Psi(\varphi)$ and relations (3.1), one can prove that conditions (3.2) split into two independent problems:

$$
\begin{aligned}
A\Phi_1(\varphi_0) + B\Phi_2(\varphi_0) &= 0,\\
A\Phi_1'(\varphi_0) + B\Phi_2'(\varphi_0) &= 0,
\end{aligned}
\tag{3.3}
$$

and

$$C\Psi_1(\varphi_0) + D\Psi_2(\varphi_0) = 0, \tag{3.4}$$
$$C\Psi_1'(\varphi_0) + D\Psi_2'(\varphi_0) = 0,$$

where "prime" stands for the first derivative with respect to φ.

Equations (3.3) describe Mode I (tension) while system (3.4) corresponds to the Mode II (shear). Characteristic equation associated with (3.3) has the following form:

$$\begin{aligned}(\gamma_2^2 - \gamma_1^2)\cos[(2-\lambda)\beta_1(\varphi_0)]\cos[(2-\lambda)\beta_2(\varphi_0)]\tan\varphi_0 \\ - (1+\gamma_1^2)\gamma_2\cos[(2-\lambda)\beta_1(\varphi_0)]\sin[(2-\lambda)\beta_2(\varphi_0)] \\ + (1+\gamma_2^2)\gamma_1\cos[(2-\lambda)\beta_2(\varphi_0)]\sin[(2-\lambda)\beta_1(\varphi_0)] = 0,\end{aligned} \tag{3.5}$$

while Eqs. (3.4) yield the following condition

$$\begin{aligned}(\gamma_2^2 - \gamma_1^2)\sin[(2-\lambda)\beta_1(\varphi_0)]\sin[(2-\lambda)\beta_2(\varphi_0)]\tan\varphi_0 \\ + (1+\gamma_1^2)\gamma_2\sin[(2-\lambda)\beta_1(\varphi_0)]\cos[(2-\lambda)\beta_2(\varphi_0)] \\ - (1+\gamma_2^2)\gamma_1\sin[(2-\lambda)\beta_2(\varphi_0)]\cos[(2-\lambda)\beta_1(\varphi_0)] = 0.\end{aligned} \tag{3.6}$$

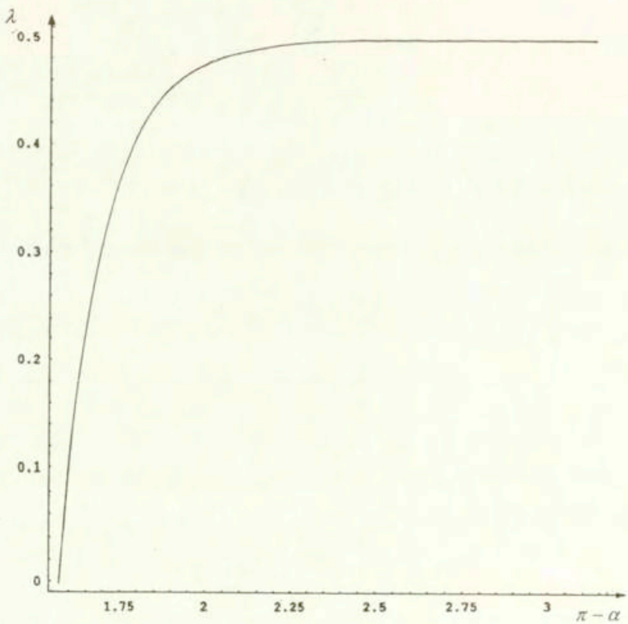

FIG. 2. Order of singularity λ versus decreasing opening angle α, Mode I, $\gamma_1 = 0.4$, $\gamma_2 = 0.3$ ($E_1/E_2 = 0.0144$), notch axis perpendicular to the axis of maximal stiffness.

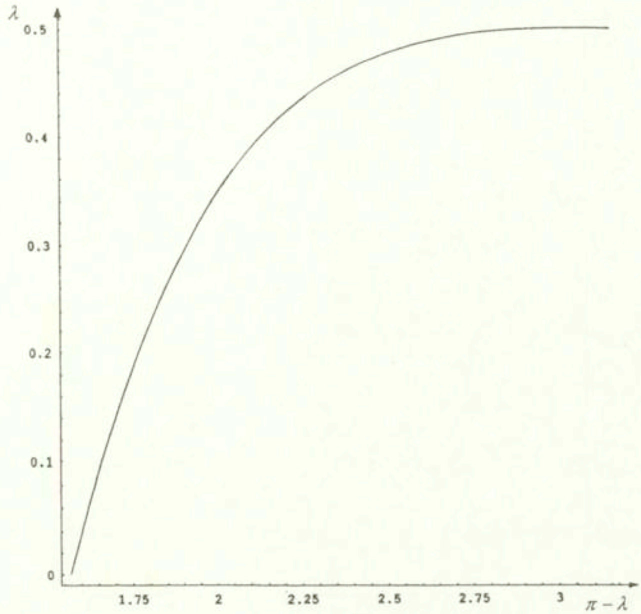

FIG. 3. Order of singularity λ versus decreasing opening angle α, Mode I, almost isotropic material $\gamma_1 = 0.99$, $\gamma_2 = 1.01$.

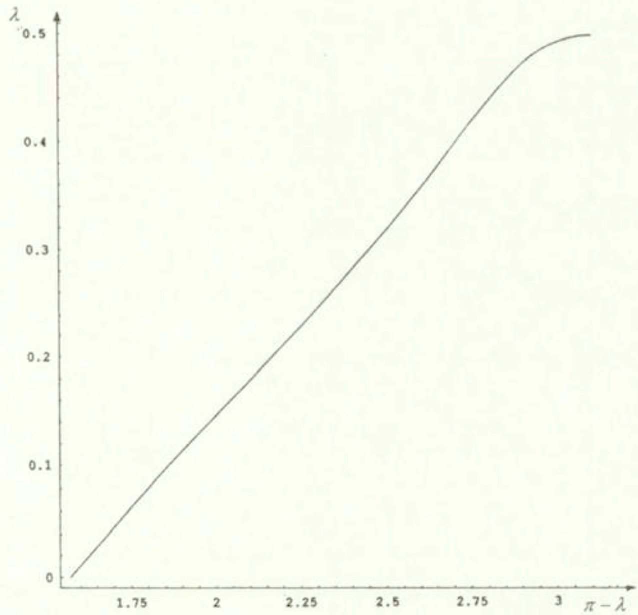

FIG. 4. Order of singularity λ versus decreasing opening angle α, Mode I, $\gamma_1 = 4$, $\gamma_2 = 3$ $(E_1/E_2 = 144)$, notch axis parallel to the axis of maximal stiffness.

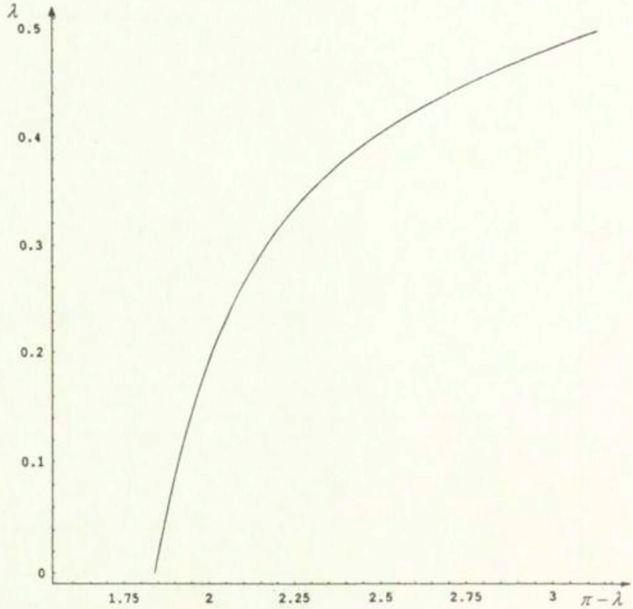

FIG. 5. Order of singularity λ versus decreasing opening angle α, Mode II, $\gamma_1 = 0.4$, $\gamma_2 = 0.3$ ($E_1/E_2 = 0.0144$), notch axis perpendicular to the axis of maximal stiffness.

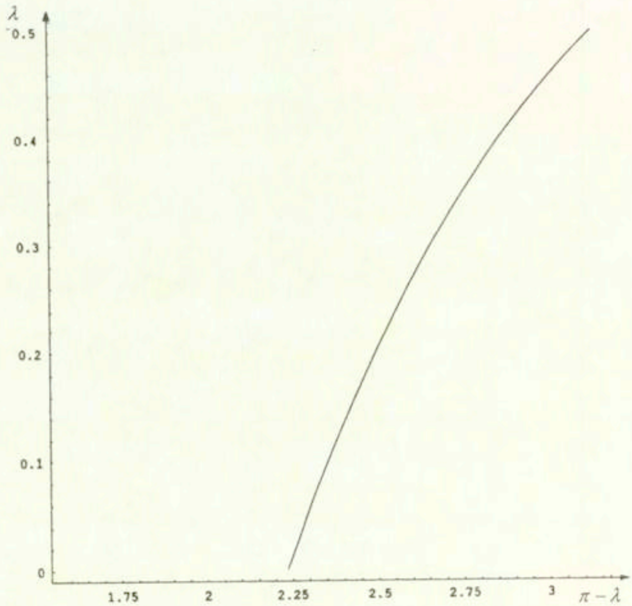

FIG. 6. Order of singularity λ versus decreasing opening angle α, Mode II, almost isotropic material $\gamma_1 = 0.99$, $\gamma_2 = 1.01$.

In Figs. 2, 3, 4 are shown, for different values of γ_2 and γ_2, contour plots (zero level only) of the function defined by the left-hand side of Eq. 3.5. The curves join the points at which Eq. (3.5) is satisfied. Similar plots for Eq. (3.6) are shown in Figs. 5, 6, 7.

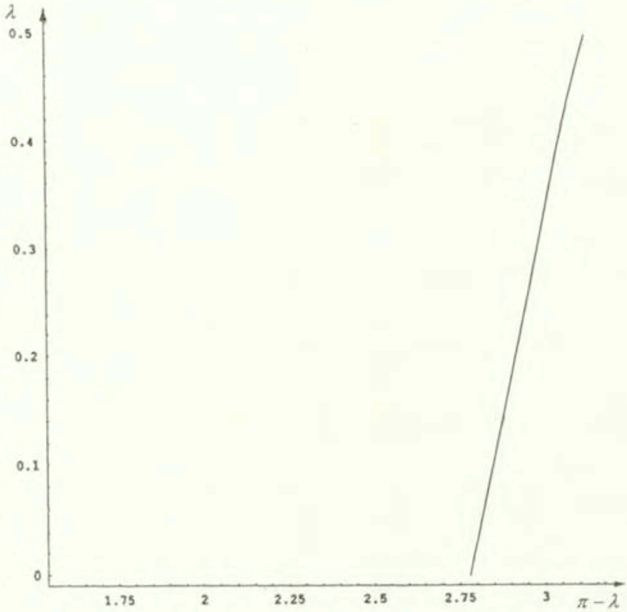

FIG. 7. Order of singularity λ versus decreasing opening angle α, Mode II, $\gamma_1 = 4$, $\gamma_2 = 3$ ($E_1/E_2 = 144$), notch axis parallel to the axis of maximal stiffness.

4. Concluding remarks

It can be readily seen that even strong anisotropy does not change qualitatively the results which had been found earlier for the isotropic case (cf. [1]). It is a proper place here to recall that the ratio of Young moduli E_1/E_2 is equal to the product of the squared gammas $\gamma_2^2\gamma_1^2$, thus, in our examples, longitudinal modulus differs from the transversal one by two orders of magnitude. The following quantitative effects can be observed: the order of singularity λ for both modes of loading is lower for the case of notches having their symmetry axes parallel to the axis of maximal stiffness, and higher for the perpendicular orientation. The isotropic case takes the intermediate position.

As it has been mentioned earlier, the problems of arbitrarily oriented V-notches and, possibly, of the stress distribution at the vicinity of the contact points of three differently oriented wedges made of the same anisotropic material (modelling polycrystalline solids), will be considered in a separate paper.

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