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Material instability in the tensile response of short-fibre-reinforced quasi-brittle composites

Dedicated to Professor Zenon Mróz on the occasion of his 70th birthday

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THIS PAPER GIVES THE CONDITIONS for the onset of instability in the tensile response of short-fibre-reinforced quasi-brittle composites whose deformation is characterised by multiple cracking and localisation. First, the tensile stress-strain relation is established analytically for a body containing multiple bridged microcracks. The material instability is examined using the classical bifurcation criterion, with an emphasis on the role played by fibre bridging in the macroscopic instability. It is found that while the microscopic instability in the bridging traction plays a major role in the macroscopic instability of the composite, it is the level of damage in the matrix that determines when the macroscopic instability is induced by the bridging instability. The satisfaction of the classical bifurcation criterion is identified with several failure modes, depending on the degree of damage in the matrix.

1. Introduction

THE FIRST SIGNS OF TENSILE DAMAGE in short-fibre-reinforced quasi-brittle matrices, such as cements, are the appearance of multiple parallel microcracks. They cause the stress-strain curve to deviate from linearity, i.e. give the composite a strain-hardening response. Bridging of the multiple microcracks by short fibres is an important mechanism for increasing the strength and toughness of these composites and for preventing a sudden loss of their overall stiffness when the microcracks coalesce and localise into large bands. The response of the material after the attainment of ultimate tensile strength is characterised by the localisation of deformation, i.e. by the continuous opening of cracks and stretching of unbroken material ligaments in a narrow localisation band.

On the micromechanical level, a lot of studies have been devoted to the bridging effect of fibres on the crack propagation and the toughening of the

composite in the context of fracture mechanics. On the other hand, there have been relatively few studies on the macroscopic constitutive behaviour of short-fibre-reinforced quasi-brittle composites, such as cementitious composites. The studies at the microlevel benefit the material technologists, whereas those at the macrolevel benefit the structural engineers. The relation between the macroscopic constitutive response and the micomechanical parameters is also vital in the material design. For quasi-brittle monolithic or fibre-reinforced composites, in which the deformation process leading to complete rupture usually involves multiple cracking, the complete constitutive behaviour is important to the analysis of propagation of a macroscopic crack, as was demonstrated by ORTIZ [1]. Multiple cracking also characterises the compressive response of concrete [2].

KARIHALOO *et al.* [3] have previously studied the complete constitutive behaviour of short-fibre-reinforced cementitious composites under unidirectional tension. In their work, the pre-peak strain-hardening is simulated by a doubly periodic array of bridged cracks whose density increases with increasing tensile load. The peak stress, i.e., the ultimate failure stress of the composite material is calculated using the law of mixtures. The post-peak tension-softening response is simulated using a single row of periodic discrete bridged cracks. The model of KARIHALOO *et al.* [3] cannot predict the transition from the strain-hardening to tension-softening. This leads to a discontinuity in the slope of the stress-strain/displacement curve at the peak load. In other words, conditions for localisation are still not clear.

In the present paper, the mechanisms for the material instability which lead to deformation localisation and tension softening will be revealed. The study will be based upon an analytical procedure which allows the conditions for the material instability in short-fibre-reinforced composites to be highlighted. The material instability is examined using the classical bifurcation criterion, with an emphasis on the role played by fibre bridging in the macroscopic instability. It is found that while the microscopic instability in the bridging traction plays a major role in the macroscopic instability of the composite, it is the level of damage in the matrix that determines when the macroscopic instability is induced by the bridging instability. The satisfaction of the classical bifurcation criterion is identified with several failure modes, depending on the degree of damage in the matrix. The phrase 'microscopic instability' is used to define the instant when the bridging stiffness momentarily vanishes.

2. General formulae for multiple cracks

The prediction of the effective elastic and fracture properties of a medium containing multiple cracks has received considerable attention. Among the stud-

ies are the solutions based upon the non-interacting approximation, when the interaction among the cracks is neglected. Under this assumption, the effective elastic properties can be expressed in explicit forms. The interactions among multiple cracks complicate the prediction of the overall material behaviour. The schemes based upon indirect considerations of crack interactions, such as the self-consistent method and the differential scheme may considerably underestimate the overall moduli, as has been pointed by WANG *et al.* [4].

In this section, we shall present an analytical approach for the calculation of the overall tensile modulus of bodies containing multiple parallel bridged cracks. For this, we shall make use of the procedures in the previous work of KARIHALOO *et al.* [3], and those in the recent works of WANG *et al.* [4, 5].

The overall (average) strain and stress of a cracked body are related via (e.g. [6])

$$\varepsilon_{ij} = C^0_{ijkl}\,\sigma_{kl} + \frac{1}{2V}\sum_N \int_{S_N} \left([u_i]\,n_j + [u_j]\,n_i\right) dS_N \tag{2.1}$$

where ε_{ij} and σ_{kl} are the average strain and stress components, respectively. u_i and n_i are the total crack opening/sliding displacement (COD/CSD) and the component of the unit vector normal to the crack faces. C^0_{ijkl} is the compliance tensor of the uncracked material. For parallel flat cracks when n_i is a constant, Eq. (2.1) can be rewritten as

$$\varepsilon_{ij} = C^0_{ijkl}\,\sigma_{kl} + \frac{1}{2V}\sum_N \left(\overline{[u_i]}\,n_j + \overline{[u_j]}\,n_i\right) S_N \tag{2.2}$$

where $\overline{[u_i]}$ is the average COD/CSD for a single crack over its faces, and S_N is its area.

We note that Eq. (2.2) can be written as

$$\varepsilon_{ij} = C^0_{ijkl}\,\sigma_{kl} + \frac{\omega}{L}\left(\overline{[u_i]}\,n_j + \overline{[u_j]}\,n_i\right) \tag{2.3}$$

where ω is a non-dimensional crack density parameter, and L is an internal length scale, which will be defined later. In the above expression, $\overline{[u_i]}$ is taken to be the COD/CSD for a representative crack.

We consider only infinitesimal deformation and rotation. Taking the time-derivative of the above equation gives

$$\dot{\varepsilon}_{ij} = C^0_{ijkl}\,\dot{\sigma}_{kl} + \frac{1}{L}\left(\overline{[u_i]}\,n_j + \overline{[u_j]}\,n_i\right)\frac{\partial\omega}{\partial\sigma_{kl}}\dot{\sigma}_{kl} + \frac{\omega}{L}\left(\frac{\partial\overline{[u_i]}}{\partial\sigma_{kl}}\,n_j + \frac{\partial\overline{[u_j]}}{\partial\sigma_{kl}}\,n_i\right)\dot{\sigma}_{kl}. \tag{2.4}$$

Here, we have assumed that the crack density is a function of the applied stress.

Strictly speaking, $\overline{[u_i]}$ is also a function of ω, which is in turn a function of σ_{kl}. However, Eq. (2.3) implies that $\overline{[u_i]}$ is a generic crack opening/sliding displacement in a representative element. In analogy with the usual non-interacting solution, we assume at this stage that this generic crack opening/sliding displacement is not related to the crack density, so that the derivative of $\overline{[u_i]}$ with respect to ω vanishes. Equation (2.4) gives the tangent compliance tensor

$$C_{ijkl} = C^0_{ijkl} + \frac{1}{L}\left(\overline{[u_i]}\, n_j + \overline{[u_j]}\, n_i\right)\frac{\partial \omega}{\partial \sigma_{kl}} + \frac{\omega}{L}\left(\frac{\partial \overline{[u_i]}}{\partial \sigma_{kl}}\, n_j + \frac{\partial \overline{[u_j]}}{\partial \sigma_{kl}}\, n_i\right) \tag{2.5}$$

whence the rate form of the constitutive relation (2.4) can be written as

$$\dot{\varepsilon}_{ij} = C_{ijkl}\, \dot{\sigma}_{kl}. \tag{2.6}$$

We now return to the determination of $\overline{[u_i]}$. For bodies containing multiple cracks, the effect of crack interactions and of any bridging tractions must be taken into account in the calculation of the crack opening displacement. Using the pseudo-traction formalism [7], the average crack opening displacement is calculated by applying a pseudo-traction on the faces of a single crack. In order to determine $\overline{[u_i]}$ and thus C_{ijkl} for a body containing randomly distributed multiple parallel bridged cracks, we shall first invoke the analytical procedures in the works of WANG *et al.* [4, 5] for two regular arrays of bridged cracks, namely, a doubly periodic rectangular array and a doubly periodic diamond-shaped array of bridged cracks, shown in Fig. 1. We consider the two-dimensional case, when the parallel cracks are perpendicular to, say, the direction 2. Following the procedures in the above works, the traction consistency condition on each crack in either of the two doubly periodic configurations is expressed as follows:

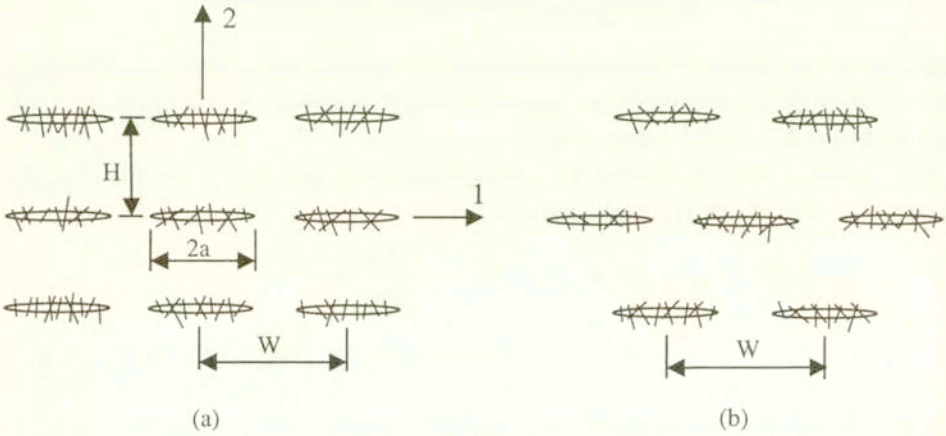

FIG. 1. Doubly periodic rectangular (a) and diamond-shaped array (b) of bridged cracks.

$$\sigma_{ij}^{p}(x) - 2\sum_{j=1}^{+\infty}\int_{0}^{a} K_{ijkl}(x,x^j)\,\sigma_{kl}^{p}(x^j)\,dx^j + p_{ij}(x) = \sigma_{ij}^{0}, \quad x \in [0,a), \tag{2.7}$$

where σ_{ij}^{p} is the pseudo-traction on the crack faces, σ_{ij}^{0} is the applied stress, p_{ij} is the bridging stress exerted by the fibres, and a is the half-length of a crack. $K_{ijkl}(x,x^j)$ is the stress influence tensor which has been described in the previous works of the authors (e.g. [4, 8]).

However, in the present paper, in order to trace the nonlinear behaviour of the material, we shall recast the traction consistency condition (2.7) in an incremental form

$$\Delta\sigma_{ij}^{p}(x) - 2\sum_{j=1}^{+\infty}\int_{0}^{a} K_{ijkl}(x,x^j)\,\Delta\sigma_{kl}^{p}(x^j)\,dx^j + \Delta p_{ij}(x) = \Delta\sigma_{ij}^{0}, \tag{2.8}$$

$$x \in [0,a).$$

For the two-dimensional case under study, the parallel cracks are perpendicular to direction 2, so that we need only the pseudo-tractions σ_{22}^{p} and σ_{12}^{p} for calculating the crack opening/sliding displacements. Following the procedure in the recent work by WANG *et al.* [4], it is found that the incremental pseudo-tractions for the two periodic arrays of cracks shown in Fig. 1 can be written as

$$\begin{Bmatrix} \Delta\sigma_{22}^{pr} \\ \Delta\sigma_{22}^{pd} \end{Bmatrix} = \begin{Bmatrix} \zeta^{r} \\ \zeta^{d} \end{Bmatrix} \Delta\sigma_{22}^{0}, \tag{2.9}$$

$$\begin{Bmatrix} \Delta\sigma_{12}^{pr} \\ \Delta\sigma_{12}^{pd} \end{Bmatrix} = \begin{Bmatrix} \eta^{r} \\ \eta^{d} \end{Bmatrix} \Delta\sigma_{12}^{0}, \tag{2.10}$$

where

$$\zeta^{r} = \left\{1 + 4\sin^2\frac{\pi a}{W}\,e^{-2\frac{H}{W}\pi}\left[1 + 2\frac{H}{W}\pi\right] - \frac{2\bar{k}_{22}\,W^2}{\pi\,a\,E'}\,\ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1}, \tag{2.11}$$

$$\zeta^{d} = \left\{1 - 4\sin^2\frac{\pi a}{W}\,e^{-2\frac{H}{W}\pi}\left[1 + 2\frac{H}{W}\pi\right] - \frac{2\bar{k}_{22}\,W^2}{\pi\,a\,E'}\,\ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1}, \tag{2.12}$$

$$\eta^{r} = \left\{1 + 4\sin^2\frac{\pi a}{W}\,e^{-2\frac{H}{W}\pi}\left[1 - 2\frac{H}{W}\pi\right] - \frac{2\bar{k}_{12}\,W^2}{\pi\,a\,E'}\,\ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1}, \tag{2.13}$$

$$\eta^d = \left\{1 - 4\sin^2\frac{\pi a}{W}\, e^{-2\frac{H}{W}\pi}\left[1 - 2\frac{H}{W}\pi\right] - \frac{2\bar{k}_{12}\, W^2}{\pi\, a\, E'}\, \ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1}. \tag{2.14}$$

$E' = E$ for plane-stress, and $E' = E/(1-\nu^2)$ for plane-strain deformtion. In deriving Eqs. (2.9) – (2.10), a linear relationship between the increnental bridging stress $\Delta p_{22}(x)$ and incremental COD/CSD has been assumed

$$\Delta p_{22}(x) = \bar{k}_{22}\, \Delta[u_2](x); \qquad \Delta p_{12}(x) = \bar{k}_{12}\, \Delta[u_1](x); \tag{2.15}$$

where $[u_1](x)$ and $[u_2](x)$ are the crack opening/sliding displacements. It is evident that $\bar{k}_{12}$ and $\bar{k}_{22}$ in the above expression should be the tangent brdging stiffnesses.

We presented above the incremental pseudo-tractions on the crack faces following the asymptotic analysis of WANG *et al.* [4, 5]. They are found to be constants (in an average sense only) and dependent upon the geometry of the crack arrays and the instantaneous tangent bridging stiffnesses. $\Delta\sigma_{22}^{pr}$ ($\Delta\sigma_{12}^{pr}$) and $\Delta\sigma_{22}^{pd}$ ($\Delta\sigma_{12}^{pd}$) are the two incremental pseudo-tractions for the douby periodic rectangular array and diamond-shaped array, respectively. Accordng to the analysis in the work by WANG *et al.* [4], these two regular patterns should represent the two extreme interactions among multiple parallel cracks, namely, the maximum "shielding" and "magnification" effects under unidirectional tension and the maximum "magnification" and "shielding" effects under in-plane shear. Based upon this analysis, WANG *et al.* [4] deduced that the overall modulus of a body containing randomly distributed multiple parallel cracks should be vithin a pair of bounds corresponding to the moduli for the doubly periodic rectangular and diamond-shaped array, respectively. Moreover, it was found that when the the terms $4\sin^2\frac{\pi a}{W}\, e^{-2\frac{H}{W}\pi}\left[1 + 2\frac{H}{W}\pi\right]$ and $4\sin^2\frac{\pi a}{W}\, e^{-2\frac{H}{W}\pi}\left[1 - 2\frac{H}{W}\pi\right]$ were neglected in the expressions (2.11) – (2.14), i.e., when the expressions (2.11) – (2.14) reduced to

$$\zeta \equiv \left\{1 - \frac{2\bar{k}_{22}\, W^2}{\pi\, a\, E'}\, \ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1}, \tag{2.16}$$

$$\eta \equiv \left\{1 - \frac{2\bar{k}_{12}\, W^2}{\pi\, a\, E'}\, \ln\left(\cos\frac{\pi a}{W}\right)\right\}^{-1}, \tag{2.17}$$

the overall moduli so calculated for low to moderate crack density were always in the middle of the range bounded by those obtained when the terms $4\sin^2\frac{\pi a}{W}\, e^{-2\frac{H}{W}\pi}\left[1 + 2\frac{H}{W}\pi\right]$ and $4\sin^2\frac{\pi a}{W}\, e^{-2\frac{H}{W}\pi}\left[1 - 2\frac{H}{W}\pi\right]$ were retained. Expressions (2.16) – (2.17) are nothing but the so-called non-interacting solution. Numerical computations of KACHANOV [6] for random discrete unbridged

parallel cracks also were found to be close to the non-interacting solution. Thus, it is reasonable to use the expressions (2.16) – (2.17) to calculate the pseudo-tractions

$$\Delta\sigma_{22}^{p} = \zeta\,\Delta\sigma_{22}^{0}, \qquad \Delta\sigma_{12}^{p} = \eta\,\Delta\sigma_{12}^{0}. \tag{2.18}$$

Having obtained the pseudo-tractions, the average crack opening/sliding displacements can be easily found

$$\begin{aligned}\left\{\begin{array}{c}\Delta\overline{[u_1]}\\ \Delta\overline{[u_2]}\end{array}\right\} &= \frac{1}{2a}\int\limits_{-a}^{+a}\left\{\begin{array}{c}\Delta[u_1](x)\\ \Delta[u_2](x)\end{array}\right\}dx \\ &= -\frac{2W^2}{\pi\,aE}\ln\left(\cos\frac{\pi a}{W}\right)\left\{\begin{array}{c}\Delta\sigma_{12}^{p}\\ \Delta\sigma_{22}^{p}\end{array}\right\}.\end{aligned} \tag{2.19}$$

The subsequent development is for plane-stress deformation condition. Substituting (2.18) into (2.19) gives

$$\left\{\begin{array}{c}\Delta\overline{[u_1]}\\ \Delta\overline{[u_2]}\end{array}\right\} = -\frac{2W^2}{\pi\,aE}\ln\left(\cos\frac{\pi a}{W}\right)\left\{\begin{array}{c}\eta\,\Delta\sigma_{12}^{0}\\ \zeta\,\Delta\sigma_{22}^{0}\end{array}\right\} \tag{2.20}$$

which can be rewritten as

$$\left\{\begin{array}{c}\dfrac{\Delta\sigma_{12}^{0}}{E}\\[2ex] \dfrac{\Delta\sigma_{22}^{0}}{E}\end{array}\right\} = \left[\begin{array}{cc}-\dfrac{1}{\dfrac{2W}{\pi a}\eta\ln\left(\cos\dfrac{\pi a}{W}\right)} & 0\\[2ex] 0 & -\dfrac{1}{\dfrac{2W}{\pi a}\zeta\ln\left(\cos\dfrac{\pi a}{w}\right)}\end{array}\right]\left\{\begin{array}{c}\dfrac{\Delta\overline{[u_1]}}{W}\\[2ex] \dfrac{\Delta\overline{[u_2]}}{W}\end{array}\right\}. \tag{2.21}$$

Expression (2.21) describes the local behaviour of the cracked material. The global constitutive behaviour of the material can be determined from (2.6), together with (2.5).

3. Analysis of material instability

In this section, we shall study the material instability in the macroscopic tensile response of the composite, especially that induced by the microscopic bridging mechanism. For this, we use the classical bifurcation criterion for discontinuity localisation across parallel planes [9]. As will be seen later, for the case studied in this paper, this criterion is equivalent to other bifurcation criteria identified by NEILSEN and SCHREYER [10] for the study of material instabilities. The classical bifurcation criterion is

$$\det[Q_{ij}] = 0 \tag{3.1}$$

where Q_{ij} is the acoustic tensor defined as

$$Q_{ij} = n_k D_{kijl} n_l. \tag{3.2}$$

D_{kijl} is the tangent stiffness tensor which is the inverse of the tangent compliance tensor C_{kijl} in Eq. (2.5).

In order to obtain the acoustic tensor (3.2), we need to calculate the tangent compliance tensor from Eq. (2.5). For this, we need the total crack opening/sliding displacement $\overline{[u_i]}$, and its partial derivative with respect to σ_k. For fibre-reinforced quasi-brittle composites, it is observed in experiments that the density of the multiple cracks increases with increasing load until it reaches a saturation level ω_s, when the localisation sets in [11]. Moreover, in these materials the damage localisation usually coincides with the pull-out of fibres from the matrix. This implies that at localisation, the partial derivative $\partial\omega/\partial\sigma_{kl}$ in Eq. (2.5) can be equated to zero. Of course, the second term in Eq. (2.5) is essential to the strain hardening description which may be found in the work of KARIHALOO and WANG [12]. The tangent compliance tensor (2.5) at the instant of localisation therefore reduces to

$$C_{ijkl} = C^0_{ijkl} + \frac{\omega_s}{L}\left(\frac{\partial\overline{[u_i]}}{\partial\sigma_{kl}} n_j + \frac{\partial\overline{[u_j]}}{\partial\sigma_{kl}} n_i\right). \tag{3.3}$$

Let us consider the localisation into a planar band under unidirectional tension σ_{22}. For the considered two-dimensional case, the conventional crack density parameter ω is defined as

$$\omega = \frac{a^2}{WH}. \tag{3.4}$$

The rate form of the stress-strain relations (2.6), after making use of Eq.(3.3), are

$$\begin{Bmatrix} \dot{\varepsilon}_{11} \\ \dot{\varepsilon}_{22} \\ \dot{\varepsilon}_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E} & -\dfrac{\nu}{E} & 0 \\ -\dfrac{\nu}{E} & \dfrac{1}{E} + 2\dfrac{\omega_s}{L}\dfrac{\partial\overline{[u_2]}}{\partial\sigma_{22}} & 0 \\ 0 & 0 & \dfrac{2(1+\nu)}{E} + 2\dfrac{\omega_s}{L}\dfrac{\partial\overline{[u_1]}}{\partial\sigma_{12}} \end{bmatrix} \begin{Bmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{Bmatrix}. \tag{3.5}$$

Calculating $\partial\overline{[u_2]}/\partial\sigma_{22}$ and $\partial\overline{[u_1]}/\partial\sigma_{12}$ from Eq. (2.20), and noting that for the considered case, $L = a$, we get the tangent compliance matrix

$$[C_{ij}] = \tag{3.6}$$

$$= \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi E}\zeta \ln\left(\cos\frac{\pi a}{W}\right) & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi E}\eta \ln\left(\cos\frac{\pi a}{W}\right) \end{bmatrix}$$

and, by inversion, the corresponding tangent stiffness matrix

$$[D_{ij}] = \tag{3.7}$$

$$= \begin{bmatrix} \frac{E\left[1 - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_t\right]}{\left[1 - \nu^2 - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_t\right]} & \frac{E\nu}{\left[1 - \nu^2 - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_t\right]} & 0 \\ \frac{E\nu}{\left[1 - \nu^2 - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_t\right]} & \frac{E}{\left[1 - \nu^2 - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_t\right]} & 0 \\ 0 & 0! & \frac{E}{2\left[(1+\nu) - 2\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_s\right]} \end{bmatrix}$$

Here, we have introduced two non-dimensional parameters

$$L_t \equiv \zeta \ln\left(\cos\frac{\pi a}{W}\right), \tag{3.8}$$

$$L_s \equiv \eta \ln\left(\cos\frac{\pi a}{W}\right), \tag{3.9}$$

where the subscripts t and s denote tension and shear, respectively.

Substituting the tangent stiffness tensor into Eq. (3.2) gives the acoustic tensor whose components in a matrix form are

$$[Q_{ij}] = \begin{bmatrix} \frac{E}{2\left[(1+\nu) - 2\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_s\right]} & 0 \\ 0 & \frac{E}{\left[1 - \nu^2 - 4\left(\frac{W}{a}\right)^2 \frac{\omega_s}{\pi} L_t\right]} \end{bmatrix}. \tag{3.10}$$

As we are interested in the unidirectional tensile case, we only discuss L_t in the sequel. We first rewrite L_t (3.8), using Eq. (2.16), and omit the subscript 22 from $\bar{k}_{22}$ for brevity,

$$L_t = \frac{\ln\left(\cos\dfrac{\pi a}{W}\right)}{1 - \dfrac{2\bar{k}\,W^2}{\pi a E}\ln\left(\cos\dfrac{\pi a}{W}\right)}. \tag{3.11}$$

The value $2a/W$ represents the ratio of the cracked area to the nominal area in the direction perpendicular to the loading direction. It therefore represents the conventional damage parameter in the context of damage mechanics. Denoting $\Omega = 2a/W$, L_t can be rewritten as

$$L_t = \frac{\ln\left(\cos\dfrac{\pi}{2}\Omega\right)}{1 - \dfrac{8\bar{k}\,a}{\pi E}\dfrac{1}{\Omega^2}\ln\left(\cos\dfrac{\pi}{2}\Omega\right)}. \tag{3.12}$$

It is seen from Eq. (3.10) that the satisfaction of the localisation criterion (3.1) requires that $L_t \to \infty$. When this condition is met, it is seen from (3.7) that the determinant of the tangent stiffness matrix, $\det[D_{ij}]$, also vanishes. $[D_{ij}]$ is symmetric, as is $[Q_{ij}]$. Therefore, the condition $L_t \to \infty$ leads to the satisfaction of all bifurcation criteria identified by NEILSEN and SCHREYER [10], namely, the classical bifurcation criterion (3.1), the general bifurcation criterion, the limit point bifurcation criterion and the loss of strong ellipticity criterion. In the following, we shall use the phrases "localisation" or "material instability" to refer to the consequences of $\det[Q_{ij}] = 0$, i.e. when $L_t \to \infty$.

4. Conditions for material instability

Several features of the material instability are revealed by the above results. First, for unbridged material ($\bar{k} = 0$), it seen from Eq. (3.12) that the satisfaction of the localisation criterion $\det[Q_{ij}] = 0$ requires that $L_t = \ln\left(\cos\dfrac{\pi}{2}\Omega\right) \to \infty$. This simply means that the damage parameter, or any effective quantity, Ω tends to 1. In this case, L_s also becomes zero. So the material loses instability both under unidirectional tension perpendicular to the crack and under in-plane shear. Thus, the bifurcation criterion is identified with the damage-induced rupture of the material.

When $\bar{k} \neq 0$, it is seen from Eq. (3.12) that L_t is determined by the tangent bridging stiffness $\bar{k}$. For short-fibre-reinforced cementitious composites, a trilinear bridging law, such as OABC shown in Fig. 2, is commonly used (e.g. [3]). This is obviously an idealisation of the actual fibre pull-out test results.

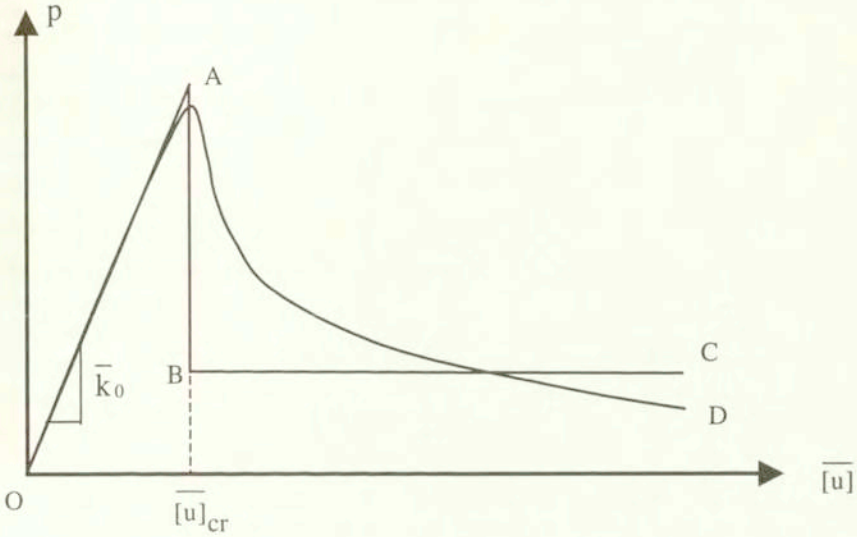

FIG. 2. An idealised trilinear bridging law OABC and a more realistic smooth bridging law OAD with continuous slope.

The problem with the idealised trilinear bridging law is the discontinuity in the tangent bridging stiffness $\bar{k}$. In real materials, especially when the average effect of randomly distributed fibres is considered, the tangent bridging stiffness varies gradually, as shown by OAD in Fig. 2. This continuous bridging traction can, for example, be described by

$$p = \bar{k}\,\overline{[u]}\,e^{-\frac{\overline{[u]}}{\overline{[u]}_{cr}}}. \tag{4.1}$$

The tangent bridging stiffness can thus be written as

$$\bar{k} = \bar{k}_0 \left\{ 1 - \frac{\overline{[u]}}{\overline{[u]}_{cr}} \right\} e^{-\frac{\overline{[u]}}{\overline{[u]}_{cr}}}, \tag{4.2}$$

where $\bar{k}_0$ is the initial tangent bridging stiffness when the fibres are bonded to the matrix (see Fig. 2). It is evident that the tangent bridging stiffness vanishes at $\overline{[u]} = \overline{[u]}_{cr}$ and it becomes negative, when $\overline{[u]} > \overline{[u]}_{cr}$. The expression (4.2) is in line with the simple local constitutive law that JIRÁSEK and BAŽANT [13] used in their study of the localisation phenomenon within the formalism of the non-local theory. The initial tangent bridging stiffness $\bar{k}_0$ can be calculated from the linear bridging model developed by LANGE-KORNBAK and KARIHALOO [14]

$$\bar{k}_0 = V_f \frac{\tau_v}{\tau_g} E_f \frac{h}{L} \tag{4.3}$$

where τ_g and τ_v are the frictional bond strength and the adhesive bond strength, respectively. h is the so-called snubbing factor, L is the length of fibres, and E_f their modulus of elasticity.

When $\bar{k} > 0$, it follows from (3.12) that L_t has the following property:

$$\Omega \to 0: \qquad L_t \to 0. \tag{4.4}$$

In other words, no instability can set in, if there is no damage in the material. Thus, as expected, the case $\Omega \to 0$ can be excluded from the instability analysis.

When $0 < \Omega < 1$, the bifurcation criterion ($L_t \to \infty$) can only be satisfied when the following condition is met (cf. (3.12)):

$$1 - \frac{8\bar{k}a}{\pi E}\frac{1}{\Omega^2}\ln\left(\cos\frac{\pi}{2}\Omega\right) = 0, \tag{4.5}$$

that is

$$\bar{k}^{cr} = \frac{\pi E}{8a}\frac{\Omega^2}{\ln\left(\cos\frac{\pi}{2}\Omega\right)} < 0 \qquad \text{for} \quad 0 < \Omega < 1 \tag{4.6}$$

with

$$\Omega \to 1: \qquad \bar{k}^{cr} \to 0. \tag{4.7}$$

The variation of the normalised $\bar{k}^{cr}$ with Ω given by (4.6) is shown in Fig. 3.

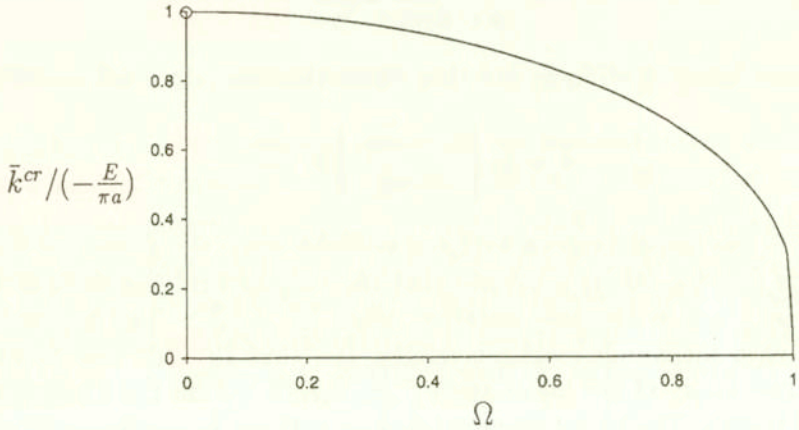

FIG. 3. Variation of normalised tangent bridging stiffness at macroscopic instability with damage parameter. No instability is possible when $\Omega = 0$. Thus the point on the axis of ordinates at 1 is excluded, as highlighted by the open circle.

From the above analysis and Fig. 3, we are able to discern several features of incipient material instability. First, it follows from Eq. (4.6) that for all values of Ω in the range $0 < \Omega < 1$, no instability can set in, if the tangent bridging stiffness $\bar{k}$ is greater than 0. In other words, no matter how much the matrix is damaged, as long as the fibres are still bonded to it ($\bar{k} > 0$) , the composite will not exhibit any instability at the macroscopic level, even when a through crack has formed in the matrix (see Fig. 4(a)). The tangent stiffness of the composite will continue to be positive. This is exactly what happens, for example, in strong continuous fibre-reinforced composites, as is demonstrated by the ACK model [15].

The second feature is that $\bar{k}$ can vanish before or after a through crack has formed, i.e., $\Omega \to 1$. Here, we discuss the formation of a through crack ($\Omega \to 1$) when the fibres are still bonded to the matrix, i.e., $\bar{k} \geq 0$. In this case, L_t (3.12) can be approximated by

$$L_t = -\frac{\pi a E}{4\bar{k}W}. \tag{4.8}$$

Here, W loses its meaning, although its appearance in the above formula simply points to the existence of an internal length scale. The expression (4.8) indicates that the bifurcation condition ($L_t \to \infty$) requires that the fibre bridging stiffness vanishes $\bar{k} = 0$. In other words, after a through crack has formed or is about to form, the macroscopic instability of the composite coincides with the (microscopic) bridging instability (see Fig. 4(b)).

The third feature is that when the localisation band is still not a through crack (i.e. $0 < \Omega < 1$), the localisation criterion can still be satisfied when $\bar{k}^{cr}$ is given by (4.6). We recall that the average crack opening displacement for a row of periodic cracks without the bridging action of fibres can be rewritten as (see, e.g. [16])

$$\Delta\sigma_m = -\frac{\pi E}{8a}\frac{\Omega^2}{\ln\left(\cos\frac{\pi}{2}\Omega\right)}\Delta\overline{[u]}. \tag{4.9}$$

When the cracks are bridged by fibres, the total instantaneous resistance of the composite material to crack opening can be written as the sum of matrix and fibre contributions

$$-\frac{\pi E}{8a}\frac{\Omega^2}{\ln\left(\cos\frac{\pi}{2}\Omega\right)} + \bar{k}. \tag{4.10}$$

Thus the condition (4.6) implies that the resistance of the composite material to crack opening displacement vanishes because the instantaneous resistance of the matrix itself to crack opening is exactly counterbalanced by the loss of the bridging resistance (see Fig. 4(c)).

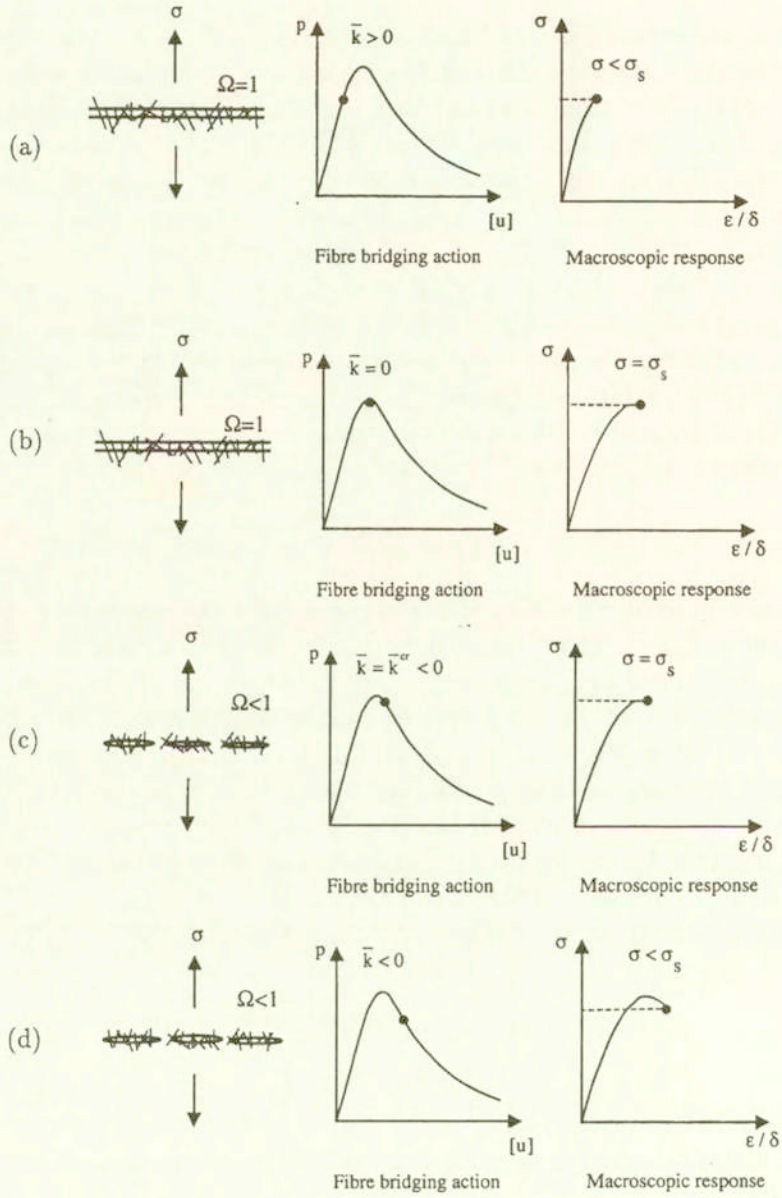

FIG. 4. Relationship between bridging stiffness and macroscopic response of the composite material. (a) no instability, when the fibres are bonded to matrix ($\bar{k} > 0$) even though a through crack has formed in it ($\Omega = 1$); (b) macroscopic instability induced by loss of bridging stiffness ($\bar{k} = 0$) and formation of a through crack ($\Omega = 1$); (c) macroscopic instability induced by a combination of matrix damage ($\Omega < 1$) and initial softening of bridging stiffness equal to $\bar{k}^{cr} < 0$; (d) macroscopic tension-softening caused by matrix damage ($\Omega < 1$) and considerable softening of bridging stiffness ($\bar{k} < 0$).

The parameter $\bar{k}^{cr}$ in (4.6) includes the physical effect of several factors on the inception of localisation. First, instability of the composite at the macrolevel is possible, when $\bar{k} < 0$, i.e., the fibres are only exerting the residual friction bridging action. The closer Ω is to 1, the smaller the absolute value of $\bar{k}^{cr}$. When $\Omega = 1$, the macroscopic instability occurs at $\bar{k} = 0$, i.e. it coincides with the microscopic (bridging) instability. Second, the dependence of the absolute value of $\bar{k}^{cr}$ on the modulus E of the uncracked matrix indicates that the stiffer the matrix, the less susceptible is the material to microscopic instability. Third, $\bar{k}^{cr}$ contains a length scale, here the half-length a of a crack. As the parameters Ω and E are scale insensitive, $\bar{k}^{cr}$ introduces a scale effect into the loss of macroscopic instability of short-fibre-reinforced materials. The macroscopic response of large specimens made from these composites will be more sensitive to vanishing of bridging stiffness than that of small specimens with the same level of damage $(0 < \Omega < 1)$.

It is seen from Fig. 3 that the absolute value of $\bar{k}^{cr}$ increases rapidly when the value of Ω deviates from 1. This means that in order to delay macroscopic instability of the composite, it is very important to prevent or delay the coalescence of the discrete microcracks. In principle, if the microcracks are somehow prevented from coalescing whilst at the same time the tangent bridging stiffness is maintained above the critical value given by (4.6), macroscopic instability of the composite cannot occur. In practice though, the discrete microcracks are likely to propagate and coalesce once the fibres begin to be pulled out, i.e., once $\bar{k}$ reaches zero. Figure 4(d) illustrates such a possibility whereby the composite exhibits tension-softening, while the cracks in the localisation band are still fragmented $(0 < \Omega < 1)$ but the tangent bridging stiffness is equal to or less than the critical value (4.6). This provides a softening model which is different from that introduced by LI *et al.* [17] in which the softening is a result of fibre pull-out from a through crack $(\Omega = 1)$.

5. Conclusions

In this paper, based upon the pseudo-traction technique and an asymptotic analysis, the tensile stress-strain relation is established analytically for short-fibre-reinforced composites containing multiple parallel bridged microcracks. This allows an analysis to be made of the macroscopic material instability in the tensile deformation process of these composites. The material instability at the macrolevel is examined using the classical bifurcation criterion, with an emphasis on the role of the bridging action of fibres. Conditions for the incipient macroscopic instability are obtained as functions of damage in the matrix, crack length, and the microscopic bridging stiffness. It is found that no macroscopic

instability is possible as along as the tangent bridging stiffness is positive, i.e., as long as the fibres remain bonded to the matrix. However, whilst the bridging instability at the microlevel plays a major role in the macroscopic instability, it is the damage in the matrix that determines when the macroscopic instability is induced by the bridging instability. The microscopic bridging instability does not necessarily induce macroscopic instability. Indeed, macroscopic instability may be delayed until the fibres are only exerting residual frictional action. Likewise, the formation of a through crack is neither a necessary nor a sufficient condition for the onset of tension-softening in the composite. The results also suggest that in order to delay macroscopic instability in the tensile response, it is very important to prevent or to delay the coalescence of the discrete microcracks that form in the strain-hardening stage.

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