# Probabilistic micromechanical description of fatigue crack initiation

*Dedicated to Professor Zenon Mróz*
*on the occasion of his 70th birthday*

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IN THIS PAPER the Mura-Nakasone model for fatigue crack initiation is extended to the situation where randomness in material microstructure (e.g. in grain size) and in the applied stress are important factors. The analysis and formulae presented provide a quantitative characterization of the effect of statistical scatter in grain size of the metals on the crack initiation time. Also, crack initiation time is analyzed for the situation where the applied stress is a narrow-band stationary stochastic process. The mean value, variance and probability distribution of the time to crack initiation are derived, and the results are illustrated graphically for real empirical data.

## 1. Introduction

IT IS WELL RECOGNIZED that the nucleation or crack initiation process often comprises a significant portion of the fatigue life in engineering structures. The notion of crack initiation is however, not unique; the definition depends on the scale of observation. For example, in the assessment of fatigue life in engineering design, the crack nucleation time may be associated with the number of cycles necessary to initiate a dominant (long) crack. However, for physicists and material scientists, the initiation time is usually regarded as the time in which the defects in crystals appear and microscopic flaws along the slip bands are generated. It is clear that, in order to gain a proper understanding of the entire fatigue phenomenon in engineering materials, an insight into the microscopic mechanisms of fatigue is of great importance. Development of quantitative microscopic models for crack initiation is also of practical significance, especially at low stress levels

(i.e., high-cycle fatigue) where the crack initiation period usually consumes the predominant part of the fatigue life.

Although the fatigue crack initiation phenomenon is not yet fully understood, a number of proposals have been put forward to explain the origins of fatigue cracks and to estimate the associated initiation time (for a comprehensive analysis of the existing results, see Suresh [1]). In more recent models, various dislocation mechanisms in the metallic structure play a dominant role. The approaches which seem to be especially physically sound are associated with the key role played by persistent slip bands or PSB (i.e., narrow bands with highly localized cyclic strain accumulation) and with modelling the forward and reverse plastic displacement (within PSB) by dislocations motion (cf., Mura, Tanaka [2], Tanaka, Mura [3], Lin, Fine, Mura [4] and Mura, Nakasone [5]).

In the model of Mura and Nakasone [5], which has methodical similarities to the Griffith theory of (macroscopic) fracture, the fatigue crack initiation life is predicted as the number of cycles necessary for the Gibbs free energy to reach the maximum value. More specifically, according to Mura and Nakasone, the accumulation of dislocations within PSB (during the load cycles) enhances the internal tensile stress which leads to an energetically unstable state of the material, and therefore, this energy has to be released via the formation of an extremely thin void, i.e., the initiation of a crack. The critical time, in terms of the number of loading cycles, is obtained by considering the balance of the elastic strain energy (enhanced by the accumulated dislocations) and the energy released via the formation of the crack embrion in the PSB. In addition to its convincing physical background and reported applications to crack initiation under contact fatigue (cf., [6]), the model of Mura and Nakasone has features which make it feasible to perform the analysis aimed at quantification of the effects of randomness present in the crack initiation mechanism. The objective of this paper is to extend the Mura and Nakasone predictions of fatigue crack initiation time to the situations when randomness in the microstructure (specifically in the grain size) and in the applied stress occur as important factors. The analysis and formulae presented in the paper provide a quantitative characterization of the crack initiation time in terms of statistical parameters (e.g., variance, higher-order moments) of the grain size and the amplitude of the randomly varying applied stress.

## 2. Basic features of Mura-Nakasone model

Construction of a quantitative model for crack initiation in metallic polycrystal materials based on the accumulations of dislocation dipoles in PSBs and making use of the change in the Gibbs free energy, is presented in detail by Mura and Nakasone in Ref. [5] (cf., also the companion paper by Venkatadryan,

*et al.* [7]). Therefore, detailed derivations will not be repeated here. However, for completeness, the general idea has been presented in what follows.

The stress $s(t)$ applied across a slip plane in the material is assumed to be uniformly periodic. This applied stress causes dislocation accumulation which changes the Gibbs free energy of the system. After a certain number of cycles, this energy reaches a maximum value which is allowed to be released by forming a crack. The change in the Gibbs free energy $G$ is (cf. [5])

$$(2.1) \qquad \Delta G = -W_1 - W_2 + 2a_0 \gamma_s$$

where $\gamma_s$ is the surface energy of each face of the crack; $a_0$ is the embryonic micro-crack size; $W_1$ is the accumulated strain energy of dislocations after $N$ cycles of loading; and $W_2$ is the elastic energy release anticipated due to the opening of an embryonic initial crack of size $a_0$. In general, $W_2$ is defined in terms of the stress intensity factors for Mode I and II, and therefore, it depends on the specific geometrical situation in question, particularly on the location of crack initiation. If the crack initiation site is far from the surface of the body, one may use the intensity factors for a crack in an infinite body. In this case, the energy $W_2$ can be represented in a simple analytical form (cf. [5]). The embryonic initial crack size $a_0$ is calculated in terms of the density of dislocations in the slip band.

Appropriate calculations give an explicit result for $\Delta G$ in the form of a quadratic function of the number of cycles $N$, i.e., $\Delta G = -A_1 N^2 + A_2 N$, $A_1 > 0$, where $A_1, A_2 > 0$ are constants related to the geometry, material properties, dislocation characteristics, etc. Hence, there exists a critical cycle number $N^*$ for which $\Delta G$ takes a maximum and the system becomes unstable, i.e., when

$$(2.2) \qquad \frac{\partial}{\partial N}(\Delta G) = 0.$$

Solution of the above algebraic equation yields the following expression for the number $N^*$ of cycles to crack initiation:

$$(2.3) \qquad N^* = C\frac{2-f}{(\Delta s - 2s_f)f}, \qquad C = \frac{\gamma_s}{h\left(\ln\dfrac{2l}{h} - \dfrac{3}{2}\right)f},$$

where the following notation is introduced (cf., Fig. 1):

| | |
|---|---|
| $f$ | – slip irreversibility factor; $0 < f < 1$, |
| $s_f$ | – frictional stress of the material, |
| $\Delta s = s_{\max} - s_{\min}$ | – the shear stress range acting on slip layer, |
| $h$ | – width of the slip band, |
| $l$ | – length of the dislocation pile-up; grain size $d = 2l$. |

Formula (2.3) for $N^*$ has been obtained in [5] for an applied shear stress. For a normally applied stress, the shear stress range can be represented as

$$(2.4) \qquad\qquad \Delta s = \frac{\Delta s_n}{2} \sin \alpha,$$

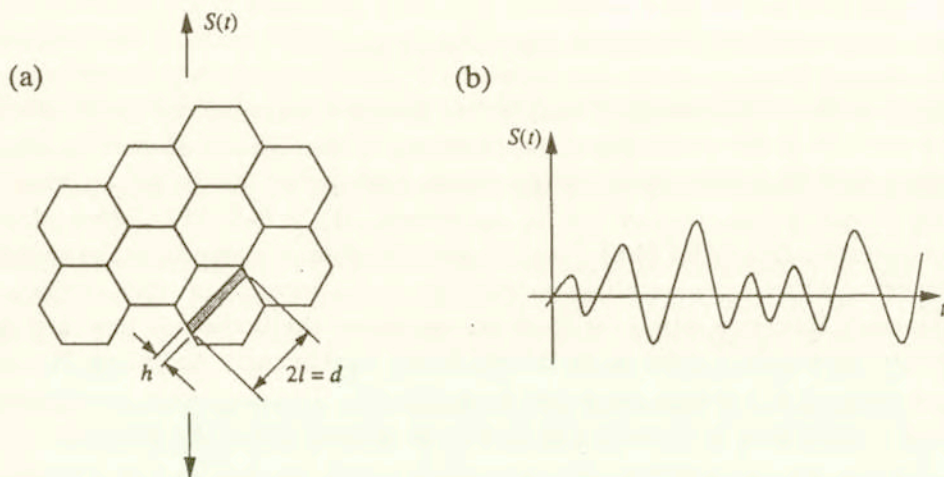where $\alpha = \pi/2 - \beta$, and $\beta$ is the angle between the slip band and the normal stress direction $s_n$.



FIG. 1. a) Schematic illustration of the crystalline structure and the PSB in a crystal grain; b) A possible trajectory of a narrowband random stress process acting on the slip plane.

The formula of Mura and Nakasone (2.3) may be regarded as an $S - N$ curve on the microscopic level. It clearly exhibits an inverse relation between the number of cycles for crack nucleation and the applied stress range. The fatigue limit can be regarded as twice the frictional stress $s_f$ of the material, i.e., no crack is initiated at stress ranges lower than $s_f$. Formula (2.3) also gives an explicit dependence of the crack initiation time on the grain size when the length of the persistent slip band $2l$ is regarded as the grain size. The nucleation time is greater for smaller grain sizes.

The frictional stress $s_f$ is the resistance of material to motion of the dislocations, and its value depends on the structure of the material. For convenience in practical applications, it has been assumed (cf., [6]) that $s_f$ can be replaced by the kinematic shear yield strength $s_y$. As far as the irreversibility factor $f$ is concerned, the experimental data indicate that values of $f$ are quite small (in the range from $10^{-4}$ for small plastic strain amplitudes to $10^{-1}$ for large strain amplitudes). In Ref. [5], the interpretation of the irreversibility factor $f$ is related to the ability to accumulate plastic deformation in the material (e.g., if

the material is purely elastic, it has no plastic deformations, and $f = 0$), and an estimate of $f$ from plastic strain is proposed. Since plastic strain depends on the applied stress amplitude $\Delta s$, the value of $f$ is, in general, a function of time. Using the experimental data reported in [8] for the tension-compression tests and the power empirical equation relating $f$ to the plastic strain $\varepsilon_p$, the authors of paper [6] give the following relationship: $f = c(\Delta s - 2s_f)^q$, where $q$ is the exponent in the relation: $f \sim (\varepsilon_p)^q$.

## 3. Effects of randomness on crack initiation time

### 3.1. General remarks

As it is seen from the short description of the model of Mura and Nakasone, the crack initiation life $N^*$ (or, $T^* = N^*/\omega_e$, where $\omega_e$ is the cyclic frequency of the applied load) depends on the basic material properties (e.g., grain size and frictional stress, which in turn depends on temperature and material hardness), the applied stress and the boundary conditions when a specific crack initiation problem is considered for a finite body. However, what has been widely recognized, an additional factor that plays a significant role is the statistical variability of the mirostructural properties. An important source of this variability is the randomness in the grain size.

In the micromechanical modelling of various phenomena (including the Mura and Nakasone model), a mean (or characteristic) grain size is usually used. However, as we have discussed previously (cf., SOBCZYK, SPENCER [9]), the mean grain diameter is not a number which can be readily derived or estimated from any single kind of metallurgical measurement. It is usually obtained by dividing the total of the mean grain diameters by the number of grains in some "representative volume". But, the mean grain size diameter always provides only a very rough information about the microstructure. Although the basic stereological relationships for the mean grain diameter are viewed as exact, the accuracy of the measurement depends on the number of grid applications and the degree of irregularity of the grains. In addition, the passage from the true spatial dimensions to their counterparts observed on planar sections involves a great loss of information (cf., [9]). Therefore, it seems to be important to incorporate into the model of Sec. 2, a more detailed information on the grain size, e.g., its statistical moments of higher order and probability distribution. Moreover, appropriate modification of the model (2.3) to include randomness in the applied stress seems to be important for wider applicability of the model.

## 3.2. Randomness in grain size

According to [5], the grain size in formula (2.3) expressing the initiation life-time, is the mean grain size $d = 2l$. Therefore, the question which arises is: what form should formula (2.3) take if a more detailed information about the grain size is to be taken into account? We mean such an extension of the initiation time formula (2.3) which accounts for a more detailed characterization of the grain size, while at the same time it preserves the information contained in formula (2.3), i.e. regarding the mean grain size. A possible approach is as follows.

Equation (2.3) can be rewritten as

$$(3.1) \qquad N^* = \varphi_{M,N}(d; \alpha), \qquad \varphi_{M,N}(d; \alpha) = \frac{A_0}{\ln d - B_0},$$

where $A_0$ and $B_0$ are the following constants:

$$(3.2) \qquad A_0 = \frac{\gamma_s(2 - f)}{hf(\Delta s - 2s_f)}, \qquad B_0 = \ln h + \frac{3}{2},$$

and $\alpha$ in (3.1) symbolizes all parameters occurring in $A_0$ and $B_0$. Let us consider the relationship

$$(3.3) \qquad N^* = \varphi(X; \alpha)$$

where $X$ is a random variable characterizing the grain size. Let us assume that the mean value $\langle X \rangle$ of $X$ is $d$, the variance of $X$ is $\sigma_X^2$, and the third and fourth central moments are denoted by $m_3$, $m_4$, respectively. Expansion of $\varphi(X; \alpha)$ in a Taylor series about the point $d = \langle X \rangle$ yields

$$(3.4) \qquad N^* = \varphi(d; \alpha) + (X - d)\frac{d\varphi}{dX}\Big|_{X=d} + \frac{1}{2}(X - d)^2\frac{d^2\varphi}{dX^2}\Big|_{X=d} + \dots.$$

Formula (3.1) is seen to take into account only the first term in (3.4); in this sense we can say that the Mura, Nakasone formula (2.3) written in the form (3.1) constitutes a zero-order approximation of $N^*$. In order to account for the variance of the grain size as well as for its higher order moments, the further terms in Eq. (3.4) should be considered. Averaging of both sides of (3.4) yields

$$(3.5) \qquad \hat{N}^* = \varphi(d; \alpha) + \frac{1}{2}\sigma_X^2\frac{d^2\varphi}{dX^2}\Big|_{X=d} + \frac{1}{6}m_3\frac{d^3\varphi}{dX^3}\Big|_{X=d} + \dots.$$

Therefore, the Mura-Nakasone formula for crack initiation time generalized in this way is

(3.6) $\quad \hat{N}^* = \varphi(d; \alpha) + \dfrac{1}{2}\sigma_X^2 \dfrac{A_0(\ln d - B_0 + 2)}{d^2(\ln d - B_0)^3}$

$$+ \dfrac{1}{6}m_3 \dfrac{A_0(2B_0 - 2\ln d - 6)}{d^3(\ln d - B_0)^4} + \ldots$$

where, according to (3.1), (3.2), the first term on the right-hand side of (3.6) is the right-hand side of the Mura, Nakasone formula (2.3).

Formula (3.6) can be written as:

(3.7) $$\hat{N}^* = \varphi_{M,N}(d; \alpha)[1 + \eta(d; \alpha, \sigma_X^2, m_3, \ldots)]$$

where

(3.8) $\quad \eta(d; \alpha, \sigma_X^2, m_3, \ldots) = \dfrac{1}{2}\sigma_X^2 \dfrac{\ln d - B_0 + 2}{d^2(\ln d - B_0)^2} + \dfrac{1}{6}m_3 \dfrac{2B_0 - 2\ln d - 6}{d^3(\ln d - B_0)^3} + \ldots$

For the case of uniform distribution of the grain size $X$ in the interval $[d - \Delta, d + \Delta]$, where $\Delta$ can be regarded as a scatter parameter, we have

(3.9) $$\langle X \rangle = d, \quad \sigma_X^2 = \dfrac{\Delta^3}{3}, \quad m_3 = d(d^2 + \Delta^2)$$

and formula (3.8) takes the form

(3.10) $\quad \eta(d; \alpha, \sigma_X^2, m_3, \ldots) = \dfrac{1}{6}\Delta^2 \dfrac{\ln d - B_0 + 2}{d^2(\ln d - B_0)^2}$

$$+ \dfrac{1}{6}(d^2 + \Delta^2)\dfrac{2B_0 - 2\ln d - 6}{d^3(\ln d - B_0)^3} + \ldots$$

In the case of lognormal distribution of the grain size (which is a common hypothesis) i.e., when the probability density of $X$ is

(3.11) $\quad f_X(x) = \dfrac{1}{\zeta x \sqrt{2\pi}} \exp\left[-\dfrac{1}{2}\left(\dfrac{\ln x - \lambda}{\zeta}\right)^2\right], \quad 0 < x < \infty$

where $\zeta > 0$ ($\lambda$ and $\zeta^2$ are the mean value and variance, respectively, of the normal random variable $\ln X$), formula (3.7) takes the form

(3.12) $\quad \hat{N}^* = \varphi_{M,N}(d; \alpha)\left[1 + \dfrac{1}{2}\left(\exp(\zeta^2) - 1\right)\dfrac{\ln d - B_0 + 2}{(\ln d - B_0)^2}\right].$

Figures 2 and 3 visualize the effect of the uniform randomness in the grain size on the number of cycles to crack initiation (along with the Mura-Nakasone predictions).
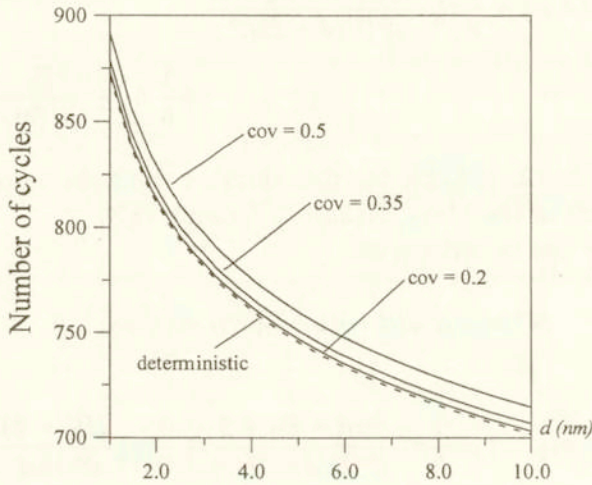
FIG. 2. Mean value of the number of cycles to crack initiation for different values of the coefficient of variation versus mean value of the grain size. Uniform distribution of the grain size is assumed.
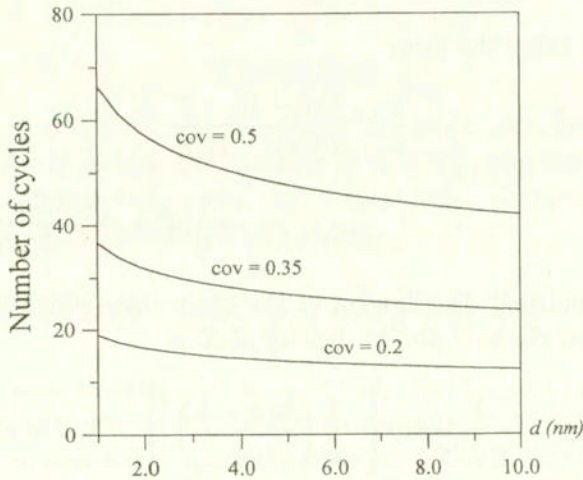


FIG. 3. Standard deviation of the number of cycles to crack initiation.

## 3.3. Randomness in applied stress

The model of Mura, Nakasone (2.3) has been built under the assumption of constant amplitude cyclic loading. However, most loads which are of interest in engineering practice show significant random variability. Moreover, when the model is used to predict the crack initiation time in such problems as fatigue

under contact stresses (cf., [6]), the stress amplitude $\Delta s$ occurring in formula (2.3) is determined by the contact pressure $p$ as a function of the horizontal spatial coordinate, the friction force and the residual stresses which depend on the morphology of the surface and properties of the material. These factors provide additional contributions to the random variability of the stress range. Extension of the model to a general randomly varying load which accounts for the interaction effects seem to be premature. However, accounting for a random loading seems to be possible if we restrict our attention to random processes which, in a sense, resemble harmonic oscillations with randomly varying amplitude, i.e., stationary narrow-band random processes. In this case, one can adopt the notions of an equivalent cycle and an equivalent stress range (cf., [10]).

Let us assume that a time-varying stress acting on the structural component and generating fatigue crack initiation is characterized by a narrow-band stationary random process $S(t)$ with the mean $m_S = 0$, standard deviation $\sigma_S$ and spectral density $g_S(\omega)$. For a narrow-band process, the spectral density $g_S(\omega)$ has significant values over a narrow band of frequencies (around certain central frequency $\omega_0$). The spectral content of such a process is conveniently characterized by the spectral moments $\lambda_k$ and the regularity factor $\beta$ (cf., [10])

$$(3.13) \qquad \lambda_k = \int\limits_{-\infty}^{+\infty} \omega^k g_S(\omega)d\omega, \qquad \beta = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}}.$$

If the process $S(t)$ is additionally Gaussian, then the regularity factor $\beta$ is the ratio of the average number of zero-crossings by process $S(t)$ to the average number of peaks. A narrow-band process has (approximately) an equal number of peaks and zero-crossings with positive (or negative) slope, so $\beta \to 1$.

The stress range $\Delta s = H_S$ occurring in formula (2.3) in the case of a random (stationary, Gaussian) $S(t)$ can be characterized in various ways. We propose here two approximations. The first approximation is a mean range:

$$(3.14) \qquad S_{\mathrm{mr}} = \langle \Delta S \rangle = \langle S_{\max} \rangle - \langle S_{\min} \rangle = 2\langle Z \rangle$$

where $Z$ is a random height of the peaks of the stress process which has a Rayleigh probability distribution with the density

$$(3.15) \qquad p(z) = \frac{z}{v^2}\exp\left(-\frac{z^2}{2v^2}\right), \qquad z \geq 0$$

with $v = \sigma_S$. After evaluating $\langle Z \rangle$ by integration with respect to the density (3.15), we have

$$(3.16) \qquad S_{\mathrm{mr}} = 2S_{\mathrm{rms}}\sqrt{\frac{\pi}{2}(1 - \varepsilon^2)}, \qquad \varepsilon = \sqrt{1 - v^2}.$$

For a narrow band process, $\varepsilon \to 0$ and one obtains

(3.17) $$S_{\mathrm{mr}} = \sqrt{2\pi}S_{\mathrm{rms}} = \sqrt{2\pi}\sigma_S.$$

Therefore, the formula (2.3) of Mura and Nakasone modified in this way is

(3.18) $$\hat{N}^* = C\frac{2-f}{\left(\sqrt{2\pi}\sigma_S - 2s_f\right)f}.$$

A second, more satisfactory modification of formula (2.3) is as follows. For a narrow-band random Gaussian stress process, the stress range $H_S$ can be quantified as being equal to $2Z$. Hence, it is a random variable which has a Rayleigh distribution $p(z)$ given by (3.15) with parameter $v = 2\sigma_S$. In this case, the time to crack initiation is a random variable $T^* = N^*/w_e$, where $w_e$ is the equivalent frequency of the stress process $S(t)$. Explicitly,

$$w_e T^* = N^* = C\frac{2-f}{(H_S - 2s_f)f} = C_1\frac{1}{H_S - 2s_f},$$

(3.19)

$$C_1 = \frac{C(2-f)}{f},$$

where $C$ is given in (2.3). The time to crack initiation is a positive quantity, i.e. $T^* > 0$. This implies that

(3.20) $$H_S - 2s_f > 0$$

since $f$ occurring in (3.19) is a parameter with its possible numerical values around 0.1 in an air environment. Therefore, to satisfy the condition (3.20) the stress range $H_S$ should have a shifted Rayleigh distribution

(3.21) $$p_{H_s}(z) = \frac{z - 2s_f}{v^2}\exp\left(-\frac{(z - 2s_f)^2}{2v^2}\right), \qquad z > 2s_f$$

or a conditional Rayleigh distribution $p_{\tilde{H}_s}(z|H_S > 2s_f) = p(z)/\left(1 - \int_0^{2s_f} p(x)dx\right)$ where $p(z)$ is given in (3.15). Our further considerations will be focused on the influence of the shifted Rayleigh distribution on the distribution of the time to crack initiation $T^*$. Due to (3.19) and (3.21), this distribution has the form

(3.22) $$p_{T^*}(t) = \frac{C_1}{v^2 t^3}\exp\left(-\frac{C_1^2}{2v^2 t^2}\right), \qquad t > 0$$

where $v = 2\sigma_S$. It is worth noting that the probability density (3.22) has no finite moments of orders higher that two. Distribution (3.22) allows us to calculate

exact values of the mean and the variance of the time to crack initiation. However, in practice, simpler formulae for mean value and variance of $N^*$ may be useful. Using the Taylor expansion of $N^*$ given by (3.19) with respect to the random variable $H_S$ around the mean value $\langle H_S \rangle = h_S$ and averaging, we obtain the following results for $\langle N^* \rangle$

$$(3.23) \qquad \langle N^* \rangle = \frac{C_1}{h_S - 2s_f} + 4\left(2 - \frac{\pi}{2}\right)\frac{C_1}{(h_S - 2s_f)^3}\sigma_S^2 + \dots$$

and the first order estimate of variance $\mathrm{var}(N^*)$

$$(3.24) \qquad \mathrm{var}(N^*) = (4 - \pi)\frac{C_1}{(h_S - 2s_f)^4}\sigma_S^2.$$

Since $H_S$ has distribution (3.21) with $v = 2\sigma_S$, hence

$$(3.25) \qquad \langle H_S \rangle = h_S = \sqrt{\pi v/2} + 2s_f = \sigma_S\sqrt{2\pi} + 2s_f.$$

It is seen that formula (3.18) accounts only for the first term in (3.23). The equivalent frequency occurring in (3.19) is (cf. [10])

$$(3.26) \qquad \omega_e = 2\pi\langle N^+(0.1) \rangle = \frac{\lambda_2}{\sigma_S} = \frac{\sigma_{\dot{S}}}{\sigma_S},$$

where $\langle N^+(0.1) \rangle$ is the expected number of zero-crossings with positive slope. Therefore,

$$(3.27) \qquad \langle T^* \rangle = \frac{1}{\omega_e}\langle N^* \rangle, \qquad \mathrm{var}(T^*) \approx \frac{1}{\omega_e^2}\mathrm{var}(N^*).$$

## 4. Numerical illustration

In order to determine numerically the effect of randomness on the crack initiation time according to the formulae derived in the previous sections, typical values of the basic parameters of the model have to be specified. The micro-mechanical parameters of the model depend upon the material under consideration, the environment, the temperature etc. For example, the friction stress $s_f$ characterizes the resistance of material to the motion of the dislocations and its value depends on the structure of the material, and usually it is affected by temperature (cf., [12]); as indicated in [6], it can be express in terms of material hardness and some empirical constants.

Here, we adopt the parameters occurring in [5]. They are as follows: $h = 0.016$ mm, $s_f = 25$ MPa $= 25$ N/mm$^2$, $\gamma_S = 35$ N/m, $f = 0.1$. The values of the parameters $\gamma_S$, $f$ are characteristic for air environment. Figures 2 and 3

visualize the effect of randomness in the grain size on the number of cycles $N^*$ to crack initiation. In these figures, the dependence of the mean value and standard deviation of the number of cycles to crack initiation on the coefficient of variation of the grain size are presented versus the mean value of the grain size, respectively. The uniform distribution of the grain size in the interval $[d - \Delta, d + \Delta]$ and the deterministic stress range $\Delta S = 100$ MPa are assumed. The mean values $\hat{N}^*$ presented in the Fig. 2 are calculated on the basis of the formulae (3.7) when only the mean value $d$ and variance $\sigma_X^2 = \Delta^2/3$ of the grain size are taken into account, i.e.

$$(4.1) \qquad \hat{N}^* = \frac{A_0}{\ln d - B_0} \left[ 1 + \frac{1}{6} \Delta^2 \frac{\ln d - B_0 + 2}{d^2(\ln d - B_0)^2} \right].$$

The dashed curve presents the deterministic number of cycles to crack initiation obtained from the formulae (2.3). Figure 3 shows the standard deviations of the number of cycles $N^* = T^*/\omega_e$ when they are calculated using the following exact probability density:

$$(4.2) \qquad f_{T^*}(t) = \left( \frac{A_0}{\omega_e t^2} \exp(A_0/\omega_e t) + B_0 \right) f_d(\exp(A_0/\omega_e t) + B_0), \quad t > 0.$$

This density is obtained in the common way (e.g. [10]) when $N^*$ in formula (2.3) is regarded as a function of random variable $d = 2l$ with probability density $f_d(x)$, where $x > h\exp (1.5)$ to satisfy the condition $T^* > 0$. It turns out, that the approximation (4.1) of the mean number of the cycles to crack initiation gives a very satisfactory agreement with the exact mean value $< N^* >_d = \int_0^\infty (t/ \omega_e) f_{T^*}(t) dt$ calculated from the exact probability density (4.2). In the case of uniform distribution of the grain size, the relative error $R = | < N^* >_d - \hat{N}^* |/ < N^* >_d$ is very small. Namely, for the considered coefficients of variation cov $= 0.2$, cov $= 0.35$ and cov $= 0.5$, the error $R$ satisfies the inequalities $R < 0.01\%$, $R < 0.15\%$ and $R < 0.8\%$, respectively.

To quantify the effect of randomness in applied stress on the basis of the formulae given above, let us assume that process $S(t)$ is Gaussian and stationary with the following correlation function and the corresponding spectral density

$$(4.3) \qquad K_S(t) = \sigma_S^2 \exp(-\alpha^2 \tau^2), \quad g_S(\omega) = \frac{\sigma_S^2}{2\alpha\sqrt{\pi}} \exp\left( -\frac{\omega^2}{2\alpha^2} \right),$$

where $\tau = t_2 - t_1$, $\sigma_S^2$ is the variance of $S(t)$ and $\alpha$ is the correlation parameter characterizing the rate of correlation decay with increase of $\tau$.

According to (3.26), the equivalent frequency is $\omega_e = 2\alpha^2$. The regularity factor, according to (3.13), is $\beta = 1.95\alpha$. To make the considered process narrow-band, we select a value for $\alpha$ for which the regularity factor $\beta$ tends to one; therefore we take $\alpha \approx 0.5$. In this case

(4.4)
$$\omega_e = \frac{\sigma_{\dot{S}}}{\sigma_S} = \frac{\sqrt{2}\alpha\beta_S}{\sigma_S} = \sqrt{2}\alpha \approx 0.7.$$

Figures 4 – 6 show the probability density (3.22) of the random crack nucleation time for various numerical values of the grain size $d$, standard deviation $\sigma_S$
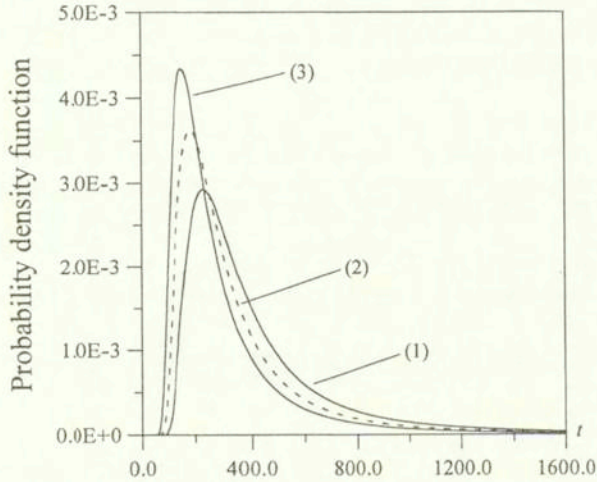


FIG. 4. Probability density of the crack initiation time for different values of grain size: (1) $d = 0.001$; (2) $d = 0.01$; (3) $d = 0.1$ mm; standard deviation of the stress process $\sigma_S = 55$ MPa.
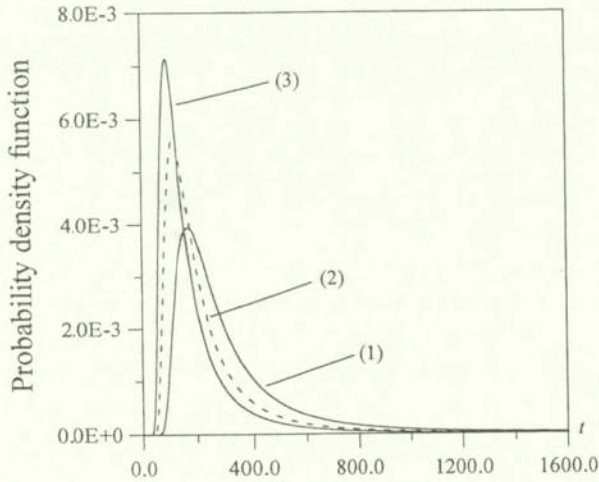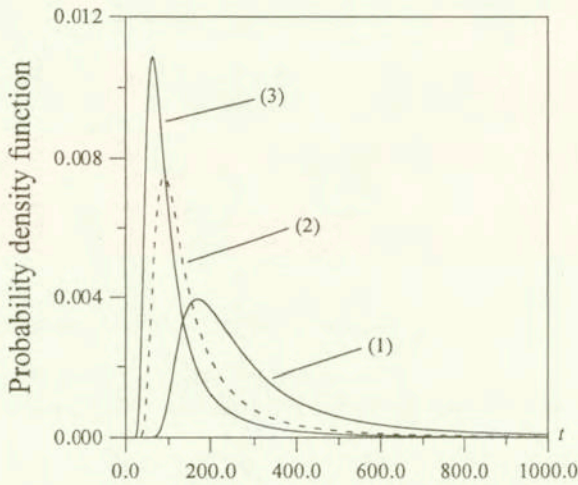


FIG. 5. Probability density of crack initiation time for different values of the standard deviation $\sigma_S$ of the stress process: (1) 55.0 MPa, (2) 75.0 MPa, (3) 95.0 MPa, grain size $d = 0.05$ mm.

of the random applied stress and width of the slip-band $h$, respectively. The characteristic behavior of the probability $P(T^* > t) = \int_t^\infty f_{T^*}(x)dx$ of no crack nucleation for fixed time $t$ should be underlined. For each figure given above, growth of the values of the parameters $d$, $\sigma_S$ and $h$ causes a decrease in the probability $P(T^* > t)$, which means an increase in the crack nucleation time. It is also interesting to notice that probability densities converge very slowly to zero in their right-hand side tails, especially for the case of the small values of the parameters $d$, $\sigma_S$ and $h$.



FIG. 6. Probability density of the crack initiation time for different values of the slip band width $h$: (1) $h = 0.016$; (2) $h = 0.032$; (3) $h = 0.048$ (nm); grain size $d = 0.05$ mm, standard deviation of the stress process $\sigma_S = 75$ MPa.

Figures 7 – 9 illustrate the exact mean value of the crack initiatin time calculated using the exact probability density (3.22), that is

$$(4.5) \qquad \langle N^* \rangle_S = \langle \omega_e T^* \rangle = \omega_e \int_0^\infty t p_{H_s}(t)dt = \frac{C_1\sqrt{2\pi}}{4\sigma_S}, \qquad C_1 = \frac{C(2-f)}{f},$$

and its approximations obtained from the formula (3.23). Curve (1) presents the zero-order approximation $\hat{N}^*_{(1)}$ of $\langle N^* \rangle_S$ when only the first term in (3.23) is accounted for. Curve (2) shows the second-order approximation $\hat{N}^*_{(2)}$ of $\langle N^* \rangle_S$ when the first and second terms in (3.23) are included. According to (3.23) and (3.25), the explicit formulae for $\hat{N}^*_{(1)}$ and $\hat{N}^*_{(2)}$ are as follows:

(4.6)
$$\hat{N}^*_{(1)} = \frac{C_1}{\sigma_S\sqrt{2\pi}},$$

(4.7)
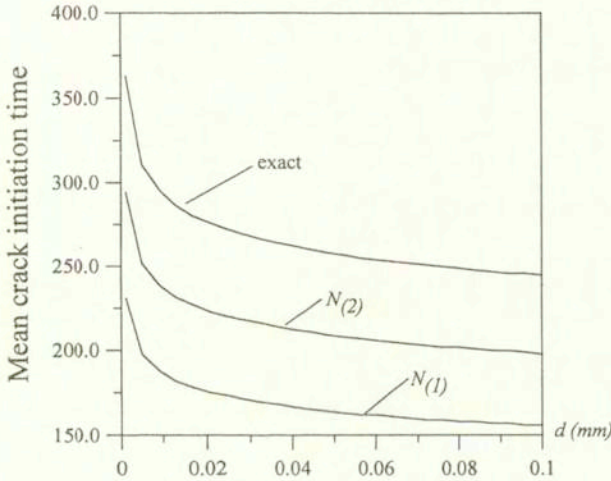$$\hat{N}^*_{(2)} = \frac{4C_1}{\sigma_S\pi\sqrt{2\pi}}.$$



FIG. 7. Exact mean value of the crack initiation and its approximations versus different grain size when random stress process with standard deviation $\sigma_S = 55$ MPa is applied. Width of the slip band $h = 0.016$ nm.
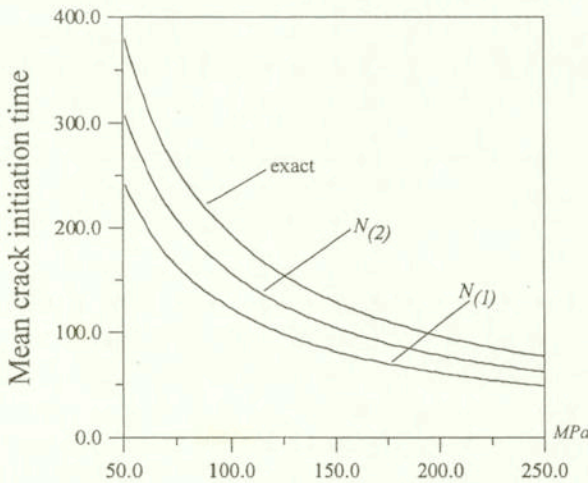


FIG. 8. Exact mean value of the crack initiation time and its approximations versus different standard deviations $\sigma_S$ of the random stress process. Grain size $d = 0.05$ mm, slip band $h = 0.016$ nm.
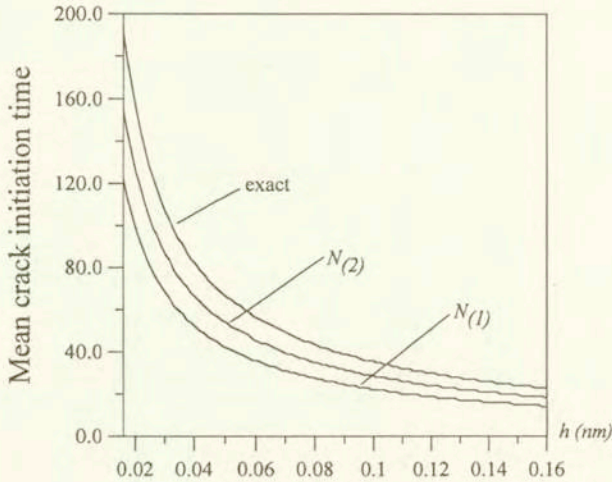
FIG. 9. Exact mean value of the crack initiation time and its approximations versus different width of the slip band $h$ when random stresss process with standard deviation $\sigma_S = 75$ MPa is applied. Grain size $d = 0.05$ mm.

A simple calculation gives the following relationships between the exact mean value $\langle N^* \rangle_S$ and its approximations

$$(4.8) \qquad \langle N^* \rangle_S = \frac{1}{2}\pi \hat{N}^*_{(1)}, \qquad \langle N^* \rangle_S = \frac{1}{8}\pi^2 \hat{N}^*_{(2)}.$$

Figures 7 – 9 and relationship (4.8) show that the approximation of the exact mean crack nucleation time, taking into account only the mean value and variance of the loading process, can not be sufficiently satisfactory. In such a case the higher order statistical moments of random loading process should be considered. Nevertheless, the approximation (4.7) seems to be a very effective and simple tool to predict the lower-bound approximation of the exact mean crack initiation time.

## 5. Conclusions

In the paper the fatigue crack initiation has been studied with a special attention given to the effects of a randomness in the grain size and the applied stress. Making use of the idea proposed for the deterministic case by Mura and Nakasone, we derived simple formulae for the statistical moments and probability distribution of a random crack initiation time. The numerical calculations for real material parameters show the sensivity of the initiation time to the random variations in the grain size or the applied stress.

For example, Fig. 3 shows that the coefficient of variation of grain size affects significantly the standard deviation of the initiation time. This indicates the

importance of the higher-order moments of the grain size in prediction of the initiation time. Figures 4 – 9 illustrate the dependence of various statistics of the initiation time on the basic material constants and characteristics of the applied stress obtained under various approximations. The formulae derived and the figures presented can be easily used for estimation of the crack initiation time for purposes of the reliability assessment.

## Acknowledgment

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