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A qualitative approach to Hooke's tensors. Part I

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The main qualitative properties of Hooke's tensors can be found in their invariant decompositions, both linear and nonlinear. The invariant nonlinear spectral decompositions are presented in the review [7] and the papers quoted therein. This paper deals with linear invariant decompositions initiated in [12] – [20]. A straightforward and complete description of all such possible decompositions is presented here in Part I. The main results are given in formulae (7.1), (7.3). The next part (to appear), Part II, will contain derivations, conclusions and unexpected applications.

1. Introduction

1.1. In this paper we will call *Hooke's tensor* any Euclidean tensor of the fourth order $\mathbf{H}$ which realises a symmetric linear transfromation of the space of symmetric second order tensors $\mathcal{S}$ into itself, or quite the same, which realises a quadratic form on $\mathcal{S}$,

$$\xi \to \mathbf{H} \cdot \xi \qquad \xi \to \xi \cdot \mathbf{H} \cdot \xi. \tag{1.1}$$

It is convenient to identify *the space of all Hooke's tensors* $\mathcal{H}$ with symetrized tensorial square of space $\mathcal{S}$

$$\mathcal{H} = \operatorname{sym} \mathcal{S} \otimes \mathcal{S}. \tag{1.2}$$

1.2. The importance of Hooke's tensors in many applications cannot be overestimated. They are, primarily, the starting point of the linear theory of elasticity, which is still, for many essential areas of science and technology, the most important part of solid state mechanics. This starting point are elasticity tensors: the *stiffness tensor* $\mathbf{S}$ and the *compliance tensor* $\mathbf{C}$ of Hooke's law

$$\sigma = \mathbf{S} \cdot \varepsilon, \quad \varepsilon = \mathbf{C} \cdot \sigma, \quad \mathbf{S} \circ \mathbf{C} = \mathbf{C} \circ \mathbf{S} = \mathbb{I}_{\mathcal{S}} \tag{1.3}$$

Quadratic forms

$$e(\varepsilon) \equiv \varepsilon \cdot \mathbf{S} \cdot \varepsilon, \qquad w(\sigma) \equiv \sigma \cdot \mathbf{C} \cdot \sigma \tag{1.4}$$

are here respectively, doubled *energy of infinitesimal deformation* ε and doubled *work of stress* σ.

The *limit tensor* **M** in quadratic limit criterion

$$\sigma \cdot \mathbf{M} \cdot \sigma \leqslant \text{const} \tag{1.5}$$

is also a Hooke's tensor. It was introduced by R. VON MISES [1] and later popularised for the orthotropy case by R. HILL [2] and others. The form $m(\sigma) \equiv \sigma \cdot \mathbf{M} \cdot \sigma$ is sometimes called the Mises stress intensity. Earlier we have demonstrated [3] that the condition (1.5) for *each* anisotropic elastic material defined by compliance tensor **C** has a uniquely defined energy meaning.

1.3. The number of theoretical papers on Hooke's tensors is staggering. Yet new ideas, approaches and results appear not very often. For example, although their symmetry with respect to rotations and mirror-reflections of the basic Euclidean space, enjoys great description by A. E. H. LOVE and W. VOIGT, it still remains interesting (see e.g. [4]). Still far from a satisfactory and effective description is the problem of complete systems of invariants of Hooke's tensors (see the key results [5]).

1.4. In the recent years, one of the more promising directions of the qualitative analysis of Hook's tensors are their invariant decompositions. In the last 15 years, particular development can be observed in the direction [6], originated, as it soon turned out, by Lord Kelvin 150 years ago. It consists in **spectral decomposition** of an individual Hooke's tensor. Symmetric operator **H** acts in space endowed with scalar product $\alpha \cdot \beta$, hence the unique spectral decomposition is the case (see [6, 7])

$$\mathbf{H} = h_1\mathbf{P}_1 + ... + h_r\mathbf{P}_r, \qquad h_1 < ... < h_r, \quad r \leqslant 6, \tag{1.6}$$

where invariants $h_1, \ldots, h_r$ are eigenvalues and tensors $\mathbf{P}_1, ..., \mathbf{P}_r$ constitute an orthogonal decomposition of unity

$$\mathbb{I}_{\mathcal{S}} = \mathbf{P}_1 + ... + \mathbf{P}_r, \qquad \mathbf{P}_k \circ \mathbf{P}_l = \begin{cases} \mathbf{P}_k & k = l, \\ \mathbf{0} & k \neq l. \end{cases} \tag{1.7}$$

The orthogonal projector $\xi \to \mathbf{P}_k \cdot \xi$ projects orthogonally $\mathcal{S}$ onto subspace $\mathcal{P}_k$ composed of proper states ω of tensor **H** which correspond to eigenvalue h_k, $\mathbf{H} \cdot \omega = h_k\omega$,

$$\mathcal{S} = \mathcal{P}_1 \dotplus ... \dotplus \mathcal{P}_r, \qquad \dim \mathcal{P}_k = \operatorname{Tr} \mathbf{P}_k. \tag{1.8}$$

Sometimes it is more convenient to put the spectral decomposition in a less rigorous form

$$\mathbf{H} = h_1 \omega_1 \otimes \omega_1 + ... + h_6 \omega_6 \otimes \omega_6, \tag{1.9}$$

without demanding the difference between the eigenvalues. Here ω_k are proper states which constitute the orthonormal basis in $\mathcal{S}$,

$$\mathbf{H} \cdot \omega_1 = h_1 \omega_1, ..., \quad \mathbf{H} \cdot \omega_6 = h_6 \omega_6, \quad \omega_k \cdot \omega_l = \delta_{kl}.$$

For example, for a *cubic crystal*, the stiffness tensor has the spectral decomposition

$$\mathbf{S} = s_1 \mathbf{P}_1 + s_2 \mathbf{P}_2 + s_3 \mathbf{P}_3, \tag{1.10}$$

while the decomposition of the space of deformations into the subspaces of proper states has the form

$$\mathcal{S} = \mathcal{P}_1 \dotplus \mathcal{P}_2 \dotplus \mathcal{P}_3, \qquad 6 = 1 + 2 + 3 \tag{1.11}$$

where, using the crystal axis we have

$$\mathcal{P}_1 \sim \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}, \quad \mathcal{P}_2 \sim \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -u-v \end{pmatrix}, \quad \mathcal{P}_3 \sim \begin{pmatrix} 0 & p & r \\ p & 0 & q \\ r & q & 0 \end{pmatrix}. \tag{1.12}$$

To this example we will return in Sec. 3, Part II of this paper (to be published in Arch. Mech.).

The invariants h_k and projectors $\mathbf{P}_k$ are isotropic functions on the space $\mathcal{H}$. So the spectral representations (1.6) are *nonlinear* invariant decompositions of Hooke's tensors. As to the history, details and benefits of this approach, see paper [6], the review [7] and the references quoted therein.

1.5. Another, completely different approach to Hooke's tensors consists in their *linear* invariant decompostions. They appeared as a natural adaptation of the classical theory of group representations (see e.g. [8, 9, 10, 11]). Thus the obtained results are given in papers [12, 13, 14, 15, 16, 17, 18, 19, 20], and possibly elsewhere. These papers are useful and have been applied, but they have certain drawbacks. Only some decompositions are described there, while the mathematical tools are unnecessarily complex, and, in any case, uknown to most of the specialists in solid mechanics. For some historical remarks see paper [4].

1.6. This paper deals with the *very linear invariant decompositions of the space of Hooke's tensors $\mathcal{H}$*. We will try to:

(1) Systematically and completely describe *all such possible* decompositions.

(2) Do this in a straightforward and possibly simple way, referring to the most basic notions of geometry and algebra (on purpose, we do neglect references to the theory of group representations, harmonic functions, etc., being quite unneeded here).

(3) Persuade the Reader that this approach is interesting, and may perhaps lead to applications, worth considering, important and even surprising (see Part II, esp. examples A trough G).

This set of problems dealt with here touches upon the theory of tensor functions and its applications. We do not describe these relations, making references to the exisiting broad reviews [21, 22, 23, 24].

2. A classical decomposition of $\mathcal{H}$ with respect to internal symmetry

2.1. Let us begin with acting on Hooke's tensors of 4! permutations of the symmetric group $\Sigma = \Sigma_4$. These are linear operations $\sigma \times \mathbf{H}$ referred to in Appendix 1. Each linear combination of operators $\sigma\times$ will be called **permutation operator**. Put them as a formal linear combination of elements of the group Σ

$$\mathfrak{p} = (a_1\sigma_1 + ... + a_{24}\sigma_{24}), \quad \mathfrak{p} \times \mathbf{H} \equiv \sum_{1}^{24} a_i (\sigma_i \times \mathbf{H}) \tag{2.1}$$

(here some of the factors a_i might of course be equal to zero).

Every Hooke's tensor $\mathbf{H}$ has the internal symmetry, usually given in the form of tediously written equations for its components

$$H_{ijkl} = H_{ijlk} = H_{jikl} = H_{klij}. \tag{2.2}$$

Properly speaking, it means that this is a tensor symmetrical to the following subgroup:

$$\Sigma_{\mathcal{H}} \equiv \{(1234), (2134), (1243), (3412), (2143), (4321), (1423), (1324)\}. \tag{2.3}$$

Let us number its elements from 1 to 8. Let us introduce the symmetrization operator with respect to this subgroup

$$\mathfrak{s}_{\mathcal{H}} \equiv \frac{1}{8}(\sigma_1 + \cdots + \sigma_8), \qquad \sigma_i \in \Sigma_{\mathcal{H}}. \tag{2.4}$$

Cleary, repeating this operation does not change the result, i.e. $\mathfrak{s}_{\mathcal{H}} \circ \mathfrak{s}_{\mathcal{H}} = \mathfrak{s}_{\mathcal{H}}$. The operation $\bigotimes^4 \mathcal{E} \ni \mathbf{A} \to \mathfrak{s}_{\mathcal{H}} \times \mathbf{A} \in \bigotimes^4 \mathcal{E}$ is therefore a projector. It projects the entire 81-dimensional space $\bigotimes^4 \mathcal{E}$ onto the 21-dimensional subspace of Hooke's tensors, $\mathcal{H} = \mathfrak{s}_{\mathcal{H}} \times \left(\bigotimes^4 \mathcal{E}\right)$.

The entire group Σ is generated by subgroup $\Sigma_{\mathcal{H}}$ and two appropriately chosen (external to it) permutations, e.g. two transpositions

$$\langle 23 \rangle \equiv (1324), \qquad \langle 24 \rangle \equiv (1432). \tag{2.5}$$

Let us introduce two convenient permutation operators

$$\mathfrak{l} \equiv id \equiv (1234), \qquad \mathfrak{c} \equiv \frac{1}{2}(\langle 23 \rangle + \langle 24 \rangle). \tag{2.6}$$

They can be considered to be basic permutation operators acting on Hooke's tensors in the following sense.

LEMMA. Permutation operator $\mathfrak{p}$ transforms each Hooke's tensor into a Hooke's tensor, i.e. $\mathfrak{p} \times \mathcal{H} \subset \mathcal{H}$, iff it can be written in the form

$$\mathfrak{p} = a\mathfrak{l} + b\mathfrak{c}. \tag{2.7}$$

P r o o f. It is evident that the property $\mathfrak{p} \times \mathcal{H} \subset \mathcal{H}$ can be equivalently given as the commutation rule $\mathfrak{p} \circ \mathfrak{s}_{\mathcal{H}} = \mathfrak{s}_{\mathcal{H}} \circ \mathfrak{p}$. Therefore it is necessary to demonstrate that the form (2.7) is sufficient and necessary for the commutation to occur.

Sufficience: Let us divide the group Σ with respect to subgroup $\Sigma_{\mathcal{H}}$ into a sum of three right layers $\Sigma_{\mathcal{H}}$, $\Sigma_{\mathcal{H}} \circ \langle 23 \rangle$, $\Sigma_{\mathcal{H}} \circ \langle 24 \rangle$ and a sum of left layers $\Sigma_{\mathcal{H}}$, $\langle 23 \rangle \circ \Sigma_{\mathcal{H}}$, $\langle 24 \rangle \circ \Sigma_{\mathcal{H}}$. The operator $\mathfrak{c}$ commutates with $\mathfrak{s}_{\mathcal{H}}$. Indeed, on both sides of the equation $\mathfrak{c} \circ \mathfrak{s}_{\mathcal{H}} = \mathfrak{s}_{\mathcal{H}} \circ \mathfrak{c}$ there is the same sum of all the 16 elements of the group Σ which do not belong to subgroup $\Sigma_{\mathcal{H}}$. Hence every operator $\mathfrak{p}$ of the form (2.7) commutates with $\mathfrak{s}_{\mathcal{H}}$.

Necessity: Let us take any permutation operator (2.1). Let us denote by $\sigma_9, \cdots, \sigma_{16}$ all the elements of the layer $\langle 23 \rangle \circ \Sigma_{\mathcal{H}}$, whereas by $\sigma_{17}, ..., \sigma_{24}$ – the elements of layer $\langle 24 \rangle \circ \Sigma_{\mathcal{H}}$. We see that for every Hooke's tensor

$$\mathfrak{p} \times \mathbf{H} = (a_1\sigma_1 + \ldots + a_{24}\sigma_{24}) \times \mathbf{H} = (a\mathfrak{l} + b\langle 23 \rangle + c\langle 24 \rangle) \times \mathbf{H} \tag{2.8}$$

where $a \equiv (a_1 + \ldots + a_8)$, $b \equiv (a_9 + \ldots + a_{16})$, $c \equiv (a_{17} + \ldots + a_{24})$. In order for the Hooke's tensor $\mathbf{H} = (\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{y} \otimes \mathbf{x} \otimes \mathbf{x})$ to remain after the operation $\mathfrak{p}\times$, a Hooke's tensor, we additionally need $b = c$. Therefore (2.1) must have the form (2.7).

2.2. Let us find the permutation operators acting on Hooke's tensors as projectors, thus fulfilling the condition

$$\mathfrak{p} \times (\mathfrak{p} \times \mathbf{H}) = \mathfrak{p} \times \mathbf{H} \quad \text{for all} \quad \mathbf{H} \in \mathcal{H}. \tag{2.9}$$

This condition is reduced to two equations $2a^2 + b^2 = 2a$, $4ab + b^2 = 2b$ and has only two solutions different from $\mathfrak{l}$, namely

$$\mathfrak{s} = \frac{1}{3}(\mathfrak{l} + 2\mathfrak{c}), \qquad \mathfrak{t} = \frac{2}{3}(\mathfrak{l} - \mathfrak{c}). \tag{2.10}$$

Operator $\mathfrak{s}$ acts upon $\mathcal{H}$ as a total symmetrization of Hooke's tensors, since

$$\mathfrak{s} \times \mathbf{H} = \frac{1}{3}((1234) + \langle 23\rangle + \langle 24\rangle) \times \mathbf{H} = \frac{1}{24}\sum_{1}^{24} \sigma_i \times \mathbf{H}. \tag{2.11}$$

It projects orthogonally Hooke's tensors onto *the subspace of totally symmetric Hooke's tensor* $\mathcal{H}_\mathfrak{s} \equiv \mathfrak{s} \times \mathcal{H}$. Since $\mathfrak{s} + \mathfrak{t} = \mathfrak{l}$ and, as it can be easily verified, for all tensors $(\mathfrak{s} \times \mathbf{H}_1) \cdot (\mathfrak{t} \times \mathbf{H}_2) = 0$, then $\mathcal{H}_\mathfrak{t} \equiv \mathfrak{t} \times \mathcal{H}$ is an orthogonal complement of $\mathcal{H}_\mathfrak{s}$ *in* $\mathcal{H}$.

We have described a classical **basic orthogonal invariant decomposition**.

$$\mathcal{H} = \mathcal{H}_\mathfrak{s} \dotplus \mathcal{H}_\mathfrak{t}, \qquad 21 = 15 + 6. \tag{2.12}$$

The dimensions follow from the direct calculation: there are 6 binding conditions for 21 components H_{ijkl} if it is to be symmetric to all permutations. The invariance of this decomposition follows from the commutation of operating of the groups $\Sigma, \mathcal{O}$ in $\mathcal{H}$. Further on we shall decompose both parts of (2.12).

Therefore, every Hooke's tensor can be uniquely given in the form of the sum

$$\mathbf{H} = \mathbf{H}_\mathfrak{s} + \mathbf{H}_\mathfrak{t} \tag{2.13}$$

of *the totally symmetric part* $\mathbf{H}_\mathfrak{s}$ and the part $\mathbf{H}_\mathfrak{t}$ orthogonal to it, Fig. 1.

EXAMPLE 1. Totally symmetric stiffness tensors

$$\mathbf{S} = \mathbf{S}_\mathfrak{s}, \quad \mathbf{S}_\mathfrak{t} = \mathbf{0} \tag{2.14}$$

describe Cauchy's elastic materials. The components of the stiffness tensor must satisfy 6 conditions $(\mathfrak{t} \times \mathbf{S})_{ijkl} = 0$, i.e.

$$2S_{ijkl} - S_{iklj} - S_{ilkj} = 0, \tag{2.15}$$

called *Cauchy's conditions*. The discussions on the formal and physical status of these conditions go 170 years back (see e.g. [25, 26, 27, 15]).

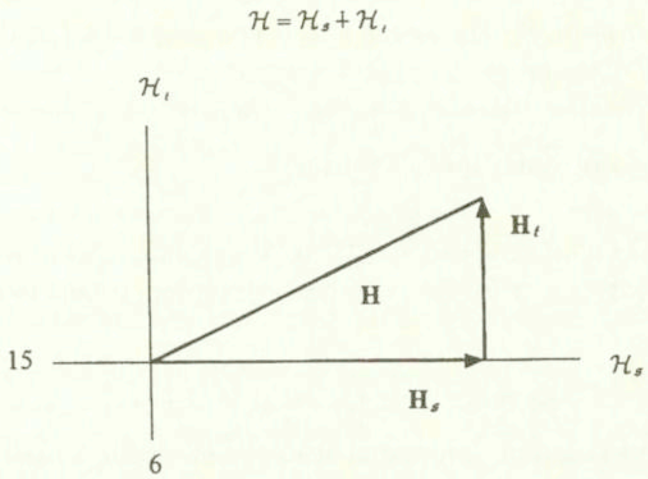

FIG. 1.

3. The decomposition of $\mathcal{H}$ with respect to external symmetry

Isotropic Hooke's tensors constitute a 2-dimensional subspace $\mathcal{H}^{is}$. By denoting its orthogonal complement as $\mathcal{H}^{an}$ we obtain the second **basic orthogonal invariant decomposition**.

(3.1) $$\mathcal{H} = \mathcal{H}^{is} \dotplus \mathcal{H}^{an}, \qquad 21 = 2 + 19.$$

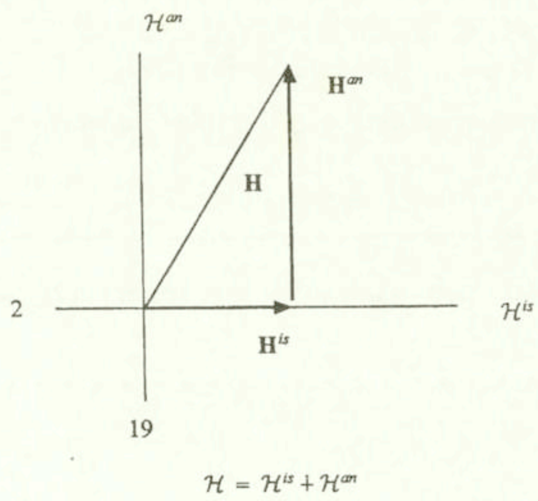

FIG. 2.

Parts of the decomposition

(3.2) $$\mathbf{H} = \mathbf{H}^{is} + \mathbf{H}^{an}$$

will be called, respectively, the *isotropic* and *anisotropic part* of Hooke's tensor **H**, Fig. 2.

4. The plane of isotropic Hooke's tensors[1]

4.1. The building blocks for isotropic Hooke's tensors are, as usual, the unity **1** and the symmmetric group Σ. Every such tensor is proportional to a respectively chosen tensor

$$\mathbb{I}_{\mathfrak{m}} \equiv \mathfrak{m} \times (\mathbf{1} \otimes \mathbf{1}), \qquad \mathfrak{m} = a\mathfrak{l} + b\mathfrak{c}. \tag{4.1}$$

All tensors of this form, proportional to the one with a fixed permutation operator $(a, b) = \text{const}$, constitute an invariant straight line spanned on $\mathbb{I}_{\mathfrak{m}}$ which will be denoted as $\mathcal{J}_{\mathfrak{m}}$. The straight lines $\mathcal{J}_{\mathfrak{m}_1}$, $\mathcal{J}_{\mathfrak{m}_2}$ are different iff operators $\mathfrak{m}, \mathfrak{m}_2$ are *independent*, i.e. are not proportional, $(a_1, b_1) \neq l(a_2, b_2)$. So we have an **infinite number of invariant decompositions of the plane** $\mathcal{H}^{is}$

$$\mathcal{H}^{is} = \mathcal{J}_{\mathfrak{m}_1} + \mathcal{J}_{\mathfrak{m}_2}, \qquad 2 = (1 + 1). \tag{4.2}$$

Scalar product on this plane is not hard to obtain:

$$\mathbb{I}_{\mathfrak{m}_1} \cdot \mathbb{I}_{\mathfrak{m}_2} = 9a_1a_2 + 6b_1b_2 + 3(a_1b_2 + a_1b_2). \tag{4.3}$$

Hence

$$|\mathbb{I}_{\mathfrak{m}}|^2 = 9a^2 + 6b^2 + 6ab.$$

The invariant decomposition (4.2) is orthogonal when

$$3a_1a_2 + 2b_1b_2 + (a_1b_2 + a_2b_1) = 0. \tag{4.4}$$

4.2. Hooke's isotropic tensors act as linear operators in $\mathcal{S}$ (hence e.g. in Hooke's law) as follows

$$\mathbb{I}_m \cdot \boldsymbol{\xi} = a(\operatorname{tr} \boldsymbol{\xi})\mathbf{1} + b\boldsymbol{\xi}. \tag{4.5}$$

Their composition has the form

$$\mathbb{I}_{\mathfrak{m}_1} \circ \mathbb{I}_{\mathfrak{m}_2} = \mathbb{I}_{\mathfrak{m}_3}, \quad \mathfrak{m}_3 = (3a_1a_2 + a_1b_2 + a_2b_1)\mathfrak{l} + (b_1b_2)\,\mathfrak{c}. \tag{4.6}$$

[1]It is sensible to gather a few classical formulae of linear isotropic elasticity in a clear geometric form.

Operation $\boldsymbol{\xi} \to \mathbb{I}_{\mathrm{m}} \cdot \boldsymbol{\xi}$ is a projection of $\mathcal{H}^{is}$ on the straight line $\mathbb{I}_{\mathrm{m}}$, i.e.

$$\mathbb{I}_{\mathrm{m}} \circ \mathbb{I}_{\mathrm{m}} = \mathbb{I}_{\mathrm{m}}, \tag{4.7}$$

if equations $3a^2 + 2ab = 2a$, $b^2 = b$ are valid, so for three and only three pairs

$$(0,1)\,, \quad \left(\frac{1}{3}, 0\right), \quad \left(-\frac{1}{3}, 1\right). \tag{4.8}$$

In other words, *only three* isotropic Hooke's tensors act as orthogonal projectors in space $\mathcal{S}$:

the unity (identity operator)

$$\mathbb{I}_{\mathcal{S}} \equiv \mathfrak{c} \times (\mathbf{1} \otimes \mathbf{1}), \qquad \mathbb{I}_{\mathcal{S}} \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi}, \tag{4.9}$$

the projector onto the straight line of spherical tensors $\mathcal{P}$

$$\mathbb{I}_{\mathcal{P}} \equiv \frac{1}{3}(\mathbf{1} \otimes \mathbf{1}), \qquad \mathbb{I}_{\mathcal{P}} \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi}_{\mathcal{P}}, \tag{4.10}$$

the projector onto the 5-dimensional subspace of deviators $\mathcal{D}$

$$\mathbb{I}_{\mathcal{D}} \equiv \left(\mathfrak{c} - \frac{1}{3}\mathfrak{l}\right) \times (\mathbf{1} \otimes \mathbf{1}), \quad \mathbb{I}_{\mathcal{D}} \cdot \boldsymbol{\xi} \equiv \boldsymbol{\xi}_{\mathcal{D}}. \tag{4.11}$$

By these definitions we have $\mathbb{I}_{\mathcal{S}} = \mathbb{I}_{\mathcal{P}} + \mathbb{I}_{\mathcal{D}}$. This corresponds to a *classical invariant irreducible decomposition of the space* $\mathcal{S}$,

$$\mathcal{S} = \mathcal{P} \dotplus \mathcal{D}, \qquad 6 = 1 + 5 \tag{4.12}$$

and, hence, to a unique decomposition of every symmetric tensor $\boldsymbol{\xi}$ into the spherical part $\boldsymbol{\xi}_{\mathcal{P}}$ and deviator $\boldsymbol{\xi}_{\mathcal{D}}$

$$\boldsymbol{\xi} = \boldsymbol{\xi}_{\mathcal{P}} + \boldsymbol{\xi}_{\mathcal{D}}, \quad \boldsymbol{\xi}_{\mathcal{P}} = \frac{\mathrm{tr}\boldsymbol{\xi}}{3}\mathbf{1}. \tag{4.13}$$

Let us consider the formulae

$$\begin{aligned} \mathbb{I}_{\mathcal{S}} \cdot \mathbb{I}_{\mathcal{S}} &= \mathrm{Tr}\, \mathbb{I}_{\mathcal{S}} = \dim \mathcal{S} = 6, \\ \mathbb{I}_{\mathcal{P}} \cdot \mathbb{I}_{\mathcal{P}} &= \mathrm{Tr}\, \mathbb{I}_{\mathcal{P}} = \dim \mathcal{P} = 1, \\ \mathbb{I}_{\mathcal{D}} \cdot \mathbb{I}_{\mathcal{D}} &= \mathrm{Tr}\, \mathbb{I}_{\mathcal{D}} = \dim \mathcal{D} = 5, \end{aligned} \tag{4.14}$$

and a useful identity

$$\mathbf{H} = \mathbb{I}_{\mathcal{S}} \circ \mathbf{H} = \mathbf{H} \circ \mathbb{I}_{\mathcal{S}}. \tag{4.15}$$

4.3. As the most natural basis on the plane $\mathcal{H}^{is}$ we consider the *orthogonal* basis

$$(4.16)\qquad (\mathbb{I}_\mathcal{P},\mathbb{I}_\mathcal{D}), \quad |\mathbb{I}_\mathcal{P}| = 1, \quad |\mathbb{I}_\mathcal{D}| = \sqrt{5}, \quad \mathbb{I}_\mathcal{P} \cdot \mathbb{I}_\mathcal{D} = 0.$$

It is reasonable to take the *orthogonal* basis

$$(4.17)\qquad (\mathbb{I}_\mathfrak{s},\mathbb{I}_\mathfrak{t}), \quad |\mathbb{I}_\mathfrak{s}| = \sqrt{5}, \quad |\mathbb{I}_\mathfrak{t}| = 2, \quad \mathbb{I}_\mathfrak{s} \cdot \mathbb{I}_\mathfrak{t} = 0$$

where

$$(4.18)\qquad \mathbb{I}_\mathfrak{s} \equiv \mathfrak{s} \times (\mathbf{1} \otimes \mathbf{1}) = \frac{1}{3}(\mathfrak{l} + 2\mathfrak{c}) \times (\mathbf{1} \otimes \mathbf{1})$$

is the isotropic totally symmetric tensor.

In classical elasticity, it was historically assumed to use the *nonorthogonal* basis

$$(4.19)\qquad (\mathbb{I}_\mathcal{P},\ \mathbb{I}_\mathcal{S}), \qquad \mathbb{I}_\mathcal{P} \cdot \mathbb{I}_\mathcal{S} = 1.$$

We have not seen the non-orthogonal basis $(\mathbb{I}_\mathcal{D},\ \mathbb{I}_\mathcal{S})$ in use.

All these bases are shown in Fig. 3.

$$\mathcal{H}^{is} = \mathcal{J}_\mathcal{P} \dot{+} \mathcal{J}_\mathcal{D} = \mathcal{J}_s \dot{+} \mathcal{J}_t = \mathcal{J}_\mathcal{P} + \mathcal{J}_\mathcal{S}$$

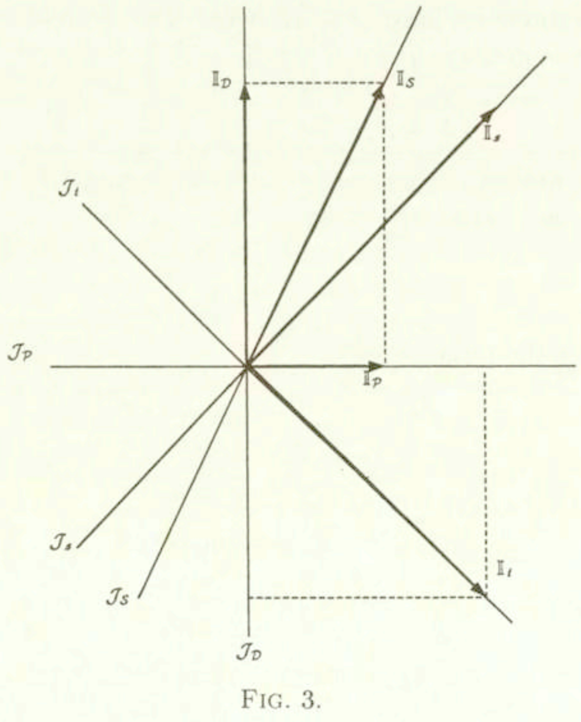

FIG. 3.

The following relations are useful:

$$
(4.20)\quad
\begin{cases}
\mathbb{I}_{\mathcal{P}} = \frac{1}{3}\mathbb{I}_{\mathfrak{s}} + \frac{1}{3}\mathbb{I}_{\mathfrak{t}} = \mathbb{I}_{\mathcal{P}}, \\
\mathbb{I}_{\mathcal{D}} = \frac{2}{3}\mathbb{I}_{\mathfrak{s}} - \frac{5}{6}\mathbb{I}_{\mathfrak{t}} = -\mathbb{I}_{\mathcal{P}} + \mathbb{I}_{\mathcal{S}},
\end{cases}
\qquad
\begin{cases}
\mathbb{I}_{\mathfrak{s}} = \frac{5}{3}\mathbb{I}_{\mathcal{P}} + \frac{2}{3}\mathbb{I}_{\mathcal{D}} = \mathbb{I}_{\mathcal{P}} + \frac{2}{3}\mathbb{I}_{\mathcal{S}}, \\
\mathbb{I}_{\mathfrak{t}} = \frac{4}{3}\mathbb{I}_{\mathcal{P}} - \frac{2}{3}\mathbb{I}_{\mathcal{D}} = 2\mathbb{I}_{\mathcal{P}} - \frac{2}{3}\mathbb{I}_{\mathcal{S}}.
\end{cases}
$$

The relations between the invariant factors of decompositions

$$
(4.21)\qquad \mathbf{H}^{is} = h_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} + h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} = h_{\mathfrak{s}}\mathbb{I}_{\mathfrak{s}} + h_{\mathfrak{t}}\mathbb{I}_{\mathfrak{t}} = k\mathbb{I}_{\mathcal{P}} + l\mathbb{I}_{\mathcal{S}}
$$

follow from the previous formulae.

EXAMPLE 2. For the stiffness tensor $\mathbf{S}$ of an elastic isotropic material, when $\mathbf{S} \in \mathcal{H}^{is}$, $\mathbf{S}^{an} = \mathbf{0}$, the invariants $(S_{\mathcal{P}}, S_{\mathcal{D}})$ are examples of Kelvin's moduli (see [6, 7]). We have $k = 3\lambda$, $l = 2\mu$ where λ, μ are *Lamé's constants.*

5. The invariant 10-dimensional subspace of anisotropic Hooke's tensors of the first type

5.1. We are now to examine the possibility of invariant divisions of 19-dimensional space $\mathcal{H}^{an}$, consiting of all anisotropic parts of Hooke's tensors. The orthogonality to $\mathcal{H}^{is}$ can be written in the form of orthogonality conditions to basis ($\mathbb{I}_{\mathcal{P}}$, $\mathbb{I}_{\mathcal{S}}$)

$$
(5.1)\qquad 3\mathbb{I}_{\mathcal{P}} \cdot \mathbf{H} = 0, \qquad \mathbb{I}_{\mathcal{S}} \cdot \mathbf{H} = 0.
$$

5.2. Let us fix the permutation operator $\mathfrak{m}$ and consider the set $\mathfrak{D}_{\mathfrak{m}}$ of Hooke's tensors of the form

$$
(5.2)\qquad \mathfrak{m} \times (\mathbf{1} \otimes \boldsymbol{\eta} + \boldsymbol{\eta} \otimes \mathbf{1}), \qquad \mathfrak{m} = \text{const},
$$

where $\boldsymbol{\eta}$ takes any value from the 5-dimensional space of *deviators* of the second order $\mathcal{D}$. Clearly, the set $\mathfrak{D}_{\mathfrak{m}}$:

- is located in $\mathcal{H}^{an}$,
- is a linear subspace,
- $\dim \mathfrak{D}_{\mathfrak{m}} = 5$,
- this subspace is invariant,
- this subspace is irreducible.

Indeed $\mathfrak{D}_{\mathfrak{m}} \subset \mathcal{H}^{an}$, since every tensor (5.2) as a result of the equation $\mathrm{tr}\boldsymbol{\eta} = 0$, is orthogonal to $\mathcal{H}^{is}$. Every linear combination of the tensors of the form (5.2) is a tensor of this form, hence $\mathfrak{D}_{\mathfrak{m}}$ is a subspace. Every basis $\eta_1, .., \eta_5$ of $\mathcal{D}$ generates a basis in $\mathfrak{D}_{\mathfrak{m}}$, so $\dim \mathfrak{D}_{\mathfrak{m}} = \dim \mathcal{D}$. If we rotate any tensor (5.2) from $\mathfrak{D}_{\mathfrak{m}}$ by any tensor $\mathbf{R} \in \mathcal{O}$

$$(5.3) \qquad \mathbf{R} * [\mathfrak{m} \times (\mathbf{1} \otimes \eta + \eta \otimes \mathbf{1})] = \mathfrak{m} \times [\mathbf{1} \otimes (\mathbf{R} * \eta) + (\mathbf{R} * \eta) \otimes \mathbf{1}],$$

it remains in $\mathfrak{D}_{\mathfrak{m}}$ so this is an invariant subspace. Finally, the irreducibility of $\mathfrak{D}_{\mathfrak{m}}$ follows immediately from the irreducibility of the space of deviators $\mathcal{D}$, being a building block for $\mathfrak{D}_{\mathfrak{m}}$.

5.3. Let us take two subspaces $\mathfrak{D}_{\mathfrak{m}_1}$ and $\mathfrak{D}_{\mathfrak{m}_2}$. It is clear that only two situations are possible:

$$(5.4) \qquad \mathfrak{D}_{\mathfrak{m}_1} \cap \mathfrak{D}_{\mathfrak{m}_2} = \{\mathbf{0}\} \qquad \text{when} \quad \mathfrak{m}_1, \mathfrak{m}_2 \ \text{are independent},$$

$$(5.5) \qquad \mathfrak{D}_{\mathfrak{m}_1} = \mathfrak{D}_{\mathfrak{m}_2} \qquad \text{when this is not the case.}$$

When the pair $(\mathfrak{m}_1, \mathfrak{m}_2)$ is an independent pair, let us take a direct sum (see Appendix 1)

$$(5.6) \qquad \mathfrak{D} \equiv \mathfrak{D}_{\mathfrak{m}_1} + \mathfrak{D}_{\mathfrak{m}_2}, \qquad 10 = (5 + 5).$$

Since for all independent $\mathfrak{m}_1$, $\mathfrak{m}_2$

$$(5.7) \qquad \mathfrak{D}_{\mathfrak{m}_1} + \mathfrak{D}_{\mathfrak{m}_2} = \mathfrak{D}_{\mathfrak{l}} + \mathfrak{D}_{\mathfrak{c}},$$

then we have obtained a *10-dimensional invariant subspace of anisotropic* Hooke's *tensors* $\mathfrak{D}$. It contains all 5-dimensional subspaces $\mathfrak{D}_m$ and can be split, in an infinite number of ways, into a sum of two such irreducible subspacess.

Grave warning: It can be easily verified that it is not so that every tensor from the found space $\mathfrak{D}$ has the form $\mathfrak{m} \times (\mathbf{1} \otimes \eta + \eta \otimes \mathbf{1})$ for certain $\mathfrak{m}$, η. To put it more forcefully, an infinite number of tensors from $\mathfrak{D}$ do not belong to any 5-dimensional subspace $\mathfrak{D}_{\mathfrak{m}}$ (some of our geometric intuitions obtained in dimensions 2 and 3 ought to be given a tight rein!).

5.4. Let us calculate the scalar product of the parts of the decomposition

$$(5.8) \qquad \mathbf{H}^{an}_{\mathfrak{D}} \equiv \mathfrak{m}_1 \times (\mathbf{1} \otimes \eta + \eta \otimes \mathbf{1}) + \mathfrak{m}_2 \times (\mathbf{1} \otimes \xi + \xi \otimes \mathbf{1}).$$

We have

$$(5.9) \qquad [\mathfrak{m}_1 \times (\mathbf{1} \otimes \eta + \eta \otimes \mathbf{1})] \cdot [\mathfrak{m}_2 \times (\mathbf{1} \otimes \xi + \xi \otimes \mathbf{1})] = [6a_1a_2 + 4(a_1b_2 + a_2b_1) + 5b_1b_2]\, \eta \cdot \xi.$$

Hence in particular

$$(5.10) \qquad |\mathfrak{m} \times (\mathbf{1} \otimes \eta + \eta \otimes \mathbf{1})|^2 = \mu(\mathfrak{m})\, |\eta|^2\,,$$

where $\mu\,(\mathfrak{m}) = 6a^2 + 8ab + 5b^2$.

Two subspaces $\mathfrak{D}_{\mathfrak{m}_1}$ and $\mathfrak{D}_{\mathfrak{m}_2}$ constitute complete invariant *orthogonal* decomposition of space $\mathfrak{D}$ iff

$$(5.11) \qquad 6a_1a_2 + 4(a_1b_2 + a_2b_1) + 5b_1b_2 = 0.$$

There is an infinite number of such decompositions.

5.5. The following particular cases of the decompositions of Hooke's tensors $\mathbf{H}^{an}_{\mathfrak{D}} \in \mathfrak{D}$ will prove to be most interesting:

standard decomposition, non-orthogonal one

$$(5.12) \qquad \mathbf{H}^{an}_{\mathfrak{D}} = (\mathbf{1} \otimes \omega + \omega \otimes \mathbf{1}) + \mathfrak{c} \times (\mathbf{1} \otimes \varrho + \varrho \otimes \mathbf{1})\,,$$

orthogonal decomposition with respect to internal symmetry

$$(5.13) \qquad \mathbf{H}^{an}_{\mathfrak{D}} = \mathfrak{s} \times (\mathbf{1} \otimes \alpha + \alpha \otimes \mathbf{1}) + \mathfrak{t} \times (\mathbf{1} \otimes \beta + \beta \otimes \mathbf{1})\,,$$

orthogonal decomposition

$$(5.14) \qquad \mathbf{H}^{an}_{\mathfrak{D}} = (\mathbf{1} \otimes \varphi + \varphi \otimes \mathbf{1}) + \left(\mathfrak{c} - \frac{2}{3}\mathfrak{l}\right) \times (\mathbf{1} \otimes \psi + \psi \otimes \mathbf{1})\,,$$

seemingly quite artificial, but its role will become evident in Part II of this paper.

The tensor $\mathbf{H}^{an}_{\mathfrak{D}}$ is an orthogonal projection of Hooke's tensor $\mathbf{H}$ onto a 10-dimensional subspace $\mathfrak{D}$. From the definition of permutation operators these relations follow immediately

$$(5.15) \qquad \begin{cases} \alpha = \omega + \varrho, \\ \beta = \dfrac{1}{2}(2\omega - \varrho), \end{cases} \qquad \begin{cases} \omega = \dfrac{1}{3}\,(\alpha + 2\beta)\,, \\ \varrho = \dfrac{2}{3}(\alpha - \beta), \end{cases}$$

$$\begin{cases} \varphi = \dfrac{1}{7}(7\alpha + 2\beta) = \dfrac{1}{3}(3\omega + 2\varrho), \\ \psi = \dfrac{2}{3}(\alpha - \beta) = \varrho. \end{cases}$$

The norm of tensor $\mathbf{H}^{an}_{\mathfrak{D}}$ from $\mathfrak{D}$ is

$$(5.16)\qquad |\mathbf{H}^{an}_{\mathfrak{D}}|^2 = \frac{14}{3}\alpha\cdot\alpha + \frac{1}{3}\beta\cdot\beta = 5\omega\cdot\omega + 9\omega\cdot\varrho + \frac{19}{4}\varrho\cdot\varrho.$$

6. The canonical decomposition of the space of Hooke's tensors

6.1. What is left to describe are the tensors that constitute in $\mathcal{H}^{an}$ the invariant orthogonal complement of the defined invariant space $\mathfrak{D}$.

We denote this complement by D, $\dim\mathsf{D} = 19 - 10 = 9$.

First, it is immediately evident that every tensor $\mathbf{D} \in \mathsf{D}$ is **totally traceless**, i.e.

$$(6.1)\qquad \operatorname{tr}\mathbf{D} = 0 \qquad \text{for every operation tr.}$$

Indeed, it should be a tensor orthogonal to $\mathcal{H}^{is}$ and to $\mathfrak{D}$. If $\mathbf{D}\cdot(\mathbf{1}\otimes\mathbf{1}) = \mathbf{1}\cdot\mathbf{D}\cdot\mathbf{1} = 0$ and $(\mathbf{1}\otimes\boldsymbol{\alpha} + \boldsymbol{\alpha}\otimes\mathbf{1})\cdot\mathbf{D} = 2\mathbf{1}\cdot\mathbf{D}\cdot\boldsymbol{\alpha} = 0$ for every $\boldsymbol{\alpha}$, then $D_{ppij} = 0$. Similarly, with the orthogonality conditions of $\mathbf{D}$ to $\mathfrak{c}\times(\mathbf{1}\otimes\mathbf{1})$ and $\mathfrak{c}\times(\mathbf{1}\otimes\boldsymbol{\alpha} + \boldsymbol{\alpha}\otimes\mathbf{1})$, we obtain $D_{ippj} = 0$. From the symmetry with respect to group $\Sigma_{\mathcal{H}}$, it follows that $\mathbf{D}$ is totally traceless.

Secondly, every tensor $\mathbf{D} \in \mathsf{D}$ is **totally symmetric**. This results from orthogonal decompositions: $\mathcal{H} = \mathcal{H}_{\mathfrak{s}} \dotplus \mathcal{H}_{\mathfrak{t}}$ and

$$(6.2)\qquad \mathcal{H}_{\mathfrak{s}} = \mathcal{J}_{\mathfrak{s}} \dotplus \mathfrak{D}_{\mathfrak{s}} \dotplus \mathsf{D}, \qquad 15 = 1 + 5 + 9,$$

$$(6.3)\qquad \mathcal{H}_{\mathfrak{t}} = \mathcal{J}_{\mathfrak{t}} \dotplus \mathfrak{D}_{\mathfrak{t}}, \qquad 6 = 1 + 5,$$

evident due to the dimensions of the parts.

6.2. Thus the invariant space D consists of **totally symmetric and traceless** tensors of the fourth order, which we call **fourth-order deviators**[2].

It can be demostrated that space D is irreducible[3].

[2]In [12, 15, 19] and possibly elsewhere, they are called, for certain reasons, *harmonic* tensors, and even the very decompositions to which they led are called *harmonic* decompositions. These names are in our approach neither necessary nor fortunate.

[3]This would be the only proof reaching deeper into the theory of representations of compact groups and that is why it will be neglected here (see e.g. [10] and the classical textbooks quoted therein).

To sum up: we obtained the following **unique canonical decomposition of the space of Hooke's tensors**

(6.4) $$\boxed{\mathcal{H} = \mathcal{H}^{is} \dotplus \mathfrak{D} \dotplus \mathsf{D}\,, \qquad 21 = 2 + 10 + 9\,.}$$

The meaning and the gravity of this decomposition and the solemnity of its name will become clear later on (see also J. P. SERRE [11]). Every Hooke's tensor is a unique sum

(6.5) $$\boxed{\mathbf{H} = \mathbf{H}^{is} + \mathbf{H}^{an}_{\mathfrak{D}} + \mathbf{H}^{an}_{\mathsf{D}}}$$

of three mutally orthogonal parts: *the isotropic part* $\mathbf{H}^{is}$, *the first anisotropic part* $\mathbf{H}^{an}_{\mathcal{D}}$ *expressed by two second-order deviators and the second anisotropic part* $\mathbf{H}^{an}_{\mathsf{D}}$ *being a fourth-order deviator*, Fig. 4.

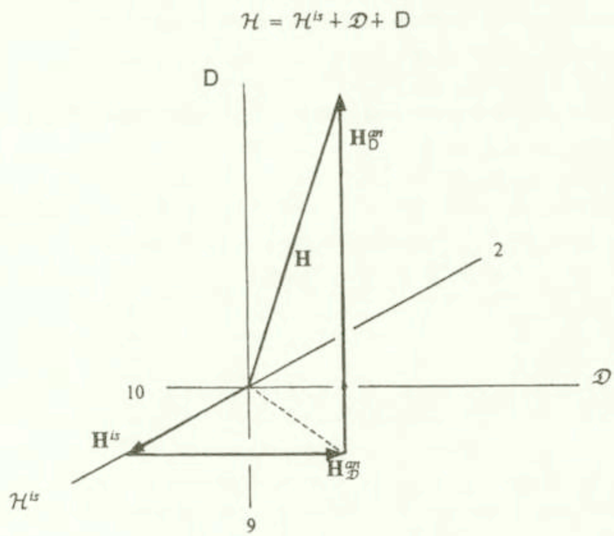

FIG. 4.

7. Summary: all the possible invariant decompositions of the space of Hooke's tensors

7.1. All that has been presented above can be summarised as follows.

Every complete invariant decomposition of the space of Hooke's tensors has the form:

(7.1) $$\boxed{\mathcal{H} = (\mathcal{J}_{\mathrm{n}_1} + \mathcal{J}_{\mathrm{n}_2}) \dotplus (\mathfrak{D}_{\mathrm{m}_1} + \mathfrak{D}_{\mathrm{m}_2}) \dotplus \mathsf{D}, \quad 21 = (1+1) + (5+5) + 9,}$$

where $\mathfrak{n}_1, \mathfrak{n}_2$) *and* $(\mathfrak{m}_1, \mathfrak{m}_2)$ *are two pairs of pairwise independent permutation operators. These pairs can be chosen arbitrarily, so there is an infinite number of such decompositions. Each of them is a split of parts* $\mathcal{H}^{is}$ *and* $\mathfrak{D}$ *of the* **unique canonical decomposition** (6.4). *By fixing operators* $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{m}_1, \mathfrak{m}_2$ *and grouping arbitrarily the irreducible spaces from* (7.1), *we can obtain every invariant decomposition of space* $\mathcal{H}$.

In particular:

1) the classical decomposition $\mathcal{H} = \mathcal{H}_{\mathfrak{s}} \dotplus \mathcal{H}_{\mathfrak{t}}$ we obtain by taking $\mathfrak{n}_1 = \mathfrak{m}_1 = \mathfrak{s}$, $\mathfrak{n}_2 = \mathfrak{m}_2 = \mathfrak{t}$ and grouping the spaces as in (6.2) and (6.3);

2) the classical decomposition $\mathcal{H} = \mathcal{H}^{is} \dotplus \mathcal{H}^{an}$ is valid for any choice of permutation operators,

$$\mathcal{H}^{an} = \mathfrak{D} \dotplus \mathsf{D}, \qquad 19 = 10 + 9. \tag{7.2}$$

7.2. The result obtained (7.1) will be given also in another form, being a handy form for someone primarily interested in applications.

The manual for linear complete invariant decomposition of any Hooke's tensor:

Let $(\mathfrak{n}_1, \mathfrak{n}_2)$ *and* $(\mathfrak{m}_1, \mathfrak{m}_2)$ *be two arbitrary chosen pairs of pairwise independent permutation operators* (2.7). *Then for any Hooke's tensor* $\mathbf{H} \in \mathcal{H}$ *there is a pair of scalars* (h_1, h_2) *a pair of second order deviators* $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, *a fourth-order deviator* $\mathbf{D}$, *such that*

$$\boxed{\mathbf{H} = h_1 \mathbb{I}_{n_1} + h_2 \mathbb{I}_{n_2} + \mathfrak{m}_1 \times (\mathbf{1} \otimes \boldsymbol{\xi}_1 + \boldsymbol{\xi}_1 \otimes \mathbf{1}) + \mathfrak{m}_2 \times (\mathbf{1} \otimes \boldsymbol{\xi}_2 + \boldsymbol{\xi}_2 \otimes \mathbf{1}) + \mathbf{D}\,.} \tag{7.3}$$

Having fixed the pairs $(\mathfrak{n}_1, \mathfrak{n}_2)$, $(\mathfrak{m}_1, \mathfrak{m}_2)$, *the correspondence*

$$\mathbf{H} \leftrightarrow (h_1, h_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{D})$$

is isotropic and one-to-one.

The choice of a decomposition, i.e. the choice of operators $(\mathfrak{n}_1, \mathfrak{n}_2)$ and $(\mathfrak{m}_1, \mathfrak{m}_2)$ depends, as we shall see, on the role of Hooke's tensor $\mathbf{H}$ in a given context. The anisotropic part of Hooke's tensor $\mathbf{H}^{an}$ is uniquely determined by pair $(\mathfrak{m}_1, \mathfrak{m}_2)$ and three deviators $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{D})$. These deviators will be called *anisotropy deviators.*

For example, we shall demonstrate the complete decomposition choosing $\mathfrak{n}_1 = \dfrac{1}{3}\mathfrak{l}$, $\mathfrak{n}_2 = \left(\mathfrak{c} - \dfrac{1}{3}\mathfrak{l}\right)$, $\mathfrak{m}_1 = \mathfrak{s}$, $\mathfrak{m}_2 = \mathfrak{t}$. Then

$$\mathbf{H} = h_{\mathcal{P}} \mathbb{I}_{\mathcal{P}} + h_{\mathcal{D}} \mathbb{I}_{\mathcal{D}} + \mathfrak{s} \times (\mathbf{1} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{1}) + \mathfrak{t} \times (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1}) + \mathbf{D}. \tag{7.4}$$

7.3. We shall find explicit isotropic functions defining $(h_1, h_2, \xi_1, \xi_2, \mathbf{D})$ by $\mathbf{H}$. We will do so, for example, for (7.4)

$$\mathbf{H} \to (h_\mathcal{P}, h_\mathcal{D}, \alpha, \beta, \mathbf{D}). \tag{7.5}$$

It is most simple to obtain the coefficients of the isotropic part $h_\mathcal{P}, h_\mathcal{D}$. Let us introduce two self-evident *linear invariants* of tensor $\mathbf{H}$

$$\mathbf{H} \cdot (3\mathbb{I}_\mathcal{P}) = \mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1} = H_{ppqq}, \tag{7.6}$$

$$\mathbf{H} \cdot \mathbb{I}_\mathcal{S} = \operatorname{Tr} \mathbf{H} = H_{pqpq}. \tag{7.7}$$

Every linear invariant of $\mathbf{H}$ has the form $\mathbf{H} \cdot \mathbb{I}_\mathrm{m}$, so it is a linear combination of these two. In particular, by scalar multiplication of (7.4) by $\mathbb{I}_\mathcal{P}$ and $\mathbb{I}_\mathcal{D}$, we have

$$h_\mathcal{P} = \frac{1}{3}\mathbf{1} \cdot \mathbf{H} \cdot \mathbf{1}, \quad h_\mathcal{D} = \frac{1}{5}(\operatorname{Tr} \mathbf{H} - h_\mathcal{P}). \tag{7.8}$$

To obtain the deviators α, β, it is convenient to use *V. V. Novozhilov's tensors*[4]

$$\mu \equiv \mathbf{H} \cdot \mathbf{1}, \quad \mu_{ij} = H_{ijpp}, \tag{7.9}$$

$$\nu \equiv (\langle 23 \rangle \times \mathbf{H}) \cdot \mathbf{1}, \quad \nu_{ij} = H_{ipjp} = H_{ippj}. \tag{7.10}$$

He observed in [27] that these are the only two tensors that can be obtained from Hooke's tensor by direct application of the trace operator.

EXAMPLE 3. If $\mathbf{C}$ is a compliance tensor, then $\varepsilon = \mathbf{C} \cdot \mathbf{1}$ is deformation, to which an elastic material reacts under hydrostatic pressure $\sigma = \mathbf{1}$.

EXAMPLE 4. In the theory of elastic waves, a role is played by the tensor S_{ippj}/ϱ, where ϱ is density and $\mathbf{S}$ is the stiffness tensor, [28] (see also example F in Part II of this paper).

Let us note that

$$h_\mathcal{P} = \frac{1}{3}\operatorname{tr}\mu, \qquad h_\mathcal{D} = \frac{1}{15}(3\operatorname{tr}\nu - \operatorname{tr}\mu). \tag{7.11}$$

7.4. Let us introduce *Novozhilov's deviators*

$$\mu_\mathcal{D} \equiv \mu - \frac{\operatorname{tr}\mu}{3}\mathbf{1}, \qquad \nu_\mathcal{D} \equiv \nu - \frac{\operatorname{tr}\nu}{3}\mathbf{1}. \tag{7.12}$$

[4] These tensors are also used in [19]; tensor ν_{ij} is called therein Voigt's tensor (for reasons unknown to me).

By taking $\mathbf{H}$ in the form (7.4) we obtain, by direct calculation, according to definition (7.9), (7.10),

$$\mu_{\mathcal{D}} = \frac{1}{3}(7\alpha + 2\beta), \qquad \nu_{\mathcal{D}} = \frac{1}{3}(7\alpha - \beta). \tag{7.13}$$

Hence

$$\alpha = \frac{1}{7}(\mu_{\mathcal{D}} + 2\nu_{\mathcal{D}}), \qquad \beta = \mu_{\mathcal{D}} - \nu_{\mathcal{D}}. \tag{7.14}$$

Finally, by substituting (7.11), (7.14) in (7.4) we obtain the fourth-order deviator $\mathbf{D}$.

Thus the entire sequence $(h_{\mathcal{P}}, h_{\mathcal{D}}, \alpha, \beta, \mathbf{D})$ *was expressed by Novozhilov's tensors* μ, ν, *as an explicit linear isotropic tensor function of* $\mathbf{H}$.

Similarly we can obtain the sequence $(h_{\mathcal{P}}, h_{\mathcal{D}}, \omega, \varrho, \mathbf{D})$ in a complete decomposition

$$\mathbf{H} = h_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} + h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} + (\mathbf{1} \otimes \omega + \omega \otimes \mathbf{1}) + \mathfrak{c} \times (\mathbf{1} \otimes \varrho + \varrho \otimes \mathbf{1}) + \mathbf{D}. \tag{7.15}$$

We shall then have

$$\omega = \frac{1}{7}(5\mu_{\mathcal{D}} - 4\nu_{\mathcal{D}}), \qquad \varrho = \frac{2}{7}(3\nu_{\mathcal{D}} - 2\mu_{\mathcal{D}}). \tag{7.16}$$

The identity resulting from these formulae

$$\begin{aligned} \mathbf{H} = h_{\mathcal{P}}\mathbb{I}_{\mathcal{P}} + h_{\mathcal{D}}\mathbb{I}_{\mathcal{D}} &+ \mathfrak{m}_1 \times (\mathbf{1} \otimes \mu_{\mathcal{D}} + \mu_{\mathcal{D}} \otimes \mathbf{1}) \\ &+ \mathfrak{m}_2 \times (\mathbf{1} \otimes \nu_{\mathcal{D}} + \nu_{\mathcal{D}} \otimes \mathbf{1}) + \mathbf{D}, \end{aligned} \tag{7.17}$$

where $\mathfrak{m}_1 = \frac{1}{7}(5\mathfrak{l} - 4\mathfrak{c})$, $\mathfrak{m}_2 = \frac{2}{7}(-2\mathfrak{l} + 3\mathfrak{c})$ expresses directly Hooke's tensor by its Novozhylov's tensors μ, ν and $\mathbf{D}$.

Let us also note formulae for the norm of Hooke's tensor

$$|\mathbf{H}|^2 = |\mathbf{H}^{is}|^2 + |\mathbf{H}^{an}_{\mathfrak{D}}|^2 + |\mathbf{H}^{an}_{\mathrm{D}}|^2 = h_{\mathcal{P}}^2 + 5h_{\mathcal{D}}^2 + \frac{14}{3}\alpha \cdot \alpha + \frac{1}{3}\beta \cdot \beta + \mathbf{D} \cdot \mathbf{D}. \tag{7.18}$$

7.5. In most papers on the linear invariant decompositions of Hooke's tensor, the starting point was, what is customary in the theory of group representation, the symmetrization $\mathfrak{s} \times \mathbf{H}$. This leads to decomposition (7.3) with

$$\mathfrak{n}_1 = \mathfrak{m}_1 = \mathfrak{s}, \qquad \mathfrak{n}_2 = \mathfrak{m}_2 = \mathfrak{t}, \tag{7.19}$$

(G. BACKUS [12], YU. I. SIROTIN [13, 14], J. JERPHAGNON, D. CHEMLA, R. BONNEVILLE [15], J. PRATZ [16], R. BAERHAIM [19]). The tools used were unnecessarily complex. This decomposition is, moreover, often quite unnatural, e.g. for anisotropic parts.

Only E. T. ONAT [17] used one of the infinite number of other decompositions, namely the case

$$\mathfrak{n}_1 = \mathfrak{m}_1 = \mathfrak{l}, \quad \mathfrak{n}_2 = \mathfrak{m}_2 = 2\mathfrak{c}. \tag{7.20}$$

It has been repeated by S. C. COWIN [18], described in [19], and proved to be convenient for S. FORTE and M. VIANELLO [4].

Nowhere is the **unique canonical decomposition** (6.4) noted or underlined, even though it is the true basis for all invariant decompositions of space $\mathcal{H}$.

Appendix 1. Notions

For the readers without everyday touch with multilinear algebra, we remind several introductory notions, necessary and sufficient to understand this paper.

1. Direct sums. Let us take any finite-dimensional *linear space* $\mathcal{L}$ with elements $\mathbf{x}, \ldots$. The subset $\mathcal{P} \subset \mathcal{L}$ is called *linear subspace* in $\mathcal{L}$ when it contains any finite linear combination of its elements. Taking two mutually independent subspaces $\mathcal{P}_1$, $\mathcal{P}_2$, i.e. those that intersect only in $\mathbf{0}$, we will call a *direct sum* of these two, a set of all $\mathbf{x} \in \mathcal{L}$ of the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1 \in \mathcal{P}_1$, $\mathbf{x}_2 \in \mathcal{P}_2$. This sum is, of course, a subspace in $\mathcal{L}$. We will use the simplest notation for it: $\mathcal{P}_1 + \mathcal{P}_2$[5]. The decomposition $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ of every element $\mathbf{x} \in \mathcal{P}_1 + \mathcal{P}_2$ is unique. This uniqueness is best regarded as a generalizing definition: the smallest subspace in $\mathcal{L}$ which contains all the subspaces of the sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ will be called their *direct sum*, denoted by $\mathcal{P}_1 + \ldots + \mathcal{P}_k$, when the representation of each element $\mathbf{x} \in \mathcal{P}_1 + \ldots + \mathcal{P}_k$ in the form

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_k, \qquad \mathbf{x}_1 \in \mathcal{P}_1, \ldots, \ \mathbf{x}_k \in \mathcal{P}_k \tag{A.1}$$

is unique, i.e. when

$$\mathbf{x} = 0 \qquad \text{iff} \qquad \mathbf{x}_1 = \mathbf{x}_2 = \ldots = \mathbf{x}_k = \mathbf{0}. \tag{A.2}$$

A direct sum of the sequence $\mathcal{P}_1, \ldots, \mathcal{P}_k$ such that

$$\mathcal{P}_1 + \ldots + \mathcal{P}_k = \mathcal{L} \tag{A.3}$$

will be called *direct division* of the space $\mathcal{L}$.

If in the space $\mathcal{L}$ a scalar product is defined, and the parts of the direct sum are mutually orthogonal, then the direct sum is called *orthogonal*. In order

[5] There are also other notations instead of + (e.g. $\dot{+}$ or $\oplus$).

to stress this we will write $\dot{+}$ instead of $+$. If $\mathcal{P}_1 \dot{+} \mathcal{P}_2 = \mathcal{L}$ then $\mathcal{P}_2$ is called *orthogonal complement* of $\mathcal{P}_1$ in $\mathcal{L}$ and we write $\mathcal{P}_2 = \mathcal{P}_1^{\perp}$.

2. Spaces of Euclidean tensors. Let $\mathcal{E}$ be our starting Euclidean space with elements $\mathbf{x}, \ldots$ called vectors with scalar product $\mathbf{xy} \equiv \mathbf{x} \cdot \mathbf{y}$. Every element $\mathbf{A} \in \bigotimes^q \mathcal{E}$ we call, as usual (at least for over fourty years [29]), *q-order Euclidean tensor*. Every tensor is a finite linear combination of *simple tensors* $\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \ldots\ldots \otimes \mathbf{a}_q$. In this paper we are only interested in two cases $\dim\mathcal{E} = 3$ and $\dim\mathcal{E} = 2$ (see Sec. 4, Part. II) and $q = 2$ and $q = 4$. Moreover, we examine only symmetric second-order tensors $\boldsymbol{\omega} \in \mathcal{S} \equiv \operatorname{sym} \bigotimes^2 \mathcal{E} \subset \bigotimes^2 \mathcal{E}$ and fourth-order tensors of the type $\mathbf{H} \in \bigotimes^2 \mathcal{S} \subset \bigotimes^4 \mathcal{E}$,

$$\boldsymbol{\omega}^{\mathrm{T}} = \boldsymbol{\omega}, \qquad \mathbf{H}^{\mathrm{T}} = \mathbf{H}. \tag{A.4}$$

It is convenient to view the set of tensors $\bigotimes^q \mathcal{E}$ as a Euclidean space endowed with a natural scalar product $\mathbf{A} \cdot \mathbf{B}$, being a bilinear operation defined on simple tensors by the recipe

$$(\mathbf{a}_1 \otimes \ldots\ldots \otimes \mathbf{a}_q) \cdot (\mathbf{b}_1 \otimes \ldots\ldots \otimes \mathbf{b}_q) \equiv (\mathbf{a}_1\mathbf{b}_1) \ldots\ldots (\mathbf{a}_q\mathbf{b}_q). \tag{A.5}$$

In every space $\bigotimes^q \mathcal{E}$ acts the group $\mathcal{O}$ of rotations and mirror reflections of the starting space $\mathcal{E}$. This action $\mathbf{A} \longrightarrow \mathbf{R}{*}\mathbf{A}$, $\mathbf{R} \in \mathcal{O}$ is by definition linear and defined on simple tensors by the recipe

$$\mathbf{R}{*}(\mathbf{a}_1 \otimes \ldots\ldots \otimes \mathbf{a}_q) \equiv \mathbf{R}\mathbf{a}_1 \otimes \ldots\ldots \otimes \mathbf{R}\mathbf{a}_q, \tag{A.6}$$

where $\mathbf{a} \longrightarrow \mathbf{Ra}$ is action of the group $\mathcal{O}$ on vectors. The subgroup $\mathcal{O}(\mathbf{A})$ consisting of all $\mathbf{R} \in \mathcal{O}$ that preserve $\mathbf{A}$, $\mathbf{R} * \mathbf{A} = \mathbf{A}$, will be called *symmetry group* of tensor $\mathbf{A}$.

The linear subspace of tensors $\mathcal{P} \subset \bigotimes{}^q\mathcal{E}$ will be called here *isotropic or invariant* subspace[6] when it is stable under group $\mathcal{O}$, $\mathcal{O} * \mathcal{P} = \mathcal{P}$ (i.e. when $\mathbf{R} * \mathbf{A} \in \mathcal{P}$ for all $\mathbf{R} \in \mathcal{O}$, $\mathbf{A} \in \mathcal{P}$). If $\mathcal{P}$ is an invariant subspace, then its orthogonal complement $\mathcal{P}^{\perp}$ is an invariant subspace too. An invariant subspace is called *irreducible* if it does not contain any proper invariant subspace. The decomposition of the examined invariant subspace $\mathcal{L} \subset \bigotimes^q \mathcal{E}$ into a direct sum of invariant subspaces $\mathcal{P}_1, \ldots, \mathcal{P}_k$

$$\mathcal{P}_1 + \ldots + \mathcal{P}_k = \mathcal{L} \tag{A.7}$$

will be called a *complete invariant decomposition* if all the subspaces $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are irreducible. Not always are we interested in complete decompositions.

[6] In multilinear algebra, the name *tensorial subspace* is rather used.

In the space $\bigotimes^q \mathcal{E}$ acts also one more group, namely the *symmetric* group Σ_q consisting of permutations $\sigma, \ldots$ We shall denote permutations in the usual way, e.g. $\sigma = (2413) \in \Sigma_4$ means $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 1$, $\sigma(4) = 3$. The action $\mathbf{A} \to \sigma \times \mathbf{A}$ is linear and defined on simple tensors by the recipe

$$\sigma \times (\mathbf{a}_1 \otimes \ldots\ldots \otimes \mathbf{a}_q) \equiv \mathbf{a}_{\sigma^{-1}(1)} \otimes \ldots\ldots \otimes \mathbf{a}_{\sigma^{-1}(q)}. \tag{A.8}$$

The actions of groups Σ_q and $\mathcal{O}$ are commutative:

$$\sigma \times (\mathbf{R} * \mathbf{A}) = \mathbf{R}*(\sigma \times \mathbf{A}). \tag{A.9}$$

REMARK. We will not really need either a more developed terminology or more profound results of the *theory of group representations*, otherwise truly beautiful.

Appendix 2. Notation

For readers more accustomed to the usual Cartesian index notation, we shall add a convenient rephrasing of the formulae:

$$\begin{aligned}
\mathbf{x}, \mathbf{y} &\leftrightarrow x_i, y_i, & \boldsymbol{\alpha}, \xi &\leftrightarrow \alpha_{ij}, \xi_{ij},\\
\mathbf{x} \otimes \mathbf{y} &\leftrightarrow x_i y_k, & \boldsymbol{\alpha} \otimes \boldsymbol{\xi} &\leftrightarrow \alpha_{ij}\xi_{kl},\\
\mathbf{x}\mathbf{y} &\leftrightarrow x_p y_p, & \boldsymbol{\alpha} \cdot \boldsymbol{\xi} &\leftrightarrow \alpha_{pq}\xi_{pq},\\
\mathbf{1} &\leftrightarrow \delta_{ij} & &\text{(Kronecker's symbol)},\\
\mathbf{A}, \mathbf{H} &\leftrightarrow A_{ijkl}, H_{ijkl}, & \mathbf{H} \cdot \boldsymbol{\omega} &\leftrightarrow H_{ijpq}\omega_{pq},\\
\mathbf{A} \circ \mathbf{H} &\leftrightarrow A_{ijpq} H_{pqkl}, & \boldsymbol{\omega} \cdot \mathbf{H} \cdot \boldsymbol{\omega} &\leftrightarrow H_{pqrs}\omega_{pq}\omega_{rs},\\
\mathbf{A} \cdot \mathbf{H} &\leftrightarrow A_{pqrs} H_{pqrs}, & \boldsymbol{\sigma} = \mathbf{H} \cdot \varepsilon &\leftrightarrow \sigma_{ij} = H_{ijpq}\varepsilon_{pq},\\
(\boldsymbol{\omega}^{\mathrm{T}})_{ij} &\equiv \omega_{ji}, & (\mathbf{H}^{\mathrm{T}})_{ijkl} &\equiv H_{klij},\\
\operatorname{tr} \boldsymbol{\omega} = \mathbf{1} \cdot \boldsymbol{\omega} &= \omega_{pp}, & \operatorname{Tr} \mathbf{A} = \mathbb{I}_{\mathcal{S}} \cdot \mathbf{A} &= A_{pqpq},\\
(\mathfrak{l} \times \mathbf{A})_{ijkl} &= A_{ijkl}, & (\mathfrak{c} \times \mathbf{A})_{ijkl} &= \frac{1}{2}(A_{ikjl} + A_{ilkj}),\\
|\boldsymbol{\omega}| &= (\omega_{pq}\omega_{pq})^{\frac{1}{2}}, & |\mathbf{H}| &= (H_{pqrs}H_{pqrs})^{\frac{1}{2}}.
\end{aligned}$$

This dictionary enables one to write any formula in this paper in Cartesian index language. For example,

$$[\mathfrak{t} \times (\mathbf{1} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \mathbf{1})]_{ijkl} = \frac{1}{3}(2\delta_{ij}\beta_{kl} + 2\beta_{ij}\delta_{kl} - \delta_{ik}\beta_{jl} - \beta_{ik}\delta_{jl} - \delta_{il}\beta_{kj} - \beta_{il}\delta_{kj})$$

(see (7.4), (2.10), (2.6)).

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