Arch. Mech., **52**, 4–5, pp. 713–735, Warszawa 2000

An approach to elastic shakedown based on the maximum plastic dissipation theorem

Dedicated to Professor Zenon Mróz on the occasion of his 70^{th} birthday

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Elastic-perfectly plastic solid structures are considered subjected to combined loads, superposition of permanent (mechanical) loads and cyclically variable loads, the latter being specified to within a scalar multiplier. The classical maximum dissipation theorem is used to derive known results of the shakedown theory, as well as a few apparently novel concepts: the shakedown limit load associated with a given (noninstantaneous) collapse mode, the mixed upper bound to the shakedown safety factor, and the mixed static-kinematic formulation of the shakedown safety factor problem. The shakedown load boundary surface is also investigated and a number of its notable features are pointed out. A simple illustrative example is presented.

Notation

A compact notation is used throughout the paper, with bold-face symbols for vectors and tensors, with the rules: $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$, $(\boldsymbol{\sigma}\cdot\mathbf{n})_i = \sigma_{ij}n_j$, $\mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} = u_i\sigma_{ij}n_j$, $(\mathbf{A} : \boldsymbol{\sigma})_{ij} = A_{ijhk}\sigma_{hk}$, $\boldsymbol{\sigma} : \mathbf{A} : \boldsymbol{\sigma} = A_{ijhk}\sigma_{ij}\sigma_{hk}$, where the indicial summation rule applies. The symbol $:=$ means equality by definition. Other symbols are defined where they appear for the first time.

1. Introduction

In this paper, known notions of (elastic) shakedown theory will be discussed from a non-traditional point of view. The motivation for such an approach to the shakedown theory is suggested by a neighbour theory of plastic limit analysis.

It is known (see e.g. [6, 8, 11, 12]) that, in the latter theory, a central role is played by the maximum plastic work theorem of HILL [7], which in fact enables us to evaluate the limit load corresponding to an arbitrarily assigned collapse mechanism of a given structure, and can thus be utilized as a departure point to develop the static and kinematic approaches to the plastic limit analysis. In particular, within the kinematic approach, the structure's plastic collapse safety factor can be determined as the minimum value, over the entire set of collapse mechanisms, of the so-called kinematic load multiplier, which can be interpreted as the structure's safety factor against a specified collapse mechanism.

In the shakedown theory (see e.g. [4, 5, 9, 10, 12]), a theorem analogous to the above maximum plastic work theorem was provided in [3] for elastic-perfectly plastic structures subjected to combined cyclic/permanent loads. This theorem is capable of providing the combined load at the shakedown limit corresponding to a specified collapse mode. It is the purpose of the present paper to show that, like in the plastic limit analysis, the latter theorem can be utilized as a starting point to develop the static and kinematic approaches to the shakedown theory. Also, the notion of structure's shakedown limit load for an assigned (noninstaneous) collapse mechanism will be introduced to show that its minimum value over the entire set of such mechanisms coincides with the structure's shakedown limit load.

The plan of the paper is as follows. In Sec. 2, the maximum plastic dissipation theorem (written in space integral form) will be recalled and applied to plastic limit analysis for demonstrative purposes. In Sec. 3, the concept of inadaptation collapse mechanism will be briefly discussed and its ingredients, such as the '(noninstantaneous) collapse mechanism' and the 'plastic strain path', pointed out for use in Sec. 4. In the next section, the maximum plastic dissipation theorem (written in time-space integral form) will be employed to derive the concept of 'shakedown limit load for assigned (noninstantaneous) collapse mechanism', for which two alternative formulations are given, static and kinematic respectively, together with the set of equations which govern the related structural problem. Section 5 will be devoted to the shakedown load boundary surface and to its essential features, showing that it plays the role of yield surface for the structure: namely, any load staying within this surface is below the shakedown limit, hence no inadaptation collapse mechanism is produced (i.e. the limit state is elastic), whereas any load on the shakedown boundary surface corresponds to a shakedown limit state, in which it is prone to an impending (noninstantaneous) collapse mechanism represented by a vector normal to the above boundary surface at the load point. In Sec. 6, the maximum dissipation theorem (still written in time-space integral form) is used to derive the classical static and kinematic formulations of the shakedown safety factor for combined cyclic/permanent loads. Alternative formulations are also provided: one is a two-stage kinematic formulation, another is a mixed static-kinematic formulation, the latter being characterized by the use of free stress variables and compatibility equations. A simple application is presented in Sec. 7. A resumé is presented, together with the conclusions, in Sec. 8.

2. Preliminary considerations

The maximum plastic dissipation theorem [6 – 8, 11, 12] written in space-integral form, reads:

$$\max_{\boldsymbol{\sigma}} \int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \, dV \quad \text{s.t. } f(\boldsymbol{\sigma}) \leq 0 \ \text{ in } V \tag{2.1}$$

where 's.t.' stands for 'subject to', $\dot{\boldsymbol{\varepsilon}}^p$ is a plastic strain rate field assigned in the structure's domain V, $f(\boldsymbol{\sigma})$ is the (convex, smooth) yield function and $\boldsymbol{\sigma}$ is an unknown stress field. It can be shown that the Euler-Lagrange equations of problem (2.1) coincide with the material plastic flow laws, i.e.

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}}, \tag{2.2}$$

$$f(\boldsymbol{\sigma}) \leq 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} f(\boldsymbol{\sigma}) = 0, \tag{2.3}$$

where $\dot{\lambda}$ is the consistency coefficient, Eqs. (2.2) and (2.3) being *enforced everywhere* in V. In other words, the optimal stress field $\boldsymbol{\sigma}$ given by problem (2.1) corresponds to $\dot{\boldsymbol{\varepsilon}}^p$ in a point-wise manner through (2.2) and (2.3).

In consideration of the arbitrariness of $\dot{\boldsymbol{\varepsilon}}^p$ in (2.1), let $\dot{\boldsymbol{\varepsilon}}^p$ be chosen to be compatible with the velocity field $\dot{\mathbf{u}}$; that is, the compatibility equations

$$\dot{\boldsymbol{\varepsilon}}^p = \nabla^s \dot{\mathbf{u}} \quad \text{in } V, \qquad \dot{\mathbf{u}} = 0 \quad \text{on } S_D, \tag{2.4}$$

are satisfied. In (2.4), ∇^s is the symmetric part of the gradient operator ∇, S_D is a part of the boundary surface $S = \partial V$, i.e. $S_D \subset S$. The notation $\dot{\boldsymbol{\varepsilon}}^p(\dot{\mathbf{u}})$ will be used in the following to mean that $\dot{\boldsymbol{\varepsilon}}^p$ is related to $\dot{\mathbf{u}}$ by Eq. (2.4). Then, applying the virtual work principle, one can write the equality

$$\int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \, dV = (\mathbf{p}, \dot{\mathbf{u}}) \tag{2.5}$$

where the internal product $(\mathbf{p}, \dot{\mathbf{u}})$ is given by

$$(\mathbf{p}, \dot{\mathbf{u}}) := \int_V \mathbf{p}_V \cdot \dot{\mathbf{u}} \, dV + \int_{S_T} \mathbf{p}_S \cdot \dot{\mathbf{u}} \, dS. \tag{2.6}$$

Here, the load $\mathbf{p}$ equilibrated by $\boldsymbol{\sigma}$ consists of volume forces $\mathbf{p}_V$ in V and surface forces $\mathbf{p}_S$ on $S_T = S \backslash S_D$, i.e. $\mathbf{p} = \{\mathbf{p}_V \,\mathrm{in}\, V,\ \mathbf{p}_S \,\mathrm{on}\, S_T\}$, and thus

$$\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{p}_V = \mathbf{0} \quad \text{in } V, \qquad \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{p}_S \quad \text{on } S_D. \tag{2.7}$$

After the above assumptitions, problem (2.1) can be recast as

$$\max_{\{\mathbf{p},\boldsymbol{\sigma}\}} (\mathbf{p}, \dot{\mathbf{u}}) \quad \text{s.t.} \begin{cases} f(\boldsymbol{\sigma}) \le 0 \text{ in } V \\ \boldsymbol{\sigma} \text{ in equilibrium with } \mathbf{p} \end{cases} \tag{2.8}$$

where $\dot{\mathbf{u}}$ is an arbitrary fixed field satisfying the boundary condition $\dot{\mathbf{u}} = \mathbf{0}$ on S_D. Problem (2.8) substantially coincides with the aforementioned result of HILL [6] and in fact it provides the *limit load* $\mathbf{p}[\dot{\mathbf{u}}]$ *corresponding to a specified plastic collapse mechanism*, $\dot{\mathbf{u}}$, of the body constrained on S_D. The optional objective value of problem (2.8) equals the overall plastic work developed through the mechanism $\dot{\mathbf{u}}$, i.e.

$$(\mathbf{p}[\dot{\mathbf{u}}], \dot{\mathbf{u}}) = W[\dot{\mathbf{u}}] := \int_V D(\dot{\boldsymbol{\varepsilon}}^p)\, dV\,, \tag{2.9}$$

where $\dot{\boldsymbol{\varepsilon}}^p = \dot{\boldsymbol{\varepsilon}}^p(\dot{\mathbf{u}})$. In the load space, the collapse mechanism $\dot{\mathbf{u}}$ is orthogonal to the structure's resistance surface at the corresponding limit load $\mathbf{p}[\dot{\mathbf{u}}]$.

If, additionally, in problem (2.8) some restriction upon the external forces $\mathbf{p}$ is introduced, then $\dot{\mathbf{u}}$ must be suitably relaxed; for instance, if $\mathbf{p} = \alpha\bar{\mathbf{p}}$, where $\bar{\mathbf{p}}$ is a specified external force distribution and the scalar α is arbitrary, one obtains

$$(\mathbf{p}, \dot{\mathbf{u}}) = \alpha\,(\bar{\mathbf{p}}, \dot{\mathbf{u}})\,. \tag{2.10}$$

This means that in the maximum plastic work theorem (2.8), the mechanism $\dot{\mathbf{u}}$ needs to be specified only through the scalar parameter a defined as

$$a := (\bar{\mathbf{p}}, \dot{\mathbf{u}})\,, \tag{2.11}$$

which represents mechanism's projection in the 'direction' of the specified load $\bar{\mathbf{p}}$. Assuming e.g. $a = 1$, problem (2.8) takes the special form

$$\alpha_p = \max_{\{\alpha,\boldsymbol{\sigma}\}} \alpha \quad \text{s.t.} \begin{cases} f(\boldsymbol{\sigma}) \le 0 \text{ in } V \\ \boldsymbol{\sigma} \text{ in equilibrium with } \alpha\bar{\mathbf{p}} \end{cases} \tag{2.12}$$

and, correspondingly, $W = \alpha_p$. Problem (2.12) is recognized as the well-known static formulation of the plastic collapse safety factor problem. Dualization gives then the related kinematic formulation, i.e.

$$\alpha_p = \min_{\dot{\mathbf{u}} \in M} W[\dot{\mathbf{u}}] \quad \text{s.t.} \ (\bar{\mathbf{p}}, \dot{\mathbf{u}}) = 1, \tag{2.13}$$

where the minimization is performed with respect to the set M of all mechanisms (2.4), and the objective function, given by (2.9), represents the kinematic load multiplier generated by $\dot{\mathbf{u}}$.

The above considerations show that the maximum plastic dissipation theorem can be taken as a starting point to derive the classical results of limit analysis. Following a reasoning similar to that developed above, known results of the shakedown theory will be derived together with a few apparently novel concepts.

3. The inadaptation collapse mechanism

In Sec. 2, the maximum plastic dissipation theorem has been used for the evaluation of the plastic limit load of a structure that undergoes a specified (plastic) collapse mechanism. The key idea consisted in introducing, into the body, a compatible plastic strain rate field, that is, a strain field capable of representing an *instantaneous* collapse mechanism of the structure. In order to apply a similar procedure for evaluating the (shakedown) limit load corresponding to a specified noninstantaneous collapse mechanism, it is necessary to give a precise meaning to the latter sort of mechanisms.

Within the shakedown context, structures are considered to be subjected to (besides, possibly, a permanent (mechanical) load, $\mathbf{p}$) to loads $\mathbf{q}^c$ allowed to vary in a given (closed) domain, say Π^c. Any path within the latter domain represents a potentially active load path, and every load condition is by the hypothesis below the plastic collapse limit value. A closed load path Π^c, repeatedly travelled by the load point, represents a cyclic load, which is potentially dangerous because plastic strain effects may cumulate progressively cycle after cycle until failure. (Among such cyclic loads, most dangerous are the load paths lying on $\partial\Pi^c$. If, as usual, Π^c is a plane, then there is just one most dangerous load path enveloping $\partial\Pi^c$). The failure produced by the cyclic load, referred to as *inadaptation collapse*, exhibits the characteristics that can be specified by explaining the straining process that manifest itself in the course of application of some cyclic load higher than the shakedown limit value. Namely, the structure subjected to a periodic load tends towards (and generally reaches after a few cycles) a steady state in which the response is characterized by stresses $\boldsymbol{\sigma}$ and plastic strain rates, $\dot{\boldsymbol{\varepsilon}}^p$, periodic as the load, such that the ratchet strain, $\Delta\boldsymbol{\varepsilon}^p$, i.e. the net plastic strain accumulated in the steady cycle, is a compatible strain field with zero ratchet displacements, $\Delta\mathbf{u}$, on the constrained boundary of the body (see e.g. [2, 13, 15]).

The above plastic straining process, related to an actual steady cycle, can thus be described by the following equations:

$$\Delta\boldsymbol{\varepsilon}^p(\mathbf{x}) = \nabla^s\Delta\mathbf{u}(\mathbf{x}) \quad \text{in } V, \qquad \Delta\mathbf{u}(\mathbf{x}) = \mathbf{0} \quad \text{on } S_D, \tag{3.1}$$

(3.2) $$\Delta \boldsymbol{\varepsilon}^p(\mathbf{x}) = \int_0^T \dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}, t)\, dt$$

in which (3.1) is like (2.4). Two types of inadaptation collapse modes can be distinguished:

– *Incremental collapse* (or *Ratchetting*), in which the ratchet strain $\Delta\boldsymbol{\varepsilon}^p$ is nonvanishing at least somewhere in the structure, what causes the structure to fail by excessive plastic strain. (It includes, as a special case, the instantaneous plastic collapse by taking $\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x},t) = \Delta\boldsymbol{\varepsilon}^p(\mathbf{x})\delta_D(t-\bar{t})$, where $\delta_D(t-\bar{t})$ is the Dirac delta centred at time $\bar{t}$.)

– *Alternating plasticity collapse* (or *Plastic shakedown*), in which the ratchet strain $\Delta\boldsymbol{\varepsilon}^p$ is vanishing everywhere in the structure which thus undergoes alternating plasticity with consequent low-cycle fatigue failure.

It is seen that both types of the inadaptation collapse, similarly to the instantaneous plastic collapse, involve a collapse mechanism with its compatible plastic strain field, $\Delta\boldsymbol{\varepsilon}^p$ (though the latter is trivial for alternating plasticity collapse modes). However, in the present case, $\Delta\boldsymbol{\varepsilon}^p$ describes a *noninstantaneous* plastic mechanism because the ratchet strain $\Delta\boldsymbol{\varepsilon}^p$ is the result of cumulating plastic contributions, each occurring at a different time within the strain cycle.

If the applied load is not higher than the (elastic) shakedown limit, no inadaptation collapse mechanism is produced, i.e. the steady-state response is purely elastic and Eqs. (3.1) and (3.2) turn out to be meaningless. However, for load values at the shakedown limit, the structure shakedown limit state can be envisaged, which is characterized by an *impending inadaptation collapse mechanism*. The latter mechanism has the same features as an actual inadaptation collapse mechanism, and thus can be represented as

(3.3) $$\Delta\mathbf{e}^p(\mathbf{x}) = \nabla^s\mathbf{v}(\mathbf{x}) \quad \text{in } V, \qquad \mathbf{v}(\mathbf{x}) = \mathbf{0} \quad \text{on } S_D,$$

(3.4) $$\Delta\mathbf{e}^p(\mathbf{x}) = \int_0^T \dot{\mathbf{e}}^p(\mathbf{x}, t)\, dt,$$

where $\dot{\mathbf{e}}^p(\mathbf{x},t)$ is some (fictitious) plastic strain rate history. Equations (3.3) and (3.4) can be derived from Eqs. (3.1) and (3.2) by considering the (positive) scalar η measuring the excess value of a load promoting the actual steady cycle (3.1) and (3.2), with respect to the related shakedown limit value. Dividing (3.1) and (3.2) by η and then taking the limit for $\eta \to 0$, one has $\dot{\boldsymbol{\varepsilon}}^p/\eta \to \dot{\mathbf{e}}^p$, $\Delta\boldsymbol{\varepsilon}^p/\eta \to \Delta\mathbf{e}^p$, $\Delta\mathbf{u}/\eta \to \mathbf{v}$, and thus Eqs. (3.3) and (3.4) are generated. In other words, Eqs. (3.3) and (3.4) represent, to within a positive factor, the *incipient* actual inadaptation collapse mechanism produced as soon as the load slightly exceeds the limit value [3, 14, 15].

A (fictitious) plastic strain rate history $\dot{\mathbf{e}}^p$ specified in V for $0 \le t \le T$ and satisfying Eqs. (3.3) and (3.4) constitutes a *kinematically admissible plastic strain cycle* after KOITER [9]. It is characterized by two essential ingredients, i.e. the *collapse mechanism* $\{\Delta\mathbf{e}^p, \mathbf{v}\}$ specified by (3.3) and the related *plastic strain path* $\dot{\mathbf{e}}^p$ of (3.4). A field $\mathbf{v}$ satisfying the boundary conditions $\mathbf{v} = \mathbf{0}$ on S_D represents a collapse mechanism in the set M of all possible collapse mechanisms. For a given collapse mechanism $\mathbf{v} \in M$, with related ratchet strain $\Delta\boldsymbol{\varepsilon}^p = \Delta\boldsymbol{\varepsilon}^p(\mathbf{v})$, there is a set, $\Lambda[\Delta\mathbf{e}^p]$ say, of infinite plastic strain paths, $\dot{\mathbf{e}}^p(\mathbf{x}, t)$, all of them satisfying (3.3). For $\Delta\mathbf{e}^p \equiv \mathbf{0}$, the set $\Lambda_0 := \Lambda[\mathbf{0}]$ collects all the cyclic plastic strain paths.

Note that, if the compatible plastic strain cycle of Eqs. (3.3) and (3.4) is applied upon a stress-free strain-free elastic structure as an imposed plastic strain history, the stress and displacement responses, $\boldsymbol{\sigma}^R$ and $\mathbf{u}^R$ say, are such that $\boldsymbol{\sigma}^R(\mathbf{x}, T) = \mathbf{0}$ in V and $\mathbf{u}^R(\mathbf{x}, T) = \mathbf{v}(\mathbf{x})$ in $V \cup S_T$.

4. The shakedown limit load for assigned collapse mechanism

An elastic-perfectly plastic structure is subjected to a combined cyclic/permanent load superposition of a periodic load $\bar{\mathbf{q}}^c(t)$, $0 \le t \le T$, and a permanent (mechanical) load $\mathbf{p}$. Let $\mathbf{v} \in M$ be a specified collapse mechanism, and let $\Delta\mathbf{e}^p = \Delta\mathbf{e}^p(\mathbf{v})$ be the related ratchet strain. The following problem is posed: find a cyclic load multiplier, $\beta > 0$, and a permanent (mechanical) load, $\mathbf{p}$, (suitably distributed over $V \cup S_T$) such that the structure subjected to the *combined* load $\mathbf{q}(t) = \mathbf{p} + \beta\,\bar{\mathbf{q}}^c(t)$, $0 \le t \le T$, be able to reach a shakedown limit state characterized by an impending inadaptation collapse mechanism complying with the assigned mechanism, $\mathbf{v} \in M$. Later on in this section, the equations governing the above problem will be established and shown to be capable of providing, besides the (shakedown) limit load $\mathbf{q}$ through the unknown pair $\{\mathbf{p}, \beta\}$, also the related plastic strain path $\dot{\mathbf{e}}^p \in \Lambda[\Delta\mathbf{e}^p(\mathbf{v})]$.

In order to solve the above problem, let the maximum plastic dissipation theorem be cast in a time-space integral form as

$$(4.1) \qquad \max_{\boldsymbol{\sigma}} \int_0^T \int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \, dV \, dt \quad \text{s.t. } f(\boldsymbol{\sigma}) \le 0 \text{ in } V \times (0, T),$$

where $\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}, t)$, $0 \le t \le T$, is a plastic strain-rate history specified in $V \times (0, T)$. In analogy to problem (2.1), it can be stated that problem (4.1) is equivalent to the material plastic flow laws, Eqs. (2.2) and (2.3), enforced everywhere in V and for every t, $0 \le t \le T$. Let $\dot{\boldsymbol{\varepsilon}}^p$ be taken as a kinematically admissible plastic strain cycle, say $\dot{\mathbf{e}}^p(\mathbf{x}, t)$, associated with a given collapse mechanism $\mathbf{v} \in M$, that is $\dot{\mathbf{e}}^p(\mathbf{x},t)$ satisfies Eq. (3.4) with $\Delta\mathbf{e}^p = \Delta\mathbf{e}^p(\mathbf{v})$, hence $\dot{\mathbf{e}}^p \in \Lambda[\Delta\mathbf{e}^p]$. Also, let the maximization operation of (4.1) be performed within the stress set given by

$$\boldsymbol{\sigma} = \mathbf{s} + \beta\,\bar{\boldsymbol{\sigma}}^c, \tag{4.2}$$

where $\mathbf{s}$ is an unknown time-independent stress field, $\beta > 0$ an unknown scalar, and $\bar{\boldsymbol{\sigma}}^c$ the elastic stress response of the structure to the cyclic load $\bar{\mathbf{q}}^c(t)$. Substituting (4.2) in (4.1) and using Eqs. (3.3) and (3.4), we obtain

$$\int_0^T\int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p\,dV\,dt = \int_V \mathbf{s} : \Delta\mathbf{e}^p\,dV + \beta\,b, \tag{4.3}$$

where $\dot{\boldsymbol{\varepsilon}}^p \equiv \dot{\mathbf{e}}^p$ and b is a scalar parameter defined as

$$b := \int_0^T\int_V \bar{\boldsymbol{\sigma}}^c : \dot{\mathbf{e}}^p\,dV\,dt, \tag{4.4}$$

which represents a measure of the plastic strain path.

It results that, with the above choices for $\dot{\boldsymbol{\varepsilon}}^p$ and $\boldsymbol{\sigma}$, problem (4.1) requires that the kinematically admissible plastic strain cycle, $\dot{\boldsymbol{\varepsilon}}^p = \dot{\mathbf{e}}^p$, should be specified only through the related collapse mechanism $\mathbf{v} \in M$ and (for $\beta > 0$) the scalar parameter b of (4.4); furthermore, ignoring for the moment that β may take zero values for certain $\mathbf{v} \in M$, the maximization operation should be done with respect to the free variables $\mathbf{s}$ and β. In fact, problem (4.1) now reads, with $b = 1$:

$$\psi[\mathbf{v}] := \max_{\{\mathbf{s},\beta\}} \left(\int_V \mathbf{s} : \Delta\mathbf{e}^p(\mathbf{v})\,dV + \beta \right) \tag{4.5}$$

$$\text{s.t. } f\left(\mathbf{s} + \beta\,\bar{\boldsymbol{\sigma}}^c\right) \le 0 \text{ in } V \times (0,T),$$

where the condition $b = 1$ amounts to normalizing $\Delta\mathbf{e}^p$ and $\mathbf{v}$. Problem (4.5) is a static formulation of the structure shakedown limit load for a given collapse mechanism, $\mathbf{v} \in M$. The following theorems can be proved.

4.1. Static theorem

For a given structure subjected to combined cyclic/permanent loads, say $\mathbf{q} = \mathbf{p} + \beta\,\bar{\mathbf{q}}^c(t)$, $0 \le t \le T$, and for a given mechanism $\mathbf{v} \in M$, the shakedown limit load $\{\hat{\mathbf{p}}, \hat{\beta}\}$ corresponding to the latter mechanism is the load which produces the maximum dissipation through the given mechanism, and thus the stress field $\mathbf{s} = \hat{\mathbf{s}}$, equilibrating $\hat{\mathbf{p}}$ and $\hat{\beta}$, solve problem (4.5); conversely, the solution to problem (4.5) provides the shakedown limit load, $\{\hat{\mathbf{p}}, \hat{\beta}\}$, corresponding to the given $\mathbf{v} \in M$.

This statement can be proved by investigating the mechanical implications of problem (4.5), which is done by the Lagrange multiplier method. Writing the relevant augmented Lagrangian functional as

$$\chi := -\int_V \mathbf{s} : \Delta\mathbf{e}^p \, dV - \beta + \int_0^T \int_V \dot{l}\, f(\mathbf{s} + \beta\bar{\boldsymbol{\sigma}}^c) \, dV \, dt \tag{4.6}$$

where $\dot{l}(\mathbf{x}, t) \geq 0$ is the Lagrange multiplier, the first variation of χ reads

$$\delta\chi = \delta\beta \left(-1 + \int_0^T \int_V \bar{\boldsymbol{\sigma}}^c : \frac{\partial f}{\partial \boldsymbol{\sigma}} \, \dot{l} \, dV \, dt \right) + \int_V \delta\mathbf{s} : \left(-\mathbf{e}^p + \int_0^T \frac{\partial f}{\partial \boldsymbol{\sigma}} \, \dot{l} \, dt \right) dV + \int_0^T \int_V \delta\dot{l}\, f(\boldsymbol{\sigma}) \, dV \, dt, \tag{4.7}$$

where $\boldsymbol{\sigma} := \mathbf{s} + \beta\, \bar{\boldsymbol{\sigma}}^c$. Since χ must take a minimum with respect to $\mathbf{s}$ and β and a maximum with respect to $\dot{l} \geq 0$, the Euler-Lagrange equations related to (4.5) read as follows:

$$f(\boldsymbol{\sigma}) \leq 0, \quad \dot{l} \geq 0, \quad \dot{l}\, f(\boldsymbol{\sigma}) = 0 \quad \text{in } V \times (0, T), \tag{4.8}$$

$$\boldsymbol{\sigma} := \mathbf{s} + \beta\, \bar{\boldsymbol{\sigma}}^c, \quad \dot{\mathbf{e}}^p := \frac{\partial f}{\partial \boldsymbol{\sigma}} \dot{l} \quad \text{in } V \times (0, T), \tag{4.9}$$

$$\int_0^T \dot{\mathbf{e}}^p \, dt = \Delta\mathbf{e}^p(\mathbf{v}) \quad \text{in } V, \qquad \int_0^T \int_V \bar{\boldsymbol{\sigma}}^c : \dot{\mathbf{e}}^p \, dV \, dt = b = 1. \tag{4.10}$$

The following can be remarked in relation to Eqs. (4.8) to (4.10):

a) The Lagrange multiplier $\dot{l}$ takes the meaning of a plastic coefficient for the (fictitious) plastic strain rate $\dot{\mathbf{e}}^p$, Eqs. (4.8) and (4.9).

b) Due to the convexity of problem (4.5), Eqs. (4.8) to (4.10) are not only necessary, but also sufficient conditions, i.e. the/a solution $(\mathbf{s}, \beta, \dot{l})$ to Eqs. (4.8) to (4.10) is such that $\{\mathbf{s}, \beta\}$ solves the problem (4.5).

c) The (fictitious) plastic strain rate history, $\dot{\mathbf{e}}^p(\mathbf{x}, t)$, constitutes a kinematically admissible plastic strain cycle complying with the given collapse mechanism $\mathbf{v} \in M$ and the scalar parameter $b = 1$, Eq. (4.10).

d) Denoting by $\mathbf{s}^E$ the elastic stress response to $\mathbf{p}$ and thus by $\boldsymbol{\sigma}^E = \mathbf{s}^E + \beta\,\bar{\boldsymbol{\sigma}}^c$ the analogous response to the load $\mathbf{q} = \mathbf{p} + \beta\,\bar{\mathbf{q}}^c$, the equality

$$\int_0^T \int_V D(\dot{\mathbf{e}}^p)\, dV\, dt = \int_0^T \int_V \boldsymbol{\sigma}^E : \dot{\mathbf{e}}^p\, dV\, dt \tag{4.11}$$

can be easily shown to hold, which means that the overall plastic work equals the external work and thus, by Koiter's theorem, the load $\mathbf{q}$ cannot be below the shakedown limit. Since, on the other hand, $\mathbf{q}$ cannot exceed this limit by Melan's theorem, it follows that $\mathbf{q}$ is a shakedown limit load and that therefore Eqs. (4.8) to (4.10) describe the related impending inadaptation collapse mechanism.

e) It can be proved that Eqs. (4.8) to (4.10) allow for a unique solution for all except for $\mathbf{s} = \mathbf{s}^E + \boldsymbol{\rho}$, where the selfstress $\boldsymbol{\rho}$ may be indeterminate in the region $V_0 \subset V$ (if any) where the assigned ratchet strain $\Delta\mathbf{e}^p = \mathbf{0}$. The proof rests on Drucker's stability postulate and on the assumption that $f(\boldsymbol{\sigma})$ is smooth, but here it is omitted for brevity.

f) Quite often in practice the load path which Eqs. (4.8) to (4.10) refer to is piecewise linear, i.e. polygonal. In such a case, the load can be regarded as one jumping from a corner to another on the polygonal path, i.e. $\mathbf{q}_{(k)} = \mathbf{p} + \beta\,\mathbf{q}^c_{(k)}$, $(k = 1, 2, ..., m)$, and Eqs. (4.8) to (4.10) can be enforced solely at times $t_{(1)}, t_{(2)}, ..., t_{(m)}$ corresponding to the *basic loads* $\mathbf{q}_{(1)}, \mathbf{q}_{(2)}, ..., \mathbf{q}_{(m)}$, and take a time-discrete form.

g) Considered that for certain $\mathbf{v} \in M$ it may result that $\hat{\beta} = 0$, the constraint $\beta \geq 0$ should be accounted for in problem (4.5), in which case Eqs. (4.8) to (4.10) remain the same as long as $\hat{\beta} > 0$, but the second equation of (4.10) vanishes when $\hat{\beta} = 0$. Since the condition $\hat{\beta} = 0$ occurs when the structure's limit state is (equivalent to) an instantaneous plastic collapse, this condition is as a rule excluded from the discussion.

Following the standard procedures (i.e. maximization of χ of Eq. (4.6) under the constraints (4.8) to (4.10), but the constraint of (4.5) being removed), the following dual problem is obtained:

$$\widetilde{W}[\mathbf{v}] = \min_{\dot{\mathbf{e}}^p} \int_0^T \int_V D(\dot{\mathbf{e}}^p)\, dV\, dt$$

$$\text{s.t.} \int_0^T \dot{\mathbf{e}}^p\, dt = \Delta\mathbf{e}^p(\mathbf{v}) \quad \text{in } V, \qquad \int_0^T \int_V \bar{\boldsymbol{\sigma}}^c : \dot{\mathbf{e}}^p\, dV\, dt = b = 1, \tag{4.12}$$

which contains only kinematic variables and $\mathbf{e}^p(\mathbf{v})$ is still fixed. This problem is the kinematic formulation of the structure's shakedown limit load for the assigned collapse mechanism, $\mathbf{v} \in M$. It provides the optimal plastic strain path associated with $\Delta \mathbf{e}^p(\mathbf{v})$, as well as the related shakedown limit load $\{\hat{\mathbf{p}}, \hat{\beta}\}$. The following can be stated.

4.2. Kinematic theorem

For a given structure subjected to combined cyclic/permanent loads, say $\mathbf{q} = \mathbf{p} + \beta \bar{\mathbf{q}}^c(t)$, $0 \leq t \leq T$, and for a given collapse mechanism, $\mathbf{v} \in M$, the optimal plastic strain path corresponding to the latter mechanism is that one which minimizes the overall plastic dissipation and thus it is the solution to problem (4.12); conversely, the solution to (4.12) provides the optimal plastic strain path related to the given collapse mechanism.

Using again the Lagrangian multiplier method, the relevant augmented functional reads:

$$\chi_1 = \int_0^T \int_V D(\dot{\mathbf{e}}^p)\, dV\, dt + \int_V \mathbf{s} : \left(\Delta \mathbf{e}^p - \int_0^T \dot{\mathbf{e}}^p\, dt \right) dV + \beta \left(1 - \int_0^T \int_V \bar{\boldsymbol{\sigma}}^c : \dot{\mathbf{e}}^p\, dV\, dt \right), \tag{4.13}$$

where $\mathbf{s}$ and β are stress-like and scalar multipliers. With a procedure similar to that used before, it can be easily realized that problem (4.12) is equivalent to problem (4.5) and that the above kinematic theorem holds good, but the proof will be omitted here for brevity.

On comparing problems (4.5) and (4.12) with each other, it is seen that they admit the same *optimal objective functionals*, that is, on considering b as a free parameter,

$$(\hat{\mathbf{p}}, \mathbf{v}) + \hat{\beta}\, b = W[\mathbf{v}, b] := \int_0^T \int_V D(\hat{\dot{\mathbf{e}}}^p)\, dV\, dt \tag{4.14}$$

where $\hat{\mathbf{p}}$ and $\hat{\beta}$ are some functionals of $\mathbf{v}$ and b. For $b = 1$, since $(\hat{\mathbf{p}}, \mathbf{v}) + \hat{\beta} = \psi[\mathbf{v}]$ and $W[\mathbf{v}, 1] = \widetilde{W}[\mathbf{v}]$, it is

$$\psi[\mathbf{v}] = W[\mathbf{v}, 1] = \widetilde{W}[\mathbf{v}]. \tag{4.15}$$

5. The structure's shakedown load boundary

For the purpose of this section, problems (4.5) and (4.12) are considered with the scalar free parameter b such that the pair $\{\mathbf{v}, b\}$ constitutes a collapse mode, whereas the related shakedown limit load is referred to by the pair $\{\mathbf{p}, \beta\}$, (instead of $\{\hat{\mathbf{p}}, \hat{b}\}$ used in Sec. 4).

Problem (4.5), or (4.12), can be used to generate, for every (noninstantaneous) collapse mode $\{\mathbf{v}, b\}$, the corresponding shakedown limit load $\{\mathbf{p}, \beta\}$. In this way, at least in principle, a surface $F[\mathbf{p}, \beta] = 0$ can be obtained in a suitable load space; if, for instance, $\mathbf{p}$ is a n-parameter load, this surface belongs to an $(n + 1)$-dimensional Euclidean space. Solving $F = 0$ with respect to β gives the equation $\beta = \beta_{\mathrm{sh}}[\mathbf{p}]$ which represents the (shakedown) limit value of β (or shakedown safety factor) for the assigned permanent load $\mathbf{p}$.

Let $W[\mathbf{v}, b]$ be the functional resulting as the optimal objective value of problem (4.5) or (4.12), as expressed by (4.14). By assumption this functional exists. The Fréchet derivatives

$$\text{(5.1)} \qquad \left.\frac{\partial W}{\partial \mathbf{v}}\right|_{\mathbf{x}} = \begin{cases} \mathbf{p}_V(\mathbf{x}) & \forall\, \mathbf{x} \in V \\ \mathbf{p}_S(\mathbf{x}) & \forall\, \mathbf{x} \in S_T \end{cases}, \qquad \frac{\partial W}{\partial b} = \beta,$$

express the sensitivity of $W[\mathbf{v}, b]$ to the changes of $\mathbf{v}$ around $\mathbf{x} \in V \cup S_T$, and with respect to b. Equation (5.1) can be obtained from Eq. (4.6) written at the optimum and by remarking that $-\chi_{\mathrm{opt}} + (\mathbf{p}, \mathbf{v}) = W[\mathbf{v}, b] = [(\mathbf{p}, \mathbf{v}) + \beta\, b]_{\mathrm{opt}}$. Note that $W[\mathbf{v}, b]$ is homogeneous of degree one, i.e. $W[m\mathbf{v}, mb] = mW[\mathbf{v}, b]\ \forall\, m > 0$, as it can be easily proved using Eqs. (4.8) to (4.10).

Let $W^*[\mathbf{p}, \beta]$ be the Legendre transform of $W[\mathbf{v}, b]$, i.e.

$$\text{(5.2)} \qquad W^*[\mathbf{p}, \beta] := (\mathbf{p}, \mathbf{v}) + \beta\, b - W[\mathbf{v}, b],$$

where $\mathbf{v}$ and b are to be meant as functionals of $\mathbf{p}$ and β obtained from (5.1), such that

$$\text{(5.3)} \qquad \left.\frac{\partial W^*}{\partial \mathbf{p}}\right|_{\mathbf{x}} = \mathbf{v}(\mathbf{x}) \quad \forall\, \mathbf{x} \in V \cup S_T, \qquad \frac{\partial W^*}{\partial \beta} = b.$$

If $\{\mathbf{p}, \beta\}$ is a shakedown load, by Melan's theorem there exists a stress field $\mathbf{s}$ in equilibrium with $\mathbf{p}$ such that the pair $\{\mathbf{s}, \beta\}$ is a feasible solution to (4.5), which implies that $(\mathbf{p}, \mathbf{v}) + \beta\, b \le W[\mathbf{v}, b]$, and thus $W^*[\mathbf{p}, \beta] \le 0$ by (5.2); but $W^*[\mathbf{p}, \beta] = 0$ if $\{\mathbf{s}, \beta\}$ solves (4.5), i.e. if $\{\mathbf{p}, \beta\}$ is a shakedown limit load, provided the collapse mode is not a trivial one. On the other hand, if $\mathbf{v} \equiv \mathbf{0}$ and $b = 0$, it is $W^* = 0$. It follows that the boundary of the shakedown load domain can be represented as a set of points $\{\mathbf{p}, \beta\}$ such that $W^* = \mu\, F[\mathbf{p}, \beta] = 0$, where $\mu \ge 0$ is a scalar. By Eq. (5.3) one then obtains

$$\text{(5.4)} \qquad \mathbf{v}(\mathbf{x}) = \mu \left.\frac{\partial F}{\partial \mathbf{p}}\right|_{\mathbf{x}} \quad \forall\, \mathbf{x} \in V \cup S_T, \qquad b = \mu\, \frac{\partial F}{\partial \beta},$$

where $\mu > 0$ is indeterminate if $F = 0$, but $\mu = 0$ if $F < 0$, i.e. if the load $\{\mathbf{p}, \beta\}$ is below the shakedown limit value.

The above result means that the surface $F[\mathbf{p},\beta] = 0$ is the boundary of the (convex) shakedown load domain and plays the role of yield surface for the structure, in the sense that for any load $\{\mathbf{p},\beta\}$ such that $F[\mathbf{p},\beta] < 0$, hence $\mu = 0$, no inadaptation collapse mechanism is produced and thus the shakedown limit state is elastic (i.e. shakedown occurs), whereas if $F[\mathbf{p},\beta] = 0$, there is a nonvanishing inadaptation collapse mechanism $\{\mathbf{v}, b\}$, given by (5.4) with $\mu > 0$, which thus lies on the external normal to this surface at point $\{\mathbf{p},\beta\}$, Fig. 1.

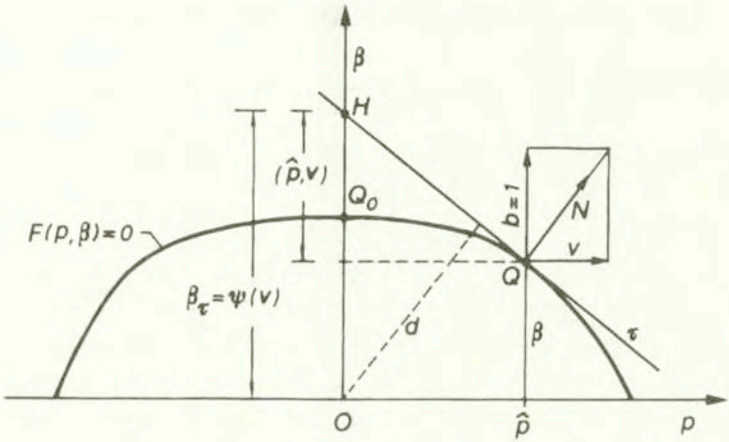

FIG. 1. Geometrical sketch representing the shakedown load boundary $F(p,\beta) = 0$ (or $\beta = \beta_{sh}(p)$ for a one-dimensional permanent load p with an arbitrarily fixed collapse mechanism $\{v, b = 1\}$ and related tangent plane τ.

If $\mathbf{p}$ is a n-parameter load, i.e. $\mathbf{p} = \alpha_1\bar{\mathbf{p}}_1 + \alpha_2\bar{\mathbf{p}}_2 + ... + \alpha_n\bar{\mathbf{p}}_n$, the shakedown boundary surface has the form $F(\alpha_1, \alpha_2, ..., \alpha_n, \beta) = 0$ and belongs to an $(n+1)$-dimensional Euclidean space. This case was discussed in [3] with results analogous to those previously expounded. In Fig. 1 the case $n = 1$ is sketched.

It is worth to note that the classical Melan's and Koiter's shakedown theorems play the role of, respectively, static and kinematic criteria in order to assess whether a given combined load $\{\mathbf{p}^*, \beta^*\}$ is within or outside the shakedown load boundary. To show this point, let $\{\mathbf{p}^*, \beta^*\}$ be a shakedown load, i.e. $F(\mathbf{p}^*, \beta^*) \le 0$. As the pair $\{\mathbf{p}^*, \beta^*\}$ is a feasible solution to (4.5) for any $\mathbf{v}$, one can write the inequality

$$(\mathbf{p} - \mathbf{p}^*, \mathbf{v}) + (\beta - \beta^*)\, b \ge 0 \tag{5.5}$$

where $(\mathbf{p}, \mathbf{v}) + \beta b = W[\mathbf{v}, b]$ is the optimal objective functional of (4.5), as specified by (4.14). Equation (5.5) holds for any load $\{\mathbf{p}^*, \beta^*\}$ such that $F(\mathbf{p}^*, \beta^*) \le 0$, with $\{\mathbf{v}, b\}$ being any fixed collapse mode, and $\{\mathbf{p}, \beta\}$ the corresponding shakedown limit load. Equation (5.5) is like the Drucker's stability postulate for a (fictitious) material endowed with a yield function $F(\mathbf{p}, \beta) \le 0$.

By Melan's theorem, a load $\{\mathbf{p}^*, \beta^*\}$ for which there exist some $\mathbf{s}^*$, equilibrating $\mathbf{p}^*$, such that $f(\mathbf{s}^* + \beta^* \bar{\boldsymbol{\sigma}}^c) \leq 0$ in $V \times (0, T)$, either is a safe shakedown load and thus $F(\mathbf{p}^*, \beta^*) < 0$, or makes the structure capable of reaching a shakedown limit state, in which case $F(\mathbf{p}^*, \beta^*) = 0$; otherwise, if such a stress $\mathbf{s}^*$ cannot be found, the load exceeds the shakedown limit, i.e. $F(\mathbf{p}^*, \beta^*) > 0$. By Koiter's theorem, a load $\{\mathbf{p}^*, \beta^*\}$ is a shakedown load, i.e. $F(\mathbf{p}^*, \beta^*) \leq 0$, if inequality (5.5) is satisfied for any $\{\mathbf{v}, b\}$, that is if

$$(\mathbf{p}, \mathbf{v}) + \beta\, b \geq (\mathbf{p}^*, \mathbf{v}) + \beta^* b \tag{5.6}$$

hence, by Eq. (4.14), if

$$\int_0^T \int_V D(\dot{\mathbf{e}}^p)\, dV\, dt \geq \int_V \mathbf{p}^* \cdot \mathbf{v}\, dV + \beta^* \int_0^T \int_V \bar{\boldsymbol{\sigma}}^c : \dot{\mathbf{e}}^p\, dV\, dt; \tag{5.6$'$}$$

otherwise, if (5.5) is violated for some $\{\mathbf{v}, b\}$, the load $\{\mathbf{p}^*, \beta^*\}$ exceeds the shakedown boundary, i.e. $F(\mathbf{p}^*, \beta^*) > 0$.

Let the collapse mode $\{\mathbf{v}, b\}$ be given and let τ be the plane tangent to the shakedown boundary surface $F[\mathbf{p}, \beta] = 0$ at the point $\{\hat{\mathbf{p}}, \hat{\beta}\}$, ($Q$ in Fig. 1), from where the collapse mechanism $\{\mathbf{v}, b\}$ departs along the (unit) external normal, $\mathbf{N}$. The distance d of τ from the origin O, that is $d = \overrightarrow{OQ} \cdot \mathbf{N}$, using (4.14) can be expressed as

$$d = \frac{1}{K}\left[(\hat{\mathbf{p}}, \mathbf{v}) + \hat{\beta}\, b\right] = \frac{1}{K} W[\mathbf{v}, b] \tag{5.7}$$

where $K := \left((\mathbf{v}, \mathbf{v}) + b^2\right)^{1/2}$. This means that d is proportional to the common optimal objective value of problems (4.5) and (4.12). Then, the abscissa of the intersection point, H, of τ with the β axis, i.e. $\beta_\tau = \overline{OH}$, is given by

$$\beta_\tau = \frac{d}{b/K} = \frac{1}{b} W[\mathbf{v}, b], \tag{5.8}$$

which for $b = 1$ reads

$$\beta_\tau = W[\mathbf{v}, 1] = \psi[\mathbf{v}]. \tag{5.9}$$

In other words, taking $b = 1$, the common optimal objective value of problems (4.5) and (4.12), $\psi[\mathbf{v}]$, equals the abscissa β_τ of point H on the β axis, Fig. 1. Moreover, $\psi[\mathbf{v}]$ is an upper bound to the shakedown safety factor, β_{sh0}, of the structure for zero permanet load, that is

$$\beta_{sh0} \leq \psi[\mathbf{v}] \quad \forall\, \mathbf{v} \in M \tag{5.10}$$

and thus

(5.11) $$\beta_{sh0} = \min_{\mathbf{v}\in M} \psi[\mathbf{v}].$$

The latter equation means that, on changing $\mathbf{v}$ in all possible ways, the plane τ changes correspondingly untill H coincides with Q_0, the latter being the intersection point of the surface $F[\mathbf{p}, b] = 0$ with the β-axis, (see Fig. 1).

6. The shakedown safety factor for combined cyclic/permanent loads

Coming back to Eq. (4.1), let the maximization operation be still performed within the stress set (4.2), but $\mathbf{s}$ being in equilibrium with a fixed permanent load, $\mathbf{p}$ say. Such a choice gives

(6.1) $$\int_0^T \int_V \sigma : \dot{\mathbf{e}}^p \, dV \, dt = \int_V \mathbf{s} : \Delta \mathbf{e}^p \, dV + \beta \, b = (\mathbf{p}, \mathbf{v}) + \beta \, b$$

where b is still given by (4.4). Since the internal product $(\mathbf{p}, \mathbf{v})$ is constant with respect to the maximum operation to be performed, Eq. (6.1) shows that the kinematically admissible plastic strain, $\dot{\mathbf{e}}^p$, must now be fixed only through the scalar parameter b. Thus, taking $b = 1$, problem (4.1), and problem (4.5) as well, become:

(6.2) $$\max_{\{\mathbf{s},\beta\}} \beta \quad \text{s.t.} \begin{cases} f(\mathbf{s} + \beta \, \bar{\sigma}^c) \le 0 \ \text{ in } V \times (0, T) \\ \mathbf{s} \text{ in equilibrium with } \mathbf{p}, \end{cases}$$

while its dual reads

(6.3) $$\begin{gathered} \min_{\{\dot{\mathbf{e}}^p, \mathbf{v}\}} \left(\int_0^T \int_V D(\dot{\mathbf{e}}^p) \, dV \, dt - (\mathbf{p}, \mathbf{v}) \right) \\ \text{s.t.} \quad \int_0^T \dot{\mathbf{e}}^p \, dt = \nabla^s \mathbf{v} \text{ in } V, \qquad \mathbf{v} = \mathbf{0} \text{ on } S_D \\ \int_0^T \int_V \bar{\sigma}^c : \dot{\mathbf{e}}^p \, dV \, dt = b = 1. \end{gathered}$$

Problems (6.2) and (6.3) are recognized as the classic continuum static and kinematic formulations of the shakedown safety factor problem for a structure subjected to combined loads $\mathbf{q} = \mathbf{p} + \beta \, \bar{\mathbf{q}}^c(t)$, their common optimal objective value, $\beta_{\text{sh}}[\mathbf{p}]$, being the shakedown safety factor in question. Therefore, it results that the maximum plastic dissipation theorem, suitably applied, enables the derivation of the static and kinematic approaches to the shakedown limit load.

The shakedown boundary surface, $F(\mathbf{p},\beta)=0$, is again considered (Fig. 2), with the tangent plane τ at point $\hat{Q}$ where the given inadaptation mechanism $\{\mathbf{v},b\}$ departs along the external normal $\mathbf{N}$. Let P be the point representing the permanent load $\mathbf{p}$, R the intersection point of τ with the straight line drawn from P parallel to the β axis, and R_0 the projection of R on the β axis; also, let Q be the intersection point of the shakedown boundary surface with the straight line PR. Obviously, it is $\overline{PQ}=\beta_{\rm sh}[\mathbf{p}]$ = shakedown safety factor for the asigned permanent load, $\mathbf{p}$. The segment PR, spanned by τ over the axis parallel to the β axis through P, has length $\overline{PR}=\overline{OH}-\overline{R_0H}$ which is a functional of $\{\mathbf{v},b\}$ which , for $b=1$, is found to read:

$$\overline{PR}=\beta_{\rm mix}[\mathbf{p},\mathbf{v}]:=\psi[\mathbf{v}]-(\mathbf{p},\mathbf{v}). \tag{6.4}$$

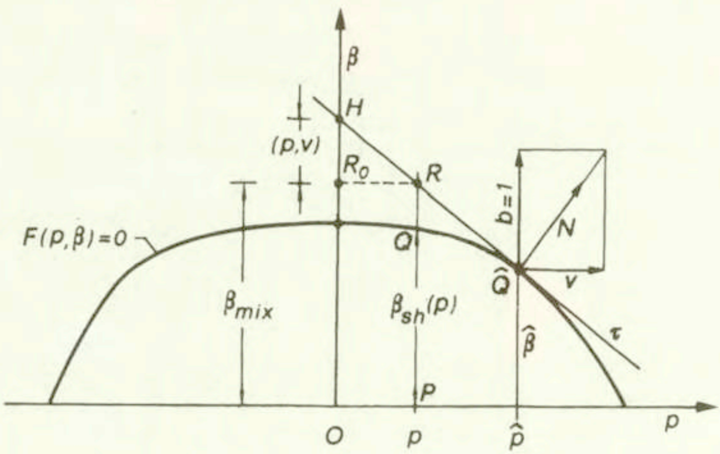

FIG. 2. Geometrical sketch representing the shakedown load boundary: for a given mechanism $\{v,b=1\}$, the segment $\overline{PR}=\beta_{\rm mix}$ spanned over a straight, p = const by the tangent plane τ is an upper bound to $\beta_{sh}(p)$.

In virtue of the convexity of $F=0$, it is geometrically clear that

$$\beta_{\rm sh}[\mathbf{p}]\le\beta_{\rm mix}[\mathbf{p},\mathbf{v}]\quad\forall\,\mathbf{v}\in M. \tag{6.5}$$

This inequality shows that $\beta_{\rm mix}[\mathbf{p},\mathbf{v}]$ is an *upper bound* (u.b.) to $\beta_{\rm sh}[\mathbf{p}]$. This upper bound will be referred to as *mixed* u.b. because $\psi[\mathbf{v}]$ on the right-hand side of (6.4) can be computed either via the static approach (4.5), or via the kinematic approach (4.12). $\beta_{\rm mix}[\mathbf{p},\mathbf{v}]$ turns out to be *more stringent* than the classical kinematic u.b., say $\beta_{\rm kin}$. In fact, denoting by $\dot{\mathbf{e}}^p(\mathbf{v})$ an arbitrary plastic strain rate cycle complying with $\mathbf{v} \in M$, one can write:

(6.6)
$$\beta_{\text{kin}}[\mathbf{p}, \dot{\mathbf{e}}^p(\mathbf{v})] = \int_0^T \int_V D\left(\dot{\mathbf{e}}^p(\mathbf{v})\right)\, dV\, dt - (\mathbf{p}, \mathbf{v})$$

and thus, remembering (4.12) and (6.4),

(6.7)
$$\beta_{\text{mix}}[\mathbf{p}, \mathbf{v}] = \min_{\{\dot{\mathbf{e}}^p \in \Lambda\,(\Delta\mathbf{e}^p(\mathbf{v}))\}} \beta_{\text{kin}}\left[\mathbf{p}, \dot{\mathbf{e}}^p\right].$$

Another formulation of the shakedown safety factor, alternative to (6.2) and (6.3) can be derived as follows. Let the minimization operation of (6.3) be performed by operating first on $\dot{\mathbf{e}}^p$ while taking $\mathbf{v}$ fixed, then by operating on $\mathbf{v}$. In this way, remembering (6.6), problem (6.3) takes the form:

(6.8)
$$\beta_{\text{sh}}[\mathbf{p}] = \min_{\mathbf{v} \in M} \min_{\{\dot{\mathbf{e}}^p \in \Lambda\,(\Delta\mathbf{e}^p(\mathbf{v}))\}} \beta_{\text{kin}}\left[\mathbf{p}, \dot{\mathbf{e}}^p\right],$$

which constitutes a *two-stage kinematic* formulation of the shakedown safety factor. By (6.7), one also can write

(6.9)
$$\beta_{\text{sh}}[\mathbf{p}] = \min_{\mathbf{v} \in M} \beta_{\text{mix}}\left[\mathbf{p}, \mathbf{v}\right],$$

which means that structure's shakedown safety factor is the smallest mixed upper bound in the set of all collapse mechanisms, $\mathbf{v} \in M$. The geometric interpretation of (6.9) is quite clear after Fig. 2; namely, changing $\mathbf{v}$ in all possible ways causes τ to change correspondingly until point $R \in \tau$, with fixed abscissa $\mathbf{p}$, coincides with Q on the surface $F = 0$. (Note that, due to the convexity of $F = 0$, point R is external to, or lies on, the surface $F = 0$.)

Another alternative formulation to problems (6.2) and (6.3) is obtained by making use of problem (4.5) to express $\beta_{\text{mix}}[\mathbf{p}, \mathbf{v}]$. According to (4.15) and (6.4), one can write

(6.10)
$$\beta_{\text{sh}}[\mathbf{p}] = \min_{\mathbf{v} \in M} \left\{ \left[\max_{\{\mathbf{s}, \beta\}} \left(\int_V \mathbf{s} : \Delta\mathbf{e}^p(\mathbf{v}) dV + \beta \right) \right.\right.$$
$$\left.\left. \text{s.t. } f(\mathbf{s} + \beta\bar{\boldsymbol{\sigma}}^c) \le 0 \ \text{ in } \ V \times (0, T) \right] - (\mathbf{p}, \mathbf{v}) \right\}$$

what represents a *mixed static-kinematic* formulation of the shakedown safety factor. Such a mixed formulation is rather novel within the shakedown theory. To authors' knowledge, formulations like (6.10) have only recently been proposed in [1, 17, 18].

Note that, if in (6.10) the stress **s** is assumed to be in equilibrium with **p**, problem (6.10) transforms into (6.2).

7. Application

In this section problem (4.5) is solved for a two-bar structure under thermal cyclic load. The aim is to illustrate the way the related shakedown load boundary can be derived starting from collapse modes instead of loading modes.

Two (parallel) bars of equal length L and cross-sections $\Omega_1 = \Omega$, $\Omega_2 = \gamma\Omega$ ($\gamma > 1$) are connected with a rigid block, Fig. 3(a). Bar 2 is maintained at a constant temperature, $T_2 = T_0$, and bar 1 undergoes cyclic temperature variations, i.e. $T_1 = T_0 - \theta\, z(t)$, $0 \le z(t) \le 1$, $\theta > 0$, Fig. 3(b).

An external permanent load $p = \alpha\bar{p}$ is applied upon the rigid block, with $\alpha \ge 0$ and $\bar{p} = (1+\gamma)\,\Omega\,\sigma_y$ = plastic limit load, σ_y being the material yield stress. Denoting by s_1 and s_2 the bars' stresses equilibrating p, it is $(s_1 + \gamma s_2)\,\Omega = \alpha\bar{p}$, hence

$$\alpha = (s_1 + \gamma s_2)/(1+\gamma)\sigma_y. \tag{7.1}$$

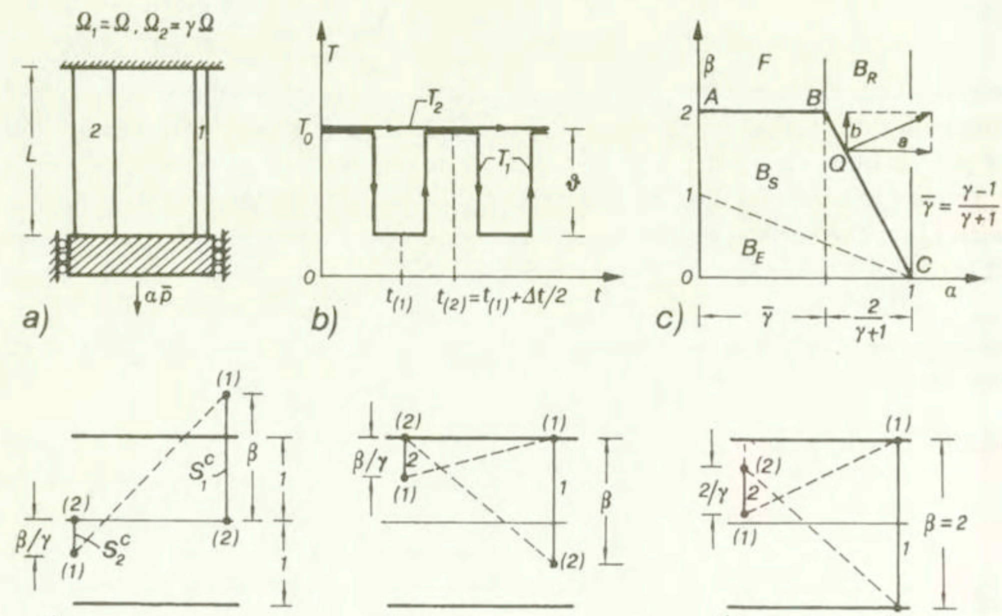

FIG. 3. Two-bar structure subjected to cyclic temperature changes and permament mechanical load: a) Geometrical configuration; b) Bar temperature histories; c) Interaction diagram; d) Elastic stress paths; e) and f) Typical locations of stress paths at the shakedown limit.

The thermo-elastic stresses are: $\sigma_1^c = \sigma_T\, z(t)$, $\sigma_2^c = -\sigma_1^c/\gamma$, where $\sigma_T := \gamma\, \alpha_T\, E\, \theta(1+\gamma)$ = max thermo-elastic stress, α_T = thermal expansion coefficient and E = Young's modulus. On setting $\beta := \sigma_T/\sigma_y$, the stress paths S_1^c and S_2^c of bars 1 and 2 are located as shown in Fig. 3(d), and the elastic stresses at times $t_{(1)}$ (at which maximum temperature reduction in bar 1 occurs, hence $z(t_{(1)}) = 1$) and $t_{(2)}$ (at which temperature T_0 occurs in bar 1, hence $z(t_{(2)}) = 0$) are: $\sigma_{1(1)}^c = \beta\sigma_y$, $\sigma_{2(1)}^c = -\beta\sigma_y/\gamma$, $\sigma_{1(2)}^c = \sigma_{2(2)}^c = 0$.

The interaction diagram of the system under study is the diagram of the (α, β) plane shown in Fig. 3(c), with the bilateral line ABC being the shakedown load boundary with equations as: $\beta = 2$ for $0 \le \alpha \le \bar{\gamma}$, where $\bar{\gamma} := (\gamma-1)/(\gamma+1)$, and $\beta = (1+\gamma)(1-\alpha)$ for $\bar{\gamma} \le \alpha \le 1$, (see [3] for more details).

Let v denote the ratchet displacement of the block, and Δe_1^p, Δe_2^p the bars' ratchet strains; obviously, $\Delta e_1^p = \Delta e_2^p = v/L$. Problem (4.5) takes the form $(b = 1)$:

$$\max_{\{s_1, s_2, \beta\}} \Phi := (s_1 + \gamma s_2)\, \Omega v + \beta, \tag{7.2}$$

subject to $\beta \ge 0$, as well as to:

$$\begin{aligned} f_1^+ &= s_1 + \beta\sigma_y - \sigma_y \le 0, \\ f_1^- &= -s_1 - \sigma_y \le 0, \\ f_2^+ &= s_2 - \sigma_y \le 0, \\ f_2^- &= -s_2 + \beta\gamma^{-1}\sigma_y - \sigma_y \le 0. \end{aligned} \tag{7.2'}$$

Here, the sign constraint $\beta \ge 0$ has been introduced in order for the analysis to include also the instantaneous plastic collapse mode.

For a fixed $v > 0$ (incremental collapse mode), the maximum is reached when s_1, s_2 and β are such that the elastic stress paths in the bars assume the positions shown in Fig. 3(e), with $f_1^+ = f_2^- = 0$. Thus, by $(7.2)'_{1-4}$ it follows:

$$s_1 = \sigma_y(1-\beta), \quad s_2 = \sigma_y, \tag{7.3}$$

$$\beta \le 2, \quad \beta \le 2\gamma. \tag{7.4}$$

Since $\beta \ge 0$ and $\gamma > 1$, the second inequality of (7.4) is certainly satisfied if $\beta \le 2$, and can thus be disregarded. Using (7.3), problem (7.2) transforms to

$$\psi(a) = \max_{\beta \ge 0} \Phi_1(\beta) = a + \left(1 - \frac{a}{1+\gamma}\right)\beta \quad \text{s.t. } \beta \le 2, \tag{7.5}$$

where $a := \bar{p}v$. Transforming (7.5) into an unconstrained problem, one can write

$$\min_{\beta \ge 0} \max_{\zeta \ge 0} = \Phi_2(\beta, \zeta) := -a + \left(\frac{a}{1+\gamma} - 1\right)\beta + \zeta(\beta - 2) \tag{7.6}$$

under only sign constraints. The Kuhn-Tucker conditions of (7.6) read

$$\frac{a}{1+\gamma} - 1 + \zeta \geq 0, \quad \beta \geq 0, \quad \beta\left(\frac{a}{1+\gamma} - 1 + \zeta\right) = 0, \tag{7.7}$$

$$\beta - 2 \leq 0, \quad \zeta \geq 0, \quad \zeta(\beta - 2) = 0. \tag{7.8}$$

The following three typical situations can be envisaged:

$$\begin{aligned} &\text{i)} && a > \gamma + 1: \quad \zeta = 0, \quad \beta = 0, \quad \alpha = 1, \quad \psi(a) = a,\\ &\text{ii)} && a = \gamma + 1: \quad \zeta = 0, \quad 0 < \beta < 2, \quad \alpha = 1 - \tfrac{\beta}{\gamma+1}, \quad \psi(a) = a,\\ &\text{iii)} && a < \gamma + 1: \quad \zeta > 0, \quad \beta = 2, \quad \alpha = \bar{\gamma}, \quad \psi(a) = 2 + \bar{\gamma}a. \end{aligned} \tag{7.9}$$

A fourth typical situation is generated for $v = 0$ (alternating plasticity collapse mode), hence $a = 0$ and $\Delta\varepsilon_1^p = \Delta\varepsilon_2^p = 0$. The maximum in (7.2) is then realized when the stress state is as that described in Fig. 3(f) with $f_1^+ = f_1^- = 0$. Thus, by $(7.2)'_{1-4}$, one has $\beta = 2$, hence $\psi(0) = 2$, and

$$s_1 = -\sigma_y, \quad \left(2\gamma^{-1} - 1\right)\sigma_y \leq s_2 \leq \sigma_y. \tag{7.10}$$

As $\alpha \geq 0$, by (7.1) and the first equation of (7.10), one has $s_2 \geq \gamma^{-1}\sigma_y$. Thus, as $2\gamma^{-1} - 1 < \gamma^{-1}$, the continued inequality in (7.10) can be rewritten as

$$\gamma^{-1}\sigma_y \leq s_2 \leq \sigma_y, \tag{7.11}$$

having in this way enforced its lower bound. Then, solving (7.1) for α and substituting in (7.11) gives

$$0 \leq \alpha \leq \bar{\gamma}. \tag{7.12}$$

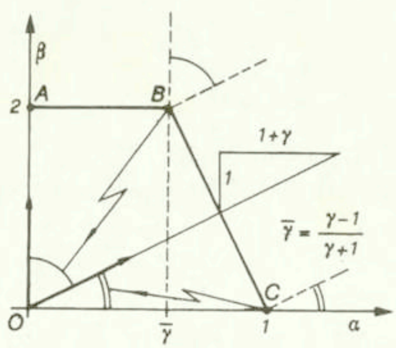

FIG. 4. Geometrical sketch representing the correspondence between collapse mechanism (v, b) and points on the shakedown load boundary curve ABC.

In the preceding considerations, the parameter $a = \bar{p}v$ has been used to generate the collapse mechanisms $v \geq 0$. The correspondence between the a values and the related limit loads is depicted in Fig. 4, where the limit load locus, ABC, is recognized to coincide with the shakedown boundary of Fig. 3(c), as expected.

8. Resumé comments and conclusions

In this paper, the maximum plastic dissipation theorem, suitably written in a time-space integral form, has been used as analytical tool to approach the shakedown theory. This enabled us not only to find the classical results of the shakedown theory, but also to derive some apparently novel concepts within this theory, such as the shakedown limit load associated with an assigned (noninstantaneous) collapse mode, the mixed upper bound to the shakedown safety factor, and the mixed static-kinematic formulation of the shakedown safety factor problem.

The shakedown limit load for the assigned collapse mode is intended as the particular combination of cyclic/permanent load under which the given structure can reach the shakedown limit state characterized by an inadaptation collapse mechanism complying with the assigned collapse mode. Dual static and kinematic problem formulations have been given to evaluate such a load. On changing the given collapse mode in all possible ways, these problem formulations can be used to determine the shakedown load boundary surface in the load space.

As already pointed out in [3], where n-dimensional permanent loads were considered, the shakedown load boundary plays the role of a yield function for the structure, in the sense that the (noninstantaneous) collapse mechanism characterizing the shakedown limit state to which the structure may report itself, obeys a plasticity flow law similar to that obeyed by the plastic strain rate tensor for the material. That is, the shakedown limit load associated with a given (noninstantaneous) collapse mechanism is that load on the shakedown load boundary surface, from where this mechanism departs along the external normal to that surface. The latter geometrical property is a generalization to the shakedown of an analogous property of plastic limit analysis, in which the (instantaneous) collapse mechanism lies on the external normal to the load resistance surface; indeed, in the absence of cyclic load, shakedown degenerates into limit plasticity and the noninstantaneous collapse modes into instantaneous collapse ones.

Another interesting property of the shakedown load boundary surface is that the tangent plane, orthogonal to the assigned collapse mode, intersects the axis drawn from a fixed point on the permanent load hyperplane, parallel to the cyclic load multiplier axis, at a point whose distance from the permanent load hyperplane constitutes a mixed upper bound to the shakedown safety factor for the structure under combined loads, i.e. superposition of cyclic loads with that fixed permanent load. This mixed upper bound, which can be computed by static, or kinematic, procedures (and for this reason it is called 'mixed'), is more stringent than the classical kinematic upper bound, since in fact it can be obtained by minimizing the latter kinematic upper bound with respect to the set of plastic strain paths complying with the given collapse mode. The above tangent plane intersects the cyclic load multiplier axis at a point whose distance from the origin equals the common optimal objective value of the two problems, afore-mentioned, dual to each other and related to the shakedown limit load for assigned collpase mode.

The shakedown safety factor can be obtained by minimizing the mixed upper bound with respect to the set of collapse modes. This gives two possible computational procedures, both being alternative to the classical static or kinematic procedures, according to whether the mixed upper bound is obtained via static or kinematic procedure. In the latter case, a two-stage kinematic formulation is obtained for the shakedown safety factor, whereas in the former case a mixed static-kinematic formulation is obtained. A notable feature of the latter formulation is that, contrary to the classical static formulation, it makes use of free stress variables and compatibility equations, and thus the equilibrium equations – which in general cause some computational problems in the shakedown analysis – are not needed. For this reason, a mixed formulation like the one given above, is expected to be suitable for numerical shakedown analysis by the finite element method. Mixed formulations for the shakedown safety factor have started to appear in the literature [1, 17, 18] only recently; they have not been fully exploited yet and deserve further investigation. This is being done in the present research work devoted to a better understanding of the matter, as well as to the application methods with their computer implementation. Methods suggested by MRÓZ and collaborators [16] may be helpful to this purpose.

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Received February 29, 2000.