

## Brake squeal: a problem of flutter instability of the steady sliding solution?

*Dedicated to Professor Zenon Mróz  
on the occasion of his 70<sup>th</sup> birthday*

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**BRAKE SQUEAL RESULTS FROM friction-induced vibrations.** This phenomenon is considered here and interpreted as a flutter instability of the steady sliding solution of an elastic solid in unilateral contact with friction with a moving obstacle. A mechanical analysis of the governing equations is given, in particular to obtain the steady sliding solution. The stability analysis of this solution is discussed. A numerical analysis by the finite element method is performed in order to compute the steady sliding solution and to discuss its stability for an automotive disk brake. The validation of the numerical procedure is examined in relation with some analytical results of the literature.

### 1. Introduction

THE PROBLEM OF BRAKE NOISES has been intensively discussed in various experimental or theoretical investigations up to the present time. Different kinds of noises and vibrations can be identified in common drum or disk brakes following their frequencies (e.g. [2, 12]). Brake squeals result from high frequency vibrations (greater than 5000 Hz) and have a relatively pure spectrum composed of a few main frequencies accompanied by some harmonics. Although a brake squeal does not affect the mechanical behaviour of the brake, it is less and less accepted by the passengers. The conditions under which a squeal occurs are relatively well known. Most experiments showed that the brake squeal is more excited at low than at high speeds (less than 30 km/h). Squeal occurs only over limited ranges of brake pressure and is most prevalent at low temperature (less than 150° C). The source of noise is attributed to the vibrations of brake components such as drum or pad and disk, and brake noises are generated mainly by the sliding phenomena.

Our objective is to present in this paper a mechanical analysis of brake squeal. The principal interpretation is the fact that brake squeal is a consequence of the flutter instability of the steady sliding solution of the pad on the disk for a disk brake. This analysis leads to the system of governing equations to be considered and enables us to determine the steady sliding solution. The fact that the disk is in rotation can be easily taken into account since the system remains in small strains. The determination of the steady sliding solution is discussed as a function of the coefficient of friction. In particular, the conditions ensuring the existence or the uniqueness of the steady sliding solution are discussed since for high friction, the steady sliding solution may be not unique or does not even exist.

The stability of a steady sliding solution is considered in the second part. Perturbed motions of the mechanical system of pad and disk near the steady sliding solution are introduced. Under the assumption of a slip regime, the dynamic solution can be simplified and the perturbed slip motions can be considered. Stability analysis with respect to these motions can be discussed and leads to the solution of a generalized eigenvalue problem with unsymmetric mass and rigidity matrices. This non-symmetry is due to the presence of friction as well as of the disk rotation. The existence of an eigenvalue with a positive real part and a non-zero imaginary part implies necessarily an instability by flutter of the steady sliding solution.

The numerical aspect of the problem is considered in the third part. The steady sliding solution can be determined by solution of the nonlinear problem in the vehicle reference by iterations. Once this solution is obtained, its stability analysis can be performed. The adopted procedure is based upon a modal basis of free vibrations. The generalized eigenvalue problem is solved in this basis of functions and the eigenvectors are generated by a finite number of the first vibration modes. The numerical procedure is first validated on the problem of contact with friction of infinite elastic layers which has been discussed analytically by ADAMS [1] and by MARTINS *et al.* [10]. Then an example of disk brake is considered.

## 2. Governing equations

The system of disk and pad is considered in the vehicle axes ( $Ox$ ), which is a Galilean reference set, since the vehicle motion is assumed to proceed at constant velocity. In this reference, the pad is an elastic solid  $P$  in small deformation. For simplicity, it is assumed that this solid is under the action of implied displacements (by the brake action) and contact forces with the disk on a potential contact surface  $S$ . If  $u^P(x, t)$  and  $\sigma^P(x, t)$  denote respectively the displacement and stress at the point  $x$  and time  $t$ , the dynamic equations for the pad  $P$  are

$$(2.1) \quad \sigma^P = L : \nabla u^P, \quad \text{Div } \sigma^P - \rho u^P_{,tt} = 0, \quad \forall x \in P, \quad \sigma^P \cdot n^P = R, \\ \forall x \in S.$$

In this expression,  $R$  is the reaction of the disk  $D$ , which is related to the relative displacement and velocity of the two solids following the unilateral contact and Coulomb's law of dry friction with a constant coefficient of friction, although more elaborated models can be proposed, e.g. [2, 13].

Let  $\omega$  be the rotation velocity of the disk around the axis  $Ox_3$ . It is convenient to consider the rotating axes  $(OX)$ , where  $OX_3$  coincides with  $Ox_3$ . The equations of motion of the solid  $D$  in axes  $(OX)$  are simply the dynamic equations of an elastic solid at small deformation in a relative reference

$$(2.2) \quad \Sigma = L : \nabla U, \quad \text{Div } \Sigma - \rho \gamma = 0, \quad \forall X \in D_o$$

with

$$(2.3) \quad \gamma = \gamma_e + \gamma_r + 2\omega k \wedge V_r = -\omega^2 r + U_{,tt} + 2\omega k \wedge U_{,t},$$

where  $D_o$  denotes the undeformed volume of the disk in these axes, and  $\Sigma = \Sigma(X, t)$ ,  $U = U(X, t)$  are respectively the stress tensor and displacement vector. It is necessary to adopt the vehicle axes  $(Ox)$  and a change of variables must be introduced. This change of variables can be expressed in terms of the cylindrical coordinates as shown in Fig. 1:

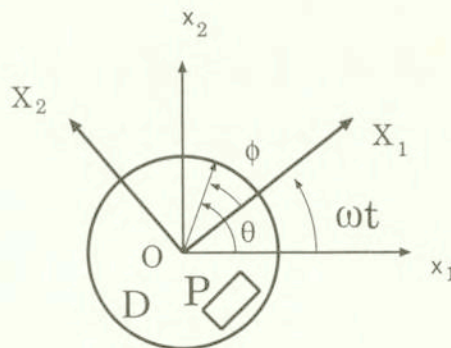


FIG. 1. A model of disk brake.

$$(2.4) \quad X = (r, \phi, z), \quad x = (r, \theta, z), \quad \theta = \phi + \omega t.$$

The change of variables

$$(2.5) \quad u(r, \theta, z, t) = U(r, \theta - \omega t, z, t), \quad U(r, \phi, z, t) = u(r, \phi + \omega t, z, t)$$

(2.6)  $\sigma(r, \theta, z, t) = \Sigma(r, \theta - \omega t, z, t), \quad \Sigma(r, \phi, z, t) = \sigma(r, \phi + \omega t, z, t)$   
gives in the Galilean axes ( $Ox$ )

$$(2.7) \quad \sigma = L : \nabla u, \quad \text{Div } \sigma - \rho \gamma = 0, \quad \forall x \in D,$$

$$(2.8) \quad \gamma = \omega^2(-r + u_{,\theta\theta} + 2k \wedge u_{,\theta}) + 2\omega(u_{,\theta t} + k \wedge u_{,t}) + u_{,tt}$$

with the boundary condition

$$(2.9) \quad \sigma \cdot n = -R, \quad \forall x \in S,$$

and implied displacements along the axis of rotation  $Ox_3$ . It is also important to give the expression of the material velocity  $v(x, t)$  in this reference system. From the expression of the velocity of a material point in the rotating axes

$$v = V_e + V_r = \omega k \wedge r + U_{,t},$$

it follows that

$$(2.10) \quad v(x, t) = \omega(k \wedge r + u_{,\theta}) + u_{,t}.$$

The conditions of unilateral contact and dry friction of the solids  $P$  and  $D$  can be written now. On the surface  $S$  of contact, which is assumed initially without the normal gap, the condition of non-penetration is

$$(2.11) \quad [u]_N = (u^P - u) \cdot n \geq 0, \quad \forall x \in S.$$

The unilateral contact condition gives

$$(2.12) \quad [u]_N \geq 0, \quad R_N \geq 0, \quad [u]_N R_N = 0, \quad \forall x \in S.$$

The relative velocity of material points in contact is

$$(2.13) \quad w(x, t) = u_{,t}^P(x, t) - v(x, t) = [u]_{,t} - \omega k \wedge r - \omega u_{,\theta}.$$

Coulomb's friction law can be written as

$$(2.14) \quad w = \nu R_T, \quad \varphi = \|R_T\| - f R_N \leq 0, \quad \nu \leq 0, \quad \nu \varphi = 0.$$

The previous expressions for  $\gamma$  and  $w$  show that a possible approximation of the solution is obtained by assuming that the relative velocity is due simply to the rotation of the disk, and one can retain in the dynamic equation of the disk only the centrifugal forces and neglect all other terms of rotation. This approximation  $\mathcal{A}$  consists in writing

$$(2.15) \quad \gamma = -\omega^2 r + u_{,tt}, \quad w(x, t) = [u]_{,t} - \omega k \wedge r.$$

### 3. Steady sliding solution

The steady state solution is obtained when  $u(x, t)$  does not depend on  $t$ . Thus, the steady state solution is governed by the following equations:

$$(3.1) \quad \sigma^P = L : \nabla u^P, \quad \text{Div } \sigma^P = 0, \quad \forall x \in P, \quad \sigma^P \cdot n = -R, \quad \forall x \in S,$$

$$(3.2) \quad \sigma = L : \nabla u, \quad \text{Div } \sigma - \rho\gamma = 0, \quad \forall x \in D, \quad \sigma \cdot n = -R, \quad \forall x \in S,$$

$$(3.3) \quad \gamma = \omega^2(-r + u_{,\theta\theta} + 2k \wedge u_{,\theta})$$

together with Eqs. (2.11), (2.12), (2.14) and the following expression of the relative velocity

$$(3.4) \quad w(x, t) = -\omega(k \wedge r + u_{,\theta}).$$

The approximation  $\mathcal{A}$  consists in solving the following equations:

$$(3.5) \quad \sigma^P = L : \nabla u^P, \quad \text{Div } \sigma^P = 0, \quad \forall x \in P, \quad \sigma^P \cdot n = -R, \quad \forall x \in S,$$

$$(3.6) \quad \sigma = L : \nabla u, \quad \text{Div } \sigma + \rho\omega^2 r = 0, \quad \forall x \in D, \quad \sigma \cdot n = -R, \quad \forall x \in S,$$

together with Eqs. (2.11), (2.12), (2.14) with  $w(x, t) = -\omega k \wedge r$ . Within this approximation, since the relative velocity is not zero, the transverse reaction  $R_T$  has the direction of  $-\tau$  with  $\tau = \frac{w}{\|w\|}$ . Finally, the steady sliding solution is given by Eqs. (2.11), (2.12) and

$$(3.7) \quad \sigma^P = L : \nabla u^P, \quad \text{Div } \sigma^P = 0, \quad \forall x \in P, \quad \sigma^P \cdot n = -R_N n + f R_N \tau, \\ \forall x \in S.$$

$$(3.8) \quad \sigma = L : \nabla u, \quad \text{Div } \sigma + \rho\omega^2 r = 0, \quad \forall x \in D, \quad \sigma \cdot n = -R_N n + f R_N \tau, \\ \forall x \in S.$$

These equations can also be written in the form

$$(3.9) \quad \int_P \nabla u^P : L : \nabla \delta u \, dV + \int_D \nabla u : L : \nabla \delta u \, dV - \int_D \omega^2 r \cdot \delta u \, dV \\ + \int_S (R_N [\delta u]_N + f R_N \tau \cdot [\delta u]_T) \, dS = 0,$$

together with (2.11) and (2.12). To understand the mathematical nature of the problem of steady sliding in the approximation  $\mathcal{A}$ , let us consider the associated

discrete problem obtained from these equations after discretization by the finite element method, for example. If  $U = (U_N, U_T, U_Z)$  denotes for each solid the degrees of freedom representing respectively the normal, tangential displacements of the contact surface and other complementary displacements in the solid, the governing equations are, for each solid,

$$\begin{bmatrix} K_{NN} & K_{NT} & K_{NZ} \\ K_{TN} & K_{TT} & K_{TZ} \\ K_{ZN} & K_{ZT} & K_{ZZ} \end{bmatrix} \begin{bmatrix} U_N \\ U_T \\ U_Z \end{bmatrix} = \begin{bmatrix} R_N \\ f[\Phi]R_N \\ F_Z \end{bmatrix},$$

where  $\Phi$  denotes the appropriate matrix. The elimination of  $U_Z$  leads to a matrix equation in terms of  $\Delta = U^P - U$

$$\begin{bmatrix} k_{NN} & k_{NT} \\ k_{TN} & k_{TT} \end{bmatrix} \begin{bmatrix} \Delta_N \\ \Delta_T \end{bmatrix} = \begin{bmatrix} R_N + F_N \\ f[\Phi]R_N + F_T \end{bmatrix}.$$

Finally, the normal displacement  $\Delta_N$  is related to the normal reaction  $R_N$  by

$$(3.10) \quad [k_{NN} - k_{NT}k_{TT}^{-1}k_{TN}][\Delta_N] = [I - f k_{NT}k_{TT}^{-1}\Phi][R_N] + [g],$$

$$(3.11) \quad \Delta_N^k \geq 0, \quad R_N^k \geq 0, \quad \Delta_N^k R_N^k = 0, \quad \forall k = 1, m.$$

This is a linear complementary problem, (cf. COTTLE *et al.* [5], ISAC [7] or KLARBRING [9]):

$$(3.12) \quad [\Delta_N] = [A][R_N] + [F], \quad \Delta_N^k \geq 0, \quad R_N^k \geq 0, \quad \Delta_N^k R_N^k = 0,$$

with

$$(3.13) \quad [A] = [k_{NN} - k_{NT}k_{TT}^{-1}k_{TN}]^{-1} - f [k_{NN} - k_{NT}k_{TT}^{-1}k_{TN}]^{-1}[k_{NT}k_{TT}^{-1}\Phi].$$

It should be recalled that for a given  $[F]$ , this problem has one and only one solution if the matrix  $[A]$  is a P-matrix. This property means that  $[A]$  satisfies the condition of P-positivity

$$(3.14) \quad [A] \text{ is a P-matrix} \Leftrightarrow \exists i \text{ such that } \sum_j X_i A_{ij} X_j > 0$$

$$\forall X \neq 0.$$

In fact, the existence of a solution is still ensured if  $[A]$  satisfies only a co-P-positivity condition:

$$(3.15) \quad [A] \text{ is a co-P-matrix} \Leftrightarrow \exists i \text{ such that } \sum_j X_i A_{ij} X_j > 0$$

$$\forall X \neq 0, \quad X \geq 0.$$

As usual, for a vector, the compact notation  $X \geq 0$  means the component-wise condition  $X_i \geq 0$  for all  $i$ . This condition is similar to the P-positivity condition but is concerned only with vectors of nonnegative components. It has been established for a matrix not necessarily symmetric that

$$(3.16) \quad \text{Positive-definiteness} \Rightarrow \text{P-positivity} \Rightarrow \text{Co-P-positivity},$$

$$(3.17) \quad \text{Positive-definiteness} \Rightarrow \text{Co-positivity} \Rightarrow \text{Co-P-positivity}.$$

For symmetric matrices, it is also true that P-positivity and positive-definiteness are equivalent,

$$(3.18) \quad \text{Positive-definiteness} \Leftrightarrow \text{P-positivity}.$$

When the full expressions (3.3), (3.4) of  $\gamma$  and  $w$  are taken into account, the discussion can be done in the same spirit and leads again to a complementary problem.

The significance of the quadratic form  $[R_N]^T[A][R_N]$  is very simple. It represents the work done by the normal reactions  $R_N \Delta_N$  of the system when subjected to the contact reaction  $R = R_N n + f R_N \tau$  and to homogeneously implied displacement. The condition of positivity states that for any non-zero distribution of normal reactions, the work done by this distribution is positive. The condition of co-P-positivity is much weaker, it states that for any non-zero and positive distribution of the normal reactions, there exists at least one strictly positive normal displacement associated with a strictly positive normal reaction.

The contribution of the rotation terms to the rigidity matrix of the solid  $D$  corresponds to some additional symmetric matrices, since the following expressions hold:

$$(3.19) \quad \int_D u_{,\theta\theta} \cdot \delta u \, dV = - \int_D u_{,\theta} \delta u_{,\theta} \, dV$$

$$(3.20) \quad \int_D (k \wedge u_{,\theta}) \cdot \delta u \, dV = \int_D \det[k, u_{,\theta}, \delta u] \, dV = \int_D - \det[k, u, \delta u_{,\theta}] \, dV \\ = \int_D \det[k, \delta u_{,\theta}, u] \, dV.$$

After partial integration with respect to  $\theta$  in the interval  $[0, 2\pi]$ , these expressions are symmetric with respect to  $(u, \delta u)$ . Under the assumption of small rotations,  $\omega$  remains sufficiently small, the contribution of these additional terms to the rigidity of solid  $D$  does not change its positivity.

The fact that the co-P-positivity condition is satisfied depends on the considered problem. Even for elastic solids at small strains, it is not difficult to give

simple examples showing that the possibility of steady sliding does not exist for a sufficiently high friction coefficient.

#### 4. Stability analysis of the steady sliding solution

The stability of the steady sliding solution is obtained from the behaviour of the perturbed motions of the system near the considered steady solution. This discussion is a priori difficult since a small perturbed motion near the steady solution is not necessarily governed by linear equations. It is well known that the unilateral contact and Coulomb's laws are non-smooth and cannot be linearized at the steady state, e.g. [3, 8, 10 – 11, 14 – 20]. It is assumed first that the steady sliding solution satisfies on the contact surface  $S$  the condition of effective contact

$$(4.1) \quad R_N(x) > 0 \quad \forall x \in S.$$

It is then expected that a small perturbed motion cannot lead to a separation of the contact at any point of  $S$ , at least at the early time. This remark enables us to consider only in-contact motions of the solids on the contact surface  $S$ , to avoid the difficulties related to the unilateral aspect.

Even in these motions, it is necessary to separate the slip regime  $w \neq 0$  from the stick regime  $w = 0$ . The problem of stick-slip motions has been much discussed in the literature. In particular, for a simple oscillator, the modification of the initial frequency has been considered, (cf. for example POPP and STELTER [18]) for a velocity-dependent coefficient of friction. The stick-slip motion has been computed for a three-dimensional oscillator by CHO and BARBER [4]. The stick-slip motions play an important role in most contact problems (cf. ZHARII [20]), in particular in the study of noise emission. However, the presence of stick-slip motions is a source of difficulty and in this stability analysis, only the slip motions will be explored. Under this restriction, the equations of motion can be effectively linearized near the steady sliding solution. Let  $u^*$  and  $\sigma^*$  be the differences

$$(4.2) \quad u^* = u_p - u_e, \quad \sigma^* = \sigma_p - \sigma_e,$$

where  $u_e, \sigma_e$  refer to the steady sliding solution, and  $u_p, \sigma_p$  to the perturbed motion. For small perturbations,  $u^*$  and  $\sigma^*$  are governed by the linearized equations at the steady sliding state. Thus, the following equations hold for the slip motions in the vicinity of the steady sliding state:

$$(4.3) \quad \sigma^{*P} = L : \nabla u^{*P}, \quad \text{Div } \sigma^{*P} - \rho u^{*P}_{,tt} = 0 \quad \forall x \in P, \quad \sigma^{*P} \cdot n = -R^*, \\ \forall x \in S,$$



$$(4.4) \quad \sigma^* = L : \nabla u^*, \quad \text{Div } \sigma^* - \rho \gamma^* = 0, \quad \forall x \in D, \quad \sigma^* \cdot n = -R^*,$$

$\forall x \in S,$

$$(4.5) \quad \gamma^* = \omega^2(u^*,_{\theta\theta} + 2k \wedge u^*,_{\theta}) + 2\omega(u^*,_{\theta t} + k \wedge u^*,_t) + u^*,_{tt},$$

$$(4.6) \quad [u^*]_N = 0.$$

In these equations, the tangent reaction  $R_T^*$  is related to the normal reaction  $R_N^*$  by the linearized expression of the equation

$$R_T = -f R_N \frac{w}{\|w\|},$$

which gives

$$(4.7) \quad R_T^* = -f R_N^* \tau - f R_N^* \left( \frac{w^*}{\|w\|} - w \frac{w \cdot w^*}{\|w\|^3} \right) \quad \text{with}$$

$$(4.8) \quad w^* = [u^*,_t] - \omega u^*,_{\theta}.$$

These equations can also be conveniently written under the variational form of the virtual work equation. For the solid  $P$ , the classical equation

$$(4.9) \quad \int_P \rho u^{*P},_{tt} \cdot \delta u \, dV + \int_P \nabla u^{*P} : L : \nabla \delta u \, dV = \int_S R^* \cdot \delta u \, dS$$

is obtained. For the solid  $D$ , the following equation holds:

$$(4.10) \quad \int_D \rho u^{*,tt} \cdot \delta u \, dV + 2\omega \int_D \rho(u^*,_{\theta t} \cdot \delta u + \det[k, u^*,_t, \delta u]) \, dV$$

$$+ \int_D \nabla u^* : L : \nabla \delta u \, dV + \omega^2 \int_D (u^*,_{\theta\theta} \cdot \delta u + 2 \det[k, u^*,_{\theta}, \delta u]) \, dV$$

$$= - \int_S R^* \cdot \delta u \, dS.$$

In this equation, it should be noted that the terms involving  $u^*,_t$  are skew-symmetric while the terms involving  $u^*,_{tt}$  or  $u$  are symmetric. If the solution is searched for in the form  $(u^*, R^*) = (d(x), r(x)) \exp st$ , the eigenvalues  $s$  and eigenvectors  $d$  must satisfy the equation:

$$(4.11) \quad s^2 \int_P \rho d^P \cdot \delta u \, dV + \int_P \nabla d^P : L : \nabla \delta u \, dV = \int_S r \cdot \delta u \, dS$$

for the solid  $P$ . For the solid  $D$ , the following equation holds:

$$(4.12) \quad s^2 \int_D \rho d^D \cdot \delta u \, dV + s \, 2\omega \int_D \rho (d^D,_{\theta} \cdot \delta u + \det[k, d^D, \delta u]) \, dV \\ + \int_D \nabla d^D : L : \nabla \delta u \, dV + \omega^2 \int_D (-d^D,_{\theta} \cdot \delta u,_{\theta} + 2 \det[k, d^D,_{\theta}, \delta u]) \, dV \\ = - \int_S r \cdot \delta u \, dS.$$

After discretization by the finite element method with the nodal shape functions  $\mathcal{N}_i(x)$

$$(4.13) \quad d(x) = \sum_{i=1}^n d_i \mathcal{N}_i(x),$$

the following matrix equations hold:

$$(4.14) \quad (s^2[M^P] + [K^P])[d^P] = [r^P], \quad (s^2[M^D] + s[G^R] + [K^D] \\ + [K^{DR}])[d^D] = -[r^D]$$

with  $[d^P] = [d_P, d_{PT}, d_N]^T$ ,  $[d^D] = [d_N, d_{DT}, d_D]^T$ , where the notation  $d_N, d_{DT}, d_D$  refers to different nodal values of solid  $D$ , respectively to the normal and tangent nodal displacements on the contact surface, and to other nodal displacements elsewhere. Note that  $[G^R]$  is skew-symmetric and  $[K^R]$  is a symmetric matrix. Thus, the effect of rotation of  $D$  is finally expressed by a gyroscopic term and by an additional symmetric rigidity. The force matrices  $[r^P]$  and  $[r^D]$  are related by

$$[r^P] = [0, r_T, r_N]^T, \quad [r^D] = [r_N, r_T, 0]^T.$$

From the expression for  $R_T^*$ , the following equation holds:

$$[r_T] = f[\Phi][r_N] + fs[A][d_{PT} - d_{DT}] + f[B_T][d_{DT}] + f[B_N][d_N],$$

where  $[A], [B_T], [B_N]$  are some appropriate square matrices. From the expression of  $[r_N]$

$$[r_N] = (s^2[M_N^P] + [K_N^P])[d^P] = -(s^2[M_N^D] + s[G_N^R] + [K_N^D] + [K_N^R])[d^D],$$

$r_T$  and  $r_N$  can be written in terms of the displacement  $[d] = [d_P, d_{PT}, d_N, d_{DT}, d_D]^T$ . Finally, the eigenvalues  $s$  and eigenvectors  $[d]$  must satisfy the condition

$$(4.15) \quad \{s^2([M] - f[m]) + s([G] - f[\ell]) + [K] + [K^R] - f[k]\} [d] = [0],$$

with

$$\begin{aligned}
 [M] &= \begin{bmatrix} M_{PP}^P & M_{PN}^P & M_{PT}^P & 0 & 0 \\ M_{TP}^P & M_{TN}^P & M_{TT}^P & 0 & 0 \\ M_{NP}^P & M_{NT}^P & M_{NN}^P + M_{NN}^Q & M_{NT}^Q & M_{ND}^Q \\ 0 & 0 & M_{TN}^Q & M_{TT}^Q & M_{TD}^Q \\ 0 & 0 & M_{DN}^Q & M_{DT}^Q & M_{DD}^Q \end{bmatrix}, \\
 [K] &= \begin{bmatrix} K_{PP}^P & K_{PN}^P & K_{PT}^P & 0 & 0 \\ K_{TP}^P & K_{TN}^P & K_{TT}^P & 0 & 0 \\ K_{NP}^P & K_{NT}^P & K_{NN}^Q + K_{NN}^P & K_{NT}^Q & K_{ND}^Q \\ 0 & 0 & K_{TN}^Q & K_{TT}^Q & K_{TD}^Q \\ 0 & 0 & K_{DN}^Q & K_{DT}^Q & K_{DD}^Q \end{bmatrix}, \\
 [m] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \Phi M_{NP}^P & \Phi M_{NT}^P & \Phi M_{NN}^P & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Phi M_{NN}^D & -\Phi M_{NT}^D & -\Phi M_{ND}^D \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 [\ell] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & -A & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -A & -\Phi G_{NN}^{DR} & A - \Phi G_{NT}^{DR} & -\Phi G_{ND}^{DR} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 [k] &= \begin{bmatrix} 0 & 0 & 0 \\ \Phi K_{NP}^P & \Phi K_{NT}^P & \Phi K_{NN}^P + B_N \\ 0 & 0 & -\Phi(K^D + K^{DR})_{NN} - B_N \\ 0 & 0 & 0 \\ & 0 & 0 \\ & B_T & 0 \\ -\Phi(K^D + K^{DR})_{NT} - B_T & -\Phi(K^D + K^{DR})_{ND} \\ & 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$[G] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{NN}^R & G_{NT}^R & G_{ND}^R \\ 0 & 0 & G_{TN}^R & G_{TT}^R & G_{TD}^R \\ 0 & 0 & G_{DN}^R & G_{DT}^R & G_{DD}^R \end{bmatrix},$$

$$[K^R] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{NN}^{DR} & K_{NT}^{DR} & K_{ND}^{DR} \\ 0 & 0 & K_{TN}^{DR} & K_{TT}^{DR} & K_{TD}^{DR} \\ 0 & 0 & K_{DN}^{DR} & K_{DT}^{DR} & K_{DD}^{DR} \end{bmatrix}.$$

This eigenvalue problem can be written as

$$(4.16) \quad (s^2 [\bar{M}] + s [\bar{Z}] + [\bar{K}])[d] = 0,$$

$$[\bar{M}] = [M] - f [m], \quad [\bar{Z}] = [G] - f [\ell], \quad [\bar{K}] = [K] + [K^R] - f [k].$$

Thus the presence of friction breaks the symmetry of the mass and rigidity matrices in the eigenvalue problem to be solved. Since the numerical solution of this non-symmetric problem of dimension  $n$  is time-consuming, a reduction of the dimension of the problem is necessary. Thus, it is interesting to consider the basis of vibration modes  $D_k$  of the ideal associated system defined by the equations (of dimension  $n$ )

$$(4.17) \quad (s_0^2 [M] + [K])[D] = [0].$$

Let  $D_k$ ,  $k = 1, m$  denote the first  $m$  vibration modes. The reduction of variable consists in searching for  $[d]$  in this basis:

$$(4.18) \quad [d] = [D_1 \ D_2 \ \dots \ D_m][q_1 \ \dots q_m]^T = [C][q].$$

Finally, the eigenvalue  $s$  and eigenvector  $[q]$  are defined by an eigenvalue problem of dimension  $m$

$$(4.19) \quad (s^2 [\mathcal{M}] + s [\mathcal{Z}] + [\mathcal{K}])[q] = [0],$$

$$[\mathcal{M}] = [C]^T [\bar{M}] [C], \quad [\mathcal{Z}] = [C]^T [\bar{Z}] [C], \quad [\mathcal{K}] = [C]^T [\bar{K}] [C].$$

The solution of (4.19) leads to the complex eigenvalues  $s_k$  and complex eigenvectors  $Q_k$ . Since  $u^*(x, t) = \mathcal{N}_i(x) \Re (\sum_k a_k C_{ij} Q_{kj} \exp s_k t)$ , it is concluded

that the steady sliding solution is unstable if there exists an eigenvalue with a positive real part. Although it is not necessary, the choice of this modal basis is interesting since the dynamic behaviour of the system is well generated by a small number of modes, in practice  $m \ll n$ .

## 5. Validation of the numerical procedure

The proposed numerical procedure is first validated on a sample problem which is the sliding of a rigid plate on an elastic infinite layer, (cf. Fig. 2). The problem was analytically discussed by MARTINS *et al.* [11] for the case of an elastic half-space, and by ADAMS [1], MOIROT [12] for the case of two elastic layers. The case of an infinite layer is interesting since a closed-form solution can be obtained for any coefficient of friction, in contrast with the case of the half-space.

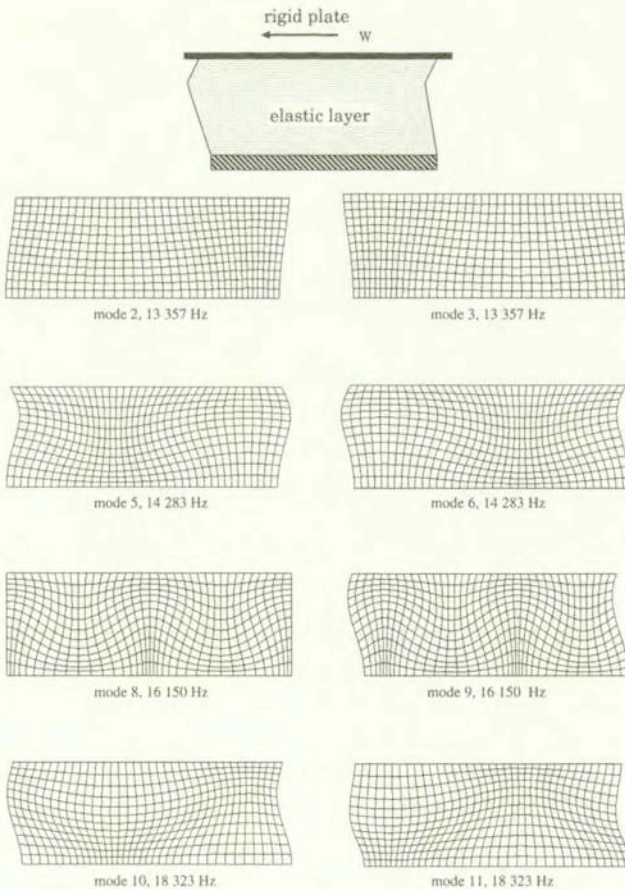


FIG. 2. The sliding problem of a rigid plate on an infinite elastic layer and the vibration basis.

### 5.1. Sliding of a rigid plate on an elastic layer

The governing equations are

$$(\lambda + 2\mu)u_{,xx} + \mu u_{,yy} + (\lambda + \mu)v_{,xy} = \rho u_{,tt},$$

$$(\lambda + 2\mu)v_{,yy} + \mu v_{,xx} + (\lambda + \mu)u_{,xy} = \rho v_{,tt},$$

with boundary conditions

$$u(x, -h, t) = v(x, -h, t) = 0,$$

and interface conditions on  $S$

$$\delta - v \geq 0, \quad \sigma_{yy} \leq 0, \quad (\delta - v)\sigma_{yy} = 0 \quad \forall x, y = 0,$$

$$\varphi = |\sigma_{xy}| - f\sigma_{yy} \leq 0, \quad W - u_{,t} = \nu\sigma_{xy}, \quad \nu \geq 0, \quad \nu\varphi = 0.$$

The steady sliding solution exists for all  $f$ ,

$$u = \delta f (y + h) \frac{\mu}{\lambda + 2\mu} \operatorname{sign}(W), \quad v = \delta(y + h).$$

The eigenvalue problem (4.16) has been discussed for a solution  $u^*$  of the form

$$u^* = \Re \exp(st) \exp(kx) U(y), \quad k \in C, \quad s \in C.$$

It has been shown that there exists  $s$  with  $\Re(s) > 0$  for any  $f > 0$ , thus the steady sliding solution with friction is always unstable. The mechanism of instability can be understood by the study of the trajectory of the eigenvector  $s(f)$  in the complex plane as a function of  $f$ . For  $f = 0$ , all modes are double. When  $f$  increases from 0, most eigenvalues remain purely imaginary while some of them are splitting into two complex, simple eigenvalues with a non-zero real part.

### 5.2. Numerical validation

The modal basis  $D_k$ ,  $k = 1, m$ , is first computed by the solution of the Eqs. (4.17). As usual, these frequencies of vibration can be obtained with great precision. The relative error of the computed frequencies as a function of the mesh size is less than 1% for the 30 first frequencies with the mesh  $12 \times 36$ . Figure 2 presents some modes and the associated frequencies of vibration.

The solution of the Eq. (4.19) for a chosen basis  $D_k$ ,  $k = 1, m$  can be done following the standard methods available for example in the Nastran code. Hessenberg's method seems to give good numerical results. It is recalled that Hessenberg's method is obtained in two steps, a reduction to a Hessenberg matrix

and an iteration by a QR algorithm. This method gives all eigenvalues and the associated eigenvectors are obtained by inverse iterations.

The variations of the frequencies as functions of the friction are computed with  $m = 200$  and with the  $12 \times 36$  mesh, and  $n = 829$ . These frequencies are obtained with error smaller than 3% compared to the exact values. It is found that good results can be obtained with a small number of modes, in practice  $m = 200$  is sufficient. These results confirmed numerically the observation that the steady sliding solution is unstable by flutter when friction is introduced.

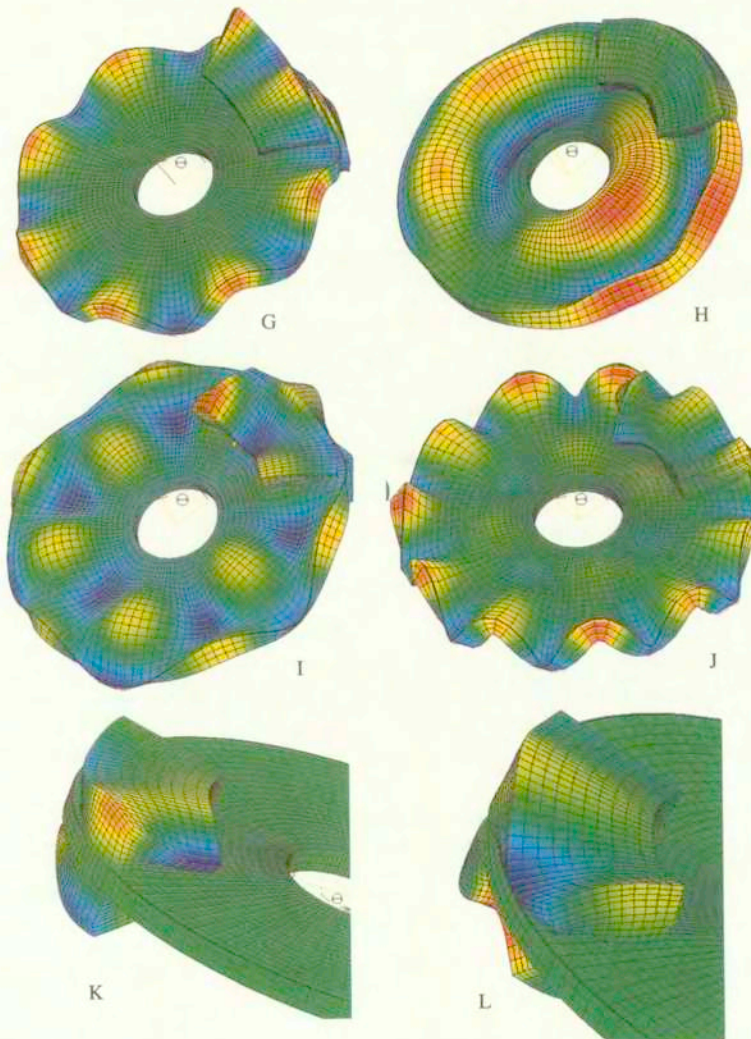


FIG. 3. Flutter instability of the steady sliding solution of the pad-disk system. The real parts of some unstable modes are presented.

## 6. Numerical results for a disk brake

A disk brake is considered with a finite element mesh using 17860 nodal points,  $n = 53580$  is the number of degrees of freedom. The pad is submitted to a uniform pressure on the upper face. Under the approximation  $\mathcal{A}$ , the steady sliding solution can be obtained within 3 iterations. The associated normal pressure is strictly positive on the whole contact surface, thus the assumption of effective contact is satisfied. The solution of (4.19) is again done using a basis of functions composed of  $m = 200$  first vibration modes of the perfect system. For  $f = 0.4$ , the unstable modes among the first 70 eigenmodes are modes 19, 24, 30, 32, 40, 48, 51, 56, 60, 63, 66, 68 of frequencies 5275, 6483, 8574, 9113, 11418, 11951, 12610, 13721, 14716, 15302, 15534, 15746, respectively. The real parts of the most unstable modes 32, 48, 51, 68, 63, 66 can be found in Fig. 3.

## 7. Concluding remarks

It is expected that, in the spirit of Hopf's bifurcation, after a flutter instability the dynamic response of the system will eventually become periodic with different phases of stick, slip and separation regimes, as it can be observed in various examples of the literature, (cf. [12 – 20]). Brake squeal results as a consequence of this periodic regime. The frequencies of the periodic responses, if they exist, are however not directly related to the flutter modes although they may remain close. From the analysis of the mechanism of flutter, our analysis leads already to some suggestions in order to impede such an instability.

## References

1. G. G. ADAMS, *Self-excited oscillations of two elastic half-spaces sliding with a constant coefficient of friction*, J. Applied Mechanics, 867-872, 1995.
2. P. CHAMBRETTE, L. JEZEQUEL, *Stability of a beam rubbed against a rotation disk*, Eur. J. Mech. A/Solids, 11, 107-138, 1992.
3. X. CHATEAU, Q. S. NGUYEN, *Buckling of elastic structures in unilateral contact with or without friction*, Eur. J. Mech. A/Solids, 10, 71-89, 1991.
4. H. CHO, J. R. BARBER, *Stability of the three-dimensional Coulomb's friction law*, Phil. Trans. Roy Soc., A 455, 839-861, London 1999.
5. R. COTTLE, J. S. PANG, R. E. STONE, *The linear complementary problem*, Academic Press, 1992.
6. S. DURAND, *Unilateral contact with bounces*, [In:] Contact Mechanics, Marseille 1995, Plenum Press, 111-118, New York 1996.
7. G. ISAC, *Complementary Problems*, Springer-Verlag, Lecture Note in Mathematics, New York 1992.



8. M. JEAN, J. J. MOREAU, *Dynamics in the presence of unilateral contact and dry friction: a numerical approach*, [In:] Unilateral problems in structural analysis, CISM course, 1–50, Springer-Verlag, Wien 1987.
9. A. KLARBRING, *Contact, friction, discrete mechanical structures and mathematical programming*, CISM Course on Contact Problems, 1997.
10. J. A. C. MARTINS, F. M. F. SIMOES, *On some sources of instable/illposedness in elasticity with Coulomb's friction*, Contact Mechanics, Marseille 1995.
11. J. A. C. MARTINS, S. BARBARIN, M. RAOUS, A. PINTO DA COSTA, *Dynamic stability of finite dimensional linear elastic system with unilateral contact and Coulomb's friction*, Comp. Meth. Appl. Mech. Engng., **177**, 3–4, 298–328, 1999.
12. F. MOIROT, *Etude de la stabilité d'un équilibre en présence de frottement de Coulomb. Application au crissement des freins à disque*, Doctorat thesis, Ecole Polytechnique, Paris 1998.
13. Z. MRÓZ, A. JARZĘBOWSKI, *Phenomenological models of contact slip*, Acta Mechanica, **102**, 59–72, 1994.
14. Q. S. NGUYEN, *Stabilité et mécanique non linéaire*, Hermès, Paris 1999.
15. V. G. OANCEA, T. A. LAURSEN, *Stability analysis of state-dependent dynamic frictional sliding*, Int. J. Nonlinear Mech., **32**, 5, 837–853, 1997.
16. J. T. ODEN, J. A. C. MARTINS, *Models and computational methods for dynamic friction phenomena*, Comp. Meth. Appl. Mech. Engng., **52**, 527–634, 1985.
17. M. OESTREICH, N. HINRICHS, K. POPP, *Bifurcation and stability analysis for a non-smooth friction oscillator*, Arch. Appl. Mech., **66**, 301–314, 1996.
18. K. POPP, P. STELTER, *Stick-slip vibrations and chaos*, Phil. Trans. Roy Soc., **A 332**, 89–105, London 1990.
19. D. VOLA, M. RAOUS, J. A. C. MARTINS, *Friction and instability of steady sliding squeal of a rubber/glass contact*, Int. J. Num. Meth. Engng., **45**, 1999.
20. O. Y. ZHARII, *Frictional contact between the surface wave and a rigid strip*, J. Applied Mechanics, **63**, 15–20, 1996.

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