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Time-dependent elastoplastic constitutive equation

Dedicated to Professor Zenon Mróz
on the occasion of his 70^{th} birthday

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Various existing constitutive equations aiming at the description of the time-dependent deformation behavior for a wide range of stress below and over the elastic limit, i.e. the yield stress, are reviewed in detail. It is suggested that the plastic stretching and the creep stretching have to be treated as independent quantities since they have substantially different physical properties, and that a stress goes out from the yield surface at a high rate of deformation since a plastic deformation is suppressed by a high viscous resistance, and then the yield surface is kept unchanged. The *subloading surface model* [1 – 3] would satisfy these requirements since it does not premise that a stress is on the yield surface even in the plastic loading process. In this article, based on the extended subloading surface model [4] which is capable of describing not only monotonic but also cyclic loading behavior, the generalized time-dependent elastoplastic constitutive equation is formulated allowing the stress go out from the yield surface by letting the plastic deformation be suppressed at a high rate of deformation and introducing the creep stretching which proceeds with time in addition to the elastic and the plastic stretching.

1. Introduction

Deformation of materials depends on a rate of deformation, i. e. time in general, and thus the description for the time-dependent deformation behavior is of importance for the analysis of practical problems in engineering. The *viscoelastic model* is applicable to the description of deformation behavior of materials at a low stress level but inapplicable to that at a stress level higher than the elastic limit, i.e. the yield stress. Then, various constitutive equations aiming at describing the time-dependent deformation behavior for a wide range of stress from the low stress level below the yield stress to the high stress level over the yield stress have been proposed. In this article they are first reviewed in detail, and then it is revealed that a satisfactory one capable of describing the time-dependent behavior of materials for any stress has not been proposed up to the present.

It should be noted that a plastic deformation based on the mutual slip between microstructures is suppressed under a deformation at a high rate causing the increase of viscous resistance acting between microstructures. Therefore, when a large deformation is induced at a high rate, the stress would go out from the yield surface since the deformation proceeds elastically, the plastic deformation being suppressed by the increase of viscous resistance, and thus the yield surface being kept unchanged. The *subloading surface model* [1 – 4] does not premise that the stress is on the *normal-yield surface* (conventional yield surface) even in the plastic loading process. In this model the subloading surface is introduced, which passes always through the current stress point even if the stress exists inside the normal-yield surface and is similar to the normal-yield surface, and it is assumed that the subloading surface approaches to the normal-yield surface in the plastic loading process. Based on this assumption, the extended consistency condition for the subloading surface is formulated, and applying the associated flow rule to the consistency condition, the plastic stretching induced by the rate of stress on or inside the normal-yield surface is formulated so as to describe the smooth *elastic-plastic transition*. It fulfills the *continuity condition* and the *smoothness condition* which are the mechanical requirements for constitutive equations [5 – 7].

The author [8] extended the initial subloading surface model [1 – 3] with the isotropic hardening so as to describe the time-dependence, while this model is limited to the description of monotonic loading behavior since the similarity-center of the normal-yield and the subloading surfaces is fixed in the origin of the stress space. Besides, based on it, the time-dependent elastoplastic constitutive equation of soils was formulated and its ability to reproduce the time-dependent behavior of real soils was verified. However, the initial subloading surface model is not capable of describing the cyclic loading behavior, in which the similarity-center of the normal-yield and the subloading surfaces are fixed in the origin of stress space. On the other hand, the extended subloading surface model [4] with the translation of the similarity-center due to the plastic deformation would be the only model capable of describing not only monotonic but also cyclic loading behavior pertinently among the existing models as was reviewed in detail in the previous article [6]. In this article, the extended subloading surface model is extended so as to describe the time-dependent deformation behavior by allowing the subloading surface to become larger than the normal-yield surface, and introducing the creep stretching which proceeds with time. That is, the generalized formulation of the time-dependent subloading surface model is given in this article. It falls within the framework of the elastoplastic-creep constitutive equation. Further, based on it, the constitutive equation of metals with the isotropic-kinematic hardening is formulated where the pertinent creep equation is formulated and its adequacy is verified by comparing it with experimental data.

2. Reviews on the existing time-dependent models

Various approaches for describing the time-dependent deformation behavior for a wide range of stress below and over the yield stress have been attempted in the past. They could be classified into the *elasto-viscoplastic model* and the *elastoplastic-creep model*. An inelastic stretching (i.e. the symmetric part of velocity gradient) is treated as a single quantity, called the *viscoplastic stretching*, in the elasto-viscoplastic model. On the other hand, it is decomposed into the *plastic stretching* and the *creep stretching* in the elastoplastic-creep model. Therefore, the elasto-viscoplastic and the elastoplastic-creep models are often called the *unified model* and the *superposition model*, respectively, by the other names. Besides, the elasto-viscoplastic model could be further classified into the *over-stress model* of PERZYNA [9 – 11] and the nonstationary flow surface model of OLSZAK and PERZYNA [12, 13].

2.1. Elasto-viscoplastic model (unified model)

The deformation can be classified into the reversible, i.e. elastic one with the *loading-path independence* and the irreversible, i.e. inelastic one without the *loading path-independence*. The latter can be classified further into the plastic and the creep (viscous) deformation. Here, it should be noted that the plastic stretching does not proceed always, being dependent on the direction of stress rate or stretching, but the creep stretching proceeds always with an elapse of time as illustrated in Fig. 1 for the decrease of stress under the stress control condition. Thus, the switching condition whether or not a stretching is generated, i.e. the *loading criterion* is required for the plastic stretching but is not required for the creep stretching. Therefore, the plastic stretching and the creep stretching have to be formulated as independent quantities different from each other contrary to the unified model, i.e. the elasto-viscoplastic model. Thus, it should be concluded that the unified model has the fundamental importance in the mechanical framework itself. However, let the basic properties of the over-stress and the nonstationary flow surface models in the unified model be further examined in detail below.

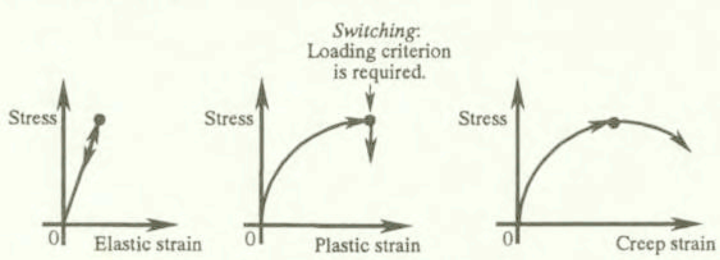

FIG. 1. The responses of the elastic, creep and plastic deformation to the decrease of stress under the stress control conditions.

2.1.1. Over-stress model. HOHENEMSER and PRAGER [14] have extended the BINGHAM'S [15] one-dimensional elasto-viscoplastic model to the three-dimensional stress state by replacing the slider with the J_2-yield condition and incorporating the potential flow rule, while the Bingham model is the modification of the Maxwell's viscoelastic model by replacing the dashpot with the parallel combination of the dashpot and the slider as the threshold for the inception of the movement of the dashpot. The *over-stress model* [9 – 11] can be regarded as the generalization of the HOHENEMSER and PRAGER'S [14] or more explicitly the PRAGER'S [16] elasto-viscoplastic model by replacing the J_2-yield condition with the general yield condition. It has the following basic structure.

i) The viscoplastic stretching is not related to the stress rate but to the stress, while it is related to the stress rate in the elastoplasticity. Thus, both the direction and the magnitude of viscoplastic stretching are independent of the stress rate but dependent of only the state of stress. This is the basic property of the Newtonian (viscous) fluid, while the viscoplastic stretching can be regarded as the viscous stretching with the threshold given by the yield condition. On the other hand, the plastic stretching is related to the stress rate in the elastoplasticity where the direction of the plastic stretching depends on the state of stress but the magnitude depends on the rate of stress.

ii) A loading criterion in terms of stress rate or stretching is not imposed to the viscoplastic stretching.

iii) The viscoplastic stretching is induced only when a stress is outside the yield surface.

Therefore, the Bingham model and the over-stress model are substantially different from the elastoplasticity. Needless to say, the over-stress model cannot reduce to the elastoplastic constitutive equation at any rate of deformation.

Thus, the over-stress model is incapable of describing

1) the plastic stretching which requires the loading criterion, while the creep stretching can be described, and

2) the inelastic deformation when a stress is on or inside the yield surface, contrary to the fact that the time-dependent behavior is generated independently of the yield condition, and thus the stress relaxation and the creep deformation cannot be predicted satisfactorily (for instance, the creep and the relaxation after a stress decreased to the interior of the yield surface by a quick unloading cannot be described).

The Bingham model and the over-stress model may be regarded as the modification of the Maxwell model by limiting the creep (viscous) stretching so as to be not generated in a low stress level below the yield stress (cf. Fig. 2). In order to cover the above defect 2), an unrealistically small yield surface is often used. This treatment results in the fact that the mechanical response reduces to the viscoelastic behavior of the Maxwell model incapable of describing a plastic deformation, however.

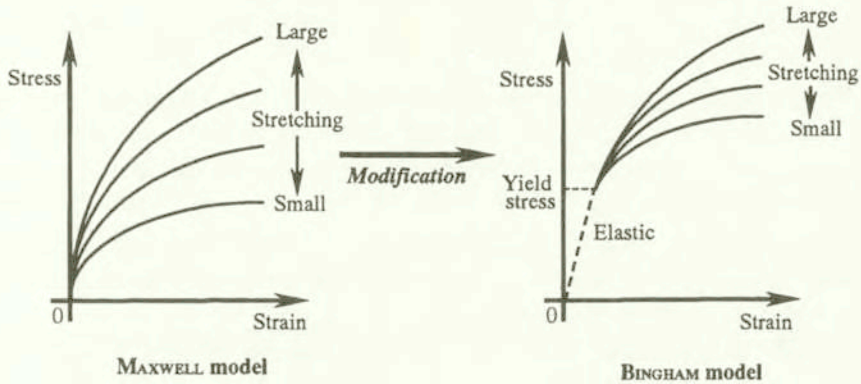

FIG. 2. The deterioration of the Maxwell viscoelastic model by BINGHAM [15].

The over-stress model could be applied to the description of the deformation of metals at high temperature, wet clays, etc. in the state of stress over the yield stress, in which creep deformation is dominant. However, the deformation of metals at room temperature, dry sands [17], etc. in which plastic deformation is dominant, cannot be described satisfactorily by this model.

2.1.2. Nonstationary flow surface model. This model [12, 13] has the following basic structure.

i) The nonstationary flow surface is incorporated, which is extended from the yield surface so as to depend on time by modifying it to include the variable describing a time-dependent alteration of the yielding property of material, in addition to the stress and internal variables, premising that viscoplastic deformation proceeds only when a stress exists on the nonstationary flow surface. Here, it should be noted that any rate tensor of stress or deformation is not incorporated in the equation of the nonstationary flow surface.

ii) The viscoplastic stretching is derived by applying the associated flow rule to the consistency condition obtained by the time-differentiation of the nonstationary flow surface. Therefore, the magnitude of the viscoplastic stretching depends on the stress rate, whilst the magnitude of the creep stretching depends on the state of stress. The stiffness tensor which relates the stretching to the stress rate is dependent of state variables, i.e. the stress, internal variables and time, but independent of any rate variable.

Therefore, this model is incapable of describing

1) the creep stretching the magnitude of which is independent of the stress rate,

2) the inelastic deformation when a stress exists inside the nonstationary flow surface and thus the stress relaxation and the creep deformation cannot be described satisfactorily (for instance, the creep and the relaxation after a stress decreased into the interior of the nonstationary flow surface by a quick unloading cannot be described), and

3) the prompt response in the alteration of stiffness (modulus) due to the abrupt change of the rate of deformation, since the nonstationary flow surface is not influenced by the rate of deformation (for instance, an abrupt increase of stress rate, i.e. an abrupt rising of stress (an almost elastic response) induced by a prompt increase of stretching cannot be described realistically). Thus, the application of this model has to be limited to the monotonic loading without a large variation of the rate of deformation.

Eventually, it should be concluded that the unified model, i.e. the elasto-viscoplastic model, is incapable of describing the time-dependent deformation behavior in the general state of stress.

2.2. Elastoplastic-creep model (superposition model)

The *elastoplastic-creep model* assumes that the stretching is additively decomposed into the elastic, the plastic and further the creep stretchings. That is, the plastic stretching and the creep stretching are formulated as independent quantities different from each other. Therefore, the fundamental defects involved in the elasto-viscoplastic model can be avoided in the elastoplastic-creep model. However, the existing models (cf. e.g. [18 – 20]) in the framework of the elastoplastic-creep model are incapable of describing the deformation behavior at a high rate realistically, since they are not taken account of the fundamental fact that a plastic deformation is suppressed with the increase of rate of deformation.

3. Time-dependent subloading surface model

The subloading surface model will be extended so as to describe time-dependent deformation behavior in the framework of the elastoplastic-creep model in this section.

3.1. Decomposition of stretching

A solid material is the assembly of solid particles, e.g. crystals in metals and soil particles in soils. Thus, the macroscopic deformation of solid materials consists of the deformations of each solid particle itself and the mutual slips between the solid particles. The deformations of each solid particle itself, exhibiting stiff and reversible characteristics, lead macroscopically to the elastic deformation exhibiting the loading path-independence at usual stress level. On the other hand, the mutual slips between the solid particles or irreversible rearrangements within them (e.g. dislocation movement) lead macroscopically to the irreversible deformation exhibiting the loading path-dependence. Then, let the stretching $\mathbf{D}$ be additively decomposed into the elastic stretching $\mathbf{D}^e$ and the inelastic stretching $\mathbf{D}^i$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^i, \tag{3.1}$$

where the elastic stretching $\mathbf{D}^e$ is given by

$$\mathbf{D}^e = \mathbf{E}^{-1}\,\overset{\circ}{\boldsymbol{\sigma}}\,. \tag{3.2}$$

$\boldsymbol{\sigma}$ is a stress, $(^\circ)$ indicates the proper corotational rate with the objectivity [21, 22] and the fourth-order tensor $\mathbf{E}$ is the elastic modulus given in Hooke's type as

$$E_{ijkl} = \left(K - \frac{2}{3}G\right)\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.3}$$

where K and G are the bulk and the shear modulus, respectively, which are functions of stress and internal state variables in general and δ_{ij} is the Kronecker's delta, i.e. $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

The mutual slips are induced by overcoming the frictional resistance and thus the macroscopic deformation due to the mutual slips has been described as the plastic deformation. Now, consider the situation that there exists a viscous medium between solid particles. The mutual slips are induced not only by overcoming the frictional resistance leading macroscopically to the plastic deformation but also with the elapse of time leading macroscopically to the creep deformation. Here, note that the creep deformation proceeds always with the elapse of time but the plastic deformation ceases when the stress becomes lower than the frictional resistance exhibiting the frictional switching, i.e. requiring the loading criterion. Therefore, the plastic and the creep deformation have to be described as independent quantities of each other. Thus, let the inelastic stretching be additively decomposed into the plastic stretching $\mathbf{D}^p$ and the creep stretching $\mathbf{D}^c$, i.e.

$$\mathbf{D}^i = \mathbf{D}^p + \mathbf{D}^c. \tag{3.4}$$

Here, note that the plastic deformation is suppressed with the increase of the deformation rate causing the viscous resistance, and that the stress can go out from the yield surface since at a high rate of deformation the elastic deformation proceeds, whilst a plastic deformation is hardly induced and the yield surface is kept unchanged.

3.2. Normal-yield and subloading surfaces

Assume the yield condition:

$$f(\hat{\boldsymbol{\sigma}}, \mathbf{H}) = F(H), \tag{3.5}$$

where

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha}. \tag{3.6}$$

The second-order tensor $\boldsymbol{\alpha}$ is the reference point on or inside the yield surface, which plays the role of the kinematic hardening variable as it translates with the plastic deformation. The scalar H and the second-order tensor $\mathbf{H}$ are isotropic and anisotropic hardening variables, respectively. Let it be assumed that the function f is homogeneous of degree one in the tensor $\hat{\boldsymbol{\sigma}}$, satisfying $f(s\hat{\boldsymbol{\sigma}}) = sf(\hat{\boldsymbol{\sigma}})$ for any nonnegative scalar s. Then, if $\mathbf{H}$ = const, the yield surface presents its similarity. An example of $\mathbf{H}$ is the rotational hardening variable for soils [23], while the kinematic hardening is not applicable to soils since the yield surface for soils always involves the origin of stress space.

Hereinafter, the elastoplastic constitutive equation will be formulated in the framework of the *unconventional plasticity* defined by DRUCKER [24] as the *extended plasticity theory* such that the interior of the yield surface is not a purely elastic domain but a plastic deformation is induced by the rate of stress inside the yield surface. Thus, the conventional yield surface is renamed as the *normal-yield surface*, since its interior is not regarded as a purely elastic domain in the present model.

Now, let the *subloading surface* [4] be introduced. This surface always passes through the current stress point $\boldsymbol{\sigma}$ and keeps the similar shape and the positioning of similarity to the normal-yield surface.

The similarity and the positioning of similarity require the followings.

i) All lines connecting the point on or within the subloading surface and the *conjugate point* on the normal-yield surface join at the specified point, i.e. the *similarity-center* $\mathbf{s}$.

ii) The *ratio* of length of an arbitrary line-element connecting two points on or inside the subloading surface and that of an arbitrary conjugate line-element connecting two *conjugate points* on or inside the normal-yield surface has the same value, called the *similarity-ratio*, denoted as R, which is also the same as the ratio of sizes of these surfaces.

Here, note that the similarity-center has to lie inside both the normal-yield and subloading surfaces, since these surfaces are not allowed to intersect to each other, whilst the subloading surface plays the role of loading surface.

The approaching degree to the normal-yield state can be described by the ratio of the size of the subloading surface to that of the normal-yield surface, i.e. the similarity-ratio R of these surfaces. The similarity-ratio will be denoted as R, while $R = 0$ corresponds to the most elastic state in which the stress coincides with the similarity-center, and $R = 1$ to the normal-yield state in which the stress exists on the normal-yield surface. Then, it holds that

(3.7) $$\boldsymbol{\sigma}_y = \frac{1}{R}\{\boldsymbol{\sigma} - (1 - R)\mathbf{s}\} \qquad (\boldsymbol{\sigma} - \mathbf{s} = R(\boldsymbol{\sigma}_y - \mathbf{s})),$$

where $\boldsymbol{\sigma}_y$ on the normal-yield surface is the *conjugate stress* of the current stress $\boldsymbol{\sigma}$ on the subloading surface (see Fig. 3).

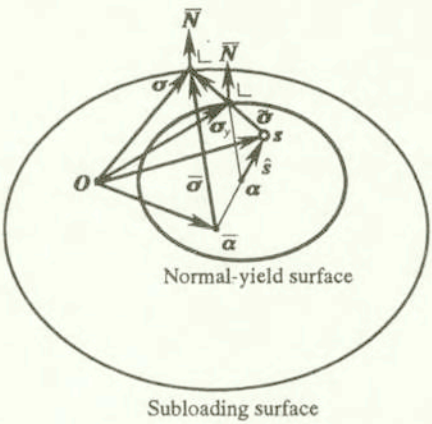

FIG. 3. The normal-yield and the subloading surfaces at a high rate of deformation.

By substituting Eq. (3.7) into Eq. (3.5) with replacing $\boldsymbol{\sigma}$ in Eq. (3.5) with $\boldsymbol{\sigma}_y$, the subloading surface is described as

(3.8) $$f(\overline{\boldsymbol{\sigma}}, \mathbf{H}) = RF(H),$$

where

(3.9) $$\overline{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \overline{\boldsymbol{\alpha}} (= R(\boldsymbol{\sigma}_y - \boldsymbol{\alpha})),$$

(3.10) $$\overline{\boldsymbol{\sigma}} \equiv \mathbf{s} - R(\mathbf{s} - \boldsymbol{\alpha}) \quad (\overline{\boldsymbol{\alpha}} - \mathbf{s} = R(\boldsymbol{\alpha} - \mathbf{s})).$$

$\overline{\boldsymbol{\alpha}}$ on or inside the subloading surface is the conjugate point of $\boldsymbol{\alpha}$ on or inside the normal-yield surface. In calculation, first R is determined from Eq. (3.8) with Eqs. (3.9) and (3.10), by substituting values of $\boldsymbol{\sigma}$, H, $\boldsymbol{\alpha}$, $\mathbf{H}$ and s, and thereafter $\overline{\boldsymbol{\alpha}}$ is found from Eq. (3.10). The four internal variables H, $\boldsymbol{\alpha}$, $\mathbf{H}$ and s are introduced in the present model.

The evolution of internal structure of materials is caused by the inelastic stretching $\mathbf{D}^i$, and thus the evolution equations of H, $\mathbf{H}$ and $\boldsymbol{\alpha}$ are homogeneous of degree one in $\mathbf{D}^i$. Here, assume that they are linear functions of $\mathbf{D}^i$, i.e.

$$
\begin{aligned}
\dot{H} &= \mathrm{tr}\{\mathbf{f}_h(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha}, \mathrm{s})\mathbf{D}^i\},\\
\overset{\circ}{\mathbf{H}} &= \mathbf{f}_H(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha}, \mathrm{s})\mathbf{D}^i, \qquad (3.11)\\
\overset{\circ}{\boldsymbol{\alpha}} &= \mathbf{f}_\alpha(\boldsymbol{\sigma}, H, \mathbf{H}, \boldsymbol{\alpha}, \mathrm{s})\mathbf{D}^i,
\end{aligned}
$$

$(\dot{\cdot})$ standing for the material time-derivative. Thus they can be additively decomposed into the plastic parts $\dot{H}^p$, $\overset{\circ}{\mathbf{H}}{}^p$, and $\overset{\circ}{\boldsymbol{\alpha}}{}^p$ and the creep parts $\dot{H}^c$, $\overset{\circ}{\mathbf{H}}{}^c$ and $\overset{\circ}{\boldsymbol{\alpha}}{}^c$ by Eq. (3.4) of decomposition of the inelastic stretching.

3.3. Evolution of similarity-ratio

It is observed from experiments that the stress asymptotically approaches the normal-yield surface in the plastic loading process $\mathbf{D}^p \neq \mathbf{0}$, and it has to be postulated as it was described in the foregoing that the plastic deformation is suppressed with the increase of the deformation rate causing the viscous resistance. The stress can go out from the normal-yield surface since at a high rate of deformation, the elastic deformation proceeds without a plastic deformation causing a variation of the normal-yield surface. Therefore, the subloading surface can expand over the normal-yield surface. Thus, let the following evolution equation of the similarity-ratio R be assumed.

$$\dot{R} = U^t \|\mathbf{D}^p\| \quad \text{for} \quad \mathbf{D}^p \neq \mathbf{0}, \qquad (3.12)$$

where U^t is additively composed of the monotonically decreasing function U_R of the similarity-ratio R and the monotonically increasing function $U_D(\geq 0)$ of the magnitude $\|\mathbf{D}\|$ of stretching, i.e.

$$U^t = U_R(R) + U_D(\|\mathbf{D}\|), \qquad (3.13)$$

satisfying

$$
\begin{aligned}
&U_R = +\infty \quad \text{for } R = 0,\\
&U_R = 0 \quad \text{for } R = 1, \qquad (3.14)\\
&(U_R < 0 \quad \text{for } R > 1)
\end{aligned}
$$

and

(3.15) $$U_D = 0 \quad \text{for } \|\mathbf{D}\| = 0.$$

$\| \, \|$ stands for the magnitude, i.e. $\|\mathbf{T}\| = \sqrt{\mathrm{tr}(\mathbf{T}\mathbf{T}^T)}$ for arbitrary tensor $\mathbf{T}$, tr( ) denoting the trace and $(\ )^T$ the transpose. Note that $\dot{R} = 0$ for $R = 1$ if $\|\mathbf{D}\| = 0$ but $\dot{R} > 0$ even for $R = 1$ if $\|\mathbf{D}\| > 0$. The function U^t is illustrated in Fig. 4.

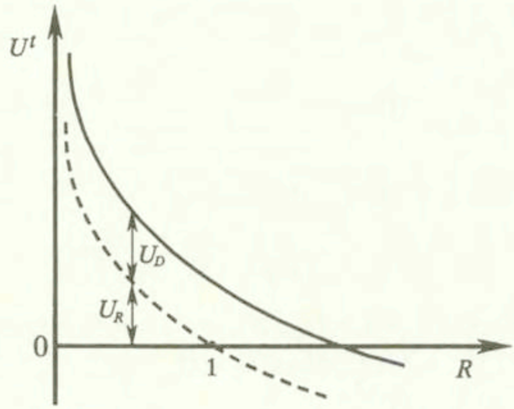

FIG. 4. The function U^t in the evolution rule of the similarity-ratio R.

3.4. The translation rule of similarity-center

The *similarity-center* $\mathbf{s}$ is required to translate with the plastic deformation in order to describe realistically the cyclic loading behavior exhibiting the Masing effect [6, 25, 26]. The translation rule of $\mathbf{s}$ is described below.

The following inequality must hold since the similarity-center $\mathbf{s}$ has to exist inside the normal-yield surface.

(3.16) $$f(\hat{\mathbf{s}}, \mathbf{H}) \le F(H),$$

where

(3.17) $$\hat{\mathbf{s}} \equiv \mathbf{s} - \boldsymbol{\alpha}.$$

Let the ultimate state $f(\hat{\mathbf{s}}, \mathbf{H}) = F(H)$ be considered, in which the similarity-center just exists on the normal-yield surface, and thus the risk that the similarity-center goes out from the normal-yield surface has to be avoided. The time-differentiation of Eq. (3.16) in the ultimate state gives:

(3.18) $$\mathrm{tr}\left[\frac{\partial f(\hat{\mathbf{s}}, \mathbf{H})}{\partial \mathbf{s}} \left(\overset{\circ}{\hat{\mathbf{s}}} + \frac{1}{F}\left\{\mathrm{tr}\left(\frac{\partial f(\hat{\mathbf{s}}, \mathbf{H})}{\partial \mathbf{H}} \overset{\circ}{\mathbf{H}}\right) - \dot{F}\right\}\hat{\mathbf{s}}\right)\right] \le 0$$
$$\text{for} \quad (\hat{\mathbf{s}}, \mathbf{H}) = F(H).$$

The inequality (3.18) or (3.20) is called the *enclosing condition of similarity-center.*

In the ultimate state $f(\hat{\mathbf{s}},\mathbf{H}) = F(H)$, the vector $\boldsymbol{\sigma}_y - \mathbf{s}(= (\boldsymbol{\sigma} - \mathbf{s})/R)$ makes an obtuse angle with the vector $\partial f(\hat{\mathbf{s}},\mathbf{H})/\partial\mathbf{s}$ which is the outward normal to the surface, called the *similarity-center surface*, that passes through the similarity-center and is similar to the normal-yield surface, provided that the normal-yield surface is convex. Noting this fact and considering the fact that the similarity-center moves only with the plastic deformation, let the following equation be assumed so as to fulfill the inequality (3.18):

$$\text{(3.19)} \qquad \overset{\circ}{\hat{\mathbf{s}}} + \frac{1}{F}\left\{\operatorname{tr}\left(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial \mathbf{H}}\right) - \dot{F}\right\}\hat{\mathbf{s}} = c\operatorname{tr}(\overline{\mathbf{N}}\mathbf{D}^i)\frac{\tilde{\boldsymbol{\sigma}}}{R},$$

provided that the inelastic stretchig $\mathbf{D}^i$ satisfies

$$\text{(3.20)} \qquad \operatorname{tr}(\overline{\mathbf{N}}\mathbf{D}^i) \geq 0$$

in the ultimate state $f(\hat{\mathbf{s}},\mathbf{H}) = F(H)$, where c is a material constant influencing the translating rate of the similarity-center and

$$\text{(3.21)} \qquad \tilde{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \mathbf{s},$$

$$\text{(3.22)} \qquad \overline{\mathbf{N}} \equiv \frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial \boldsymbol{\sigma}} \right\| \quad (\|\overline{\mathbf{N}}\| = 1)$$

which is the outward-normal of the subloading surface. The translation rule of the similarity-center is now derived from Eq. (3.19) as follows:

$$\text{(3.23)} \qquad \overset{\circ}{\mathbf{s}} = c\operatorname{tr}(\overline{\mathbf{N}}\mathbf{D}^i)\frac{\tilde{\boldsymbol{\sigma}}}{R} + \overset{\circ}{\boldsymbol{\alpha}} + \frac{1}{F}\left\{F'\dot{H} - \operatorname{tr}\left(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial \mathbf{H}}\overset{\circ}{\mathbf{H}}\right)\right\}\hat{\mathbf{s}},$$

where

$$\text{(3.24)} \qquad F' \equiv \frac{dF}{dH}.$$

It is conceivable that the similarity-center $\mathbf{s}$ approaches the current stress $\boldsymbol{\sigma}$ as can be seen from the simple case of the nonhardening state ($\overset{\circ}{\boldsymbol{\alpha}} = \overset{\circ}{\mathbf{H}} = \mathbf{0}, \dot{F} = 0$), although the evolution rule (3.23) is assumed to fulfill the requirement (3.18) in the ultimate state $f(\hat{\mathbf{s}},\mathbf{H}) = F(H)$. Here, note that $\overset{\circ}{\mathbf{s}}$ can be additively decomposed into the plastic part $\overset{\circ}{\mathbf{s}}{}^p$ and the creep part $\overset{\circ}{\mathbf{s}}{}^c$ by the additive decomposition of $\mathbf{D}^i$, $\dot{H}$, $\overset{\circ}{\mathbf{H}}$ and $\overset{\circ}{\boldsymbol{\alpha}}$.

3.5. Plastic stretching

The time-differentiation of Eq. (3.8) is given by

$$\text{(3.25)}\qquad \operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\right)-\operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\overline{\boldsymbol{\alpha}}}\right)+\operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\mathbf{H}}\right) = \dot{R}F+RF'\dot{H}.$$

By substituting Eq. (3.12) into Eq. (3.28) one has the *extended consistency condition* for the subloading surface:

$$\text{(3.26)}\qquad \operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\boldsymbol{\sigma}}\right)-\operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\overline{\boldsymbol{\sigma}}}\right)+\operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\mathbf{H}}\right) = U^t\|\mathbf{D}^p\|F+RF'\dot{H},$$

where

$$\text{(3.27)}\qquad \overset{\circ}{\overline{\boldsymbol{\alpha}}}=R\overset{\circ}{\boldsymbol{\alpha}}+(1-R)\overset{\circ}{\mathbf{s}}-U^t\|\mathbf{D}^p\|\hat{\mathbf{s}}$$

from Eqs. (3.10) and (3.12). $\overset{\circ}{\overline{\boldsymbol{\alpha}}}$ can be additively decomposed into the plastic part $\overset{\circ}{\overline{\boldsymbol{\alpha}}}{}^p$ and the creep part $\overset{\circ}{\overline{\boldsymbol{\alpha}}}{}^c$ by the additive decomposition of $\overset{\circ}{\boldsymbol{\alpha}}$ and $\overset{\circ}{\mathbf{s}}$.

Now assume the *associated flow rule*

$$\text{(3.28)}\qquad \mathbf{D}^p=\lambda\overline{\mathbf{N}}\quad(\lambda>0),$$

where λ is the positive proportionality factor. Substitution of Eq. (3.28) into the extended consistency condition (3.26) leads to

$$\text{(3.29)}\qquad \lambda=\frac{\operatorname{tr}(\overline{\mathbf{N}}\,\overset{\circ}{\boldsymbol{\sigma}})-\operatorname{tr}(\overline{\mathbf{N}}\,\overset{\circ}{\overline{\boldsymbol{\alpha}}}{}^c)-\left\{\dfrac{F'}{F}\dot{H}^c-\dfrac{1}{RF}\operatorname{tr}\left(\dfrac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\overset{\circ}{\mathbf{H}}{}^c\right)\right\}\operatorname{tr}(\overline{\mathbf{N}}\overline{\boldsymbol{\sigma}})}{\overline{M}{}^t_p},$$

where

$$\text{(3.30)}\qquad \overline{M}{}^t_p\equiv\operatorname{tr}\left[\overline{\mathbf{N}}\left(\left\{\frac{F'}{F}h^p-\frac{1}{RF}\operatorname{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\mathbf{H}}\mathbf{h}^p\right)+\frac{U^t}{R}\right\}\overline{\boldsymbol{\sigma}}+\overline{\mathbf{a}}^p\right)\right].$$

h^p, $\mathbf{h}^p$ and $\overline{\mathbf{a}}^p$ are functions of the stress, plastic internal state variables and $\overline{\mathbf{N}}$ of homogeneous degree one, while these functions are related to the plastic parts $\dot{H}^p$, $\overset{\circ}{\mathbf{H}}{}^p$ and $\overset{\circ}{\overline{\boldsymbol{\alpha}}}{}^p$ of the internal variables $\dot{H}$, $\overset{\circ}{\mathbf{H}}$ and $\overset{\circ}{\overline{\boldsymbol{\alpha}}}$ by

$$\text{(3.31)}\qquad h^p\equiv\frac{\dot{H}^p}{\lambda},\quad \mathbf{h}^p\equiv\frac{\overset{\circ}{\mathbf{H}}{}^p}{\lambda},$$

(3.32) $$\overline{\mathbf{a}}^p \equiv \frac{\overset{\circ}{\overline{\boldsymbol{\alpha}}^p}}{\lambda} = R\mathbf{a}^p + (1-R)\mathbf{z}^p - U^t\hat{\mathbf{s}},$$

(3.33) $$\mathbf{a}^p \equiv \frac{\overset{\circ}{\boldsymbol{\alpha}^p}}{\lambda},$$

(3.34) $$\mathbf{z}^p \equiv \frac{\overset{\circ}{\mathbf{s}^p}}{\lambda} = c\tilde{\boldsymbol{\sigma}} + \mathbf{a}^p + \frac{1}{F}\left\{F'h^p - \mathrm{tr}\left(\frac{\partial f(\hat{\mathbf{s}},\mathbf{H})}{\partial\mathbf{H}}\mathbf{h}^p\right)\right\}\hat{\mathbf{s}}$$

since these rate variables include λ in homogeneous degree one.

The stretching is given from Eqs. (3.1), (3.2), (3.4), (3.28) and (3.29) as

(3.35) $$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\overline{\mathbf{N}}\,\overset{\circ}{\boldsymbol{\sigma}}) - \mathrm{tr}(\overline{\mathbf{N}}\,\overset{\circ}{\overline{\boldsymbol{\alpha}}^c}) - \left\{\frac{F'}{F}\dot{H}^c - \frac{1}{RF}\mathrm{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\,\overset{\circ}{\mathbf{H}^c}\right)\right\}\mathrm{tr}(\overline{\mathbf{N}}\overline{\boldsymbol{\sigma}})}{\overline{\mathbf{M}}_p^t}\mathbf{N} + \mathbf{D}^c.$$

The positive proportionality factor in the associated flow rule (3.35) is expressed in terms of the stretching $\mathbf{D}$, replacing λ by Λ, as follows:

(3.36) $$\Lambda = \frac{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D}) - \mathrm{tr}(\overline{\mathbf{N}}\,\overset{\circ}{\overline{\boldsymbol{\alpha}}^c}) - \left\{\frac{F'}{F}\dot{H}^c - \frac{1}{RF}\mathrm{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\,\overset{\circ}{\mathbf{H}^c}\right)\right\}\mathrm{tr}(\overline{\mathbf{N}}\overline{\boldsymbol{\sigma}}) - \mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D}^c)}{\overline{M}_p^t + \mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\overline{\mathbf{N}})}.$$

The inverse expression of Eq. (3.35) is given from Eqs. (3.1), (3.2), (3.4), (3.28) and (3.36) as

(3.37) $$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}\mathbf{D} - \frac{\mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D}) - \mathrm{tr}(\overline{\mathbf{N}}\,\overset{\circ}{\overline{\boldsymbol{\alpha}}^c}) - \left\{\frac{F'}{F}\dot{H}^c - \frac{1}{RF}\mathrm{tr}\left(\frac{\partial f(\overline{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\,\overset{\circ}{\mathbf{H}^c}\right)\right\}\mathrm{tr}(\overline{\mathbf{N}}\overline{\boldsymbol{\sigma}}) - \mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\mathbf{D}^c)}{\overline{M}_p^t + \mathrm{tr}(\overline{\mathbf{N}}\mathbf{E}\overline{\mathbf{N}})} \cdot \mathbf{E}\mathbf{N} - \mathbf{E}\mathbf{D}^c.$$

Note that the stretching $\mathbf{D}$ cannot be expressed analytically in terms of the stress rate $\overset{\circ}{\boldsymbol{\sigma}}$ since the right-hand side of Eq. (3.35) includes $\mathbf{D}$ but inversely, the stress rate $\overset{\circ}{\boldsymbol{\sigma}}$ is expressed analytically in terms of the stretching $\mathbf{D}$ as seen in Eq. (3.37). Besides, the constitutive equation (3.35) or (3.37) is of the so-called *rate-nonlinearity* since it includes the magnitude of stretching $\|\mathbf{D}\|$. The rate-nonlinearity would be the basic property of the constitutive equation for time-dependent deformation for a wide range of stress below and over the yield stress in which the stiffness modulus depends on the rate of deformation.

3.6. Loading criterion

A loading criterion would not be necessary for the creep stretching since it proceeds always with the elapse of time. On the other hand, taking the fact that Λ has to be positive, let the following loading criterion for the plastic stretching be assumed [27].

$$\begin{aligned} &\mathbf{D}^p \neq \mathbf{0} : \Lambda > 0, \\ &\mathbf{D}^p = \mathbf{0} : \Lambda \leq 0. \end{aligned} \tag{3.38}$$

It should be noted that the constitutive equation formulated in the foregoing reduces to the time-independent subloading surface model when $\|\mathbf{D}\| \to 0$ and $\mathbf{D}^c = \mathbf{0}$. Here, note that the time-independent elastoplastic deformation would hold approximately in the case of the moderate rate of deformation for which the function U_D in the plastic modulus and the creep stretching are negligible.

4. Constitutive equation of metals

Based on the equations formulated in the preceding section, the constitutive equation for metals will be formulated in this section.

We adopt the subloading surface of the von Mises type with isotropic kinematic hardening [27], while the hardening variable $\sqrt{2/3}\|\mathbf{D}^p\|$ is extended so as to account for the hardening due to the creep deformation:

$$f(\overline{\boldsymbol{\sigma}}) = \sqrt{\frac{3}{2}}\|\overline{\boldsymbol{\sigma}}^*\| \quad (\mathbf{H} = \mathbf{0}), \tag{4.1}$$

$$\overset{\circ}{\boldsymbol{\alpha}} = \mathbf{a}\,\mathrm{tr}(\mathbf{N}\mathbf{D}^i)(= \mathbf{a}\|\mathbf{D}^i\|), \qquad \mathbf{a} \equiv k_1\frac{\overline{\boldsymbol{\sigma}}}{R} - k_2\boldsymbol{\alpha}, \tag{4.2}$$

$$F = F_0[1 + h_1\{1 - \exp(-h_2 H)\}], \tag{4.3}$$

$$\dot{H} = \sqrt{\frac{2}{3}}\mathrm{tr}(\overline{\mathbf{N}}\mathbf{D}^i)\left(\dot{H}^p = \sqrt{\frac{2}{3}}\mathrm{tr}(\overline{\mathbf{N}}\mathbf{D}^p)\left(= \sqrt{\frac{2}{3}}\lambda = \sqrt{\frac{2}{3}}\|\mathbf{D}^p\|\right) \right. \\ \left. \dot{H}^c = \sqrt{\frac{2}{3}}\mathrm{tr}(\overline{\mathbf{N}}\mathbf{D}^c)\right), \tag{4.4}$$

where

$$\overline{\boldsymbol{\sigma}}^* \equiv \overline{\boldsymbol{\sigma}} - \overline{\sigma}_m \mathbf{I}, \qquad \overline{\sigma}_m \equiv \frac{1}{3}\mathrm{tr}\overline{\boldsymbol{\sigma}}. \tag{4.5}$$

k_1, k_2, h_1 and h_2 are material constants, and F_0 is the initial value of F.

Let the function U_R satisfying Eq. (3.14) and the function U_D satisfying Eq. (3.15) be given by

$$U_R = -u \ln R, \tag{4.6}$$

and

$$U_D = \xi_1 \ln(1 + \xi_2 \|\mathbf{D}\|), \tag{4.7}$$

where u, ξ_1 and ξ_2 are the material constants.

The following generalization of Norton's creep law to the multi-axial case with the temperature effect by ODQVIST [28, 29] has been often used for the relation of creep stretching and stress.

$$\mathbf{D}^c = \sqrt{\frac{3}{2}}\overline{C}\left(\frac{\|\boldsymbol{\sigma}^*\|}{\sigma_c}\right)^{\overline{n}} \exp\left(-\frac{Q_c}{R_c T}\right)\frac{\boldsymbol{\sigma}^*}{\|\boldsymbol{\sigma}^*\|}, \tag{4.8}$$

where $\overline{C}$ is the material parameter, $\overline{n}$ is the material constant, σ_c is the stress-valued parameter, Q_c is the activation energy, R_c is the gas constant, T is the absolute temperature and $\boldsymbol{\sigma}^*$ is the deviatoric stress. However, $\overline{C}$ cannot be a material constant with an objectivity since it depends on the selection of σ_c. Assuming that the direction of the creep stretching is the outward-normal of the subloading surface, let Eq. (4.8) be modified for the present model as follows:

$$\mathbf{D}^c = \sqrt{\frac{3}{2}} C R^n \exp\left(-\frac{Q_c}{R_c T}\right)\overline{\mathbf{N}}, \tag{4.9}$$

where C and n are material constants, whilst Eq. (4.9) fulfills Eq. (3.20).

The following relations are derived from Eqs. (4.1) – (4.4).

$$\overline{\mathbf{N}} = \frac{\overline{\boldsymbol{\sigma}}^*}{\|\overline{\boldsymbol{\sigma}}^*\|}, \quad \left\|\frac{\partial f(\overline{\boldsymbol{\sigma}})}{\partial \overline{\boldsymbol{\sigma}}}\right\| = \sqrt{\frac{2}{3}}, \quad F' = F_0 h_1 h_2 \exp(-h_2 H), \tag{4.10}$$

$$h^p = \sqrt{\frac{2}{3}}.$$

The measured and predicted uniaxial loading behavior under various axial rates of deformation of $2_{1/4}$Cr-1Mo steel (SA 387, Gr. 22) at 600° C (test data after INOUE *et al.* [30]) are shown in Fig. 5. The material constants and initial values in the prediction are selected as follows:

Isotropic hardening: $h_1 = 0.3$, $h_2 = 20$;

Kinematic hardening: $k_1 = 0$, $k_2 = 0$;
Evolution rate of similarity ratio: $u = 1,500$, $\xi_1 = 30$, $\xi_2 = 5,000$ s;
Evolution of similarity center: $c = 0$;
Elastic moduli: $K = 140,000$ MPa, $G = 100,000$ MPa;
Creep: $C\exp\{-Q_c/(R_cT)\} = 0.00002\ \mathrm{s}^{-1}$, $n = 4$;
Initial values: $F_0 = 305$ MPa, $\boldsymbol{\alpha}_0 = \mathbf{0}$ MPa, $\mathbf{s}_0 = \mathbf{0}$ MPa (initial isotropy).

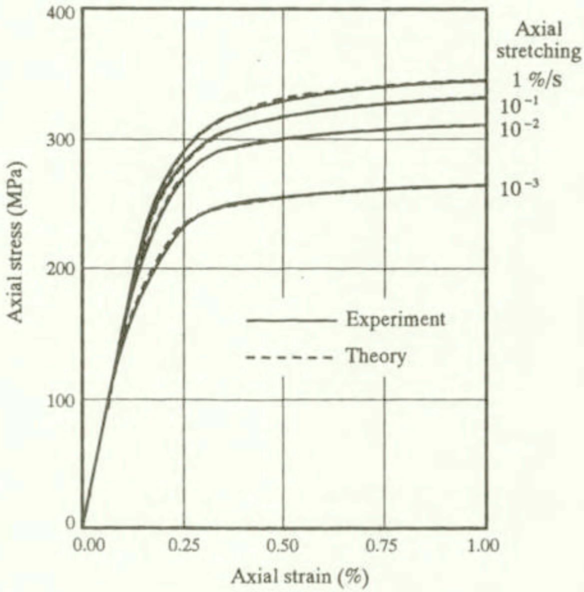

FIG. 5. Uniaxial loading behavior under various rates of deformation (test data after INOUE *et al.* [31]).

The difference of deformation behavior due to the rate of deformation is predicted quite well. The phenomenon that the rise of axial stress is weakened as the axial stretching decreases is predicted well by selecting the high value 4 for the material constant n on the influence of the similarity-ratio R.

5. Concluding remarks

The subloading surface model which would be the only model capable of describing pertinently the cyclic loading behavior among the existing models, is extended so as to describe the time-dependence in this article. The creep stretching is added to the elastic and the plastic ones in the algebraic sum. Thus, it falls within the framework of the elastoplastic-creep constitutive equation, which could be regarded as a natural extension of the traditional elastoplastic constitutive equation.

The novel features of the present constitutive equation are as follows:

1) The subloading surface is extended so as to be able to become larger than the normal-yield surface at a high rate of deformation. Here, it should be noted that there would not generally exist any surface which bounds the state of stress in the stress space. Therefore, the concept of the bounding surface of DAFALIAS [31] must be abandoned in the general deformation process with time-dependence, while in reality a stress goes out from the bounding surface at a high rate of deformation.

2) The plastic stretching is formulated so as to be suppressed by the increase of the rate of deformation through the evolution rule of the similarity-ratio. Thus, a quick response to an abrupt variation of the rate of deformation or stress is described.

3) The novel loading criterion is incorporated, which is defined by the sign of the proportionality factor in terms of stretching in the associated flow rule.

4) The Norton-Odqvist creep equation of metals is modified so as to have the objective material constant.

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