On Prandtl's lifting equation arising in wear mechanics

*Dedicated to Professor Zenon Mróz
on the occasion of his 70th birthday*

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A SLIDING WEAR CONTACT between a rigid punch and an elastic halfplane in presence of a thin aggregate film composed of solid debris and a lubricant fluid is studied. The model is based on any wear criterion and constitutive law of the film suggested by micromechanics approximation. The mechanical system is governed by the evolution of the volume fraction of debris, considered as the internal state variable. The key step of iterative computations for solving the nonlinear system of equations is based on the solution of the fundamental linear integro-differential equation for the compressive normal stress (the W-equation). Uniqueness of the solution of the integro-differential equation is then proved. It is shown that there is a profound relationship between the latter equation and Prandtl's lifting equation in aerodynamics: both equations can be solved numerically by Chebyshev's series, and experimentally by similar electrical setups. Mathematically, it is found that both equations are related to real and imaginary components of some complex potential, respectively, and to weakly adjoint integro-differential operators.

1. Introduction

A SLIDING WEAR CONTACT between two elastic solids is typically a nonlinear problem, because of two reasons: on the one hand, the contact problem itself is nonlinear even if there is no wear; on the other hand, the presence of debris or detached particles changes the load transfer conditions at the interface between two contacting solids. Following a terminology introduced in GODET [18, 19], the interface is called "third-body" and should be considered as an aggregate film composed of different particles and a lubricant fluid, having some nonlinear macroscopic rheology, which is not yet known. Different terminologies, interface or third-body, are simply a matter of scales used in describing the macroscopic

or mesoscopic behavior. On a thiner microscopic scale, damage and microcracking at the asperities are wear mechanisms feeding the third-body. Conditions for microscopic wear mechanisms to develop, depend on the macroscopic contact stresses, which are determined by the rheology of the third-body, the internal variable of which is related to the wear rate. This is a fully coupling problem at different levels.

In the literature, formulations of wear contact problems generally ignore this coupling aspect. For instance, by assuming a perfect contact between sound solids (Galin [14], Galin and Goriacheva [15]), one ignores the debris life in the contact zone or considers that the detached particles are removed instantaneously from the contact interface.

The need of an understanding of third-body processes in order to model and predict wear on a macroscopic scale is expressed by several authors: Godet [18, 19], Georges [16], Singer and Wahl [29], Berthier [3, 4], Berthier et al. [5], Meng and Ludema [24]. For a comprehensive review of wear mechanisms on a microscopic scale, see Ko [21]. A large amount of models are based on experimental observations and depend on the test conditions. Most of them are derived from Archard's relation [2]. Such models cannot be predictive when the operating conditions cannot be close to the common use conditions of machine components.

Experiments on wear friction contact, as observed in Stribeck's curves, provide a relation between the friction coefficient $\mu = \tau/p$ and the lubricant coefficient $L = \eta V/p$ (where η is the fluid viscosity, V the relative velocity, p the pressure, τ the shear stress). Three regimes are observed in Stribeck's curves (Fig. 1): (I) Coulomb's friction law with constant μ; (II) instable regime occuring probably in earthquakes (Scholz [28]); (III) hydrodynamic regime, for instance $\mu(L) \simeq a + bL$, as proposed in Dang Van's criterion [8], or $\mu(L) \simeq bL$, as in viscous laminar flow corresponding to mild wear. Stribeck's curves clearly suggest
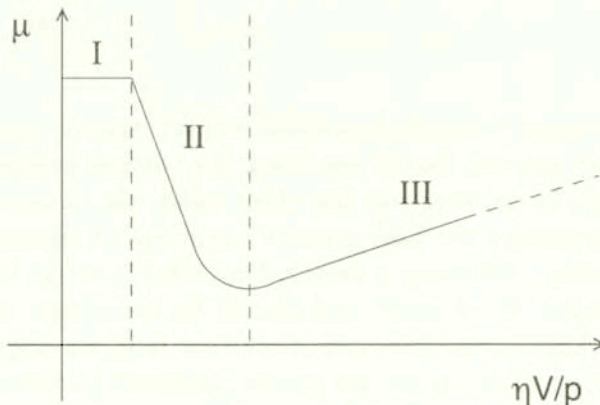

Fig. 1. Stribeck's curve.

an interaction between the debris and the elastic solids, through the evolution of some internal state variables governing the third-body, which is the proportion of solids' debris, defined either by volume fractions or by mass fractions of species. By a simple theory of mixture, one can get an idea on the third-body: constitutive laws, free energy, dissipation, etc...That approach is recently provided in DRAGON-LOUISET [11] for a wear model of regime (III), in the presence of incompressible fluid. A similar approach was given by STUPKIEWICZ and MRÓZ [31] for modeling abrasive wear.

The content of this paper is as follows. In the first part, for the consistency of the paper, we reconsider briefly the general equations of the contact-sliding mild wear model based on micromechanical considerations, given in [11]. The model is applied to the contact-sliding between a circular *rigid* solid and an *elastic* halfplane $\Omega_2 = \{x, y \leqslant -e(x) \simeq 0\}$ in presence of fluid (Fig. 2).
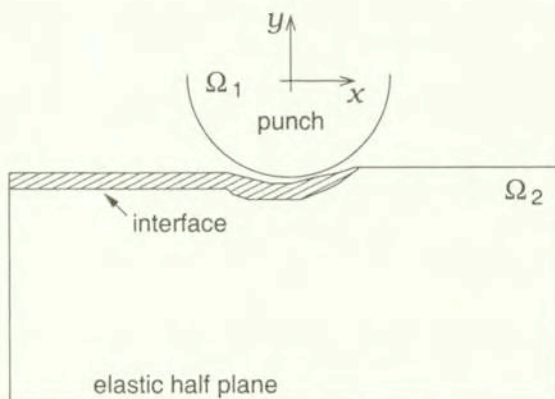

FIG. 2. A punch sliding on an elastic halfplane and their interface.

The coupled nonlinear equations of equilibrium are based on the following ideas:

i. The microscopic wear mechanisms occuring at the asperities level, describing the detachment of particles from sound solids to the third-body, will be modeled on the macroscopic scale. The wear criterion and wear rate will be given in a general form.

ii. The third-body on a mesoscopic scale, in somewhat of a thin film thickness $e(x)$ made of an aggregate of debris and fluid, considered as an *open* thermodynamical system, with mass transfer characterized by the wear rate $v(x)$ feeding the interface, at the surface Γ between sound material and the third-body. Parallel in-flow and out-flow of a two-phase aggregate occur in the third-body which behaves under shear load like a "viscous fluid". In

considering the volume fraction $\varphi(x)$ of debris, as suitable internal variables of the contact zone film, the conservation laws of mass (solid and fluid) provide the relations between the volume fraction, the wear rate and the thickness.

iii. The two-phase aggregate has a specific rheology to determine, depending on the volume fraction.

The coupled nonlinear equilibrium equations of the mechanical system, elastic solid, rigid punch and interface, provided that some reasonable assumptions are made, have the property that, at any step of the iterative computations, only *one* fundamental linear boundary integro-differential equation has to be solved (the W-equation).

   In the second part of the paper, we shall focus our analysis on the latter equation. Using methods of complex representations of potentials, we establish a relation between the W-equation and the well-known Prandtl lifting equation in aerodynamics. This relation suggests us similar methods for solving both equations, by Chebyshev's series expansion, using functions of the first kind and the second kind and also an interesting means for solving *experimentally* the W-equation, just as the Prandtl equation has been solved experimentally in the past by Malavard's electrical analogy.

2. The interface model for computational mechanics

   Having in mind the plane strain contact-sliding wear model for a rigid punch and an elastic foundation, described later, we consider a thin interface made of a two-phase mixture of solid debris and fluid. The interface extends along Ox, $-a \leqslant x \leqslant a$, with the wake interface $x \leqslant a$ (Fig. 2). For simplicity, to any meso-scale quantity $f(x, y)$ defined in the third-body $-a \leqslant x \leqslant a$, $-e(x) \leqslant y \leqslant 0$, we denote the corresponding average over the thickness $e(x)$ at x, by the same notation $f(x)$. Physically, the third-body is characterized by a proportion of solid debris s and fluid f. It can be characterized either by mass fractions or by volume fractions $\varphi_s(x)$, $\varphi_f(x) = 1 - \varphi_s(x)$. The volume fraction, being a geometrical description of the third-body, allows consideration of *stick* phenomena. For example, the shear flow is impossible for compact hexagonal arrangement of circular debris of equal radius, where $\varphi_s = 62\%$, Dragon-Louiset [11]. Hence, geometrical considerations allow the possibility of a threshold value of internal variables based on the volume fractions, beyond which stick phenomenon occurs.

   Before analysing the kinematics of the third-body, we first need a macroscopic description of wear condition, i.e. the detachment of particles feeding the interface.

2.1. Macroscopic wear criterion and wear rate

The most general form of wear criterion, was provided by DRAGON-LOUISET and STOLZ [12] who described in a thermodynamical manner the local quantities involved in the wear phenomena and proposed a wear criterion (DRAGON-LOUISET [11]) similar to the well-known *energy release rate* in Fracture Mechanics, applicable to elastic-brittle materials. Let us drop the index 2 for quantities defined in the elastic solid Ω_2, denoted Ω;

$$(2.1) \qquad g = \mathbf{n} \cdot \sigma \cdot \nabla \mathbf{u} \cdot \mathbf{n} - \rho \psi \qquad \text{(elastic solid)},$$

$$(2.2) \qquad g^3 = \mathbf{n} \cdot \sigma^3 \cdot \nabla \mathbf{u}^3 \cdot \mathbf{n} - \rho \psi^3 \qquad \text{(damaged material adjacent to } \Gamma),$$

with $\mathbf{n}$ the outward unit vector normal to the boundary Γ of Ω, with σ and σ^3 the stresses, $\mathbf{u}$ and $\mathbf{u}^3$ the displacements vectors, ψ and ψ^3 the free energies of sound and damaged material respectively, ρ (or ρ_s) the density of the solid. The displacements and the stress vectors $\sigma \cdot \mathbf{n}$ are continuous across Γ. Then, assuming the existence of a threshold energy g^s, the wear criterion is $G(\sigma) = g - g^3 - g^s < 0$, when the elastic solid does not lose material, and $G(\sigma) = g - g^3 - g^s \geqslant 0$ when it does. The wear rate is assumed to be given a priori by $v = F(g, g^3) = F(\sigma)$ when the wear criterion is verified. In this analysis wear and frictional energies are dissociated: solids can slide without being affected by wear and the loss of material.

This model is completed by the evaluation of average quantities such like stress and strain on the mesoscopic scale (depending on volume fraction of particles, the presence of a lubricant, chemical reactions, ...). Some models studied in DRAGON-LOUISET [11] provide explicitly the evaluation of g^3 in terms of stress and strain of Ω and the rheology of the third-body.

2.2. Conservation of mass

The volume fraction of solid particles φ_s is simply denoted φ. In the steady-state case, the one-dimensional conservation laws of mass, solid and fluid, can be written respectively as:

$$(2.3) \qquad \frac{\partial}{\partial x} \left[e(x)\varphi(x)\rho_s v_x(x) \right] - \alpha \rho_s v(x) = 0 \quad \text{(solid)},$$

$$(2.4) \qquad \frac{\partial}{\partial x} \left[e(x)(1 - \varphi(x))\rho_f v_x(x) \right] = 0 \quad \text{(fluid)},$$

where $v_x(x)$ is the x-coordinate of the mean velocity of solid debris or fluid, equal to $-V/2$; $r_s = \alpha \rho_s v(x)$ is the source term coming from the detachment of debris at rate $v(x)$ feeding the interface through Γ, and α is interpreted as the part of

debris which diffuses toward the third-body. The remaining part $1 - \alpha$ being imprisoned in the asperities is moving out the contact zone, without making any contribution to the third-body rheology, ρ_s and ρ_f are densities of solid and fluid, respectively.

Since up-stream there is no wear, we have $\varphi(a) = 0$. Hence, by assuming α and ρ_s constant, we obtain from (2.3) the relation between the wear rate $v(x)$, the volume fraction $\varphi(x)$ and the thickness $e(x)$

$$(2.5) \qquad \varphi(x) = -\frac{2\alpha}{Ve(x)} \int_a^x v(x)\, dx \quad (x \leqslant a).$$

For either incompressible fluid (ρ_f constant) or negligible variation of $\rho_f(x)$ along the interface, Eq. (2.4) can be reduced to $\dfrac{\partial}{\partial x}[e(x)(1 - \varphi(x))v_x(x)] \simeq 0$. Now, by assuming classical *quasi-linear* one-dimensional Stokes flow inside the third-body, between the fixed wall $v_x(x,0) = 0$ and the sliding one $v_x(x,-e) = -V$, we get the constant mean value $v_x(x) = -V/2$ as mentioned above. This means that two-dimensional fluid flows near the end points of the contact interface are disregarded. We then obtain

$$(2.6) \qquad e(x) = \frac{e_0}{1 - \varphi(x)}.$$

Equations (2.5) and (2.6) are equivalent forms of the mass conservation laws of solid particles and fluid, respectively. Provided that $v(x)$ is known, Eqs. (2.5) and (2.6) yields an integral equation for determining $\varphi(x)$ and then $e(x)$. The volume fraction can be written as:

$$(2.7) \qquad \varphi(x) = \frac{B(x)}{1 + B(x)} \quad \text{with} \quad B(x) = -\frac{2\alpha}{Ve_0} \int_a^x v(x)dx.$$

$B(x)$ is a monotonic function of x, increasing as x decreases. Down-stream, the volume fraction is constant and equal to its maximum value $\varphi_{\max} = \varphi(-a)$. Generally the latter quantity is very small. A good approximation justified by the smallness of $\dfrac{4a\alpha}{Ve_0} \max |v|$ is simply given by (2.5) with approximate $e(x) \simeq e_0$, or $\varphi(x) = B(x)$.

2.3. Rheology of the third-body

We assume that the compressive contact stress is given by the uniaxial law

$$(2.8) \qquad g[\varphi]\sigma_{yy} = c_1(u_y^+ - u_y^-), \quad \text{with} \quad g(\varphi) > 0 \quad \text{and} \quad c_1 = \frac{E}{2(1 - \nu^2)},$$

where c_1 is introduced for later use, E is the Young modulus, ν is the Poisson's ratio, $(+)$ sign for the punch and $(-)$ for the elastic halfplane. An explicit form can be given for the function $g[\varphi]$ by micromechanical considerations. For example, by making use of Reuss's model of aggregate based on strain additivity or stress homogeneity, on the mesoscopic scale, the first law takes the form

$$(2.9) \qquad \left[\frac{\varphi}{K} + \frac{1-\varphi}{K_f} \right] \frac{e_0}{1-\varphi} \sigma_{yy} = u_y^+ - u_y^- ,$$

where K is the stiffness of the detached solid particles, K_f is the stiffness of the fluid. For small φ, it follows from (??) and linearization $g[\varphi] = c_1 e_0 (\varphi/K + 1/K^f)$. The incompressible case $K_f = \infty$ is considered in [11]. The model (2.9) is rigorous when the fluid viscosity is infinitely small. A different model was given by STUPKIEWICZ and MRÓZ [31] for studying abrasive wear due to asperities. The mesoscopic stresses are considered as microscopic stress averages. As a matter of fact, their model of contact stress additionality corresponds to Voigt's model of aggregate and is therefore the *dual* model to (2.9).

The second law describes the viscous behavior of the thin film under shear load. As suggested by experiments, the shear stress can be assumed in the form $(\sigma_{xy}^- = \sigma_{xy})$

$$(2.10) \qquad \sigma_{xy} = m[\varphi] \left(\frac{du_x^+}{dt} - \frac{du_x^-}{dt} \right) \simeq m[\varphi]V,$$

where the relative velocity is approximated by V (the elastic velocity is negligible) and $m[\varphi]$ is a material constant which can be evaluated by micromechanics. As shown later, this approximation justified by in-service conditions of wear allows for the decoupling of equations.

Again, an explicit form of the function $m[\varphi]$ can be provided by classical models of solid dispersion in viscous fluid. The viscosity coefficient of the mixture is given by Einstein's law $\eta[\varphi] = \eta_0(1 + 2.5\varphi)$ (see LANDAU and LIFCHITZ [22]), so that $\sigma_{xy} = \eta[\varphi] \left(\dfrac{du_x^+}{dt} - \dfrac{du_x^-}{dt} \right) /e(x)$ and using (2.6) and linearization, it follows $m[\varphi] = \eta_0(1 + 1.5\varphi)/e_0$.

Finally, the third-body model is a medium having a *hybrid* behavior of an elastic solid in compression and a viscous fluid in shear or a plastic solid with constant threshold (φ and V held fixed). It looks like a ball bearing, capable of transmitting a compressive force, but having some resistance in sliding. The volume fraction φ appears explicitly in the wear equations, making it possible to have a quantitative and predictive analysis of wear, for a given mechanical system. The macroscopic interface between a rigid punch and an elastic solid is characterized by properties summarized in the following box.

INTERFACE MODEL ON THE MACROSCOPIC SCALE

- Wear criterion $G(\sigma) \geqslant 0$ and wear rate $v = H(G(\sigma))F(\sigma)$, with H the Heaviside function

$$\begin{cases} \text{if} \;\; G(\sigma) < 0, & \text{no wear and:} \; v = 0, \\ \text{if} \;\; G(\sigma) \geqslant 0, & \text{wear rate:} \; v = F(\sigma) > 0. \end{cases}$$

- Internal state variable $\varphi(x)$ and mass conservation laws

$$\tfrac{\partial}{\partial x}\left[e(x)\varphi(x)v_x(x)\right] - \alpha v(x) = 0 \qquad \text{(solid)},$$

$$\tfrac{\partial}{\partial x}\left[e(x)(1 - \varphi(x))v_x(x)\right] = 0 \qquad \text{(fluid)}.$$

- Constitutive laws

$$\sigma_{xy} = m[\varphi]V, \qquad \text{with } m[\varphi] > 0,$$

$$g[\varphi]\sigma_{yy} = c_1(u_y^+ - u_y^-), \quad \text{with } g[\varphi] > 0 \text{ and } c_1 = \frac{E}{2(1 - \nu^2)}.$$

3. Statement of the problem

We consider a rigid circular punch of radius R, defined by the equation $y = f(x) = (x - x_0)^2/(2R)$ sliding on an elastic halfplane, (Fig. 2). In the previous section, we gave the description of the third-body on the mesoscopic scale, $-a \leqslant x \leqslant a$, $-e(x) \leqslant y \leqslant 0$. Here the interface is considered on the macroscopic scale and is defined by $y = 0$. The elastic body Ω is the halfplane $\{x, y \leqslant 0\}$. The vertical displacement of the upper third-body surface (Γ_1, $y = 0$) is

(3.1) $$u_y^+(x) = \delta + f(x) \quad (\delta < 0),$$

where δ is the imposed position of the punch and $f(x)$ its profile. The vertical displacement of the lower surface (Γ_2, $y = -e(x)$) is u_y^- equal to the displacement of the halfplane boundary. The strain ε_{yy} of the interface medium is $\varepsilon_{yy} = (u_y^+ - u_y^-)/e$.

For the present, neither the compressive load nor the contact zone $-a \leqslant x \leqslant a$ have been specified. The punch position x_0 is yet unknown. The following

assumption is only made $-a \leqslant x_0 \leqslant a$. The integral equations for boundary fields in plane strain are given by GALIN [13] (*pv* means "principal value") ($u_x^- = u_x$, $u_y^- = u_y$)

$$(3.2) \qquad c_1\, u_x'(x) = c_2\, \sigma_{yy}(x) + pv\, \frac{1}{\pi} \int\limits_{-a}^{a} \sigma_{xy}(t) \frac{dt}{t-x},$$

$$(3.3) \qquad c_1\, u_y'(x) = -c_2\, \sigma_{xy}(x) + pv\, \frac{1}{\pi} \int\limits_{-a}^{a} \sigma_{yy}(t) \frac{dt}{t-x},$$

where $c_1 = E/[2(1-\nu^2)]$, $c_2 = (1-2\nu)/[2(1-\nu)]$ and the prime (') means differentiation. Equations (3.2) and (3.3) establish the relations between tangential gradients of displacement, normal stress σ_{yy} and shear σ_{xy} (cf. also BUI [7] for the reciprocal relations). Setting $\varepsilon_{xx}(x) = u_x'(x)$ in (3.2), i.e. the longitudinal strain parallel to the boundary, substituting (2.8), (2.10), (3.1) in (3.3), we obtain the set of equations:

$$(3.4) \qquad c_1\, \varepsilon_{xx}(x) = c_2\, \sigma_{yy}(x) + pv\, \frac{1}{\pi} \int\limits_{-a}^{a} \sigma_{xy}(t) \frac{dt}{t-x},$$

$$(3.5) \qquad c_1\, f'(x) - \frac{d}{dx}\left[g[\varphi](x)\sigma_{yy}(x)\right] = -c_2\, m[\varphi](x)V + pv\, \frac{1}{\pi} \int\limits_{-a}^{a} \sigma_{yy}(t) \frac{dt}{t-x},$$

$$(3.6) \qquad \sigma_{yy}(-a) = \sigma_{yy}(a) = 0,$$

with additional equations given previously as

$$(3.7) \qquad \sigma_{xy} = m[\varphi]V,$$

$$(3.8) \qquad \varphi(x) = -\frac{2\alpha}{V e_0} \int\limits_{a}^{x} v(x)\, dx,$$

$$(3.9) \qquad v = H(G)F(\sigma).$$

The boundary conditions (3.6) come from the continuity assumption of stresses (no normal load at the fluid interface outside the punch area). In the energy release rate approach, G is expressed in terms of the elastic strain energy densities on both sides of the phase changes line. It depends on σ_{xy}, σ_{yy}, ε_{xx} of the sound

material and on the stresses σ_{xx}, σ_{yy} of the damaged zone adjacent to the interface line. Its expression depends on particular models considered on the mesoscopic scale. Once the expression of $G(\sigma_{xy}, \sigma_{yy}, \varepsilon_{xx}, \varphi)$ in terms of its arguments is explicitly known, it can be evaluated by quantities given by Eq. (3.4) to (3.9).

Let us specify now the data for the whole mechanical system for solving the nonlinear system of Eqs. (3.4) – (3.9). Instead of giving the total compressive force, we *specify the contact zone* $-a \leqslant x \leqslant a$ and then determine the corresponding compression load given by

$$(3.10) \qquad P(a) = -\int_{-a}^{a} \sigma_{yy}(t)\, dt,$$

the shear force

$$(3.11) \qquad T(a) = -\int_{-a}^{a} \sigma_{xy}(t)\, dt,$$

and the punch position x_0.

It is worth noticing that the key step of the iterative computation is to determine the normal stress σ_{yy} and that for each iteration, until convergence, only one equation, namely (3.5), has to be solved. Other quantities of interest for the wear criterion (3.9) are derived explicitly from φ and σ_{yy} by computations of integrals. For detailed analysis of the nonlinear algorithm, we refer to Dragon-Louiset [11].

Here we shall focus on this linear integro-differential equation for the normal stress (3.5), which is referred to as the W-equation.

4. The W-equation

4.1. The step (0) algorithm

For a given volume fraction $\varphi(x)$ and a given contact zone $-a \leqslant x \leqslant a$, we consider the function $\sigma_{yy}^{(0)}$ satisfying the following equations, with $m[\varphi]V = (\eta_0/e_0)V$

$$(4.1) \qquad pv\, \frac{1}{\pi}\int_{-a}^{a} \sigma_{yy}^{(0)}(t)\frac{dt}{t-x} = c_1 f'(x) + c_2 \frac{\eta_0}{e_0} V,$$

$$(4.2) \qquad \sigma_{yy}^{(0)}(\pm a) = 0.$$

With a Hölderian function in the right-hand side, (4.1) is a classical Hilbert's equation, treated in Muskhelishvili [25]. The corresponding homogeneous

equation possesses a non-zero solution of the type $C(a^2 - x^2)^{-\frac{1}{2}}$. Hence, to obtain a bounded solution (in fact vanishing at $x = \pm a$), there is a consistency condition to be satisfied

$$(4.3) \qquad \int_{-a}^{a} \left[c_1 f'(t) + c_2 \frac{\eta_0}{e_0} V \right] (a^2 - t^2)^{-\frac{1}{2}} \, dt = 0,$$

where $f'(x) = (x - x_0)/R$. From the consistency condition (4.3) we obtain $x_0 = c_2 \eta_0 V R/(c_1 e_0)$, which is independent of a, hence a good candidate for starting the step (0) algorithm, provided that $x_0 \leqslant a$. The last condition can be satisfied by choosing appropriately the velocity V. The solution of (4.1) is then given by MUSKHELISHVILI [25],

$$(4.4) \qquad \sigma_{yy}^{(0)}(x) = -(a^2 - x^2)^{\frac{1}{2}} pv \frac{1}{\pi} \int_{-a}^{a} \left[c_1 f'(t) + c_2 \eta_0 \frac{V}{e_0} \right] (a^2 - t^2)^{-\frac{1}{2}} \frac{dt}{t - x},$$

or $\sigma_{yy}^{(0)}(x) = -(c_1/R)(a^2 - x^2)^{\frac{1}{2}}$. This is a Hertzian distribution of load with the corresponding linear force given by

$$(4.5) \qquad P_0(a) = -\int_{-a}^{a} \sigma_{yy}^{(0)}(t) \, dt = \frac{\pi c_1 a^2}{2R}.$$

If the first term of (3.5), $\dfrac{d}{dx} [g[\varphi](x) \sigma_{yy}(x)]$ is small in comparison with the remaining ones, the zero-order solution $\sigma_{yy}^{(0)}(x)$ provides a good approximation of the actual solution, except near the end points $x = \pm a$ where the derivative $\dfrac{d}{dx} \left[\sigma_{yy}^{(0)}(x) \right]$ is singular (this is mathematically a singular perturbation problem not addressed here). In what follows, we outline the method of solving (3.5) using Chebyshev's series expansion suggested by the analogy with Prandtl's equation.

4.2. Chebyshev's series solution

Having determined the approximate center position x_0, it is advantageous to solve the W-equation for the actual stress $\sigma_{yy}(x)$, written in the form

$$(4.6) \qquad \frac{d}{dx} \left(g[\varphi](x) \sigma_{yy}(x) \right) + pv \frac{1}{\pi} \int_{-a}^{a} \sigma_{yy}(t) \frac{dt}{t - x}$$

$$= \frac{c_1}{R}(x - x_0) + c_2 m[\varphi](x) V \equiv b(x).$$

We set the following change of variables: $t = -a\cos(\theta)$, $x = -a\cos(\omega)$, with $\theta \in [0, \pi]$. The function $\sigma_{yy}(\omega)$ satisfying (4.6) is expanded in truncated Fourier sine series, ANDERSON [1]:

$$(4.7) \qquad \sigma_{yy}(\theta) = \sum_{n=1}^{N} A_n \sin(n\theta), \quad 0 \leqslant \theta \leqslant \pi.$$

Using the Chebyshev identity of the first kind

$$\int_0^\pi \frac{\sin(n\theta)\sin(\theta)}{\cos(\theta) - \cos(\omega)} \, d\theta = -\pi\cos(n\omega),$$

and by setting $b(x) = b(x = -\cos(\omega))$, we obtain

$$(4.8) \qquad \sum_{n=1}^{N} aA_n \cos(n\omega)\sin(\omega) + \sum_{n=1}^{N} A_n \frac{d\left[g(\omega)\sin(n\omega)\right]}{d\omega}$$

$$-a\sin(\omega)b(\omega) = 0, \quad 0 < \omega < \pi.$$

Generally the linear system (4.8) is solved by the collocation method. This method however is unsatisfactory in the choice somewhat arbitrary of the collocation points, $0 < \omega_k < \pi$ $(k = 1, .., N)$. It is of interest to consider instead the Galerkin method. In order to use the Fourier 2π-periodic functions, the functions appearing in the left-hand side of (4.8), defined only in the interval $0 < \omega < \pi$, must be extended to $-\pi < \omega < \pi$ in such a way that the extended functions are *even*. For instance, the first term $\cos(n\omega)\sin(\omega)$ of (4.8) is odd and has to be extended to $-\pi < \omega < \pi$ by the function $sgn(\omega)\cos(n\omega)\sin(\omega)$. Only odd functions have to be extended in this way, by multiplication with the sign-function $sgn(\omega)$. Then, by using the Fourier cosine functions $\cos(k\omega)$, we obtain the set of equations for A_n

$$(4.9) \qquad \sum_{n=1}^{N}\left(A_n \int_0^\pi \left\{a\cos(n\omega)\sin(\omega) + \frac{d}{d\omega}[g(\omega)\sin(n\omega)]\right\}\cos(k\omega)\, d\omega\right)$$

$$-\int_0^\pi a\sin(\omega)b(\omega)\cos(k\omega)\, d\omega = 0, \qquad k = 1, .., N.$$

Remark that the solution (4.7) contains the factor $\sin(\theta)$, hence the normal stress $\sigma_{yy}(x)$ has the square root behavior $\sigma_{yy}(x) \simeq (a^2 - x^2)^{\frac{1}{2}}$ as $|x| \to a$. It can be shown that the zero-order solution $\sigma_{yy}^{(0)}(x)$ is an approximation of the first term

of the series (4.7). The method outlined here is an adaptation of the one given in Dragon-Louiset [11].

4.3. Uniqueness considerations

Having determined a solution satisfying (3.6) and the corresponding total load $P(a)$ by (3.10), we are concerned with the uniqueness of the solution of the W-equation (4.6). It is necessary to specify the space of functions to which the solution belongs. We need the class of Hölderian functions of Muskhelishvili to give a sense to Cauchy integrals, satisfying $\sigma_{yy}(\pm a) = 0$ and regular in $-a \leqslant x \leqslant a$. Suppose that there are two solutions σ^1, σ^2, corresponding to the same total load $P(a)$. Then, $\Sigma = \sigma^1 - \sigma^2$ satisfies the homogeneous W-equation with $b(x) = 0$, $\Sigma(\pm a) = 0$ and $\int_{-a}^{a} \Sigma(t)\, dt = 0$. As shown below, the uniqueness theorem states that $\Sigma(x) \equiv 0$, provided that $g[\varphi] > 0$.

4.4. Proof of uniqueness

Let us prove the theorem by considering the complex function $F(z)$ defined in the upper complex plane $z = x + iy$, $(y \geqslant 0)$:

$$(4.10) \qquad F(z) = \frac{1}{2i\pi} \int_{-a}^{a} i\Sigma(t) \ln(z-t)\, dt = \Phi(x,y) + i\Psi(x,y),$$

$$x, y \text{ in } \Omega^+ = \{x, y \geqslant 0\},$$

$$(4.11) \qquad F'(z) = -\frac{1}{2i\pi} \int_{-a}^{a} i\Sigma(t)\frac{dt}{t-z} = v_x(x,y) - i\, v_y(x,y),$$

$$(4.12) \qquad \Phi(x,0^+) = \frac{1}{2\pi} \int_{-a}^{a} \Sigma(t) \ln|x-t|\, dt,$$

$$(4.13) \qquad v_y(x,0^+) = \frac{\partial \Phi}{\partial y}(x,0^+) = \frac{1}{2}\Sigma(x), \quad \text{for } 0 \leqslant |x| \leqslant a,$$

$$(4.14) \qquad v_y(x,0^+) = \frac{\partial \Phi}{\partial y}(x,0^+) = 0, \quad \text{for } |x| \geqslant a.$$

The density $\Sigma(x)$ is assumed to satisfy the following equation with some $H(x)$:

$$(4.15) \qquad g[\varphi](x)\Sigma(x) - \frac{1}{\pi} \int_{-a}^{a} \Sigma(t) \ln|t-x|\, dt = H(x), \quad 0 \leqslant |x| \leqslant a.$$

Differentiating (4.15), we find that $\Sigma(x)$ satisfies the W-equation

$$(4.16) \qquad \frac{d}{dx}\left[g[\varphi](x)\Sigma(x)\right] + pv\,\frac{1}{\pi}\int_{-a}^{a}\Sigma(t)\frac{dt}{t-x} = H'(x).$$

Now Eq. (4.15) can be written equivalently as

$$(4.17) \qquad 2g[\varphi](x)\frac{\partial\Phi}{\partial y}(x,0^+) - 2\Phi(x,0^+) = H(x), \quad \text{on } 0 \leqslant |x| \leqslant a.$$

To investigate the uniqueness, we consider the homogeneous Eq. (4.16) for $\Sigma(x)$, with $H' = 0$, or the homogeneous boundary condition (4.17), with $H = 0$, for the harmonic function $\Phi(x,y)$ in Ω^+

$$(4.18) \qquad g[\varphi](x)\frac{\partial\Phi}{\partial y}(x,0^+) - \Phi(x,0^+) = 0, \quad 0 \leqslant |x| \leqslant a.$$

The function $\Phi(x,y)$ is the real part of an holomorphic function $F(z)$, regular at infinity, because in view of $\int_{-a}^{a}\Sigma(t)\,dt = 0$, the logarithmic part of $F(z)$ vanishes at infinity. Hence $|F(z)| \simeq O(1/|z|)$ and $|F'(z)| \simeq O(1/|z^2|)$. Rewriting (4.18) with the outward unit normal to Ω^+, $\partial/\partial y = -\partial/\partial n$, we obtain

$$(4.19) \qquad -\frac{\partial\Phi}{\partial n} - \frac{\Phi}{g[\varphi](x)} = 0, \quad 0 \leqslant |x| \leqslant a, \quad y = 0^+.$$

Integrating $\Phi\partial\Phi/\partial n$ on the whole boundary of the upper-half plane $\partial\Omega^+$, noticing that $\Phi\partial\Phi/\partial n = 0$ for $|x| \geqslant a$, $\Phi\partial\Phi/\partial n \simeq O(1/|z^3|)$ at infinity, we obtain on the one hand, since $g > 0$

$$\int_{\partial\Omega^+}\Phi\frac{\partial\Phi}{\partial n}\,ds = \int_{-a}^{a}\Phi\frac{\partial\Phi}{\partial n}\,dx = -\int_{-a}^{a}\frac{\Phi^2}{g}\,dx \leqslant 0,$$

and on the other hand, since Φ is harmonic :

$$\int_{\partial\Omega^+}\Phi\frac{\partial\Phi}{\partial n}\,ds = \int_{\Omega^+}|\nabla\Phi|^2\,d\Omega \geqslant 0.$$

We conclude that $\Phi = 0$ is the unique solution for the homogeneous boundary condition (4.17) and that $\Sigma = 0$.

5. Prandtl's lifting equation in aerodynamics

5.1. Prandtl's equation

Let us recall the well-known Prandtl equation for calculating the circulation distribution of vortex $\Gamma(x)$ along a finite wing $[-a, a]$:

$$(5.1) \qquad \frac{1}{\pi}\frac{\Gamma(x)}{R(x)} - pv\,\frac{1}{\pi}\int_{-a}^{a}\frac{d\Gamma(y)}{y - x} = -4V_\infty J(x)$$

where $R(x) > 0$ is the radius of the Kutta-Joukovski circle, corresponding to the section profile of the wing at x, V_∞ is the up-stream velocity and $J(x)$ is the local geometric angle of attack of the wing. Here the derivative of $\Gamma(x)$ appears inside the Cauchy integral. Equation (5.1) is an integro-differential equation, in which the unknown is $\Gamma(x)$; all the other quantities are known, ANDERSON [1], MANDEL [23]. It is well-known that the solution of is (5.1) unique.

It is worth reconsidering the method actually used for solving the Prandtl's equation. Putting $y = -a\,\cos(\theta)$, $x = -a\,\cos(\omega)$, Eq. (5.1) is solved for $\Gamma(y)$ by truncated Fourier sine series

$$(5.2) \qquad \Gamma(\theta) = \sum_{n=1}^{N} B_n \sin(n\theta), \quad 0 \leqslant \theta \leqslant \pi.$$

The pv-integral at the station ω can be written in a simple form. Using Chebyshev's formula of the second kind

$$(5.3) \qquad \int_{0}^{\pi}\frac{\cos(n\theta)}{\cos(\theta) - \cos(\omega)}\,d\theta \equiv \pi\frac{\sin(n\omega)}{\sin(\omega)},$$

we obtain

$$(5.4) \qquad \frac{1}{\pi R(\omega)}\sum_{n=1}^{N} B_n \sin(n\omega) + \sum_{n=1}^{N}\frac{n}{a}B_n\frac{\sin(n\omega)}{\sin(\omega)} = -4V_\infty J(\omega), \quad 0 < \omega < \pi.$$

Choosing N different stations $\omega_1, \omega_2, \ldots, \omega_N$ for a collocation method, Eq. (5.4) provides a linear system of N independent algebraic equations with N unknowns, $B_1, B_2, \ldots, B_N$

$$(5.5) \qquad \frac{1}{\pi R(\omega_k)}\sum_{n=1}^{N} B_n \sin(n\omega_k) + \sum_{n=1}^{N}\frac{n}{a}B_n\frac{\sin(n\omega_k)}{\sin(\omega_k)}$$

$$= -4V_\infty J(\omega_k), \quad (k = 1, .., N).$$

In this fashion, the system (5.5) provides the actual solution of Prandtl's equation as used in aerodynamics. Finally, we remark the analogy between both methods used in Secs. 4.2 and 5.1, respectively. As a matter of fact, there is a profound relationship between the W-equation and Prandtl's lifting equation.

5.2. Malavard's analogical method

Let us recall first how Prandtl's Eq. (5.1) can be solved *experimentally* by analogous Malavard's method, MANDEL [23]. An electric conducting medium occupies the halfplane $y \geqslant 0$. The potential $\Phi(x,0)$ is the trace on $y = 0$ of a harmonic function $\Phi(x,y)$, regular in the halfplane $y > 0$. The wing position is divided into N equal segments, centered at x_k, each of them being connected to a resistance $\rho(x_k)$, while the other end of the resistance is subjected to the potential $E(x_k)$. For simplicity, we shall omit the indices k of x_k. In order to choose these quantities, we introduce the complex potential $F(z)$, $z = x + i\,y$ and consider the values of potential and velocity $v_y = \partial\Phi/\partial y$ on $y = 0$, $x \leqslant a$

$$F(z) = \frac{1}{2i\pi} \int\limits_{-a}^{a} \ln(z - t)\, d\Gamma(t),$$

$$\Phi(x, 0^+) = -\frac{1}{2}\Gamma(x), \quad \text{for } |x| \leqslant a,$$

$$\Phi(x, 0^+) = 0, \quad \text{for } |x| \geqslant a,$$

$$v_y(x) = \frac{1}{2\pi}\, pv \int\limits_{-a}^{a} \frac{d\Gamma(t)}{t - x}.$$

The electric potential $\Phi = 0$ is applied to the segments $|x| \geqslant a$. We set the potential $E(x) = 2\pi R V_\infty J(x)$ to the resistance $\rho(x) = \pi R(x) > 0$. Then we measure the electric current $c = v_y(x)$ provided by Ohm's law $E - \Phi = \rho c$, corresponding to (5.1)

$$(5.6) \qquad 2E - 2\Phi = \rho 2c \quad \Leftrightarrow \quad 4V_\infty J(x)\pi R(x) + \Gamma(x)$$

$$= \pi R(x)\frac{1}{\pi}\, \mathbf{pv} \int\limits_{-a}^{a} \frac{d\Gamma(t)}{t - x}.$$

The measurement of the potential Φ at the other end of the resistance provides the distribution of line vortex with density $\Gamma'(x)$.

5.3. Experimental setup for solving the W-equation

By integrating the W-equation for the stress $S(x)$

$$(5.7) \qquad \frac{d}{dx}[g(x)S(x)] + pv\frac{1}{\pi}\int_{-a}^{a} S(t)\frac{dt}{t-x} = h(x),$$

with respect to x in the interval $[a, x]$, taking account of $S(a) = 0$, we obtain

$$(5.8) \qquad g(x)S(x) - \frac{1}{\pi}\int_{-a}^{a} S(t)\ln|t-x|\,dt = -\frac{1}{\pi}\int_{-a}^{a} S(t)\ln|t-a|\,dt + H(x),$$

where $H(x) = \int_{a}^{x} h(x)\,dx$. We can write

$$(5.9) \qquad g(x)S(x) - \frac{1}{\pi}\int_{-a}^{a} S(t)\ln|t-x|\,dt = H(x) + C$$

$$(5.10) \qquad C = -\frac{1}{\pi}\int_{-a}^{a} S(t)\ln|t-a|\,dt.$$

The constant C is not known beforehand, since it depends on the solution $S(t)$. Suppose that for a given C, there is a solution $S(x; C)$. Equation (5.10) expresses the implicit condition for determining C

$$(5.11) \qquad C = -\frac{1}{\pi}\int_{-a}^{a} S(t; C)\ln|t-a|\,dt.$$

Let us show how (5.11) can be exploited experimentally.

We introduce the complex potential $F(z)$ of a distribution of line source (or sink) with density $S(t)$. The corresponding values of potential and velocity are

$$F(z) = \frac{1}{2i\pi}\int_{-a}^{a} iS(t; C)\ln(z-t)\,dt \equiv \Phi(x,y) + i\,\Psi(x,y),$$

$$F'(z) = \frac{1}{2i\pi}\int_{-a}^{a} iS(t; C)\frac{dt}{z-t} = -\frac{1}{2i\pi}\int_{-a}^{a} iS(t; C)\frac{dt}{t-z} = v_x(x,y) - i\,v_y(x,y),$$

$$\Phi(x, 0^+; C) = \frac{1}{2\pi} \int_{-a}^{a} S(t; C) \ln|x - t| \, dt,$$

$$v_y(x, 0^+) = \frac{1}{2} S(x), \quad y = 0, \ x \leqslant a,$$

$$v_y(x, 0^+) = 0, \quad y = 0, \ x \geqslant a.$$

Here, the infinite lines $y = 0$, $|x| \geqslant a$ are stream lines $v_y = \partial\Phi/\partial y = 0$. On the line $y = 0$, $-a \leqslant x \leqslant a$, (5.9) can be rewritten as

$$\Phi(x) - E(x) = \rho(x)c(x) \quad \Leftrightarrow \quad g(x)S(x) - \frac{1}{\pi} \int_{-a}^{a} S(t) \ln|t - x| \, dt$$

$$= H(x) + C,$$

with the electric potential $E(x) = -(H(x) + C)/2$, applied to the resistance $\rho(x) = g(x) > 0$, the current $c(x) = v_y(x) = S(x)/2$ and the potential $\Phi(x) = \frac{1}{2\pi} \int_{-a}^{a} S(t) \ln|x - t| \, dt$. Far from the resistance, the potential is set to zero. The compatibility condition (5.11) is written as

$$-\frac{C}{2} = \Phi(a, 0^+; C).$$

Hence to satisfy the above condition, experimentally, one adjusts the potential $E(x)$ by varying C in such a manner that the measured potential $\Phi(a, 0^+; C)$ at $x = a$ is equal to $-C/2$. Since $H(a) = 0$, we see that $E = -C/2 = \Phi(a, 0^+; C)$ or equivalently $c(a) = 0$. Hence the adjustment of C is done in such a way that there is *no current* in the resistance at $x = a$.

5.4. Relation between the Prandtl's equation and the W-equation

As shown in the above sections, the Prandtl's equation and the W-equation are related together through complex potentials of line vortex of density $\Gamma'(x)$ and line source or sink of density $S(t)$, respectively. This does not mean that Eqs. (5.1) and (5.7) are adjoint equations, in the strictly mathematical sense. As a matter of fact, there is only a weak relation of the adjoint type.

Consider the set of functions

$$B = \left\{ \sin\left(\frac{\pi x}{2a}\right), \cos\left(\frac{2\pi x}{2a}\right), \sin\left(\frac{3\pi x}{2a}\right), \cos\left(\frac{4\pi x}{2a}\right), \ldots \right\}.$$

If $\Sigma_k(x)$ is the kth element, we get $\Sigma_k''(x) = -(k^2\pi^2/(4a^2))\Sigma_k(x)$ and $\Sigma_k'(\pm a) = 0$. Denote the Prandtl operator by $P_k[\Gamma]$

$$P_k[\Gamma] := \frac{4a^2}{k^2\pi^3}\frac{\Gamma(x)}{R(x)} - \frac{1}{\pi}\, pv \int_{-a}^{a} \frac{d\Gamma(t)}{t-x}$$

where the first term in the left-hand side of (5.1) is divided by $\dfrac{k^2\pi^2}{4a^2}$. Similarly, denote the W-operator by $W[S]$

$$W[S] := \frac{d}{dx}\,(g(x)S(x)) + pv\,\frac{1}{\pi}\int_{-a}^{a} S(t)\frac{dt}{t-x}.$$

The kernels of Cauchy integrals of $P_k[\Gamma]$, $W[S]$ are *adjoint* and there is a permutation of functions and derivatives. Then we obtain an equivalence of the adjoint type

$$\int_{-a}^{a} \Sigma_k(x)W[S](x)\,dx = \int_{-a}^{a} \Sigma_k(x)h(x)\,dx$$

$$\Leftrightarrow \quad \int_{-a}^{a} \Sigma_k'(x)P_k[S](x)\,dx = -\int_{-a}^{a} \Sigma_k'(x)h'(x)\,dx.$$

However, there is an incomplete equivalence in weak forms, because: 1. we make use of only one particular function $\Sigma_k(x)$; 2. the set B is not complete.

6. Conclusion

The mechanics of sliding mild wear contact between a rigid punch and an elastic halfplane considered in this work is characterized by the following features: (1) any wear criterion and wear rate can be used; (2) the constitutive behavior of the thin aggregate film composed of solid debris and a lubricant fluid, on the mesoscopic scale, can have any general form; it is characterized by an elastic law in compression and a fluid viscosity in shear; (3) the interface on the macroscopic scale is characterized by the elastic law in compression and a plastic law in shear; (4) the mechanical system is governed by the volume fraction of solid debris, which satisfies a nonlinear system of equations.

It was shown that the key step of the iterative algorithm for solving the latter nonlinear system is the linear integro-differential equation for the normal contact

stress. There is a profound relation between this equation and Prandtl's lifting equation found in aerodynamics: both equations can be computed using similar Chebyshev's series of the first and the second kind respectively, and can be solved experimentally by similar electrical setups. These equations are related to real and imaginary components of some complex potential respectively, and to weakly adjoint integro-differential operators.

By describing the formation of debris, using for instance the wear energy release rate criterion, and their evolution via the balance equation of mass, by averaging the behavior law of the aggregate film, via a micromechanical model, we were led to a predictive model of mild wear. There are still remaining questions about an effective solution of the nonlinear system of equations. Such questions, particularly mathematical aspects on the *convergence* of the nonlinear algorithm, will be addressed in a forthcoming paper, DRAGON-LOUISET [11].

Acknowledgements

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