Arch. Mech., **52**, 4–5, pp. 523–545, Warszawa, 2000

Extremum and saddle-point theorems for elastic solids with dissipative displacement discontinuities

Dedicated to Professor Zenon Mróz
on the occasion of his 70^{th} birthday

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In a number of engineering situations concerning structures made of quasi-brittle, concrete-like materials, all nonlinearities can be reasonably confined to a locus of possible displacement discontinuities. This locus has a lower dimensionality (by one) with respect to the problem domain; it encompasses joints, cracks, fracture process zones (described by cohesive crack models) and their possible propagation paths. Linear elasticity is assumed everywhere else for overall analysis purposes. With reference to a very broad class of interface models, i.e. of (holonomic or nonholonomic, inviscid or time-dependent) relationships between displacement jumps and tractions across that locus, the (possibly multiple, if any) solutions of the initial-boundary-value problem of structural analysis are shown herein to be characterized by duality pairs of extremum and min-max properties.

1. Introduction

1.1. A widely accepted idealized interpretation of fracture processes in quasi-brittle solids and structures (e.g. concrete dams) rests on the "cohesive crack" concept. This model is characterized by the following features: the (two or three-dimensional) open domain Ω where the analysis problem is defined, contains a discontinuity locus (one or two-dimensional, respectively), say Γ_d, across which displacement discontinuities $\mathbf{w}$ may occur; along Γ_d tractions $\mathbf{p}$ are related to relative displacements $\mathbf{w}$ by an "interface constitutive law" which exhibits a softening (unstable) behaviour up to vanishing of the strength; outside of Γ_d the material behaviour is assumed to be linear elastic; deformations are "small" in the sense that equilibrium relations are not influenced by configuration changes, and kinematic compatibility equations are linear. At a certain stage of the structural response to a given loading history, the locus Γ_d generally encompasses three parts: a portion ("process zone") where the two faces interact by tractions **p**; a part formed by actual cracks, where there is no interaction; and a portion of virgin material, where no displacement discontinuities has arisen yet.

For the fracture analysis problem based on the above idealizations, the present paper is intended to provide two general variational formulations with the following novel features.

First, a functional of kinematically admissible displacement fields is constructed *by time integrations over the time interval* T of interest, and by space integrations over the domain Ω, over its free and constrained boundary (Γ_p and Γ_u, respectively) and over the locus Γ_d of possible discontinuities. It is proven that the solution (if any) is characterized by the absolute minimum (with a value which can be determined *a priori*) of the above functional and of suitable variants of it.

Second, a functional of statically admissible stress fields is generated over the time interval T and over Ω, Γ_p, Γ_u and Γ_d, and its absolute minimum (at a known value) is proven to characterize the solutions of the boundary-initial value problem.

The two minimum principles are shown to generate two further computationally more attractive saddle-point theorems, and to reduce to the potential and complementary energy principles of elasticity when the discontinuity locus vanishes.

1.2. The present study was motivated by a research project on structural problems in dam engineering. The safety assessment of large concrete dams nowadays often rests on overall three-dimensional analyses in which all nonlinearities can be confined to surfaces where displacement discontinuities may occur (or can be realistically assumed to possibly occur), usually accompanied by energy dissipation. A variety of localised dissipative phenomena need to be allowed for: frictional contact and asperity smoothing on artificial joints and existing cracks; quasi-brittle fracture processes (primarily on concrete-foundation interface). Several nonholonomic path-dependent interface models have been proposed for the computer simulation of the response to loads of dams and many other engineering structures: rigid-plastic, elasto-plastic; plastic damage; viscoplastic; etc., softening and non-associativity being recurrent features together with their possible computationally challenging consequences such as overall instabilities and path branching.

The often reasonable hypothesis of linear behaviour outside the locus of possible dissipative discontinuities makes computationally attractive and competitive a variety of space discretizations: finite element, boundary element, and meshless methods. Some representative contributions to interface modelling and to limit-state analysis of dams and of similar structures can be found, e.g., in [5, 15, 24, 30, 31], and in [4, 16, 28], respectively. The abundant literature on these topics is surveyed in recent treatises such as [3, 22].

This paper aims at providing a unifying theoretical framework for the above mentioned varieties of interface models and analysis methods, and at contributing to bridge the present gap between structural mechanics and a mathematical research stream on variational principles for initial-boundary-value (i.b.v.) problems.

Such research trend appears so far to be rather separate from the one, fostered by the developments in engineering plasticity, on nonlinear boundary value problems in rates or in finite steps: earlier within the validity range of Drucker's postulate of material stability, (see e.g. [8, 17, 23, 25]), and later outside of it (see e.g. 18, 26]).

The origin of the methodological approach of concern herein can be traced in the adjoint operator method proposed in the fifties for the symmetrization of any non-symmetric linear operator [29], and later extended to classes of nonlinear operators ([20, 37]) and integral operators [36]. This method implies additional unknowns without physical meaning and generally does not lead to extremum characterizations of solutions. Variational formulations of i.b.v. linear problems have been established by this approach, and also by another approach (proposed by Gurtin [21] in 1964) which involves convolution and is deprived of the above disadvantages. The latter approach was further developed for a variety of i.b.v. linear problems in [33, 34, 35, 38] and in [12, 13, 27] for boundary integral formulations of viscoelasticity, dynamics and heat conduction.

A general methodology for variational formulation of any, linear or nonlinear, problem was proposed by E. TONTI [39], and developed for special categories of operators and mechanical situations, such as linear convection-diffusion [32], structural stability [1], nondifferentiable operators [2], quasi-static plasticity [9, 11], elastoplastic dynamics [14].

The present results can be regarded as a further engineering-oriented application of Tonti's approach to variational formulations of i.b.v. problems in the presence of (dissipative, nonholonomic, possibly time-dependent) constitutive models.

2. Problem formulation

The solid or structure referred to herein occupies a region Ω with a boundary Γ. The boundary is supposed to be *smooth*, i.e. the normal direction is uniquely defined everywhere (a formally convenient restriction, easily removed whenever necessary). Symbols Γ_u and Γ_p will denote the parts of Γ where displacements and surface tractions are imposed, respectively, with $\Gamma = \Gamma_u \cup \Gamma_p$ and $\Gamma_u \cap \Gamma_p = \emptyset$. In an orthogonal Cartesian reference system, $\mathbf{x} = \{x_i, i = 1, 2, 3\}$ is the position vector of a material point in Ω.

The domain Ω is assumed to contain an *a priori* known discontinuity locus, say Γ_d, across which displacement jumps $\mathbf{w}$ may occur. The locus Γ_d is assumed: (i) to be smooth in the above sense; (ii) to have no intersections with the constrained boundary Γ_u. The latter hypothesis makes simpler some developments in Sec. 4.1 and can be relaxed to the requirement that $\mathbf{w}$ on Γ_d and displacement $\bar{\mathbf{u}}$ on Γ_u are independent over a possible portion of Γ_d which is also part of Γ_u.

The external actions (barred symbols) assigned at any instant t $(0 \le t \le \bar{t})$ over a given time interval $T = [0, \bar{t}\,]$, are: volume forces $\bar{b}_i(\mathbf{x}; t)$ in Ω; imposed (say thermal) strains $\bar{\theta}_{ij}(\mathbf{x}; t)$ in Ω; displacements $\bar{u}_i(\mathbf{x}; t)$ on Γ_u; tractions $\bar{p}_i(\mathbf{x}; t)$ on Γ_p. Inertia forces are regarded as negligible. The material behaviour may be history- and time-dependent (e.g. viscoplastic) or, as a special case, inviscid (then the variable t is to be considered as an event-ordering parameter along the "quasi-static" evolution of the system).

For convenience, but conceptually without any loss of generality, two-dimensional plane-stress situations are referred to wherever it is desirable to make the problem dimensionally explicit (namely: $i, j = 1, 2$).

Under the assumption of small strains and displacements (linear kinematics), the equilibrium and compatibility equations, everywhere except on Γ_d, read (the index summation convention is adopted):

$$\sigma_{ij,j} + \bar{b}_i = 0 \quad \text{in} \quad \Omega' \times T \tag{2.1}$$

$$\sigma_{ij} n_j = \bar{p}_i \quad \text{on} \quad \Gamma_p \times T \tag{2.2}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad \Omega' \times T \tag{2.3}$$

$$u_i = \bar{u}_i \quad \text{on} \quad \Gamma_u \times T \tag{2.4}$$

where: $\Omega' = \Omega - \Gamma_d$; σ_{ij} and ϵ_{ij} are components of the stress and the strain tensors, respectively; $(.)_{,j} = \partial(.)/\partial x_j$; n_j are the components of the unit outward normal to the surface Γ.

The material constitutive model in Ω' is assumed to be linear-elastic; namely, in "direct" and "inverse" form, respectively:

$$\sigma_{ij} = D_{ijhk}\left(\epsilon_{hk} - \bar{\theta}_{hk}\right); \qquad \epsilon_{ij} = C_{ijhk}\sigma_{hk} + \bar{\theta}_{ij} \qquad \text{in} \quad \Omega'. \tag{2.5}$$

In Eqs. (2.5) $D_{ijhk} = C^{-1}_{ijhk}$ is the elastic tensor, endowed with the usual symmetry properties and positive definiteness:

$$D_{ijhk} = D_{jihk} = D_{hkij}, \tag{2.6}$$

(2.7) $$D_{ijhk}(\mathbf{x})\epsilon_{ij}\epsilon_{hk} > 0, \quad \text{for any } \epsilon_{ij} \text{ except for } \epsilon_{ij} = 0.$$

The discontinuity line Γ_d is conceived as an interface (smooth) between faces Γ_d^- and Γ_d^+, on which local Cartesian reference are chosen, n_i^- and n_i^+ being the outward unit normal to Γ_d^- and Γ_d^+, respectively (Fig. 1). Let $\mathbf{T}$ be the (orthogonal) matrix which transforms vectors from the local reference $\{\mathbf{n}^-, \mathbf{t}^-\}$ on Γ_d^- to the global reference system. We can write for any $\mathbf{x} \in \Gamma_d$:

(2.8) $$\left\{ \begin{array}{c} w_1 \\ w_2 \end{array} \right\} = \left\{ \begin{array}{c} u_1^+ \\ u_2^+ \end{array} \right\} - \left\{ \begin{array}{c} u_1^- \\ u_2^- \end{array} \right\} = \mathbf{T} \left\{ \begin{array}{c} w_n \\ w_t \end{array} \right\} \quad \text{on } \Gamma_d \times T,$$

(2.9) $$\left\{ \begin{array}{c} p_1 \\ p_2 \end{array} \right\} = \left\{ \begin{array}{c} p_1^- \\ p_2^- \end{array} \right\} = - \left\{ \begin{array}{c} p_1^+ \\ p_2^+ \end{array} \right\} = \mathbf{T} \left\{ \begin{array}{c} p_n \\ p_t \end{array} \right\} \quad \text{on } \Gamma_d \times T.$$

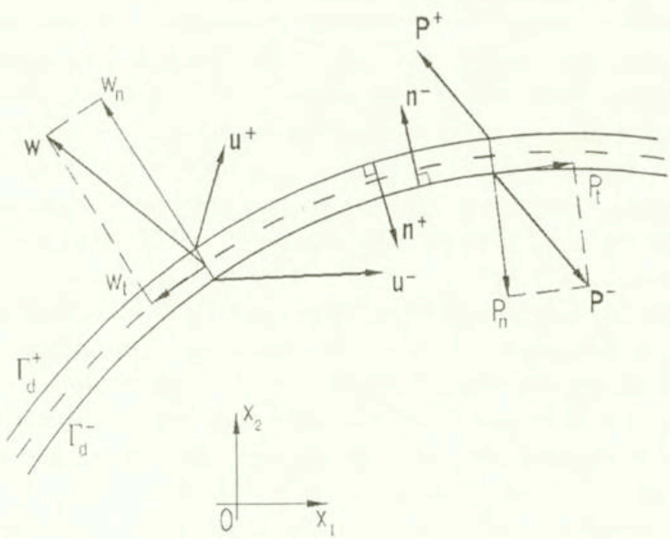

FIG. 1. Illustration of symbols and references for tractions and displacements on the discontinuity locus Γ_d.

Equation (2.8a) defines the kinematic (displacement) discontinuity $\mathbf{w}$ in the global reference; Eq. (2.8b) relates w_i to the normal ("opening", "mode I") relative displacement w_n and to the tangential ("sliding", "mode II") relative displacement w_t, represented in the local reference system on Γ_d^-. Equations (2.9) identify the interface tractions p_i with those on face 1, express interface equilibrium and relate vector p_i (which describes tractions on Γ_d^-, with components p_1 and p_2 in the global reference) to its normal p_n and tangential p_t components with respect to the locus (line) Γ_d. Tractions and stresses along the discontinuity locus Γ_d are linked by Cauchy equilibrium equations:

$$\sigma_{ij} n_j^- = p_i \quad \text{on} \quad \Gamma_d^- \times T, \qquad \sigma_{ij} n_j^+ = -p_i \quad \text{on} \quad \Gamma_d^+ \times T. \tag{2.10}$$

Static $\{p_n \, p_t\}$ and kinematic $\{w_n \, w_t\}$ variables on the discontinuity line Γ_d are linked to each other by an interface model, see e.g. [5, 15, 24, 30, 31]. The generally nonlinear time- and path-dependent ("nonholonomic", irreversible, dissipative) constitutive models along interface Γ_d will be expressed in the following compact form (direct and inverse, respectively):

$$p_i(t) = f_i[w_j(\tau); 0 \le \tau \le t]; \quad w_i(t) = g_i[p_j(\tau); 0 \le \tau \le t] \quad \text{on } \Gamma_d \times T. \tag{2.11}$$

It is worth noting that in many practical situations under proportional, monotonic loading histories, inviscid interface models may be interpreted as history-independent ("holonomic") to overall analysis purposes, namely: $p_i(t) = f(w_i(t))$, and $w_i(t) = g(p_i(t))$. This short notation will be used in what follows for all interface models, whether holonomic or nonholonomic (or partly so, when detachment occurs). Clearly, both in holonomic and nonholonomic interface models, material instability (softening and/or nonassociativity induced by internal friction and/ or damage) is expected, together with consequent multi-value nature of the dependences symbolically expressed by Eqs. (2.11). It is worth noting that these dependences usually can not be described by functionals or functions, but may be mathematically formulated as a problem with multiplicity of solutions (if any exist). Typical formulations of this kind are linear or nonlinear complementarity problems, or sequences of them for nonholonomic models. For applications to the very particular case of hydraulic fracture (an important issue today for oil industries), the interface model can accomodate the crack pressurization, by means of convenient provisions, not dealt with here explicitly.

We assume, for simplicity, that the *initial conditions* are homogeneous (preceded by an undisturbed statical regime):

$$u_i = \epsilon_{ij} = \sigma_{ij} = 0 \qquad \text{on} \ \ \Omega' \cup \Gamma, \quad \text{at} \quad t = 0 \tag{2.12}$$

$$w_i = 0 \qquad \text{on} \ \ \Gamma_d \ \ \text{at} \quad t = 0. \tag{2.13}$$

The b.i.v. problem defined by Eqs. (2.1) - (2.13) will be referred to, in what follows, as *problem P*.

3. Admissible fields

3.1. Consider a stress field history $\sigma^*_{ij}(\mathbf{x},t)$ which is defined in space over Ω and in time over T, and satisfies the equilibrium equations in Ω' and on Γ_p with the actual statical data $\bar{b}_i(\mathbf{x},t)$ and $\bar{p}_i(\mathbf{x},t)$, respectively, and on the discontinuity locus Γ_d between tractions, namely:

$$\sigma^*_{ij,j} + \bar{b}_i = 0 \quad \text{in} \quad \Omega' \times T, \tag{3.1}$$

$$\sigma^*_{ij} n_j = \bar{p}_i \quad \text{on} \quad \Gamma_p \times T, \tag{3.2}$$

$$p^*_i = p^{*-}_i = \sigma^{*-}_{ij} n^-_j = -\sigma^{*+}_{ij} n^+_j = -p^{*+}_i \quad \text{on} \quad \Gamma_d \times T, \tag{3.3}$$

where: $\mathbf{n}^- = \{n^-_i\}$ and $\mathbf{n}^+ = \{n^+_i\}$ denote the unit vectors directed as the outward normals to Γ^-_d and Γ^+_d, respectively; superscripts − and + mark stresses near these two faces of locus Γ_d, like in Eq. (2.10). The corresponding (generally non-compatible) time-histories of strains ϵ^*_{ij} in Ω' and of relative displacement jumps $\mathbf{w}^*$ on Γ_d, can be derived through the Hookean constitutive law $(2.5)_2$ and the interface constitutive law $(2.11)_2$, respectively. They read:

$$\epsilon^*_{ij} = C_{ijhk}\sigma^*_{hk} + \bar{\theta}_{ij} \quad \text{in} \quad \Omega' \times T, \tag{3.4}$$

$$w^*_i = g_i(p^*_j) \quad \text{on} \quad \Gamma_d \times T. \tag{3.5}$$

All fields (in space and time) which satisfy the above conditions will be called henceforth *statically admissible* (and marked by asterisks). It is worth noting that, by this definition, σ^*_{ij} is required not only to balance the given loads and fulfill equilibrium everywhere, but also to comply with the constraints implied by the interface model Eq. (3.5) (e.g. p^*_n cannot exceed the current tensile strength, say $\bar{\sigma}$).

A stress distribution $\sigma^{**}_{ij}(\mathbf{x},t)$ is defined as *self-equilibrated* when it satisfies the following *homogeneous* equilibrium equations:

$$\sigma^{**}_{ij,j} = 0 \quad \text{in} \quad \Omega' \times T, \tag{3.6}$$

$$\sigma^{**}_{ij} n_j = 0 \quad \text{on} \quad \Gamma_p \times T, \tag{3.7}$$

$$\sigma^{**-}_{ij} n^-_j = -\sigma^{**+}_{ij} n^+_j \quad \text{on} \quad \Gamma_d \times T. \tag{3.8}$$

Note that Eqs. (3.3) and (3.8) enforce traction continuity across Γ_d, an equilibrium requirement which could have been alternatively formulated by Eqs. (3.1) and (3.6), respectively, by substituting in them Ω for $\Omega' = \Omega - \Gamma_d$.

3.2. Now let us consider a time-history of the strain field $\epsilon^o_{ij}(\mathbf{x},t)$ derived, by means of the geometric compatibility operator, from a displacement field $u^o_i(\mathbf{x},t)$

which satisfies the assigned kinematic boundary conditions and exhibits a "jump" $\mathbf{w}^o$ on the discontinuity locus Γ_d:

$$\epsilon^o_{ij} = \frac{1}{2}(u^o_{i,j} + u^o_{j,i}) \qquad \text{in} \quad \Omega' \times T, \tag{3.9}$$

$$u^o_i = \bar{u}_i \qquad \text{on} \quad \Gamma_u \times T, \tag{3.10}$$

$$w^o_i = u^{o+}_i - u^{o-}_i \qquad \text{on} \quad \Gamma_d \times T. \tag{3.11}$$

Let stresses σ^o_{ij} in Ω' and interface tractions p^o_i on Γ_d be derived through the constitutive law $(2.5)_1$ from ϵ^o_{ij} and through the interface law $(2.11)_1$ from w^o_i, respectively. They are generally not equilibrated and read:

$$\sigma^o_{ij} = D_{ijhk}(\epsilon^o_{hk} - \bar{\theta}_{hk}) \qquad \text{in} \quad \Omega' \times T, \tag{3.12}$$

$$p^o_i = \sigma^{o-}_{ij} n^-_j = f_i(w^o_j) \qquad \text{on} \quad \Gamma_d \times T. \tag{3.13}$$

Time histories of fields (marked by ^{o}) which satisfy the above definitions will be referred to as *kinematically admissible*. Clearly, besides geometric compatibility, also the constitutive model on Γ_d, Eq. (3.13) is generally expected to set some constraints (e.g. nonegative opening displacement $w_n \geq 0$).

A strain field $\epsilon^{oo}_{ij}(\mathbf{x}, t)$ is here defined as *self-compatible* if it can be derived by means of the compatibility operator from a displacement field $u^{oo}_i(\mathbf{x}, t)$, which is generally discontinuous by w^{oo}_i across Γ_d and vanishes on Γ_u, i.e. satisfies the *homogeneous* compatibility equations:

$$\epsilon^{oo}_{ij} = \frac{1}{2}(u^{oo}_{i/j} + u^{oo}_{j/i}) \qquad \text{in} \quad \Omega' \times T, \tag{3.14}$$

$$u^{oo}_i = 0 \qquad \text{on} \quad \Gamma_u \times T, \tag{3.15}$$

$$w^{oo}_i = u^{oo+}_i - u^{oo-}_i \qquad \text{on} \quad \Gamma_d \times T. \tag{3.16}$$

4. Auxiliary problems

4.1. Elastic responses to imposed displacement jumps on Γ_d

A first auxiliary linear elastic problem will be referred to as *imposed discontinuity* problem and denoted by P^d and superscript d. It is defined by the following set of governing equations, where the input data are represented by displacement discontinuities $w^*_i(\mathbf{x}, t)$, called henceforth *statically admissible* inasmuch they are derived through the interface model on Γ_d, Eq. (3.5), from a *statically admissible* stress field history $\sigma^*_{ij}(\mathbf{x}, t)$ according to the definition of Sec. 3. The formulation of the above problem reads:

$$
\text{Problem } P^d: \quad
\begin{aligned}
&(4.1) & \sigma^d_{ij,j} &= 0 & &\text{in } \Omega' \times T\\
&(4.2) & \sigma^d_{ij} n_j &= 0 & &\text{on } \Gamma_p \times T\\
&(4.3) & \epsilon^d_{ij} &= \frac{1}{2}\left(u^d_{i,j} + u^d_{j,i}\right) & &\text{in } \Omega' \times T\\
&(4.4) & u^d_i &= 0 & &\text{on } \Gamma_u \times T\\
&(4.5) & \epsilon^d_{ij} &= C_{ijhk}\sigma^d_{hk} & &\text{in } \Omega' \times T\\
&(4.6) & w^d_i &= -w^*_i & &\text{on } \Gamma_d \times T
\end{aligned}
$$

Another auxiliary linear elastic problem, say $\tilde{P}^d$, is formulated as follows for later use:

$$
\text{Problem } \tilde{P}^d: \quad
\begin{aligned}
&(4.7) & \tilde{\sigma}^d_{ij,j} &= 0 & &\text{in } \Omega' \times T\\
&(4.8) & \tilde{\sigma}^d_{ij} n_j &= 0 & &\text{on } \Gamma_p \times T\\
&(4.9) & \tilde{\epsilon}^d_{ij} - \epsilon^*_{ij} &= \frac{1}{2}\left(\tilde{u}^d_{i,j} + \tilde{u}^d_{j,i}\right) & &\text{in } \Omega' \times T\\
&(4.10) & \tilde{u}^d_i &= -\bar{u}_i & &\text{on } \Gamma_u \times T\\
&(4.11) & \tilde{\epsilon}^d_{ij} &= C_{ijhk}\tilde{\sigma}^d_{hk} & &\text{on } \Omega' \times T\\
&(4.12) & \tilde{w}^d_i &= -w^*_i & &\text{on } \Gamma_d \times T
\end{aligned}
$$

It is worth noting that in the above problem $\tilde{P}^d$ there are two kinds of (fictitious) input data: (i) the kinematic fields $(-\epsilon^*_{ij})$ and $(-w^*_i)$ which arise, through the inverse constitutive laws Eqs. $(2.5)_2$ and $(2.11)_2$, and through sign inversion, from statically admissible fields σ^*_{ij} and p^*_i, respectively, in the sense of Sec. 3.1; (ii) the actual histories of boundary displacements $\bar{u}_i$, reversed in sign. If u^{ew}_i, ϵ^{ew}_{ij}, σ^{ew}_{ij} represent the fictitious linear elastic response of the solid to the actual time histories of external actions $\bar{b}_i$, $\bar{p}_i$, $\bar{u}_i$ and $\bar{\theta}_{ij}$ in the absence of displacement discontinuities (i.e. with $w_i = 0$ on $\Gamma_d \times T$), the following relationships hold for the solutions of the above two auxiliary linear problems P^d and $\tilde{P}^d$:

$$
(4.13) \qquad \tilde{u}^d_i = u^d_i - u^{ew}_i; \qquad \tilde{w}^d_i = w^d_i; \qquad \tilde{\epsilon}^d_{ij} = \epsilon^d_{ij} + \epsilon^*_{ij} - \epsilon^{ew}_{ij};
$$

$$
\tilde{\sigma}^d_{ij} = \sigma^d_{ij} + \sigma^*_{ij} - \sigma^{ew}_{ij}.
$$

These relationships are readily justified by substituting Eqs. (4.13) into Eqs. (4.7) – (4.12) of problem $\tilde{P}^d$ and by applying the effect superposition, account being taken of the definitions of the fields marked by (*).

4.2. Elastic responses to imposed tractions on Γ_d

The following auxiliary problem, referred to henceforth as *imposed interaction* problem P^s, concerns the linear elastic response to only the traction history

$-p_i^o$ on Γ_d generated, through the interface model and sign inversion, by any kinematically admissible strain field ϵ_{ij}^o (in the sense of Sec. 3):

$$\text{Problem } P^s: \quad \begin{aligned} &(4.14) & \sigma_{ij,j}^s &= 0 && \text{in } \Omega' \times T \\ &(4.15) & \sigma_{ij}^s n_j &= 0 && \text{on } \Gamma_p \times T \\ &(4.16) & \epsilon_{ij}^s &= \frac{1}{2}\left(u_{i,j}^s + u_{j,i}^s\right) && \text{in } \Omega' \times T \\ &(4.17) & u_i^s &= 0 && \text{on } \Gamma_u \times T \\ &(4.18) & \sigma_{ij}^s &= D_{ijhk}\epsilon_{hk}^s && \text{in } \Omega' \times T \\ &(4.19) & p_i^s &= -p_i^o && \text{on } \Gamma_d \times T \end{aligned}$$

A further auxiliary problem, indicated by $\tilde{P}^s$, is formulated as follows:

$$\text{Problem } \tilde{P}^s: \quad \begin{aligned} &(4.20) & \tilde{\sigma}_{ij,j}^s - \sigma_{ij,j}^o - \bar{b}_i &= 0 && \text{in } \Omega' \times T \\ &(4.21) & \tilde{\sigma}_{ij}^s n_j &= \sigma_{ij}^o n_j - \bar{p}_i && \text{on } \Gamma_p \times T \\ &(4.22) & \tilde{\epsilon}_{ij}^s &= \frac{1}{2}\left(\tilde{u}_{i,j}^s + \tilde{u}_{j,i}^s\right) && \text{in } \Omega' \times T \\ &(4.23) & \tilde{u}_i^s &= 0 && \text{on } \Gamma_u \times T \\ &(4.24) & \tilde{\sigma}_{ij}^s &= D_{ijhk}\tilde{\epsilon}_{hk}^s && \text{in } \Omega' \times T \\ &(4.25) & \tilde{p}_i^s &= -p_i^o + \sigma_{ij}^{o-} n_j^- && \text{on } \Gamma_d \times T \end{aligned}$$

In the above problem $\tilde{P}^s$, the input data consist of: (i) fields $(-\sigma_{ij}^o)$ and $(-p_i^o)$ arising, through the direct constitutive laws Eqs. $(2.5)_1$ and $(2.11)_1$ and sign inversion, from any kinematically admissible fields ϵ_{ij}^o and w_i^o, respectively, satisfying the compatibility equations (3.9) – (3.11); (ii) the actual histories of body forces $\bar{b}_i$ and boundary tractions $\bar{p}_i$, reversed in sign. The solutions of the two above linear problems P^s and $\tilde{P}^s$ and the kinematically admissible fields (u_i^o, w_i^o, ϵ_{ij}^o, σ_{ij}^o) generating them, are related to each other. These relationships involve also the solution (marked by superscripts ep) to the linear elastic analysis of the solid supposed to be endowed with a tractionless discontinuity locus Γ_d (i.e. $p_i = 0$ along Γ_d), in the presence of the actual external actions. The above relationships, counterparts of Eqs. (4.13), read:

$$(4.26) \qquad \tilde{u}_i^s = u_i^s + u_i^o - u_i^{ep}; \quad \tilde{p}_i^s = p_i^s + \sigma_{ij}^{o-} n_j^- - p_i^{ep}; \quad \tilde{\epsilon}_{ij}^s = \epsilon_{ij}^s + \epsilon_{ij}^o - \epsilon_{ij}^{ep};$$
$$\tilde{\sigma}_{ij}^s = \sigma_{ij}^s + \sigma_{ij}^o - \sigma_{ij}^{ep}.$$

It is worth noting that the input quantities in problems $\tilde{P}^d$ and $\tilde{P}^s$ substantially differ: in fact, their geometric compatibility suffices for zero solution to the former problem, they must vanish identically for zero solution to the latter.

5. Extremum theorems

5.1. An extended complementary energy theorem

Consider the following functionals (dimensionally energy×time) of a stress field history $\sigma^*_{ij}(\mathbf{x};t)$, statically admissible in the sense of Sec. 3.1:

$$(5.1)\qquad F_c^I\left[\sigma^*_{ij}\right] = \int\limits_T \left\{ \frac{1}{2}\int\limits_\Omega \sigma^*_{ij} C_{ijhk}\sigma^*_{hk} d\Omega + \int\limits_\Omega \bar{\theta}_{ij}\sigma^*_{ij} d\Omega - \int\limits_{\Gamma_u} \bar{u}_i n_j \sigma^*_{ij} d\Gamma \right.$$

$$\left. + \int\limits_{\Gamma_d} g_i\left(p^*_h\right)\left(p^*_i - \sigma^{ew}_{ij} n^-_j\right) d\Gamma + \frac{1}{2}\int\limits_{\Omega'} \sigma^d_{ij}(\sigma^*_{rs}) C_{ijhk} \sigma^d_{hk}(\sigma^*_{rs}) d\Omega \right\} dt$$

$$(5.2)\qquad F_c^{II}\left[\sigma^*_{ij}\right] = \int\limits_T \left\{ \frac{1}{2}\int\limits_{\Omega'} \tilde{\sigma}^d_{ij} C_{ijhk} \tilde{\sigma}^d_{hk} d\Omega \right\} dt + F_c^0$$

where

$$(5.3)\qquad F_c^0 = \int\limits_T \left\{ \frac{1}{2}\int\limits_\Omega \sigma^{ew}_{ij}\epsilon^{ew}_{ij} d\Omega + \int\limits_\Omega \bar{\theta}_{ij}\sigma^{ew}_{ij} d\Omega - \int\limits_{\Gamma_u} \bar{u}_i n_j \sigma^{ew}_{ij} d\Gamma \right\} dt.$$

According to definitions given in Sec. 4.1, σ^{ew}_{ij} are the elastic stresses in the solid under the assigned time history of the actual external actions $\bar{b}_i$, $\bar{p}_i$, $\bar{u}_i$ and $\bar{\theta}_{ij}$, with $w_i \equiv 0$ on $\Gamma_d \times T$; σ^d_{ij} and $\tilde{\sigma}^d_{ij}$ denote the stress solutions to the linear auxiliary problems P^d, Eqs. (4.1) – (4.6), and $\tilde{P}^d$, Eqs. (4.7) – (4.12), respectively.

Some meaningful properties of the above functionals are formulated below by three statements proved in what follows: a lemma, a generalized complementary energy theorem and a corollary (propositions 1, 2 and 3, respectively).

PROPOSITION 1. For the same time history of a statically admissible stress field σ^*_{ij}, functionals F_c^I, Eq. (5.1), and F_c^{II}, Eq. (5.2), attain the same value.

P r o o f. The difference functional $F_c^I - F_c^{II}$ can be written as follows by using Eq. $(4.13)_4$:

$$(5.4) \qquad F_c^I - F_c^{II} = \int\limits_T \Bigg\{ \int\limits_\Omega \bar{\theta}_{ij}\sigma_{ij}^* d\Omega - \int\limits_{\Gamma_u} \bar{u}_i n_j \sigma_{ij}^* d\Gamma + \int\limits_{\Gamma_d} g_i(p_h^*)\left(p_i^* - \sigma_{ij}^{ew} n_j^-\right) d\Gamma - \int\limits_{\Omega'} \left(\sigma_{ij}^* - \sigma_{ij}^{ew}\right) C_{ijhk}\sigma_{hk}^d d\Omega$$

$$- \int\limits_\Omega \sigma_{ij}^{ew}\epsilon_{ij}^{ew} d\Omega - \int\limits_\Omega \bar{\theta}_{ij}\sigma_{ij}^{ew} d\Omega + \int\limits_{\Gamma_u} \bar{u}_i n_j \sigma_{ij}^{ew} d\Gamma + \int\limits_\Omega \sigma_{ij}^{ew} C_{ijhk}\sigma_{hk}^* d\Omega \Bigg\} dt.$$

As consequences of the virtual work principle, the third and fourth inner integrals drop out from Eq. (5.4) and the last inner integral can be expressed as:

$$(5.5) \qquad \int\limits_\Omega \sigma_{ij}^{ew} C_{ijhk}\sigma_{hk}^* d\Omega = \int\limits_\Omega u_i^{ew}\bar{b}_i d\Omega + \int\limits_{\Gamma_p} u_i^{ew}\bar{p}_i d\Gamma + \int\limits_{\Gamma_u} \bar{u}_i\sigma_{ij}^* n_j d\Gamma - \int\limits_\Omega \bar{\theta}_i \sigma_{ij}^* d\Omega.$$

Thus, the difference functional becomes:

$$(5.6) \qquad F_c^I - F_c^{II} = \int\limits_T \Bigg\{ - \int\limits_\Omega \sigma_{ij}^{ew}\epsilon_{ij}^{ew} d\Omega - \int\limits_\Omega \bar{\theta}_{ij}\sigma_{ij}^{ew} d\Omega + \int\limits_{\Gamma_u} \bar{u}_i n_j \sigma_{ij}^{ew} d\Gamma + \int\limits_\Omega u_i^{ew}\bar{b}_i d\Omega + \int\limits_{\Gamma_p} u_i^{ew}\bar{p}_i d\Gamma \Bigg\} dt$$

and is easily seen to vanish because of the virtual work principle again. □

PROPOSITION 2. A statically admissible stress field σ_{ij}^* is a (or the) solution of the original problem P, Eqs. (2.1) – (2.13), if and only if it minimizes (absolute minimum) the functional F_c^I, Eq. (5.1), provided a solution exists. By virtue of Proposition 1, the same stress field σ_{ij}^* minimizes also functional F_c^{II}, Eq. (5.2).

P r o o f. With reference to Eq. (5.2), consider the difference:

$$(5.7) \qquad F_c^{II} - F_c^0 = \frac{1}{2}\int\limits_T \int\limits_\Omega \tilde{\epsilon}_{ij}^d \tilde{\sigma}_{ij}^d d\Omega dt.$$

The inner integral (5.7) represents the elastic strain energy due to: imposed strains $-\bar{\theta}_{ij} - C_{ijhk}\sigma^*_{hk}$; imposed relative displacements $-g_i(p^*_h)$; boundary displacements $\bar{u}_i$ on Γ_u. Therefore, the above difference, Eq. (5.7), cannot be negative. The functional F^{II}_c attains its minimum value F^0_c if and only if there exists a kinematically admissible strain field ϵ^o_{ij} and a statically admissible stress field σ^*_{ij} such that:

$$\epsilon^o_{ij} = C_{ijhk}\sigma^*_{hk} + \bar{\theta}_{ij} \qquad \text{in} \quad \Omega' \times T, \tag{5.8}$$

$$w^o_i = g_i(p^*_h) \qquad \text{on} \quad \Gamma_d \times T. \tag{5.9}$$

In fact, Eqs. (5.8) and (5.9) mean that the geometrically compatible fields (^{o}) and the statically admissible fields (*) are related to each other through the constitutive models everywhere and, hence, satisfy all the governing relationships of the original problem P, i.e. represent its solution or one of its solutions. □

PROPOSITION 3. The actual problem P has at least one solution if and only if the functional F^I_c (and, hence, F^{II}_c as well) attains, at the global minimum, the value F^0_c, Eq. (5.3).

P r o o f. If at the minimum it turns out that:

$$F^{II}_c > F^0_c \tag{5.10}$$

then no statically admissible stress field σ^*_{ij} exists such that the constitutive models, Eqs. (5.8) and (5.9), are fulfilled. Then the original problem P has no solution. □

REMARK. The nonlinear (in particular, e.g., softening) nature of the interface models $f_i(w_j)$ may imply nonconvexity of functionals F^I_c and F^{II}_c. However, by virtue of Proposition 3, possible local minima of them do not characterize solutions to the actual nonlinear i.b.v. problem (only absolute minima do).

5.2. An extended potential energy principle

Let us consider now the following generally nonconvex functionals of the kinematically admissible displacement field history $u^o_i(\mathbf{x};t)$ (cf. Sec. 3.2):

(5.11) $$F_p^I[u_i^o] = \int_T \Bigg\{ \frac{1}{2}\int_{\Omega'} \left(\epsilon_{ij}^o - \bar{\theta}_{ij}\right) D_{ijhk} \left(\epsilon_{hk}^o - \bar{\theta}_{hk}\right) d\Omega - \int_{\Omega'} \bar{b}_i u_i^o d\Omega - \int_{\Gamma_p} \bar{p}_i u_i^o d\Gamma + \int_{\Gamma_d} f_i\left(w_j^o\right)\left(w_i^o - w_i^{ep}\right) d\Gamma + \frac{1}{2}\int_{\Omega'} \epsilon_{ij}^s(u_r^o) D_{ijhk} \epsilon_{hk}^s(u_r^o) d\Omega \Bigg\} dt.$$

(5.12) $$F_{pe}^{II}[u_i^o] = \frac{1}{2}\int_T \int_{\Omega'} \tilde{\epsilon}_{ij}^s D_{ijhk} \tilde{\epsilon}_{hk}^s d\Omega dt + F_p^0$$

having set:

(5.13) $$F_p^0 = \int_T \Bigg\{ \frac{1}{2}\int_{\Omega'} \left(\epsilon_{ij}^{ep} - \bar{\theta}_{ij}\right) D_{ijhk} \left(\epsilon_{hk}^{ep} - \bar{\theta}_{hk}\right) d\Omega - \int_{\Omega'} \bar{b}_i u_i^{ep} d\Omega - \int_{\Gamma_p} \bar{p}_i u_i^{ep} d\Gamma \Bigg\} dt.$$

Here (see Sec. 4.2) w_i^{ep} represent the displacement discontinuities in the elastic solid under the actual external actions $\bar{b}_i, \bar{p}_i, \bar{u}_i$ and $\bar{\theta}_{ij}$, with $p_i = 0$ on $\Gamma_d \times T$; ϵ_{ij}^s and $\tilde{\epsilon}_{ij}^s$ denote the strain solutions to the linear auxiliary problems P^s, Eqs. (4.14) – (4.19), and $\tilde{P}^s$, Eqs. (4.20) – (4.25), respectively. As counterparts to the three theorems established in Sec. 5.1, three further propositions are proven below.

PROPOSITION 4. Functionals F_p^I and F_p^{II}, Eqs. (5.11) and (5.12), attain a common value for the same time history of a kinematically admissible displacement field u_i^o.

P r o o f. In view of Eqs. $(4.26)_3$ we may express the difference $F_p^I - F_p^{II}$ as the following functional:

$$F_p^I - F_p^{II} = \int_T \left\{ - \int_{\Omega'} \bar{b}_i u_i^o d\Omega - \int_{\Gamma_p} \bar{p}_i u_i^o d\Gamma - \int_{\Omega'} \bar{\theta}_{ij} D_{ijhk} \epsilon_{hk}^o d\Omega \right. \tag{5.14}$$

$$+ \int_{\Gamma_d} f_i(w_h^o)\,(w_i^o - w_i^{ep})\, d\Gamma - \int_{\Omega'} \left(\epsilon_{ij}^o - \epsilon_{ij}^{ep}\right) D_{ijhk} \epsilon_{hk}^s d\Omega - \int_{\Omega'} \sigma_{ij}^{ep} \epsilon_{ij}^{ep} d\Omega$$

$$\left. + \int_{\Omega'} \bar{b}_i u_i^{ep} d\Omega + \int_{\Gamma_p} \bar{p}_i u_i^{ep} d\Gamma + \int_{\Omega'} \epsilon_{ij}^{ep} D_{ijhk} \epsilon_{hk}^o d\Omega \right\} dt.$$

As it can be readily seen by applying once again the virtual work principle, in the above difference functional the sum of the fourth and fifth inner integral vanishes, whereas the last integral becomes:

$$\int_{\Omega'} \epsilon_{ij}^{ep} D_{ijhk} \epsilon_{hk}^o d\Omega = \int_{\Omega'} \bar{b}_i u_i^o d\Omega + \int_{\Gamma_p} \bar{p}_i u_i^o d\Gamma + \int_{\Gamma_u} \sigma_{ij}^{ep} n_j \bar{u}_i d\Gamma \tag{5.15}$$

$$+ \int_{\Omega'} \bar{\theta}_{ij} D_{ijhk} \epsilon_{hk}^o d\Omega.$$

As a consequence of the above remarks, the difference functional may be given the following expression which turns out to vanish:

$$F_p^I - F_p^{II} = \int_T \left\{ - \int_{\Omega'} \sigma_{ij}^{ep} \epsilon_{ij}^{ep} d\Omega + \int_{\Gamma_u} \bar{u}_i n_j \sigma_{ij}^{ep} d\Gamma + \int_{\Omega'} \bar{b}_i u_i^{ep} d\Omega \right. \tag{5.16}$$

$$\left. + \int_{\Gamma_p} \bar{p}_i u_i^{ep} d\Gamma \right\} dt.$$

□

PROPOSITION 5. A kinematically admissible displacement field u_i^o is a (or the) solution of the problem P, Eqs. (2.1) - (2.13), if and only if it minimizes (absolute minimum) the functional F_p^I, Eq. (5.11), provided a solution exists. By virtue of Proposition 4, the same displacement field u_i^o minimizes functional F_p^{II}, Eq. (5.12), as well.

P r o o f. Let us consider the difference

$$F_p^{II} - F_p^0 = \frac{1}{2} \int_T \int_{\Omega'} \tilde{\sigma}_{ij}^s \tilde{\epsilon}_{ij}^s d\Omega dt. \tag{5.17}$$

The inner integral of this functional represents the elastic strain energy in the solid if it were subjected to the following (fictitious) external actions: volume forces $\tilde{b}_i = -\left[D_{ijhk}(\epsilon_{hk}^o - \bar{\theta}_{hk})\right]_{,j} - \bar{b}_i$ in Ω', tractions $\tilde{p}_i = D_{ijhk}(\epsilon_{hk}^o - \bar{\theta}_{hk})n_j - \bar{p}_i$ on Γ_p; tractions $-f_i(w_r^o) + D_{ijhk}(\epsilon_{hk}^o - \bar{\theta}_{hk})n_j$ on Γ_d.

The functional F_p^{II} reaches its minimum F_p^0 if and only if the (fictitious, statical) external actions vanish in the auxiliary problem $\tilde{P}^s$ of Sec. 4.2; namely if and only if:

$$\left[D_{ijhk}(\epsilon_{hk}^o - \bar{\theta}_{hk})\right]_{,j} + \bar{b}_i = 0 \qquad \text{in} \quad \Omega' \times T, \tag{5.18}$$

$$D_{ijhk}(\epsilon_{hk}^o - \bar{\theta}_{hk})n_j - \bar{p}_i = 0 \qquad \text{on} \quad \Gamma_p \times T, \tag{5.19}$$

$$D_{ijhk}(\epsilon_{hk}^o - \bar{\theta}_{hk})n_j - f_i(w_r^o) = 0 \qquad \text{on} \quad \Gamma_d \times T. \tag{5.20}$$

When Eqs. (5.18), (5.19) and (5.20) hold, since the very definition of the kinematically admissible fields (^{o}) implies that Eqs. (3.9) – (3.11) are fulfilled, then the whole set of governing equations of the original problem P, Eqs. (2.1) – (2.13), is satisfied by the field u_i^o which, hence, represents a (or the) solution of it. □

Proposition 6. Problem P has at least one solution if and only if functional F_p^I (and, hence, also F_p^{II}), reaches the value F_p^0, Eq. (5.13), as its global minimum.

P r o o f. Like for Proposition 3, suppose that $F_p^{II} > F_p^0$ at the absolute minimum of functional F_p^{II}, then no kinematically admissible displacement field u_i^o (satisfying Eqs. (3.9) – (3.11)) exists such that equilibrium and constitutive laws, Eqs. (5.18) and (5.20), are complied with at the same time. This means that there is no solution to the original problem P. □

Remark. It is worth noting that functional F_p^I, Eq. (5.11), reduces to total potential energy of linear elasticity when either Γ_d vanishes or there is no interaction along it, i.e. $f_i(w_j) = 0$. Clearly, analogous specification (to complementary energy) can be noticed for F_c^I, Eq. (5.1).

6. Saddle-point theorems

For their applications, a drawback of the preceding theorems (Proposition 2 and 5) lies in the need to evaluate the last term of Eq. (5.1) and Eq. (5.11) through the solution of the elastic auxiliary problem for every admissible stress field $\boldsymbol{\sigma}^*$ or for every compatible strain field $\boldsymbol{\epsilon}^o$, respectively. In other words, applications of Propositions 2 and 5 would require to find Green functions of the elastic problem. However, the aforementioned last two terms represent elastic energies which can be evaluated, using classical elasticity principles, by maximization of suitable functionals in additional new variables, as shown in what follows.

6.1. A min-max extended complementary energy principle

With reference to Proposition 2, a functional F_c^d to be maximized in order to obtain the last term of (5.1) can be formulated by means of the principle of virtual work as follows (at any instant t):

$$\text{(6.1)} \qquad \frac{1}{2}\int\limits_{\Omega'} \sigma_{ij}^d\left(\sigma_{rs}^*\right) C_{ijhk}\sigma_{hk}^d\left(\sigma_{rs}^*\right) d\Omega = -\frac{1}{2}\int\limits_{\Omega'} \sigma_{ij}^d\left(\sigma_{rs}^*\right) C_{ijhk}\sigma_{hk}^d\left(\sigma_{rs}^*\right) d\Omega + \int\limits_{\Gamma_d} g_i\left(p_h^*\right)\sigma_{ij}^{d-} n_j^- d\Gamma.$$

The right-hand side of (6.1) is readily recognized to be the (changed in sign) value of the complementary energy at the solution of the elastic b.v. problem for imposed relative displacements $-g_i\left(p_j^*\right)$ at instant t. This means that, by virtue of the minimum principle of complementary energy, σ_{ij}^{d**} being any *self-equilibrated* stress field,

$$\text{(6.2)} \qquad \frac{1}{2}\int\limits_{\Omega'} \sigma_{ij}^d\left(\sigma_{rs}^*\right) C_{ijhk}\sigma_{hk}^d\left(\sigma_{rs}^*\right) \mathrm{d}\Omega = \max_{\sigma_{ij}^{d**}}\left\{F_c^d\left[\sigma_{ij}^{d**},\sigma_{ij}^*\right]\right\}$$

where

$$\text{(6.3)} \qquad F_c^d\left[\sigma_{ij}^{d**},\sigma_{ij}^*\right] = \int\limits_T \left\{-\frac{1}{2}\int\limits_{\Omega'} \sigma_{ij}^{d**} C_{ijhk}\sigma_{hk}^{d**} d\Omega + \int\limits_{\Gamma_d} g_i\left(p_h^*\right)\sigma_{ij}^{d**-} n_j^- d\Gamma\right\} dt.$$

By substitution of Eq. (6.2), the functional (5.1) is transformed into the following new functional:

$$\mathcal{F}_c\left[\sigma^*_{ij},\sigma^{d**}_{ij}\right]=\int\limits_T\left\{\frac{1}{2}\int\limits_\Omega \sigma^*_{ij}C_{ijhk}\sigma^*_{hk}d\Omega+\int\limits_\Omega \bar{\theta}_{ij}\sigma^*_{ij}d\Omega-\int\limits_{\Gamma_u}\bar{u}_i n_j\sigma^*_{ij}d\Gamma\right.$$
$$+\int\limits_{\Gamma_d} g_i\left(p^*_h\right)\left(p^*_i-\sigma^{ew}_{ij}n^-_j\right)d\Gamma-\frac{1}{2}\int\limits_{\Omega'}\sigma^{d**}_{ij}C_{ijhk}\sigma^{d**}_{hk}d\Omega$$
$$\left.+\int\limits_{\Gamma_d} g_i\left(p^*_h\right)\sigma^{d**-}_{ij}n^-_j d\Gamma\right\}dt. \tag{6.4}$$

As a consequence of the above remarks the following theorem can now be stated:

PROPOSITION 7. A statically admissible stress field σ^*_{ij} and a self-equilibrated stress field σ^{d**}_{ij} are a (or the) solution of problem P and problem P^d, respectively, if and only if both the following conditions are satisfied:

(i) the two fields make stationary (minimum with respect to σ^*_{ij} and maximum with respect to σ^{d**}_{ij}) the functional F_c, Eq. (6.4)

(ii) the saddle-point value of the functional $\mathcal{F}_c$, Eq. (6.4), is equal to F^0_c, Eq. (5.3).

It is worth noting that: (a) the new functional $\mathcal{F}_c$ does not require the preliminary solution of an elastic auxiliary problem (nor the evaluation of the stress Green function due to distorsions on Γ_d), (b) the new formulation of the original i.b.v. problem P involves with respect to it, a double number of unknown fields.

6.2. A min-max extended total potential energy principle

Following a path of reasoning analogous to the one in the preceding section, a new min-max principle is established below from Proposition 5. A functional F^s_p to be maximized in order to obtain the last term of Eq. (5.11) (at any instant t) may be generated simply by using the principle of virtual work as follows:

$$\frac{1}{2}\int\limits_{\Omega'}\epsilon^s_{ij}\left(\epsilon^o_{rs}\right)D_{ijhk}\epsilon^s_{hk}\left(\epsilon^o_{rs}\right)d\Omega=-\frac{1}{2}\int\limits_{\Omega'}\epsilon^s_{ij}\left(\epsilon^o_{rs}\right)D_{ijhk}\epsilon^s_{hk}\left(\epsilon^o_{rs}\right)d\Omega$$
$$+\int\limits_{\Gamma_d} f_i\left(w^o_h\right)w^s_i d\Gamma. \tag{6.5}$$

It is easy to recognize that the right-hand side of Eq. (6.5) represents the (changed in sign) value, at instant t, of the total potential energy at the solution of the elastic problem (at instant t) with imposed tractions $-f_i\left(w_j^o\right)$ on Γ_d. This means that, by virtue of the minimum principle of total potential energy, we can write:

$$\frac{1}{2}\int\limits_{\Omega'} \epsilon_{ij}^s\left(\epsilon_{rs}^o\right) D_{ijhk}\epsilon_{hk}^s\left(\epsilon_{rs}^o\right) d\Omega = \max_{u_i^{soo}}\left\{F_p^s\left[u_i^{soo}, u_i^o\right]\right\} \tag{6.6}$$

having set:

$$F_p^s\left[u_i^{soo}, u_i^o\right] = \int\limits_T \left\{-\frac{1}{2}\int\limits_{\Omega'} \epsilon_{ij}^{soo} D_{ijhk}\epsilon_{hk}^{soo} d\Omega + \int\limits_{\Gamma_d} f_i(w_h^o) w_i^{soo}) d\Gamma\right\} dt. \tag{6.7}$$

Here ϵ_{ij}^{soo} denotes any *self-compatible* strain field in Ω', i.e. any strain field which can be derived from a displacement field u_i^{soo} satisfying the homogeneous boundary conditions $u_i^{soo} = 0$ on Γ_u. By substitution of Eq. (6.6), functional F_p^I, Eq. (5.11), can be transformed into the following new functional:

$$\begin{aligned}\mathcal{F}_p\left[u_i^o, u_i^{soo}\right] = \int\limits_T \Bigg\{ & \frac{1}{2}\int\limits_{\Omega'} \left(\epsilon_{ij}^o - \bar{\theta}_{ij}\right) D_{ijhk}\left(\epsilon_{hk}^o - \bar{\theta}_{hk}\right) d\Omega - \int\limits_{\Omega'} \bar{b}_i u_i^o d\Omega \\ & - \int\limits_{\Gamma_p} \bar{p}_i u_i^o d\Gamma + \int\limits_{\Gamma_d} f_i(w_j^o)\left(w_i^o - w_i^{ep}\right) d\Gamma - \frac{1}{2}\int\limits_{\Omega'} \epsilon_{ij}^{soo} D_{ijhk}\epsilon_{hk}^{soo} d\Omega \\ & + \int\limits_{\Gamma_d} f_i(w_j^o) w_i^{soo} d\Gamma \Bigg\} dt\end{aligned} \tag{6.8}$$

and the following statement can be asserted:

PROPOSITION 8. A kinematically admissible displacement field u_i^o and a displacement field u_i^{soo} vanishing on Γ_u are a (or the) solution of problem P and problem P^s, respectively, if and only if both the following conditions are satisfied:

(i) the two fields make stationary (minimum with respect to u_i^o and maximum respect to u_i^{soo}) the functional $\mathcal{F}_p$, Eq. (6.8);

(ii) the saddle-point value of the functional $\mathcal{F}_p$, Eq. (6.8), is equal to F_p^0 Eq. (5.13).

It is worth noting that in the functional (6.8), both fields $\epsilon_{ij}^o = \epsilon_{ij}^o(u_j^o)$ and $\epsilon_{ij}^{soo} = \epsilon_{ij}^{soo}\left(u_h^{soo}\right)$ are geometrically compatible. Remark similar to the one pointed out at the end of Proposition 7 hold here, namely: the new functional does not require the preliminary solution of an elastic auxiliary problem, nor the evaluation of Green functions for strains due to imposed stresses; on the other hand, the new formulation concerns a double number of unknown fields.

7. Conclusions

The theoretical results achieved in this paper consist of duality pairs of extremum and saddle-point theorems, which characterize the nonlinear response in time to external action histories of linear elastic solids and structures containing loci where displacement discontinuities may occur, according to very general interface laws. These constitutive laws include mathematical models for, e.g.: frictional contacts on interfaces with asperities; fracture process zones in quasi-brittle materials; artificial joints in large concrete and masonry dams; delamination in laminates and debonding in composites.

The computational possibilities and some meaningful applications of the present results are being investigated elsewhere. As an illustrative example and a closing remark, we sketchily outlined below a path-of-reasoning apt to apply the last Proposition 8 (min-max theorem derived from the generalized potential energy principle of Sec. 5.2, Proposition 5).

Consider a four-point-bending or a four-point-shear test on a concrete specimen, as simulated in [16]. The loading history amounts to the monotone increase in time t of the displacement imposed by the testing device. A simulation centered on the above selected present results would encompass the following phases.

(a) A conventional finite element discretization in space is envisaged, account being taken of the conjectured locus Γ_d of possible crack propagation paths, starting from the specimen notches, so that nodal displacements represent degrees of freedom, gathered in vector $\mathbf{U}$.

(b) The duration time T of the test is subdivided into intervals $\Delta t_n = t_n - t_{n-1}$, over which the time-dependent $\mathbf{U}(t)$ can be modelled by interpolations (which may be generalized functions (distributions), so that customary time-integration schemes can be recovered as special cases).

(c) First, the time integral of elastic energy F_p^0, Eq. (5.13), is computed by a single once-for-all linear analysis and turns out to be proportional to t_n^3 at the instants t_n in the preselected sequence.

(d) Over the time step $\Delta t_n = t_n - t_{n-1}$, the nonlinear interface model $f_i(w_j)$ is made explicit so that $\mathcal{F}_p$, Eq. (6.8), becomes a functional of the variable vectors $\mathbf{U}_n^o$ and $\mathbf{U}_n^{soo}$ which govern, in space and in time, the fields u_i^o and u_i^{soo}, respectively, e.g. with stepwise constant modelling in time (the dimensionality of each of these vectors is the same as that of the d.o.f. vector $\mathbf{U}$).

(e) The step solutions, say $\hat{U}_n^o$ and $\hat{U}_n^{soo}$, are characterized by the min-max value $\hat{\mathcal{F}}_p = F_p^0$ of $\mathcal{F}_p$. In the test referred to, here three solutions are expected, a symmetric and two nonsymmetric ones, if T exceeds a bifurcation threshold, [16]. In the impractical case of force control, no solution would be expected after a time T, a circumstance denoted by $\hat{\mathcal{F}}_p > F_p^0$. The saddle-point stationarity conditions are represented by generally nonlinear systems of algebraic equations and inequalities. The transformation of a min-max into a min problem (see e.g. [7]) might be computationally advantageous.

The solution characterizations by stationarity established in this paper for the continuum nonlinear initial-boundary value problems concerning elastic solids with dissipative interfaces, are seen to provide a theoretical framework for the discretization in space and time, in view of the numerical approximate solution of these problems.

Acknowledgements

A grant from the Italian Ministry of University and Research (MURST) for a research project on integrity assessment of large dams ("Cofinanziamento" 1999-2000) is gratefully acknowledged.

References

1. S. Alliney, A. Tralli, *Extended variational formulations and F.E. methods in the stability analysis of non-conservative mechanics problems*, Comp. Meth. Appl. Mech. Engng., **51**, 209–219, 1986.
2. G. Auchmuty, *Variational principles for operator equations and initial value problems*, Nonlinear Analysis, Theory, Methods and Applications, **12**, 531–564, 1988.
3. Z. P. Bazant, J. Planas, *Fracture and size effect in concrete and other quasibrittle materials*, CRC Press, Bocaraton 1998.
4. G. Bolzon, G. Cocchetti, G. Maier, G. Novati, G. Giuseppetti, *Boundary element and finite element fracture analysis of dams by the cohesive crack model: a comparative study*, International Workshop on Dam and Fracture and Damage, Chambery, France, 16-18 March, 1994, [In:] Dam Fracture and Damage, E. Bourdarot, J. Mazars, V. Saouma, [Eds.], 69–78, Balkema, Rotterdam, Brookfield.
5. G. Bolzon, A. Corigliano, *A discrete formulation for elastic solids with damaging interfaces*, Comp. Meth. Appl. Mech. Engng., **140**, 329–359.
6. G. Bolzon, G. Maier, F. Tin-Loi, *Holonomic and non-holonomic simulation of quasi-brittle fracture: a comparative study of mathematical programming approaches*, [In:] Fracture Mechanics of Concrete Structures, F. H. Wittmann [Ed.], Aedificatio Publishers, Freiburg, 885–898, 1995.
7. J. H. Bramble, J. E. Pasciak, *A preconditioning technique for indefinite systems resulting from mixed approximation of elliptic problems*, Mathematics of Computation, **50**, 1–17, 1988.
8. M. Capurso, G. Maier, *Incremental elastoplastic analysis and quadratic optimization*, Meccanica, **4**, 107–116, 1970.

9. A. CARINI, *Colonnetti's minimum principle extension to generally nonlinear materials*, Int. J. Solids Structures, **33**, 121–144, 1996.

10. A. CARINI, *Saddle-point principles for general nonlinear material continua*, J. Applied Mechanics (ASME), **64**, 1010–1014, 1997.

11. A. CARINI and O. DE DONATO, *A comprehensive energy formulation for general non-linear material continua*, J. Applied Mechanics (ASME), **64**, 353–360, 1997.

12. A. CARINI, M. DILIGENTI, G. MAIER, *Boundary integral equation analysis in linear viscoelasticity: variational and saddle point formulations*, Computational Mechanics, **8**, 87–98, 1991.

13. A. CARINI, M. DILIGENTI, G. MAIER, *Symmetric boundary integral formulations of transient heat conduction: saddle-point theorems for BE analysis and BE-FE coupling*, Arch. Mech., **49**, 253–283, 1997.

14. A. CARINI, F. GENNA, *Some variational formulations for continuum nonlinear dynamics*, J. Mech. Phys. Solids, **46**, 1253–1277, 1998.

15. I. CAROL, P. C. PRAT and C. M. LOPEZ, *Normal/shear cracking model: application to discrete crack analysis*, J. Engng. Mech., ASCE, **123**, 1–9, 1997.

16. Z. CEN, G. MAIER, *Bifurcation and instability in fracture of cohesive softening structure: a boundary element analysis*, Fatigue and Fracture Engineering of Material and Structures, **15**, 911–928, 1992.

17. G. CERADINI, *A maximum principle for the analysis of elastic-plastic systems*, Meccanica, **1**, 77–82, 1966.

18. C. COMI, A. CORIGLIANO, G. MAIER, *Dynamic analysis of elastoplastic-softening discretized structures*, Proc. ASCE, J. Engng. Mech., **118**, 2352–2375, 1992.

19. C. COMI, G. MAIER, U. PEREGO, *Generalized variable finite element modelling and extremum theorems in stepwise holonomic elastoplasticity with internal variables*, Comp. Meth. Appl. Mech. Engng., **96**, 133–171, 1992.

20. B. A. FINLAYSON, *The method of weighted residual and variational principles with applications in fluid mechanics, heat and mass transfer*, Academic Press, New York 1972.

21. M. E. GURTIN, *Variational principles for linear initial-value problems*, Quart. Appl. Math., **22**, 252–256, 1964.

22. B. L. KARIHALOO, *Fracture Mechanics and Structural Concrete*, Longman Scientific & Technical, Harlow, Great Britain 1998.

23. W. T. KOITER, *General theorems for elastic-plastic solids*, [In:] Progress in Solid Mechanics, I. N. SNEDDON and R. HILL [Eds.], **1**, Ch. IV, 167–221, 1964.

24. H. LOFTI, P. SHING, *Interface model applied to fracture of masonry structures*, J. Struc. Engng., ASCE, **120**, 63–80, 1994.

25. G. MAIER, *Some theorems for plastic strain rates and plastic strains*, Journal de Mécanique, **8**, 5–19, 1969.

26. G. MAIER, *A minimum principle for incremental elastoplasticity with nonassociated flow laws*, J. Mech. Phys. Solids, **18**, 319–330, 1970.

27. G. MAIER, M. DILIGENTI, A. CARINI, *A variational approach to boundary element elastodynamic analysis and extension to multidomain problems*, Comp. Meth. in Appl. Mech. Engng., **92**, 193–212, 1991.

28. G. MAIER, G. NOVATI, Z. CEN, *Symmetric Galerkin boundary element method for quasi-brittle fracture and frictional contact problems*, Computational Mechanics, **13**, 74–89, 1993.

29. P. M. MORSE and H. FESHBACH, *Methods of theoretical physics*, **I**, McGraw-Hill 1953.

30. Z. MRÓZ, SHEN XINPU, *Analysis of progressive interface failure under monotonic loading*, [In:] Microstructure and Mechanical Properties of New Engineering Materials, B. Y. XU, M. TOKUDA, X. C. WANG [Eds.], International Academic Publisher, Beijing, 109–114, 1999.

31. Z. MRÓZ, G. GIAMBANCO, *An interface model for analysis of deformation behaviour of discontinuity*, Int. J. Num. Anal. Meth. Geomech., **20**, 1–33, 1996.

32. M. ORTIZ, *A variational formulation for convection-diffusion problems*, Int. J. Engng. Sci., **23**, 717–731, 1985.

33. P. RAFALSKI, *Orthogonal projection method. I. Heat conduction boundary problem*, Bull. Acad. Polon. Sci., Sér. Sci. Techn., **17**, 63–67, 1969.

34. P. RAFALSKI, *Orthogonal projection method. II. Thermoelastic problem*, Bull. Acad. Polon. Sci., Sér. Sci. Techn., **17**, 69–74, 1969.

35. R. REISS, E. J. HAUG, *Extremum principles for linear initial-value problems of mathematical physics*, Int. J. Engng. Sci., **16**, 231–251, 1978.

36. P. D. ROBINSON, P. K. YUAN, *Bi-variational methods for linear integral equations with non-symmetric kernels*, SIAM, J. Numer. Anal., **23**, 1230–1240, 1986.

37. J. J. TELEGA, *Variational principles for rate boundary-value problems in non-associated plasticity*, ZAMM, **60**, 71–82, 1980.

38. E. TONTI, *On the variational formulation for linear initial value problems*, Annali di Matematica Pura ed Applicata, **45**, 331–359, 1973.

39. E. TONTI, *Variational formulation for every nonlinear problem*, Int. J. Engng. Sci., **22**, 1343–1371, 1984.

Received August 24, 2000.