

## Spatial localization of the error of constitutive law for the identification of defects in elastic bodies

*Dedicated to Professor Zenon Mróz  
on the occasion of his 70<sup>th</sup> birthday*

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THE ERROR OF CONSTITUTIVE LAW (ECL) is a cost functional currently used in inverse problems for identifying interior distribution of coefficients of partial differential equations from overspecified boundary conditions. In previous works, different authors have shown that the ECL enables a good spatial localization of the perturbations of the coefficients.

The purpose of this paper is to justify this spatial localization property. The result is obtained for the elliptic equations of elasticity using boundary integral representations of the solutions and comparing the linear and the perturbed solution of the problem.

### 1. Introduction

THE IDENTIFICATION of spatially distributed elastic moduli from overspecified displacement-force boundary measurements is an inverse problem denoted as *generalized elastic tomography* (for an overview see [4]). From a practical point of view this means that an interior distribution of elastic moduli can not be accessed by direct measurements and has to be recovered from boundary information. This can be for example the case, if the initial distribution of elastic moduli is affected by damage.

A series of research papers has been devoted in last years to the solution of different aspects of the problem. The uniqueness and the identifiability results have been obtained in most cases using the observation equation established initially by CALDERON [5] for the linearized electric tomography problem. These

results relate the Dirichlet-to-Neumann data map to integral representations of the distribution of elastic moduli.

IKEHATA [9] and NAKAMURA and UHLMANN [19, 20] solved the linearized isotropic problem (2 Lamé moduli) under different hypothesis of regularity of the moduli. The case of the anisotropic elasticity has been discussed by CONSTANTINESCU [7] who constructed a non-uniqueness example based on previous work by KOHN and VOGELIUS [13] and proposed a reconstruction technique for some cases.

Recently IKEHATA [10, 11] gave estimation of the size of inclusion in an integral formulation using the solution of nonperturbed and perturbed linearized direct problem and the Dirichlet-to-Neumann data map.

It is interesting to remark that mathematical results are based on the observation equation and the Dirichlet-to-Neumann data map, whether numerical reconstruction techniques are based on the minimization of error functionals. One of these functionals is the error of constitutive law (ELC), which has the physical dimensions of energy and measures actually the difference between the estimated strain, stress and deformation energy using kinematically and statically admissible fields. Therefore it provides a simple and robust alternating direction minimization algorithm.

This error functional can be first retrieved in the works of LADEVÈZE and LEGUILLON [17] as a measure of the error of finite element computations. For the second time it has been utilized as a cost functional for identifying spatially distributed coefficients in partial differential equations. It has been used for electric conductivities by KOHN *et al.* [12, 14], for anisotropic elastic moduli with static measurements by CONSTANTINESCU: for two-dimensional problems [6, 7] and for elastic plates [8], or for elastic moduli and modal measurements by LADEVÈZE *et al.* [18] and BEN ABDALLAH *et al.* [1, 2].

An intriguing result of the numerical reconstruction using the ELC has been the good spatial localization of the defects or perturbations of the elastic moduli through the error of constitutive law.

The purpose of this paper is to give an formal justification for the spatial localization property of the ELC.

In a first part the solution of the perturbed direct problem is split in a zero order and a first order term. The perturbed, the zero and first order expression of the strains and stresses are represented using a boundary integral formulation. Using these formulations one can estimate the differences between the different solutions as a function of the distance to the boundary or the support of the perturbation. A similar result has been obtained by ISAACSON and ISAACSON [15] for the inverse electrical problem. Their result is based on the complete solution of an circular domain with a centered inclusion.

As a direct consequence, the estimations for the spatial distribution of the

error of constitutive law as functions of the same distances to the boundary show that the error of constitutive law is negligible outside the defects and far from the boundary.

In the final section some numerical computations of the distribution of the error of constitutive law will illustrate the localization property.

## 2. The direct and the inverse problem

Let us consider an elastic body under the hypothesis of small strains and rotations. The body occupies a domain  $\Omega$  in its reference configuration.

The displacement, strain and stress fields respectively, denoted by  $\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}$ , are subject, considering the absence of a body force, to the following set of equations:

$$(2.1) \quad \begin{aligned} \boldsymbol{\varepsilon} &= \frac{1}{2}(\nabla + \nabla^T)\mathbf{u}, \\ \boldsymbol{\sigma} &= \mathbf{L} : \boldsymbol{\varepsilon}, \\ \operatorname{div} \boldsymbol{\sigma} &= \mathbf{0}, \end{aligned}$$

where  $\mathbf{L}$  represents the fourth order tensor of elastic moduli.

The *direct elasticity problem* consists in computing a solution of the system of partial differential Eq. (2.1) with known elastic moduli  $\mathbf{L}$  and given one of the following boundary conditions on  $\partial\Omega$ : prescribed displacements  $\mathbf{u}|_{\partial\Omega} = \mathbf{w}$  or prescribed traction  $\boldsymbol{\sigma}\mathbf{n}|_{\partial\Omega} = \mathbf{t}$ . The pairs of corresponding boundary conditions  $(\mathbf{u}, \mathbf{t})$  can generally be described in terms of the Dirichlet-to-Neumann data map:

$$(2.2) \quad \Lambda_{\mathbf{L}} : \mathbf{w} \longrightarrow \Lambda_{\mathbf{L}}(\mathbf{w}) = \mathbf{t}$$

which maps a given boundary displacement at the corresponding boundary traction.

The *inverse elasticity problem* attempts to determine the *unknown* elastic moduli  $\mathbf{L}$  from the partial knowledge of the Dirichlet-to-Neumann data map  $\Lambda_{\mathbf{L}}$ . From a practical point of view; this means that the elastic coefficients have to be determined from a series of overdetermined boundary conditions, i.e. simultaneously known displacements  $\mathbf{u}$  and traction  $\mathbf{t}$ .

Let us suppose that the elastic moduli  $\mathbf{L}$  can be expressed as:

$$\mathbf{L} = \mathbf{L}_0 + \eta\mathbf{L}_1, \quad \forall \mathbf{x} \in \Omega,$$

where  $\eta \in \mathbb{R}$  is small parameter. We shall further assume that the support of  $\mathbf{L}_1$  is strictly included in the interior of  $\Omega$ .

In the sequel we shall consider the following series expansion of the displacement field  $\mathbf{u}$ :

$$\mathbf{u} = \mathbf{u}_0 + \eta \mathbf{u}_1 + o(\eta^2)$$

and of the stress field  $\sigma$ :

$$\sigma = \sigma_0 + \eta \sigma_1 + o(\eta^2),$$

and will inspect the spatial distribution of the various differences of stress terms, remarking that a similar reasoning would hold for the distribution of strains.

The *zero order* displacement  $\mathbf{u}_0$  is a solution of the following equation:

$$(2.3) \quad \operatorname{div} (\mathbf{L}_0 : \nabla \mathbf{u}_0) = \mathbf{0}$$

with the boundary conditions:

$$(2.4) \quad \mathbf{u}_0|_{\partial\Omega} = \mathbf{w} \quad \text{or} \quad \sigma_0 \mathbf{n}|_{\partial\Omega} = \mathbf{t}.$$

The stress field  $\sigma_0$  can be expressed using a classical integral representation:

$$(2.5) \quad \sigma_0(\mathbf{x}) = - \int_{\partial\Omega} \mathbf{P}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}_0(\mathbf{y}) dS_y + \int_{\partial\Omega} \mathbf{Q}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot (\sigma_0(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})) dS_y$$

with  $\mathbf{P}_{\mathbf{L}_0}$  and  $\mathbf{Q}_{\mathbf{L}_0}$  vectors fields computed from the Green function of the domain (see Appendix).

If  $\sigma_0$  has been determined with the imposed displacement  $\mathbf{w}$ , we can further write:

$$(2.6) \quad \sigma_0[\mathbf{w}](\mathbf{x}) = - \int_{\partial\Omega} \mathbf{P}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{w}(\mathbf{y}) dS_y + \int_{\partial\Omega} \mathbf{Q}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \Lambda_{\mathbf{L}_0}(\mathbf{w})(\mathbf{y}) dS_y.$$

The *first order* displacement  $\mathbf{u}_1$  is a solution of the following equation:

$$(2.7) \quad \operatorname{div} (\mathbf{L}_0 : \nabla \mathbf{u}_1) = - \operatorname{div} (\mathbf{L}_1 : \nabla \mathbf{u}_0)$$

with the boundary conditions:

$$(2.8) \quad \mathbf{u}_1|_{\partial\Omega} = \mathbf{0} \quad \text{or} \quad \sigma_1 \mathbf{n}|_{\partial\Omega} = \mathbf{0}.$$

The first order perturbation displacement is therefore generated by the body force term corresponding to stresses created by the zero displacement and the perturbation of the elastic moduli.

The Dirichlet-to-Neuman data map corresponding to this problem will be denoted by  $\Lambda_{\mathbf{L}_0, \mathbf{L}_1}$ .

The stress field  $\sigma_1$  can also be expressed using a classical integral representation:

$$(2.9) \quad \sigma_1(\mathbf{x}) = \int_{\Omega} \mathbf{L} : \nabla_{\mathbf{x}} \mathbf{G}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \operatorname{div}_{\mathbf{y}}(\mathbf{L}_1 : \nabla \mathbf{u}_0(\mathbf{y})) \, dV_{\mathbf{y}} \\ - \int_{\partial\Omega} \mathbf{P}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}_1(\mathbf{y}) \, dS_{\mathbf{y}} + \int_{\partial\Omega} \mathbf{Q}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot (\sigma_1(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})) \, dS_{\mathbf{y}}$$

with  $\mathbf{P}_{\mathbf{L}_0}$  and  $\mathbf{Q}_{\mathbf{L}_0}$  the Green function of the domain (see Appendix).

Similarly, we can rewrite the *perturbed* system of Eq. (2.8) in the form:

$$(2.10) \quad \operatorname{div}(\mathbf{L}_0 : \nabla \mathbf{u}) = - \operatorname{div}(\eta \mathbf{L}_1 : \nabla \mathbf{u})$$

which yields the following integral representation of the stress field:

$$(2.11) \quad \sigma(\mathbf{x}) = \int_{\Omega} \mathbf{L}_0 : \nabla_{\mathbf{x}} \mathbf{G}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \operatorname{div}_{\mathbf{y}}(\eta \mathbf{L}_1 : \nabla \mathbf{u}(\mathbf{y})) \, dV_{\mathbf{y}} \\ - \int_{\partial\Omega} \mathbf{P}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \, dS_{\mathbf{y}} + \int_{\partial\Omega} \mathbf{Q}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot (\sigma(\mathbf{y}) \cdot \mathbf{n}(\mathbf{y})) \, dS_{\mathbf{y}}.$$

Let us now compare different stress fields obtained from the same prescribed boundary displacement  $\mathbf{w}$ .

The difference between the *nonlinear* and the *zero order* solution,  $\sigma[\mathbf{w}]$  and  $\sigma_0[\mathbf{w}]$ , can be written after integration by parts as:

$$(2.12) \quad \sigma[\mathbf{w}](\mathbf{x}) - \sigma_0[\mathbf{w}](\mathbf{x}) \\ = (fp) \int_{\Omega} (\mathbf{L}_0 : \nabla_{\mathbf{x}} \nabla_{\mathbf{y}} \mathbf{G}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y})) \cdot (\mathbf{L}_1 : \nabla \mathbf{u}(\mathbf{y})) \, dV_{\mathbf{y}} \\ + \int_{\partial\Omega} \mathbf{Q}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot (\Lambda_{\mathbf{L}}(\mathbf{w}) - \Lambda_{\mathbf{L}_0}(\mathbf{w})) \, dS_{\mathbf{y}},$$

where  $(fp)$  denotes that the finite part of the integral.

A close inspection of the integrals shows their behavior for  $\mathbf{x} \in \Omega \setminus \operatorname{supp} \mathbf{L}_1$ . For a three-dimensional problem:

- the first term behaves as  $|\mathbf{x} - \mathbf{y}|^{-3}$ . As the inclusion lies in the interior of the body,  $\operatorname{supp} \mathbf{L}_1 \subset \Omega$ , it follows that the integral decreases as  $r^{-3}$  where  $r = d(\mathbf{x}, \operatorname{supp} \mathbf{L})$ ,
- the second term behaves as  $|\mathbf{x} - \mathbf{y}|^{-2}$ , it then follows that the integral decreases as  $r^{-2}$  with  $r = d(\mathbf{x}, \partial\Omega)$ .

For two-dimensional problems, the integral decreases as  $r^{-2}$  with the distance from the inclusion ( $r = d(\mathbf{x}, \text{supp}\mathbf{L})$ ) and as  $r^{-1}$  with the distance from the boundary ( $r = d(\mathbf{x}, \partial\Omega)$ ).

A direct consequence of this behavior, whether in two or three dimensions, is that the stress difference  $\sigma[\mathbf{w}] - \sigma_0[\mathbf{w}]$  is negligible outside the support of the perturbation of the elastic moduli  $\mathbf{L}_1$  and far from the boundaries.

The difference between the *nonlinear* stress term and the *first order* solution,  $\sigma[\mathbf{w}]$  and  $\sigma_0[\mathbf{w}] + \eta\sigma_1[\mathbf{0}]$ , can be expressed in a similar way and one obtains:

$$(2.13) \quad \begin{aligned} \sigma[\mathbf{w}](\mathbf{x}) - (\sigma_0[\mathbf{w}](\mathbf{x}) + \eta\sigma_1[\mathbf{0}](\mathbf{x})) \\ = (fp) \int_{\Omega} (\mathbf{L}_0 : \nabla_{\mathbf{x}} \nabla_{\mathbf{y}} \mathbf{G}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y})) \cdot (\eta\mathbf{L}_1(\mathbf{y}) : \nabla(\mathbf{u}(\mathbf{y}) - \mathbf{u}_0)) dV_{\mathbf{y}} \\ + \int_{\partial\Omega} \mathbf{Q}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot (\Lambda_{\mathbf{L}}(\mathbf{w}) - \Lambda_{\mathbf{L}_0}(\mathbf{w}) - \eta\Lambda_{\mathbf{L}_0, \mathbf{L}_1}(\mathbf{0})) dS_{\mathbf{y}}. \end{aligned}$$

As before, due to the singular behavior of the Green function and its derivatives, one can remark that this stress difference is also negligible outside the support of the perturbation of the elastic moduli  $\mathbf{L}_1$  and far from the boundary, since the behavior of the integrals is similar to the one before.

One can observe that the technique developed before is related to the *multiple reciprocity method* [21, 22] which allows to transform the volume integral corresponding to a body force into an infinite series of boundary integrals using the solutions of a series of auxiliary problems. The development of the series has been stopped here at its first term in order to obtain the results presented in the sequel.

### 3. Spatial defect localization of the error of constitutive law

The error of constitutive law is an error functional which can be defined for triplet  $(\mathbf{L}, \epsilon, \sigma)$  of a priori independent fields of elastic moduli, strains and stresses respectively, as:

$$(3.1) \quad \begin{aligned} \mathcal{J}(\mathbf{L}, \epsilon, \sigma) &= \int_{\Omega} \left| \mathbf{L}^{-\frac{1}{2}} : \sigma - \mathbf{L}^{\frac{1}{2}} : \epsilon \right|^2 dV \\ &= \int_{\Omega} (\sigma : \mathbf{L}^{-1} : \sigma - 2\sigma : \epsilon + \epsilon : \mathbf{L} : \epsilon) dV \\ &= \int_{\Omega} \left( \sigma_{ij} L_{ijkl}^{-1} \sigma_{kl} - 2\sigma_{ij} \epsilon_{ij} + \epsilon_{ij} L_{ijkl} \epsilon_{kl} \right) dV. \end{aligned}$$

The symetry and the positive definiteness of  $\mathbf{L}$  assure the existence of  $\mathbf{L}^{-\frac{1}{2}}$ .

The error of constitutive law can also be expressed in one of the following equivalent forms:

$$\begin{aligned}
 (3.2) \quad \mathcal{J}(\mathbf{L}, \epsilon, \sigma) &= \int_{\Omega} (\sigma - \mathbf{L} : \epsilon) : \mathbf{L}^{-1} : (\sigma - \mathbf{L} : \epsilon) \, dV \\
 &= \int_{\Omega} (\mathbf{L}^{-1} : \sigma - \epsilon) : \mathbf{L} : (\mathbf{L}^{-1} : \sigma - \epsilon) \, dV.
 \end{aligned}$$

Let us consider the corresponding elastic solutions  $(\mathbf{u}_L[\mathbf{w}], \epsilon_L[\mathbf{w}], \sigma_L[\mathbf{w}])$  and  $(\mathbf{u}_L[\mathbf{t}], \epsilon_L[\mathbf{t}], \sigma_L[\mathbf{t}])$  determined by the elasticity tensor  $\mathbf{L}$  in the absence of body forces and the boundary displacement  $\mathbf{w}$  respectively boundary traction  $\mathbf{t}$ .

For these fields one can write the error of constitutive law from (3.2) as:

$$\begin{aligned}
 (3.3) \quad \mathcal{J}(\mathbf{L}, \epsilon_L[\mathbf{w}], \sigma_L[\mathbf{t}]) &= \int_{\Omega} (\mathbf{L}^{-1} : \sigma_L[\mathbf{t}] - \epsilon_L[\mathbf{w}]) : \mathbf{L} : (\mathbf{L}^{-1} : \sigma_L[\mathbf{t}] - \epsilon_L[\mathbf{w}]) \, dV \\
 &= \int_{\Omega} (\epsilon_L[\mathbf{w}] - \epsilon_L[\mathbf{t}]) : \mathbf{L} : (\epsilon_L[\mathbf{w}] - \epsilon_L[\mathbf{t}]) \, dV.
 \end{aligned}$$

After integration by parts, using first the equilibrium equations and then the boundary conditions for both solutions, one obtains:

$$\begin{aligned}
 (3.4) \quad \mathcal{J}(\mathbf{L}, \epsilon_L[\mathbf{w}], \sigma_L[\mathbf{t}]) &= \int_{\partial\Omega} (\mathbf{u}_L[\mathbf{w}] - \mathbf{u}_L[\mathbf{t}]) : (\sigma_L[\mathbf{w}] \cdot \mathbf{n} - \sigma_L[\mathbf{t}] \cdot \mathbf{n}) \, dS \\
 &= \int_{\partial\Omega} (\mathbf{w} - \mathbf{u}_L[\mathbf{t}]) \cdot (\sigma_L[\mathbf{w}]\mathbf{n} - \mathbf{t}) \, dS.
 \end{aligned}$$

One can see that if  $(\mathbf{w}, \mathbf{t})$  is a measurement pair, i.e.  $\mathbf{t} = \Lambda_L(\mathbf{w})$ , then  $\mathbf{u}_L[\mathbf{w}] = \mathbf{u}_L[\mathbf{t}]$ , consequently the ECL vanishes as expected:  $\mathcal{J}(\mathbf{L}, \epsilon_L[\mathbf{w}], \sigma_L[\mathbf{t}]) = 0$ .

The last boundary integral provides another physical interpretation for the ELC as the mechnaical work provided by the error in displacements on the error of forces.

Let us now assume that  $(\mathbf{w}, \mathbf{t})$  is a measurement pair, i.e.  $\mathbf{t} = \Lambda_L(\mathbf{w})$  and compute the zero order approximations  $\mathbf{u}_0[\mathbf{w}]$  and  $\mathbf{u}_0[\mathbf{t}]$ . Using the solution of the unperturbed problem  $\mathbf{u}[\mathbf{w}] = \mathbf{u}[\mathbf{t}]$ , we obtain the following expressions for the error of constitutive law computed for the zero order fields:

$$\begin{aligned}
 (3.5) \quad \mathcal{J}(\mathbf{L}_0, \epsilon_0[\mathbf{w}], \sigma_0[\mathbf{t}]) &= \int_{\Omega} (\epsilon_0[\mathbf{w}] - \epsilon_0[\mathbf{t}]) : \mathbf{L}_0 : (\epsilon_0[\mathbf{w}] - \epsilon_0[\mathbf{t}]) \, dV \\
 &= \int_{\Omega} (\epsilon_0[\mathbf{w}] - \epsilon[\mathbf{w}]) : \mathbf{L}_0 : (\epsilon_0[\mathbf{w}] - \epsilon[\mathbf{w}]) \, dV \\
 &\quad + \int_{\Omega} (\epsilon[\mathbf{t}] - \epsilon_0[\mathbf{t}]) : \mathbf{L}_0 : (\epsilon[\mathbf{t}] - \epsilon_0[\mathbf{t}]) \, dV.
 \end{aligned}$$

It is seen, on the one hand that, this expression is the one minimized in the alternating direction implicit algorithm [7] and, on the other hand, that the ELC is now expressed in terms of the difference between the perturbed and the zero order solution of the direct problem which have been previously computed.

The integrand in the ECL is therefore negligible far from the support of the inclusion and the boundary. More precisely, in the three-dimensional problem they behave as:

- $r^{-6}$  as a function of the distance to the support of the inclusion  $r = d(\mathbf{x}, \text{supp}\mathbf{L})$ , and as
- $r^{-4}$  as a function of the distance to the boundary  $r = d(\mathbf{x}, \partial\Omega)$ .

For two-dimensional problems, the decrease is in  $r^{-4}$  from the inclusion ( $r = d(\mathbf{x}, \text{supp}\mathbf{L})$ ) and in  $r^{-2}$  from the boundary ( $r = d(\mathbf{x}, \partial\Omega)$ ).

#### 4. Numerical examples

The good spatial localization of defects of the error of constitutive law have been reported in a series of papers [1, 2, 6, 7, 8].

We shall present in the next example some localization results of the error of constitutive law. The case treated here has been computed for a square inclusion in an isotropic elastic body. The distribution of the Young modulus is represented in Fig. 1.

First a series of displacement-force measurements  $(\mathbf{w}_i, \mathbf{t}_i)$ ,  $i = 1, N$  have been simulated by a direct numerical computation for the perturbed distribution of elastic moduli  $\mathbf{L} = \mathbf{L}_0 + \eta\mathbf{L}_1$ . These measurements correspond to the prescribed distributed parabolic pressures at different locations around the body.

Then a series of first order solutions  $\mathbf{u}_0[\mathbf{w}_i]$ ,  $i = 1, N$  and  $\mathbf{u}_0[\mathbf{t}_i]$ ,  $i = 1, N$  have been computed from the previous measurements  $(\mathbf{w}_i, \mathbf{t}_i)$ ,  $i = 1, N$  with the unperturbed moduli  $\mathbf{L}_0$ .



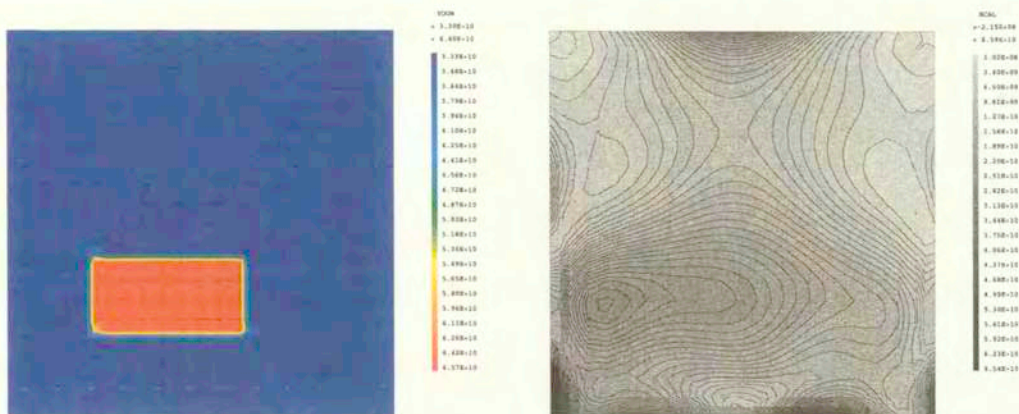


FIG. 1. Spatial distribution of the perturbed moduli  $\mathbf{L} = \mathbf{L}_0 + \eta \mathbf{L}_1$  for a square inclusion (left) and the corresponding error of constitutive law given by a parabolic pressure distribution in the middle of the upper surface (right).

For the measurement  $i$ ,  $(\mathbf{w}_i, \mathbf{t}_i)$ , the error in the constitutive law is then defined as (see Eq. (3.5)):

$$\mathcal{J}(\mathbf{L}_0, \epsilon_0[\mathbf{w}_i], \sigma_0[\mathbf{t}_i]).$$

In Fig. 2 we have presented the spatial distribution of the inclusion and the distribution of ECL corresponding to only one measurement. It is interesting to see that with only one measurement, a pressure peak appearing in the middle of the upper side of the domain, the inclusion has practically been localized.

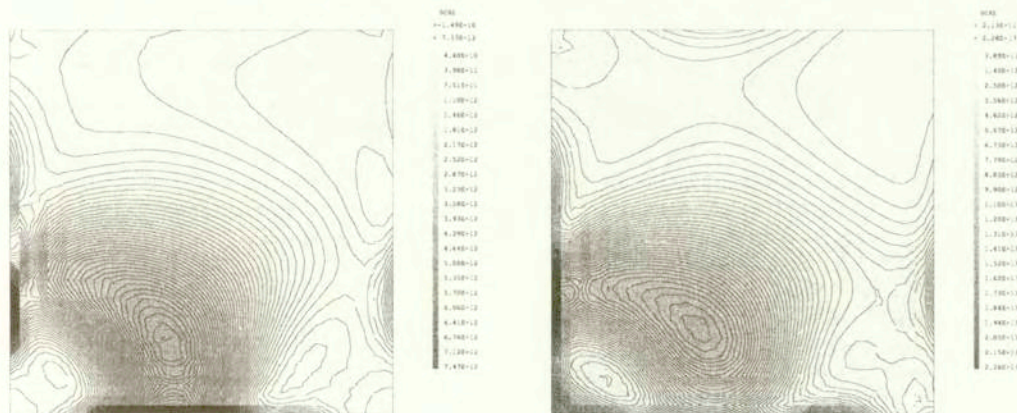


FIG. 2. Spatial distribution of cumulated error of constitutive law for a series of measurements on the lower side (left) and all sides (right) of the square.

If one continues to add the errors corresponding to other measurements, i.e.:

$$\sum_{i=1, N} \mathcal{J}(\mathbf{L}_0, \epsilon_0[\mathbf{w}_i], \sigma_0[\mathbf{t}_i])$$

changing the location of the parabolic pressure distribution along the boundary, one can see that the spatial localization has not been preserved.

A close inspection of the isolines shows that the gradients around the inclusion are steep, what is in agreement with the theoretical results.

## 5. Conclusion

In this paper we have shown that the difference between the nonlinear and zero and, respectively, first order approximations of the stresses are negligible far from the perturbation of the elastic moduli and the boundary of the body. As a direct consequence, one can explain the good spatial localization property of the error of constitutive law observed during previous numerical experiments.

It is obvious that, using similar integral equation techniques, localization results could be proved for the inverse electric or the inverse plate problem.

## Acknowledgements

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## Appendix. Integral representation formulas [3]

The fundamental solution or Green function  $\mathbf{G}_{\mathbf{L}_0}$  is defined on an open  $E$  compatible with  $\Omega$  by the equations:

$$(A.1) \quad \operatorname{div} \mathbf{L}_0 \nabla_y \mathbf{G}_{\mathbf{L}_0}^k(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) \mathbf{e}_k = \mathbf{0}.$$

For  $x \in \Omega$  it can shown that the elastic stress field can be expressed as:

$$(A.2) \quad \sigma_{ij}(\mathbf{x}) = \int_{\partial\Omega} \mathbf{L}_0 : \nabla_x \mathbf{G}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{t} dS_y - \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{L}_0 : \nabla_x \mathbf{G}_{\mathbf{L}_0}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{w} dS_y \\ + \int_{\Omega} \mathbf{L}_0 : \nabla_x \mathbf{G}_{\mathbf{L}_0} \cdot \mathbf{b} dV_y,$$

where  $\mathbf{w}$ ,  $\mathbf{t}$ ,  $\mathbf{b}$  denote respectively the vector of boundary displacement, boundary traction and body forces.  $\mathbf{n}$  represents the unit outward normal the domain. The

following notations will be used for the third order tensor fields:

$$(A.3) \quad \mathbf{Q}_{L_0}(\mathbf{x}, \mathbf{y}) = \mathbf{L}_0 : \nabla_{\mathbf{x}} \mathbf{G}_{L_0}(\mathbf{x}, \mathbf{y}),$$

$$(A.4) \quad \mathbf{P}_{L_0}(\mathbf{x}, \mathbf{y}) = \mathbf{n} \cdot \mathbf{L}_0 : \nabla_{\mathbf{x}} \mathbf{G}_{L_0}(\mathbf{x}, \mathbf{y}) = \mathbf{n} \cdot \mathbf{L}_0 : \nabla_{\mathbf{x}} \mathbf{Q}_{L_0}(\mathbf{x}, \mathbf{y}).$$

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