Arch. Mech., **52**, 4–5, pp. 489–509, Warszawa 2000

Hencky's elasticity model with the logarithmic strain measure: a study on Poynting effect and stress response in torsion of tubes and rods

Dedicated to Professor Zenon Mróz
on the occasion of his 70^{th} birthday

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Hencky's elasticity model is a finite strain elastic constitutive equation derived by replacing the infinitesimal strain measure in the classical strain-energy function of infinitesimal isotropic elasticity with Hencky's logarithmic strain measure. Anand [1, 2] has demonstrated that, with only the two classical Lamé elastic constants measurable at infinitesimal strains, predictions of the just-mentioned simple model for a wide class of materials for moderately large deformations may be in better agreement with experimental data than other known finite elasticity models. The deformation modes considered in Anand's work are simple tension and compression, simple shear, and simple torsion and combined extension-torsion of solid cylinders, etc. Here, we indicate some remarkable properties of this Hencky model and, mainly, we investigate the large deformation responses of this model in torsion of cylindrical tubes and rods with free ends. It is pointed out that if in inelastic modeling, especially in modeling of metal plasticity, the widely-used hypoelastic formulation for the elastic rate of deformation is required to be exactly integrable to an elastic relation, as it should be, then the resulting elastic relation is just the Hencky model, and, further, this model is hyperelastic and the only possible one. In the main aspect, i.e. for the torsion of cylindrical tubes and rods with free ends, we derive explicit analytical solutions for the case of compressible small deformations and for the case of incompressible large deformations. The results derived show, in a clear and direct manner, the second order effects, including the well-known Poynting effect regarding the length change in the axial direction. It is noticeable that, with only the material properties measurable at infinitesimal strains, the Hencky model can predict the just-mentioned second order effects, in particular the Poynting effect, and its predictions are in good accord with experiments reported in the literature.

1. Introduction

Let $\tilde{\mathbf{e}}$ be the infinitesimal strain measure. Then the classical strain-energy function of infinitesimal isotropic elasticity is of the form

$$W = \frac{1}{2}\Lambda(\mathrm{tr}\tilde{\mathbf{e}})^2 + G\mathrm{tr}\tilde{\mathbf{e}}^2 \,. \tag{1.1}$$

Here and henceforth, Λ and G are the Lamé elastic constants, and $\mathrm{tr}\mathbf{S}$ is used to denote the trace of a second order tensor $\mathbf{S}$, i.e. $\mathrm{tr}\mathbf{S} := S_{ii}$ in a Cartesian frame.

Hencky's logarithmic strain measure $\mathbf{h}$ (cf. HENCKY [14 – 16])[1] is defined by

$$\mathbf{h} = \frac{1}{2}\ln\mathbf{B} = \frac{1}{2}\sum_{i=1}^{3}(\ln b_i)\mathbf{n}_i \otimes \mathbf{n}_i \,, \tag{1.2}$$

where $\mathbf{B}$ is the left Cauchy-Green tensor and b_i and $\mathbf{n}_i$ are the three eigenvalues (possibly repeated) and the three corresponding orthonormal eigenvectors of $\mathbf{B}$. Considering possible multiple eigenvalues and the uniqueness problems that may occur when the triplet $\mathbf{n}_i \otimes \mathbf{n}_i$ in Eq. (1.2) is used, it has been proved more convenient to use the following alternate formulation

$$\mathbf{h} = \frac{1}{2}\sum_{\sigma=1}^{m}(\ln b_\sigma)\mathbf{B}_\sigma \,, \tag{1.3}$$

where m is the number of distinct eigenvalues and the $\mathbf{B}_\sigma$ are the corresponding eigenprojections. (For detail see e.g. XIAO, BRUHNS and MEYERS [31].)

Replacing now the infinitesimal strain measure $\tilde{\mathbf{e}}$ in Eq. (1.1) with Hencky's strain measure $\mathbf{h}$ given by Eq. (1.2), an isotropic scalar function is obtained:

$$W = \frac{1}{2}\Lambda(\mathrm{tr}\mathbf{h})^2 + G\mathrm{tr}\mathbf{h}^2 \,. \tag{1.4}$$

Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor and $\mathbf{I}$ the identity tensor. Then we derive a finite strain isotropic elastic constitutive equation as follows:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \frac{\partial W}{\partial \mathbf{h}} = \Lambda(\mathrm{tr}\mathbf{h})\mathbf{I} + 2G\mathbf{h} \,, \tag{1.5}$$

where $\boldsymbol{\tau}$ $(:=J\boldsymbol{\sigma})$ with $J = \sqrt{b_1 b_2 b_3}$ is the Kirchhoff stress tensor. Evidently, the constitutive equation (1.5), which establishes a linear relation between the Kirchhoff stress and the Hencky strain measure, is a direct generalization of the classical Hooke's law.

ANAND [1, 2] has demonstrated that, with only the two classical Lamé elastic constants Λ and G measurable at infinitesimal strains, predictions of the simple

[1] Although it seems that Hencky independently introduced the logarithmic strain measure in 1928 and was the first to make an extensive use of this strain measure in studying finite elastic deformations of rubbers etc. in a series of works, it had been introduced earlier by several other researchers, including Imbert in 1880 and Ludwik in 1909, *et al.* For detail, refer to a survey by CURNIER and RAKOTOMANANA [7].

elastic equation (1.5) for a wide class of materials for moderately large[2] deformations, where the principal stretch falls within the range (0.7; 1.3), may be in better agreement with experimental data than other known finite elasticity models. In Anand's work a number of basic deformation modes are considered, including simple tension and compression, simple shear, and simple torsion and combined extension-torsion of incompressible solid cylinders, etc.

The finite elasticity equation (1.5) will be called *Hencky's elasticity model.* In Sec. 2, we shall indicate some remarkable properties of this model, and, mainly, in the other sections, besides the basic deformation modes considered by ANAND [1, 2], we further study stress and deformation response of this model, in particular the Poynting effect, in torsion of cylindrical tubes and rods with free ends.

2. Some remarkable properties of the Hencky model

First, we show that the Hencky model (1.5) is a finite elasticity model in Green's sense, i.e. a finite hyperelasticity model, and that its strain-energy function is just given by Eq. (1.4). This fact has been pointed out in XIAO, BRUHNS and MEYERS [29] by virtue of the integrability conditions for the hypoelasticity model with the logarithmic stress rate. In what follows we supply a short alternative proof.

In fact, for the elastic material defined by the Hencky model (1.5), the specific stress power per unit reference volume is given by

$$\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}) = \mathrm{tr}(\frac{\partial W}{\partial \mathbf{h}}\mathbf{D}), \tag{2.1}$$

where $\mathbf{D}$ is the stretching tensor, or the tensor of rate of deformation. Applying a formula recently derived in XIAO, BRUHNS and MEYERS [30, 32], we infer that the following relation holds:

$$\mathbf{D} = \dot{\mathbf{h}} + \mathbf{h}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\mathbf{h}, \tag{2.2}$$

where $\boldsymbol{\Omega}^{\log}$ is a skewsymmetric tensor, called the logarithmic spin. Since $\boldsymbol{\tau}$ and $\mathbf{h}$ are coaxial, as shown by Eq. (1.5), we have

$$\mathrm{tr}(\boldsymbol{\tau}(\mathbf{h}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\mathbf{h})) = 0.$$

Thus, substituting Eq. (2.2) into Eq. (2.1) and utilizing the equality just given, we derive

$$\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}) = \mathrm{tr}(\frac{\partial W}{\partial \mathbf{h}}\dot{\mathbf{h}}),$$

[2]We note that the notion of "moderately large" deformations is generally not precisely defined. Here and in what follows we take the definition of ANAND [1, 2]. With reference to the behaviour of most metallic materials these stretches are indeed "moderate", if not "large".

in example

$$\dot{W} = \mathrm{tr}(\boldsymbol{\tau}\mathbf{D})\,. \tag{2.3}$$

The latter relation clearly shows that the material time derivative of the scalar function W given by Eq. (1.4) furnishes the specific stress power of the elastic material defined by Hencky's model (1.5). Thus, Hencky's model is a finite hyperelasticity model and its strain-energy function is exactly the scalar function W given by Eq. (1.4).

Second, we indicate a relation of the model (1.5) with inelastic modeling. In inelastic modeling, especially in modeling of metal plasticity, the hypoelastic equation of grade zero

$$\overset{\circ}{\boldsymbol{\tau}} = 2G\mathbf{D}^e + \Lambda(\mathrm{tr}\mathbf{D}^e)\mathbf{I}\,, \tag{2.4}$$

and its inverse

$$\mathbf{D}^e = \frac{1}{2G}\,\overset{\circ}{\boldsymbol{\tau}} - \frac{\Lambda}{2G(3\Lambda+2G)}(\mathrm{tr}\dot{\boldsymbol{\tau}})\mathbf{I}\,, \tag{2.5}$$

are widely used to formulate the linear relation between the elastic rate of deformation, $\mathbf{D}^e$, and an objective stress rate $\overset{\circ}{\boldsymbol{\tau}}$. In the sense of self-consistency, for each process of purely elastic deformation, i.e. for $\mathbf{D}^e = \mathbf{D}$, the above rate equation should be exactly integrable to deliver an elastic relation between a strain measure and the stress $\boldsymbol{\tau}$. Very recently, the present authors (BRUHNS, XIAO and MEYERS [6] and XIAO, BRUHNS and MEYERS [33]) have demonstrated that among the rate type Eqs. (2.4) or (2.5) with all possible corotational stress rates (cf. XIAO, BRUHNS and MEYERS, [32]) and other well-known stress rates, there is one and only one that is exactly integrable to deliver an elastic relation, and that the unique integrable rate-type equation of the form (2.4) or (2.5) exactly results in the Hencky model (1.5).

It turns out that the unique integrable hypoelastic formulation (2.4) or (2.5) is just an equivalent rate form of the Hencky model. Thus, the Hencky model is incorporated as a basic constituent in inelastic modeling.

Finally, it is evident that the potential W given by (1.4) is convex in the logarithmic strain measure $\mathbf{h}$, and hence fulfills Hill's inequality with the logarithmic strain measure (see HILL [17, 18]; see also OGDEN [21] and ŠILHAVÝ [27]).

As mentioned before, it has been confirmed by Anand's impressive work (ANAND [1, 2]) that the model (1.5) with the potential (1.4) and therefore its rate form (2.4) or (2.5) should have wide applicability for elastic behaviour of isotropic materials at moderately large deformations. In fact, it has been incorporated into commercial packets of finite element codes and widely used in numerical computation and simulation, see, e.g., BONET and WOOD [5] for detail.

In the succeeding sections, further study is provided for responses of this model in torsion of cylindrical tubes and solid cylinders.

3. Kinematics for simultaneous extension, inflation and torsion

For elastic materials, the finite deformation mode at issue and the corresponding stress response have been studied by many researchers, in particular including the well-known Poynting effect (cf. POYNTING [22, 23]). The general analyses in this aspect are presented in, e.g. RIVLIN [24, 25], RIVLIN and SAUNDERS [26], GREEN and SHIELD [10], GREEN and ZERNA [12], GREEN and ADKINS [13], TRUESDELL and NOLL [28], and OGDEN [21], *et al.* Investigations of the Poynting effect for elastic materials with particular strain-energy functions, such as neo-Hookean materials, Mooney-Rivlin materials and second order elasticity etc., can be found in, e.g. MURNAGHAN [20], RIVLIN [24, 25], GREEN and SHIELD [10], and GREEN [11], *et al.* Experimental data are available in, e.g. POYNTING [22, 23], RIVLIN and SAUNDERS [26], FOUX [8], FREUDENTHAL and RONAY [9], and BILLINGTON [4], *et al.* Recent development in this respect and related recent literature can be found in the monograph by ANTMAN [3]. Very recently, JIANG and OGDEN [19] have made a comprehensive study of interesting axial shear deformations of compressible elastic circular cylindrical tubes, in which some related references are incorporated.

Consider a fixed rectangular Cartesian coordinate system, $(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with the origin O at the midpoint of the axis of the cylinder under consideration and $\mathbf{e}_3$ in the direction of the just-mentioned axis. Accordingly, let $(O; \mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_3)$ be a fixed cylindrical polar coordinate system. With reference to the two fixed systems, a typical particle P in the cylinder has the coordinates (X_1, X_2, X_3) and (R, Θ, Z), respectively, i.e. the position vector $\overrightarrow{OP}$ of the particle P is given by

$$\mathbf{X} = \overrightarrow{OP} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3 \tag{3.1}$$

with

$$X_1 = R\cos\Theta, \quad X_2 = R\sin\Theta, \quad X_3 = Z\,. \tag{3.2}$$

At a current strained state, the foregoing particle P moves to p. With reference to the afore-mentioned two fixed systems, the point p has the coordinates (x_1, x_2, x_3) and (r, θ, z). For the deformation at issue, we have

$$\mathbf{x} = \overrightarrow{Op} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{3.3}$$

with

$$\begin{aligned} x_1 &= r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z\,, \\ r &= r(R), \quad \theta = \Theta + \psi Z, \quad z = \lambda Z\,, \end{aligned} \tag{3.4}$$

where ψ is the angle of twist per unit initial length and λ the axial extension ratio. With Eq. (3.2), Eq. (3.4) may be recast in the form

$$
\begin{aligned}
x_1 &= \frac{r}{R}(X_1 \cos\psi Z - X_2 \sin\psi Z)\,, \\
x_2 &= \frac{r}{R}(X_2 \cos\psi Z + X_1 \sin\psi Z)\,, \\
x_3 &= \lambda Z\,.
\end{aligned}
\tag{3.5}
$$

We now introduce a moving system of cylindrical polar coordinates $(O; \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3)$ by

$$
\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_1 \cos(\Theta + \psi Z) + \mathbf{e}_2 \sin(\Theta + \psi Z)\,, \\
\mathbf{e}_\theta &= -\mathbf{e}_1 \sin(\Theta + \psi Z) + \mathbf{e}_2 \cos(\Theta + \psi Z)\,.
\end{aligned}
\tag{3.6}
$$

Utilizing Eqs. (3.5) and (3.2), we obtain the deformation gradient

$$
\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{e}_j,
$$

referred to the fixed system $(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Then, with reference to the moving system given by Eq. (3.6) we arrive at

$$
\begin{aligned}
\mathbf{F} = r' \cos\psi Z \mathbf{e}_r \otimes \mathbf{e}_r + \frac{r}{R}\cos\psi Z \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda \mathbf{e}_3 \otimes \mathbf{e}_3 \\
- r' \sin\psi Z(\mathbf{e}_r \otimes \mathbf{e}_\theta - \mathbf{e}_\theta \otimes \mathbf{e}_r) + \psi r \mathbf{e}_\theta \otimes \mathbf{e}_3\,.
\end{aligned}
\tag{3.7}
$$

Throughout the paper we denote $r' = \partial r/\partial R$. Hence, the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is given by

$$
\begin{aligned}
\mathbf{B} = r'^2 \mathbf{e}_r \otimes \mathbf{e}_r + r^2(R^{-2} + \psi^2)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda^2 \mathbf{e}_3 \otimes \mathbf{e}_3 \\
+ \lambda\psi r(\mathbf{e}_\theta \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\theta)\,.
\end{aligned}
\tag{3.8}
$$

The three eigenvalues of $\mathbf{B}$ are as follows:

$$
b_1 = r'^2, \quad b_{2,3} = \frac{1}{2}(\lambda^2 + \frac{r^2}{R^2} + \psi^2 r^2 \pm \sqrt{u}\,)
\tag{3.9}
$$

with

$$
u = \left((\lambda + \frac{r}{R})^2 + \psi^2 r^2\right)\left((\lambda - \frac{r}{R})^2 + \psi^2 r^2\right),
\tag{3.10}
$$

and their corresponding subordinate eigenprojections are given by

$$
\mathbf{B}_1 = \mathbf{e}_r \otimes \mathbf{e}_r, \quad \mathbf{B}_2 = (\bar{\mathbf{B}} - b_3 \bar{\mathbf{I}})/(b_2 - b_3), \quad \mathbf{B}_3 = (\bar{\mathbf{B}} - b_2 \bar{\mathbf{I}})/(b_3 - b_2)\,,
\tag{3.11}
$$

where

$$\bar{\mathbf{I}} = \mathbf{I} - \mathbf{e}_r \otimes \mathbf{e}_r, \quad \bar{\mathbf{B}} = \mathbf{B} - \mathbf{e}_r \otimes \mathbf{e}_r.$$

Then, the Hencky strain tensor $\mathbf{h}$ defined by Eq. (1.3) is given by

$$\mathbf{h} = \frac{1}{2}(\ln b_1 \; \mathbf{B}_1 + \ln b_2 \; \mathbf{B}_2 + \ln b_3 \; \mathbf{B}_3).$$

Hence, we have

$$\mathbf{h} = \lambda\psi r \, \frac{\ln b_2 - \ln b_3}{2(b_2 - b_3)} \, (\mathbf{e}_\theta \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\theta) + \frac{(b_2 - \lambda^2)\ln b_2 + (\lambda^2 - b_3)\ln b_3}{2(b_2 - b_3)} \, \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{(\lambda^2 - b_3)\ln b_2 + (b_2 - \lambda^2)\ln b_3}{2(b_2 - b_3)} \, \mathbf{e}_3 \otimes \mathbf{e}_3 + \ln r' \mathbf{e}_r \otimes \mathbf{e}_r \, . \tag{3.12}$$

4. The governing equations and the boundary conditions

With reference to the moving system (3.6), for the deformation at issue the Cauchy stress tensor $\boldsymbol{\sigma}$ is of the form

$$\boldsymbol{\sigma} = \sigma_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{zz}\mathbf{e}_3 \otimes \mathbf{e}_3 + \sigma_{z\theta}(\mathbf{e}_3 \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_3) \, . \tag{4.1}$$

From the stress-deformation relation (1.5) and Eq. (3.12), we obtain the normal stress components

$$J\sigma_{rr} = \Lambda \ln J + 2G \ln r' \, , \tag{4.2}$$

$$J\sigma_{\theta\theta} = \Lambda \ln J + G \, \frac{(b_2 - \lambda^2)\ln b_2 + (\lambda^2 - b_3)\ln b_3}{b_2 - b_3} \, , \tag{4.3}$$

$$J\sigma_{zz} = \Lambda \ln J + G \, \frac{(\lambda^2 - b_3)\ln b_2 + (b_2 - \lambda^2)\ln b_3}{b_2 - b_3} \, , \tag{4.4}$$

and the shear stress component

$$J\sigma_{z\theta} = G\lambda\psi r \, \frac{\ln b_2 - \ln b_3}{b_2 - b_3} \, , \tag{4.5}$$

where b_2 and b_3 are given by Eqs. (3.9) and (3.10), and

$$J = \det\mathbf{F} = \sqrt{b_1 b_2 b_3} = \frac{\lambda r r'}{R} \, . \tag{4.6}$$

Noting that each non-vanishing stress component depends merely on r or, equivalently, on R, with reference to the moving system (3.6) the equations of equilibrium are reduced to the single one

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 . \tag{4.7}$$

Consider a cylindrical tube with free ends. Assume, further, that there are no tractions exerted on the inner and outer surfaces. Then, the boundary conditions are of the forms:

$$\sigma_{rr}|_{R=R_1} = 0, \quad \sigma_{rr}|_{R=R_0} = 0 , \tag{4.8}$$

$$\int_{R_0}^{R_1} r r' \sigma_{zz} dR = 0 . \tag{4.9}$$

In the above, R_1 and R_0 are used to denote the initial inner and outer radii. When a solid cylinder or a rod with free ends is treated, the conditions (4.8) should be replaced by

$$\sigma_{rr}|_{R=R_1} = 0, \quad r|_{R=0} = r(0) = 0 . \tag{4.10}$$

Finally, the resultant twisting moment at two ends is given by

$$M = 2\pi \int_{R_0}^{R_1} r^2 r' \sigma_{z\theta} dR . \tag{4.11}$$

In Eqs. (4.9) and (4.11), $R_0 = 0$ for solid cylinders and rods.

For the problem at issue, the unknowns are the non-vanishing stress components σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and $\sigma_{z\theta}$, as well as the axial extension ratio λ and the current radial coordinate r, each of which is a function of R and ψ. The stress-deformation relations (4.2) – (4.5), the equation of equilibrium (4.7) and the boundary conditions (4.8) and (4.9) (and (4.9) and (4.10), respectively, for a solid cylinder) constitute a coupled system of differential equations for these unknowns. It doesn't seem easy to derive from this system an analytical solution for the general case. In the subsequent sections, we shall provide a small deformation solution for the general case of compressible tubes and rods, and a (moderate) large deformation solution for the case of incompressible tubes and rods.

5. Small deformation solutions for compressible tubes and rods

Suppose that the angle of twist ψ is small, but not necessarily infinitesimal. Both the extension ratio λ and the dimensionless deformation quantity r/R are close to 1, i.e.

$$\tilde{\lambda} = \lambda - 1, \quad \tilde{\alpha} = \frac{r}{R} - 1 \tag{5.1}$$

are small. In fact, the latter two are of the same order of magnitude as ψ^2. Hence we have the asymptotic expressions

$$\begin{aligned}
\tilde{\lambda} &= \mathcal{O}(\psi^2), \quad \tilde{\alpha} = \mathcal{O}(\psi^2),\\
b_2 - b_3 &= \sqrt{u} = 2\psi R + \mathcal{O}(\psi^3),\\
b_2 &= 1 + \psi R + \frac{1}{2}\psi^2 R^2 + \tilde{\lambda} + \tilde{\alpha} + \mathcal{O}(\psi^3),\\
b_3 &= 1 - \psi R + \frac{1}{2}\psi^2 R^2 + \tilde{\lambda} + \tilde{\alpha} + \mathcal{O}(\psi^3),\\
\ln b_2 &= \psi R + \tilde{\lambda} + \tilde{\alpha} + \mathcal{O}(\psi^3),\\
\ln b_3 &= -\psi R + \tilde{\lambda} + \tilde{\alpha} + \mathcal{O}(\psi^3).
\end{aligned}$$

Here and henceforth, the notation $\mathcal{O}(x)$ stands for a small quantity of the same order of magnitude as the small quantity x. Then, utilizing the above asymptotic expressions and neglecting small quantities of higher orders than ψ (for the shear stress component $\sigma_{z\theta}$) or ψ^2 (for the other stress components except $\sigma_{z\theta}$), from Eqs. (3.9), (3.10) and (4.2) – (4.6) we derive

$$\sigma_{rr} = 2G\,\frac{\partial(R\tilde{\alpha})}{\partial R} + \Lambda\left(\tilde{\lambda} + \tilde{\alpha} + \frac{\partial(R\tilde{\alpha})}{\partial R}\right), \tag{5.2}$$

$$\sigma_{\theta\theta} = 2G(\frac{1}{4}\psi^2 R^2 + \tilde{\alpha}) + \Lambda\left(\tilde{\lambda} + \tilde{\alpha} + \frac{\partial(R\tilde{\alpha})}{\partial R}\right), \tag{5.3}$$

$$\sigma_{zz} = 2G(-\frac{1}{4}\psi^2 R^2 + \tilde{\lambda}) + \Lambda\left(\tilde{\lambda} + \tilde{\alpha} + \frac{\partial(R\tilde{\alpha})}{\partial R}\right), \tag{5.4}$$

$$\sigma_{z\theta} = G\psi R. \tag{5.5}$$

Substituting Eqs. (5.2) and (5.3) into Eq. (4.7) and using $r = R + R\tilde{\alpha}$, we obtain the differential equation

$$(\Lambda + 2G)\left(\frac{\partial^2(R\tilde{\alpha})}{\partial R^2} + \frac{\partial\tilde{\alpha}}{\partial R}\right) = \frac{1}{2}\,G\psi^2 R. \tag{5.6}$$

The general solution of the above equation is given by

$$\tilde{\alpha} = k_2 R^{-2} + k_1 + \frac{1}{16}\frac{G}{\Lambda + 2G}\,\psi^2 R^2, \tag{5.7}$$

with the two parameters k_1 and k_2 depending merely on ψ.

Now we consider the boundary conditions. For a tube with the initial outer and inner radii R_1 and R_0, Eqs. (4.8), (4.9), (5.2) and (5.4) yield

$$R = R_1,\ R_0: \quad (\Lambda + 2G)\frac{\partial(R\tilde{\alpha})}{\partial R} + \Lambda(\tilde{\lambda} + \tilde{\alpha}) = 0\,, \tag{5.8}$$

$$\int_{R_0}^{R_1} \left(2G(-\frac{1}{4}\psi^2 R^3 + R\tilde{\lambda}) + \Lambda(R\tilde{\lambda} + R\tilde{\alpha} + R\,\frac{\partial(R\tilde{\alpha})}{\partial R})\right) dR = 0\,. \tag{5.9}$$

For a rod with the initial radius R_1, Eqs. (4.9), (4.10), (5.2) and (5.4) produce

$$R = R_1: \quad (\Lambda + 2G)\frac{\partial(R\tilde{\alpha})}{\partial R} + \Lambda(\tilde{\lambda} + \tilde{\alpha}) = 0\,, \tag{5.10}$$

$$R = R_0: \quad R\tilde{\alpha} = 0\,, \tag{5.11}$$

$$\int_{0}^{R_1} \left(2G(-\frac{1}{4}\psi^2 R^3 + R\tilde{\lambda}) + \Lambda(R\tilde{\lambda} + R\tilde{\alpha} + R\,\frac{\partial(R\tilde{\alpha})}{\partial R})\right) dR = 0\,. \tag{5.12}$$

For a tube with the initial outer and inner radii R_1 and R_0, from Eqs. (5.7) – (5.9) we can determine k_1, k_2, $\tilde{\alpha}$ and $\tilde{\lambda}$, and then the stress components. The final results are as follows:

$$\lambda - 1 = \frac{1}{8}\psi^2(R_1^2 + R_0^2)\,, \tag{5.13}$$

$$r = R + \frac{1}{16}\psi^2\left(\frac{G}{\Lambda + 2G}R^3 - \frac{\Lambda + 3G}{\Lambda + 2G}(R_1^2 + R_0^2)R - \frac{2\Lambda + 3G}{\Lambda + 2G}R_1^2R_0^2R^{-1}\right)\,, \tag{5.14}$$

$$\sigma_{rr} = \frac{1}{8}\,\frac{2\Lambda + 3G}{\Lambda + 2G}\,G\psi^2\left(R^2 - (R_1^2 + R_0^2) + R_1^2R_0^2R^{-2}\right)\,, \tag{5.15}$$

$$\sigma_{\theta\theta} = \frac{1}{8}\,\frac{2\Lambda + 3G}{\Lambda + 2G}\,G\psi^2\left(3R^2 - (R_1^2 + R_0^2) - R_1^2R_0^2R^{-2}\right)\,, \tag{5.16}$$

$$\sigma_{zz} = \frac{1}{8}\,\frac{\Lambda + 4G}{\Lambda + 2G}\,G\psi^2(R_1^2 + R_0^2 - 2R^2)\,, \tag{5.17}$$

$$\sigma_{z\theta} = G\psi R\,. \tag{5.18}$$

For a rod with the initial radius R_1, from Eqs. (5.7) and (5.10) - (5.12) we can determine k_1, k_2, $\tilde{\alpha}$ and $\tilde{\lambda}$, and then the stress components. The final results for this case are:

$$\lambda - 1 = \frac{1}{8}\psi^2 R_1^2, \tag{5.19}$$

$$r = R + \frac{1}{16}\psi^2\left(\frac{G}{\Lambda+2G}R^3 - \frac{\Lambda+3G}{\Lambda+2G}R_1^2 R\right), \tag{5.20}$$

$$\sigma_{rr} = \frac{1}{8}\,\frac{2\Lambda+3G}{\Lambda+2G}\,G\psi^2(R^2 - R_1^2), \tag{5.21}$$

$$\sigma_{\theta\theta} = \frac{1}{8}\,\frac{2\Lambda+3G}{\Lambda+2G}\,G\psi^2(3R^2 - R_1^2), \tag{5.22}$$

$$\sigma_{zz} = \frac{1}{8}\,\frac{\Lambda+4G}{\Lambda+2G}\,G\psi^2(R_1^2 - 2R^2), \tag{5.23}$$

$$\sigma_{z\theta} = G\psi R. \tag{5.24}$$

It is of interest to note that Eqs. (5.19) - (5.23) are also obtainable by setting $R_0 = 0$ in Eqs. (5.13) - (5.17). Thus, Eqs. (5.13) - (5.18) supply the unified solution for both tubes and rods. On the other hand, the solution for incompressible tubes and rods are derivable by evaluating the limits of Eqs. (5.13) - (5.18) when $\Lambda \to \infty$. The results then are:

$$\lambda - 1 = \frac{1}{8}\psi^2(R_1^2 + R_0^2), \tag{5.25}$$

$$r = R - \frac{1}{16}\psi^2((R_1^2 + R_0^2)R + 2R_1^2R_0^2R^{-2}), \tag{5.26}$$

$$\sigma_{rr} = \frac{1}{4}G\psi^2(R^2 - (R_1^2 + R_0^2) + R_1^2R_0^2R^{-2}), \tag{5.27}$$

$$\sigma_{\theta\theta} = \frac{1}{4}G\psi^2(3R^2 - (R_1^2 + R_0^2) - R_1^2R_0^2R^{-2}), \tag{5.28}$$

$$\sigma_{zz} = \frac{1}{8}G\psi^2(R_1^2 + R_0^2 - 2R^2), \tag{5.29}$$

$$\sigma_{z\theta} = G\psi R. \tag{5.30}$$

Now, substituting the results for the rod (Eqs. (5.19) - (5.24)) into Eq. (4.11), and neglecting terms of higher order than ψ^3, we obtain the resultant twisting moment

$$M = \frac{\pi}{2} G R_1^3 \gamma \left\{ 1 - \frac{1}{48} \gamma^2 \frac{9\Lambda + 17G}{\Lambda + 2G} \right\}. \tag{5.31}$$

In the limit for incompressible materials ($\Lambda \to \infty$), this result turns over to the simple relation

$$\frac{M}{G R_1^3} = \frac{\pi}{2} \gamma \left\{ 1 - \frac{3}{16} \gamma^2 \right\} \tag{5.32}$$

for the dimensionless twisting moment, where here and in what follows the shear strain γ is related to the angle of twist, ψ, through $\gamma = \psi R_1$.

Equation (5.13) indicates that the axial strain $\epsilon_n = \tilde{\lambda}$ is proportional to the square of the angle of twist thus explaining the Poynting effect. According to the experimental data for solid cylindrical specimens of highly filled polyurethane rubber by FREUDENTHAL and RONAY [9], two $\epsilon_n - \gamma$ curves (cf. Fig. 2 therein; the other two curves for creep tests are not included here) for two different strain rates are of the forms:

$$\epsilon_n = 0.095\gamma^2, \quad \epsilon_n = 0.140\gamma^2.$$

The average of the above two curves is

$$\epsilon_n = \tilde{\lambda} = 0.1175\gamma^2.$$

The curve (5.19), i.e. $\epsilon_n = \tilde{\lambda} = 0.125\gamma^2$, predicted by the Hencky model, is in good accord with the above average experimental curve. As a comparison, the prediction from a neo-Hookean model is contrasted with the prediction (5.19) from the Hencky model. By setting $C_2 = 0$ in the classical formula (7.16) given in RIVLIN [25], the $\epsilon_n - \gamma$ relation predicted by the neo-Hookean model assumes the form:

$$\epsilon_n = \frac{1}{12}\gamma^2 = 0.0833\gamma^2.$$

It may be seen that the Hencky model is in better accord with experiments than the neo-Hookean model.

6. Large deformation solution for incompressible tubes and rods

The incompressible Hencky elasticity model assumes the form

$$\boldsymbol{\sigma} = p\mathbf{I} + 2G\mathbf{h}\,, \tag{6.1}$$

with the hydrostatic pressure $p = p(R)$ and the incompressibility condition

$$J = \det\mathbf{F} = 1\,. \tag{6.2}$$

For torsion of tubes and rods with free ends under (moderate) large deformations, the two dimensionless deformation quantities $\tilde{\lambda}$ and $\tilde{\alpha}$ defined by Eq. (5.1) are small and of the same order of magnitude. Then, we have the asymptotic expressions

$$b_2 - b_3 = \sqrt{u} = \psi R\sqrt{4+\psi^2R^2} + \frac{2\psi R}{4+\psi^2R^2}\,\tilde{\lambda} + \frac{6\psi R + 2\psi^3R^3}{\sqrt{4+\psi^2R^2}}\,\tilde{\alpha} + \mathcal{O}(\tilde{\lambda}^2),$$

$$b_2 = 1 + \frac{1}{2}\,\psi^2R^2 + \frac{1}{2}\,\psi R\sqrt{4+\psi^2R^2} + \left(1 + \frac{\psi R}{\sqrt{4+\psi^2R^2}}\right)\tilde{\lambda} + \left(1 + \psi^2R^2 + \frac{3\psi R + \psi^3R^3}{\sqrt{4+\psi^2R^2}}\right)\tilde{\alpha} + \mathcal{O}(\tilde{\lambda}^2),$$

$$b_3 = 1 + \frac{1}{2}\,\psi^2R^2 - \frac{1}{2}\,\psi R\sqrt{4+\psi^2R^2} + \left(1 - \frac{\psi R}{\sqrt{4+\psi^2R^2}}\right)\tilde{\lambda} + \left(1 + \psi^2R^2 - \frac{3\psi R + \psi^3R^3}{\sqrt{4+\psi^2R^2}}\right)\tilde{\alpha} + \mathcal{O}(\tilde{\lambda}^2),$$

$$\ln b_2 = 2\ln\left(\sqrt{1+\frac{1}{4}\psi^2R^2} + \frac{1}{2}\psi R\right) + \left(1 - \frac{\psi R}{\sqrt{4+\psi^2R^2}}\right)\tilde{\lambda} + \left(1 + \frac{\psi R}{\sqrt{4+\psi^2R^2}}\right)\tilde{\alpha} + \mathcal{O}(\tilde{\lambda}^2),$$

$$\ln b_3 = 2\ln\left(\sqrt{1+\frac{1}{4}\psi^2R^2} - \frac{1}{2}\psi R\right) + \left(1 + \frac{\psi R}{\sqrt{4+\psi^2R^2}}\right)\tilde{\lambda} + \left(1 - \frac{\psi R}{\sqrt{4+\psi^2R^2}}\right)\tilde{\alpha} + \mathcal{O}(\tilde{\lambda}^2).$$

Using Eq. (4.6) and neglecting quantities of orders higher than $\tilde{\lambda}$ or $\tilde{\alpha}$, we reduce the condition (6.2) to

$$R\frac{\partial\tilde{\alpha}}{\partial R} + 2\tilde{\alpha} + \tilde{\lambda} = 0. \qquad (6.3)$$

Then, we derive

$$\tilde{\alpha} = \frac{1}{R^2}K - \frac{1}{2}\tilde{\lambda}, \qquad (6.4)$$

where $K = K(\psi)$ is a quantity of the same order of magnitude as $\tilde{\lambda}$.

On the other hand, replacing $\Lambda \ln J$ in Eqs. (4.2) – (4.4) with p and noting the condition (6.2), we obtain stress-deformation relations. Then, utilizing Eq. (6.4) and the asymptotic expressions given before and neglecting the quantities of higher orders than $\tilde{\lambda}$ or $\tilde{\alpha}$, from the just-mentioned stress-deformation relations we derive the reduced relations

$$\frac{1}{G}\sigma_{rr} = \frac{1}{G}p - \tilde{\lambda} - \frac{1}{2}\omega^{-2}(\psi^2 K), \tag{6.5}$$

$$\begin{aligned}\frac{1}{G}\sigma_{\theta\theta} = \frac{1}{G}p + \frac{2\omega}{\sqrt{1+\omega^2}}\,\mathrm{sh}^{-1}\omega + \Bigg(\frac{1-2\omega^2}{3(1+\omega^2)} \\ -\frac{1+2\omega^2}{\omega(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega\Bigg)\frac{3}{2}\,\tilde{\lambda} \\ +\frac{1}{4}\omega^{-2}\left(\frac{1+2\omega^2}{1+\omega^2} + \frac{1+2\omega^2}{\omega(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega\right)(\psi^2 K),\end{aligned} \tag{6.6}$$

$$\begin{aligned}\frac{1}{G}\sigma_{zz} = \frac{1}{G}p - \frac{2\omega}{\sqrt{1+\omega^2}}\,\mathrm{sh}^{-1}\omega + \Bigg(\frac{1+4\omega^2}{3(1+\omega^2)} \\ +\frac{1+2\omega^2}{\omega(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega\Bigg)\frac{3}{2}\,\tilde{\lambda} \\ +\frac{1}{4}\omega^{-2}\left(\frac{1}{1+\omega^2} - \frac{1+2\omega^2}{\omega(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega\right)(\psi^2 K).\end{aligned} \tag{6.7}$$

In the above and henceforth, $\mathrm{sh}^{-1}\omega$ is used for the inverse hyperbolic sine function, i.e.

$$\mathrm{sh}^{-1}\omega = \ln(\omega + \sqrt{1+\omega^2}), \quad \omega = \frac{1}{2}\,\psi R. \tag{6.8}$$

Moreover, Eq. (4.5) for the shear stress component $\sigma_{z\theta}$ is reduced to

$$\begin{aligned}\frac{1}{G}\sigma_{z\theta} = \frac{2\mathrm{sh}^{-1}\omega}{\sqrt{1+\omega^2}} - \left(\frac{\omega}{1+\omega^2} - \frac{1+2\omega^2}{(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega\right) \\ \cdot\left(\frac{3}{2}\,\tilde{\lambda} - \frac{1}{4}\omega^{-2}(\psi^2 K)\right).\end{aligned} \tag{6.9}$$

Substituting Eqs. (6.5) – (6.6) into Eq. (4.7) and replacing r with R, we derive the differential equation governing the pressure p as follows:

$$\begin{aligned}\frac{R}{G}\frac{\partial p}{\partial R} = \frac{2\omega}{\sqrt{1+\omega^2}}\,\mathrm{sh}^{-1}\omega + \left(\frac{1}{1+\omega^2} - \frac{1+2\omega^2}{\omega(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega\right) \\ \cdot\left(\frac{3}{2}\,\tilde{\lambda} - \frac{1}{4}\omega^{-2}(\psi^2 K)\right).\end{aligned} \tag{6.10}$$

Then, integrating the above equation, we arrive at

$$\frac{1}{G}p = K_0 + (\mathrm{sh}^{-1}\omega)^2 + P(\omega)\tilde{\lambda} + Q(\omega)(\psi^2 K), \tag{6.11}$$

where $K_0 = K_0(\psi)$, and

$$P(\omega) = \frac{3}{2}\frac{\mathrm{sh}^{-1}\omega}{\omega\sqrt{1+\omega^2}}, \tag{6.12}$$

$$Q(\omega) = \frac{1}{12}\frac{1}{\omega^2} + \frac{1}{3}\ln\omega - \frac{4\omega^4 + 2\omega^2 + 1}{12\omega^3\sqrt{1+\omega^2}}\mathrm{sh}^{-1}\omega. \tag{6.13}$$

Then, substituting Eqs. (6.5), (6.7) and (6.10) into the conditions (4.8) and (4.9), we deduce

$$(P_1 - 1)\tilde{\lambda} + (Q_1 - \frac{1}{2\omega_1^2})(\psi^2 K) + K_0 = -(\mathrm{sh}^{-1}\omega_1)^2, \tag{6.14}$$

$$(P_0 - 1)\tilde{\lambda} + (Q_0 - \frac{1}{2\omega_0^2})(\psi^2 K) + K_0 = -(\mathrm{sh}^{-1}\omega_0)^2, \tag{6.15}$$

$$U_0^1\tilde{\lambda} + V_0^1(\psi^2 K) + 2(\omega_1^2 - \omega_0^2)K_0 = S_0^1, \tag{6.16}$$

where $\omega_1 = \frac{1}{2}\psi R_1$, $\omega_0 = \frac{1}{2}\psi R_0$ and the following notations are used

$$Z_0 = Z(\omega_0), \quad Z_1 = Z(\omega_1), \quad Z_0^1 = Z_1 - Z_0,$$

for each function $Z(\omega) \in \{P(\omega), Q(\omega),\ S(\omega), U(\omega), V(\omega)\}$, where $P(\omega)$ and $Q(\omega)$ are given by Eqs. (6.12) – (6.13) and $S(\omega)$, $U(\omega)$ and $V(\omega)$ by

$$S(\omega) = 6\omega\sqrt{1+\omega^2}\mathrm{sh}^{-1}\omega - (3 + 2\omega^2)(\mathrm{sh}^{-1}\omega)^2 - 3\omega^2, \tag{6.17}$$

$$U(\omega) = 9(\mathrm{sh}^{-1}\omega)^2 + 4\omega^2 - \frac{6\omega}{\sqrt{1+\omega^2}}\mathrm{sh}^{-1}\omega, \tag{6.18}$$

$$V(\omega) = \frac{2}{3}\omega^2\ln\omega + \frac{4 - \omega^2 - 2\omega^4}{3\omega\sqrt{1+\omega^2}}\mathrm{sh}^{-1}\omega. \tag{6.19}$$

From the linear system consisting of Eqs. (6.14) – (6.16), the three functions of the angle of twist ψ, $(\tilde{\lambda}, \psi^2 K, K_0)$, can be determined. The results are as follows:

$$\tilde{\lambda} = \frac{\Delta_1}{\Delta_0}, \quad \psi^2 K = \frac{\Delta_2}{\Delta_0}, \quad K_0 = \frac{\Delta_3}{\Delta_0}, \tag{6.20}$$

where Δ_i, $i = 0, 1, 2, 3$, are the four determinants

$$(6.21)\qquad \Delta_0 = \begin{vmatrix} P_1 - 1 & Q_1 - 0.5\omega_1^{-2} & 1 \\ P_0 - 1 & Q_0 - 0.5\omega_0^{-2} & 1 \\ U_0^1 & V_0^1 & 2\omega_1^2 - 2\omega_0^2 \end{vmatrix},$$

$$(6.22)\qquad \Delta_1 = \begin{vmatrix} -(\mathrm{sh}^{-1}\omega_1)^2 & Q_1 - 0.5\omega_1^{-2} & 1 \\ -(\mathrm{sh}^{-1}\omega_0)^2 & Q_0 - 0.5\omega_0^{-2} & 1 \\ S_0^1 & V_0^1 & 2\omega_1^2 - 2\omega_0^2 \end{vmatrix},$$

$$(6.23)\qquad \Delta_2 = \begin{vmatrix} P_1 - 1 & -(\mathrm{sh}^{-1}\omega_1)^2 & 1 \\ P_0 - 1 & -(\mathrm{sh}^{-1}\omega_0)^2 & 1 \\ U_0^1 & S_0^1 & 2\omega_1^2 - 2\omega_0^2 \end{vmatrix},$$

$$(6.24)\qquad \Delta_3 = \begin{vmatrix} P_1 - 1 & Q_1 - 0.5\omega_1^{-2} & -(\mathrm{sh}^{-1}\omega_1)^2 \\ P_0 - 1 & Q_0 - 0.5\omega_0^{-2} & -(\mathrm{sh}^{-1}\omega_0)^2 \\ U_0^1 & V_0^1 & S_0^1 \end{vmatrix}.$$

Then, all non-vanishing stress components and the deformation quantity $\tilde{\alpha}$ are obtainable as functions of R and ψ.

For a rod, the second condition in Eq. (4.10) yields Eq. (5.11). From the latter and Eq. (6.4) we infer that $K = 0$, i.e.

$$(6.25)\qquad K = 0, \quad \tilde{\alpha} = -\frac{1}{2}\tilde{\lambda}.$$

Thus, Eqs. (6.14) and (6.16) are reduced to

$$(6.26)\qquad (P_1 - 1)\tilde{\lambda} + K_0 = -(\mathrm{sh}^{-1}\omega_1)^2,$$

$$(6.27)\qquad U_1\tilde{\lambda} + 2\omega_1^2 K_0 = S_1.$$

Then we derive

$$(6.28)\qquad \tilde{\lambda} = \frac{-\omega_1^2\sqrt{1+\omega_1^2} + 2\omega_1(1+\omega_1^2)\mathrm{sh}^{-1}\omega_1 - \sqrt{1+\omega_1^2}(\mathrm{sh}^{-1}\omega_1)^2}{2\omega_1^2\sqrt{1+\omega_1^2} - 3\omega_1\mathrm{sh}^{-1}\omega_1 + 3\sqrt{1+\omega_1^2}(\mathrm{sh}^{-1}\omega_1)^2},$$

$$(6.29)\qquad K_0 = -(\mathrm{sh}^{-1}\omega_1)^2 - \left(\frac{3}{2}\frac{\mathrm{sh}^{-1}\omega_1}{\omega_1\sqrt{1+\omega_1^2}} - 1\right)\tilde{\lambda}.$$

Substituting Eqs. (6.25), (6.29), (6.12) and (6.13) into Eqs. (6.11), (6.5) – (6.7) and (6.9), we obtain the stress components

$$(6.30)\qquad \frac{1}{G}\sigma_{rr} = (\mathrm{sh}^{-1}\omega)^2 - (\mathrm{sh}^{-1}\omega_1)^2 + \frac{3}{2}\left(\frac{\mathrm{sh}^{-1}\omega}{\omega\sqrt{1+\omega^2}} - \frac{\mathrm{sh}^{-1}\omega_1}{\omega_1\sqrt{1+\omega_1^2}}\right)\tilde{\lambda},$$

$$(6.31)\qquad \frac{1}{G}\sigma_{\theta\theta} = (\mathrm{sh}^{-1}\omega)^2 - (\mathrm{sh}^{-1}\omega_1)^2 + \frac{2\omega}{\sqrt{1+\omega^2}}\,\mathrm{sh}^{-1}\omega$$

$$+\frac{3}{2}\left(\frac{1}{1+\omega^2} - \frac{\omega\mathrm{sh}^{-1}\omega}{(1+\omega^2)^{3/2}} - \frac{\mathrm{sh}^{-1}\omega_1}{\omega_1\sqrt{1+\omega_1^2}}\right)\tilde{\lambda},$$

$$(6.32)\qquad \frac{1}{G}\sigma_{zz} = (\mathrm{sh}^{-1}\omega)^2 - (\mathrm{sh}^{-1}\omega_1)^2 - \frac{2\omega}{\sqrt{1+\omega^2}}\,\mathrm{sh}^{-1}\omega$$

$$+\frac{3}{2}\left(\frac{1+2\omega^2}{1+\omega^2} + \frac{2+3\omega^2}{\omega(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega - \frac{\mathrm{sh}^{-1}\omega_1}{\omega_1\sqrt{1+\omega_1^2}}\right)\tilde{\lambda},$$

$$(6.33)\qquad \frac{1}{G}\sigma_{z\theta} = \frac{2\mathrm{sh}^{-1}\omega}{\sqrt{1+\omega^2}} + \frac{3}{2}\left(\frac{1+2\omega^2}{(1+\omega^2)^{3/2}}\,\mathrm{sh}^{-1}\omega - \frac{\omega}{1+\omega^2}\right)\tilde{\lambda}.$$

Moreover, substituting Eqs. $(5.1)_2$, (6.25) and (6.33) into Eq. (4.11) and neglecting small quantities of higher order than $\tilde{\lambda}$, we obtain the resultant twisting moment

$$M = 2\pi\int_0^{R_1} R^2\left(\frac{2\mathrm{sh}^{-1}\omega}{\sqrt{1+\omega^2}} - \left(\frac{\mathrm{sh}^{-1}\omega}{(1+\omega)^{3/2}} + \frac{\omega}{1+\omega^2}\right)\tilde{\lambda}\right)dR.$$

Hence we have

$$(6.34)\qquad \frac{M}{GR_1^3} = \pi\left\{\frac{2\mathrm{sh}^{-1}\omega_1}{\sqrt{1+\omega_1^2}} - \left(\frac{1}{\omega_1} - \frac{2\mathrm{sh}^{-1}\omega_1}{\omega_1^2\sqrt{1+\omega_1^2}} + \frac{(\mathrm{sh}^{-1}\omega_1)^2}{\omega_1^3}\right)(1+\tilde{\lambda})\right\},$$

the dimensionless twisting moment.

When ψ is small, substituting the asymptotic expressions

$$\sqrt{1+\omega^2} = 1 + \frac{1}{2}\omega^2 + \mathcal{O}(\omega^4), \quad \mathrm{sh}^{-1}\omega = \omega - \frac{1}{6}\omega^3 + \mathcal{O}(\omega^5)$$

into Eq. (6.28), again Eq. (5.19) can be derived. In a similar manner, from Eqs. (6.25) and (6.30) – (6.33) again one can derive Eqs. (5.20) – (5.24). Generally, from Eqs. (6.28) – (6.33) we know that the shear stress component $\sigma_{z\theta}$ is an odd function of ω, while any quantity except $\sigma_{z\theta}$ is an even function of ω. Accordingly, the general expansions with respect to ω are of the forms:

$$(6.35)\qquad \sigma_{z\theta} = \sum_{t=1}^{\infty} A_{2t-1}\,\omega^{2t-1},$$

$$(6.36)\qquad \beta = \sum_{t=1}^{\infty} B_{2t}\,\omega^{2t},$$

where $\beta \in \{\tilde{\lambda}, \tilde{\alpha}, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}\}$. In the above, the coefficients A_{2t-1} and B_{2t} are independent of ψ and R_1.

7. Results and conclusion

Numerical results have been determined for the axial strain and the resultant twisting moment of a rod at large deformations from Eqs. (6.28) and (6.34), respectively. The corresponding curves of the axial strain $\tilde{\lambda}$ versus ω_1 and the dimensionless resultant twisting moment $M/(GR_1^3)$ versus ω_1 are depicted in Figs. 1 and 2.

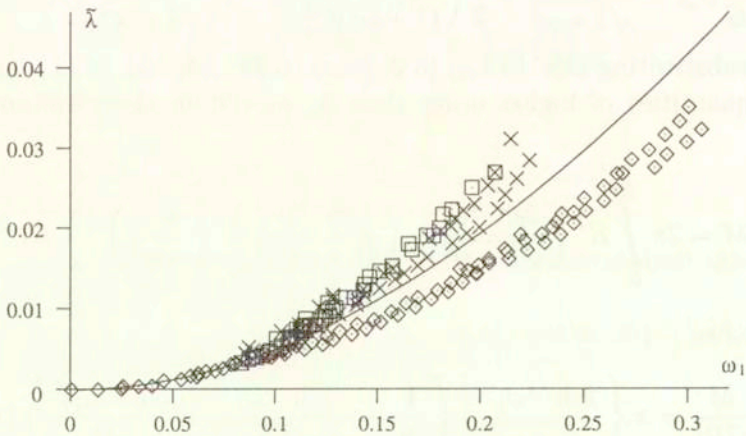

FIG. 1. Axial strain $\tilde{\lambda}$ versus maximum shear strain ω_1.

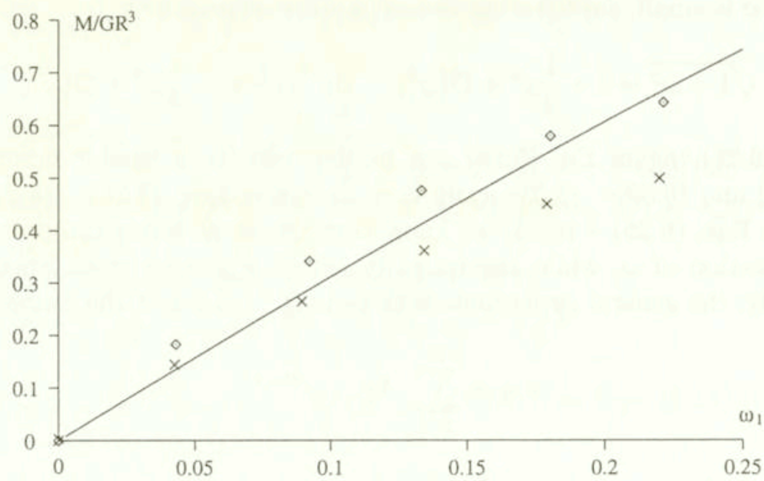

FIG. 2. Dimensionless twisting moment M/GR_1^3 versus maximum shear strain ω_1.

We note that ω_1 herein through (cf. Eq. $(6.8)_2$)

$$\omega_1 = \frac{1}{2}\,\psi R_1 = \frac{1}{2}\,\gamma$$

is the normally used maximum shear strain at the outer surface of the rod. In both figures the experimental data of FREUDENTHAL and RONAY [9] for solid cylindrical specimens of highly filled polyurethane rubber at two different strain rates are also incorporated. It can be seen that the $\tilde{\lambda} - \frac{1}{2}\gamma$ relation as well as the twisting moment-$\frac{1}{2}\gamma$ relation predicted by the Hencky model compare favorably with these experiments.

It has to be emphasized further, that even for the moderate large strains under consideration the small deformation solutions for the axial strain (cf. Eq. (5.19)) as well as the twisting moment (cf. Eq. (5.32)) fit fairly well to these data. This finally supports Anand's statement (ANAND [2]) that the Hencky model Eq. (1.5) "*... should be of wide applicability ... for the analysis of general engineering problems involving moderately large elastic strains.*"

Acknowledgement

This work was completed under the financial support of the Deutsche Forschungsgemeinschaft (DFG), contract No.: Br 580/26-1. This support is gratefully acknowledged.

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Received December 3, 2000; revised version June 6, 2000.