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# Irreducible representations for constitutive equations of anisotropic solids III: crystal and quasicrystal classes $D_{2m+1h}$ and $D_{2md}$

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A SIMPLE, UNIFIED PROCEDURE is applied to derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In this part, our concern is for the crystal classes and quasicrystal classes $D_{2m+1h}$ and $D_{2md}$ for all integers $m \geq 1$.

## 1. Introduction

WE CONTINUE OUR STUDY of irreducible nonpolynomial representations for anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In the final part, we are concerned with the crystal and quasicrystal classes $D_{2m+1h}$ and $D_{2md}$ for all $m \geq 1$. These classes are more complicated than those considered before, since we have to draw distinction between the reflection symmetry with respect to a plane and the two-fold rotation symmetry with respect to an axis.

As it has been done in the preceding two parts, we shall apply the unified procedure outlined in Sec. 3 in Part I of this series (see XIAO, BRUHNS and MEYERS [21]. Henceforth the just-mentioned reference will simply be referred to as Part I) to derive the desired results. For notations and preliminaries and for some relevant references, refer to Sec. 1 – 3 in Part I for detail.

## 2. Crystal and quasicrystal classes $D_{2m+1h}$

The classes at issue take forms

$$(2.1)\qquad D_{2m+1h}(\mathbf{n},\mathbf{e}) = \left\{(-1)^k\mathbf{R}_{\mathbf{n}}^{\theta_k},\ (-1)^k\mathbf{R}_{\boldsymbol{\tau}_k}^{\pi} \mid \theta_k = \frac{2k\pi}{4m+2},\right.$$
$$\left.\boldsymbol{\tau}_k = \mathbf{R}_{\mathbf{n}}^{\theta_k/2}\mathbf{e},\ k = 0, 1, \cdots, 4m+1\right\}.$$

They include the crystal class $D_{3h}$ as the particular case when $m = 1$. Note that each $\boldsymbol{\tau}_{2r+1}$ and each $\boldsymbol{\tau}_{2r}$, respectively, correspond to the reflection with respect to the $\boldsymbol{\tau}_{2r+1}$-plane and the two-fold rotation about an axis in the direction of $\boldsymbol{\tau}_{2r}$, where $r = 0, 1, \cdots, 2m$. Accordingly, we shall call each $\boldsymbol{\tau}_{2r+1}$ and each $\boldsymbol{\tau}_{2r}$ a *reflection axis vector* and a *two-fold rotation axis vector* of the group $D_{2m+1h}$. In particular, the two orthonormal vectors $\mathbf{e}$ $(= \boldsymbol{\tau}_0)$ and $\mathbf{e}'$ $(= \boldsymbol{\tau}_{2m+1})$ are a two-fold rotation axis vector and a reflection axis vector of $D_{2m+1h}$, respectively. Throughout this section, $\mathbf{v}$ is used to represent a two-fold rotation axis vector of $D_{2m+1h}$, i.e. $\mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1 \cdots, 2m\}$.

Throughout, for any given vector $\mathbf{z}$ we shall use $\mathbf{z}'$ to denote the vector $\mathbf{n} \times \mathbf{z}$, i.e.

$$\mathbf{z}' = \mathbf{n} \times \mathbf{z}.$$

A useful fact for the group $D_{2m+1h}$ is: if $\boldsymbol{\tau}$ is a two-fold rotation (resp. reflection) axis vector, then $\boldsymbol{\tau}'$ is a reflection (resp. two-fold rotation) axis vector.

Let $Y = (\mathbf{J}_1, \cdots, \mathbf{J}_s)$, where each $\mathbf{J}_\alpha$ is a skewsymmetric tensor or a symmetric tensor. Then the identities

$$(2.2) \qquad f(\mathbf{Q}_0 Y \mathbf{Q}_0^T) = f(Y), \qquad \mathbf{F}(\mathbf{Q}_0 Y \mathbf{Q}_0^T) = \mathbf{Q}_0 \mathbf{F}(Y) \mathbf{Q}_0^T, \qquad \mathbf{Q}_0 = \pm \mathbf{I},$$

for every scalar-valued function $f(Y)$ and every skewsymmetric and every symmetric tensor-valued function $\mathbf{F}(Y)$. From this fact and the fact that the group $D_{2m+1h}$ and the central inversion $-\mathbf{I}$ generate the centrosymmetrical group $D_{4m+2h}$, each invariant $f(Y)$ and each form-invariant function $\mathbf{F}(Y)$ under the group $D_{2m+1h}$ turns out to be an invariant and a form-invariant function under the larger group $D_{4m+2h}$ $(\supset D_{2m+1h})$. As a result, for the five sets of variables, $(\mathbf{W})$, $(\mathbf{A})$, $(\mathbf{W}, \boldsymbol{\Omega})$, $(\mathbf{W}, \mathbf{A})$ and $(\mathbf{A}, \mathbf{B})$, results for functional bases and skewsymmetric and symmetric tensor generating sets relative to the group $D_{2m+1h}$, as well as related invariants from the scalar products, can be obtained from the corresponding results given in Sec. 4 in Part I by the replacement of $m$ with $2m + 1$. Thus, in what follows, for the foregoing five sets of variables, we only need to derive vector generating sets and their related invariants from the scalar products. Moreover, according to Sec. 4 (**xiii**) in Part I, we can omit the set $(\mathbf{u}, \mathbf{v}, \mathbf{r})$ of three vector variables. Finally, for each of the sets of three variables, $(\mathbf{u}, \mathbf{W}, \boldsymbol{\Omega})$ and $(\mathbf{u}, \mathbf{A}, \mathbf{B})$, from Theorem 2.1 and Theorem 3.3 in XIAO [20] we know that it suffices to supply a vector generating set and a functional basis.

#### 2.1. Single variables

(i) A single vector $\mathbf{u}$

$$V \quad \{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \mathring{\mathbf{u}}, \eta_{2m}(\mathring{\mathbf{u}})\} \; (\equiv V''_{2m+1}(\mathbf{u}))$$

Skw $\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge \eta_{2m}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{N}\}$ $(\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u}))$

Sym $\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}})\}$
$(\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u}))$

$R$ $(\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n}), \overset{\circ}{\mathbf{r}}\cdot\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{r}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{u}})$;

$\beta_{2m+1}(\overset{\circ}{\mathbf{u}})(\mathrm{tr}\mathbf{HN}), (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{Hn}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Hn}$;

$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{C}}\mathbf{n}$;

$\{(\mathbf{u}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\}$ $(\equiv I''_{2m+1}(\mathbf{u}))$.

The proof for the above results is as follows. According to Theorem 3 in XIAO [17], isotropic functional bases and generating sets of the extended variables $(\mathbf{u}, \eta_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$ supply anisotropic functional bases and generating sets of the vector variable $\mathbf{u}$ under the group $D_{2m+1h}$. Then, applying the related results for isotropic functions, we know that the three presented sets $I''_{2m+1}(\mathbf{u})$, $V''_{2m+1}(\mathbf{u})$ and $\mathrm{Skw}''_{2m+1}(\mathbf{u})$ supply a desired functional basis, a desired vector generating set and a desired skewsymmetric tensor generating set, respectively. Moreover, a desired symmetric tensor generating set is formed by the six generators in the presented set $\mathrm{Sym}''_{2m+1}(\mathbf{u})$ as well as the generator $\mathbf{G}=\eta_{2m}(\overset{\circ}{\mathbf{u}})\otimes\eta_{2m}(\overset{\circ}{\mathbf{u}})$. Here the decomposition formula $\mathbf{u}=(\mathbf{u}\cdot\mathbf{n})\mathbf{n}+\overset{\circ}{\mathbf{u}}$ (see (2.15) in Part I) is used.

We need to show that the generator $\mathbf{G}$ is redundant. In fact, $\mathbf{G}$ is a 2-dimensional symmetric tensor defined on the $\mathbf{n}$-plane. When the two vectors $\overset{\circ}{\mathbf{u}}$ and $\eta_{2m}(\overset{\circ}{\mathbf{u}})$ on the $\mathbf{n}$-plane are linearly independent, the three symmetric tensors $\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}$, $\overset{\circ}{\mathbf{u}}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}})$ and $\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$ constitute a basis of the space of all symmetric tensors defined on the $\mathbf{n}$-plane. Hence $\mathbf{G}$ is redundant. $\mathbf{G}$ is obviously redundant when $\overset{\circ}{\mathbf{u}}$ and $\eta_{2m}(\overset{\circ}{\mathbf{u}})$ are linearly dependent.

(ii) A single skewsymmetric tensor $\mathbf{W}$

$V$ $\{\alpha_{2m+1}(\mathbf{Wn})\mathbf{n}, \eta_{2m}(\mathbf{Wn}), \mathbf{W}\eta_{2m}(\mathbf{Wn}), \mathbf{W}^2\eta_{2m}(\mathbf{Wn})\}$ $(\equiv V''_{2m+1}(\mathbf{W}))$

$R$ $(\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), \overset{\circ}{\mathbf{r}}\cdot\eta_{2m}(\mathbf{Wn}), \eta_{2m}(\mathbf{Wn})\cdot\mathbf{Wr}, \eta_{2m}(\mathbf{Wn})\cdot\mathbf{W}^2\mathbf{r}$.

According to Theorem 3 in XIAO [17], a vector generating set for form-invariant vector-valued functions of the skewsymmetric tensor variable $\mathbf{W}$ under $D_{2m+1h}$ is obtainable from an isotropic vector generating set for the extended variables $(\eta_{2m}(\mathbf{Wn}), \mathbf{W}, \mathbf{n}\otimes\mathbf{n})$. By applying the related results for isotropic functions we know that the latter is just given by the presented set $V''_{2m+1}(\mathbf{W})$. Moreover, by considering $\mathbf{W}_1=\mathbf{E}(\mathbf{n}+\mathbf{e})$ and $\mathbf{W}_2=\mathbf{E}(\mathbf{n}+\mathbf{e}')$, we deduce that each of the four generators in the set $V''_{2m+1}(\mathbf{W})$ are irreducible.

(iii) A single symmetric tensor $\mathbf{A}$

$$V \quad \{\eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_e\eta_m(\mathbf{q}(\mathbf{A})), \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + J(\mathbf{A})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\}\ (\equiv V''_{2m+1}(\mathbf{A}))$$

$$R \quad \overset{\circ}{\mathbf{r}}\cdot\eta_m(\mathbf{q}(\mathbf{A})), \eta_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{A}_e\,\overset{\circ}{\mathbf{r}}, (\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{r}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\overset{\circ}{\mathbf{r}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + J(\mathbf{A})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\mathbf{r}\cdot\mathbf{n}).$$

We show that the presented set $V''_{2m+1}(\mathbf{A})$ obeys the criterion (2.3) given in Part I. The case when $\overset{\circ}{\mathbf{A}} = \mathbf{O}$ can be treated easily. Let $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$ and $\overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0}$. Then we have

$$D = \operatorname{rank} V''_{2m+1}(\mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_e\eta_m(\mathbf{q}(\mathbf{A}))\} \\ \qquad\qquad = 2 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0, \\ \operatorname{rank}\{\eta_m(\mathbf{q}(\mathbf{A}))\} = 1 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0, \end{cases}$$

$$\Gamma(\mathbf{A}) \cap D_{2m+1h} = \begin{cases} C_{1h}(\mathbf{n}) \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0, \\ C_{2v}(\mathbf{v}, \mathbf{n}, \mathbf{n}\times\mathbf{v}) \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0. \end{cases}$$

Let $\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \neq 0$ and let $D = \operatorname{rank} V''_{2m+1}(\mathbf{A})$. Then we have

$$D \geq \begin{cases} \operatorname{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 3 \text{ if } \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \neq 0, \\ \operatorname{rank}\{\mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_e\eta_m(\mathbf{q}(\mathbf{A}))\} = 3 \text{ if } \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ \qquad\qquad \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0, \\ \operatorname{rank}\{\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A}))\} = 3 \text{ if } \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \\ \qquad\qquad = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0,\ J(\mathbf{A}) \neq 0, \\ \operatorname{rank}\{\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 2 \text{ if } \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = J(\mathbf{A}) = 0, \end{cases}$$

$$\Gamma(\mathbf{A}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}\times\mathbf{v}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\ \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = J(\mathbf{A}) = 0.$$

Let $\overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0}$ and $\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0$. Then, we have

$$\operatorname{rank} V''_{2m+1}(\mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A}))\} = 3 \text{ if } J(\mathbf{A}) \neq 0, \\ \operatorname{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 1 \text{ if } J(\mathbf{A}) = 0, \end{cases}$$

and

$$\Gamma(\mathbf{A}) \cap D_{2m+1} = C_2(\mathbf{v}) \text{ if } \overset{\circ}{\mathbf{A}}\, \mathbf{n} \neq \mathbf{0},\ J(\mathbf{A}) = \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n}) = 0.$$

It is readily understood that the four cases for $\mathbf{A}$ considered above exhaust all cases for $\mathbf{A}$. Thus, from the above results and Table 1 in Sec. 2 in Part I, we conclude that the presented set $V''_{2m+1}(\mathbf{A})$ obeys the criterion (2.3) in Part I and hence supplies a desired vector generating set. Further, by considering the two tensors $\mathbf{A}_1 = \mathbf{n} \vee (\mathbf{e} + \boldsymbol{\tau}_1)$ and $\mathbf{A}_2 = \mathbf{e} \vee \boldsymbol{\tau}_1$, we deduce that each of the five generators in the set $V''_{2m+1}(\mathbf{A})$ are irreducible.

### 2.2. $D_{2m+1h}$-irreducible sets of two variables

(iv) The $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

$V$ $\quad V''_{2m+1}(\mathbf{u}) \cup V''_{2m+1}(\mathbf{v})\ (\equiv V''_{2m+1}(\mathbf{u}, \mathbf{v}))$

Skw $\quad \mathrm{Skw}''_{2m+1}(\mathbf{u}) \cup \mathrm{Skw}''_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \wedge \mathbf{v}, |\mathbf{u} \cdot \mathbf{n}|^{2m-1}(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \wedge \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})$
$+|\mathbf{v} \cdot \mathbf{n}|^{2m-1}(\mathbf{v} \cdot \mathbf{n})\mathbf{n} \wedge \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}\ (\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}))$

Sym $\quad \mathrm{Sym}''_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}''_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v}, |\mathbf{u} \cdot \mathbf{n}|^{2m-1}(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})$
$+|\mathbf{v} \cdot \mathbf{n}|^{2m-1}(\mathbf{v} \cdot \mathbf{n})\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}\ (\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}))$

$R$ $\quad \mathbf{r} \cdot V''_{2m+1}(\mathbf{u}, \mathbf{v}), \mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{z}), \mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{z}), \mathbf{z} = \mathbf{u}, \mathbf{v};$

$\mathbf{u} \cdot \mathbf{H}\mathbf{v}; \mathbf{u} \cdot \overset{\circ}{\mathbf{C}}\, \mathbf{v};$

$|\mathbf{u} \cdot \mathbf{n}|^{2m-1}(\mathbf{u} \cdot \mathbf{n})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}) \cdot \mathbf{H}\mathbf{n} + |\mathbf{v} \cdot \mathbf{n}|^{2m-1}(\mathbf{v} \cdot \mathbf{n})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) \cdot \mathbf{H}\mathbf{n};$

$|\mathbf{u} \cdot \mathbf{n}|^{2m-1}(\mathbf{u} \cdot \mathbf{n})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}) \cdot \overset{\circ}{\mathbf{C}}\, \mathbf{n} + |\mathbf{v} \cdot \mathbf{n}|^{2m-1}(\mathbf{v} \cdot \mathbf{n})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) \cdot \overset{\circ}{\mathbf{C}}\, \mathbf{n};$

$I''_{2m+1}(\mathbf{u}) \cup I''_{2m+1}(\mathbf{v}) \cup \{(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}), \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{v}}\}\ (\equiv I''_{2m+1}(\mathbf{u}, \mathbf{v})).$

To prove the above results, we first work out the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{v})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{u}, \mathbf{v}$. Evidently, we have $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq C_1$ for $\mathbf{z} = \mathbf{u}, \mathbf{v}$. The latter implies that either $-\mathbf{R}^{\pi}_{\mathbf{n}}$ or $\mathbf{R}^{\pi}_{\mathbf{v}}$ or $-\mathbf{R}^{\pi}_{\mathbf{v}'}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ of the vector $\mathbf{z}$. Hence we derive: $\mathbf{z} = a\mathbf{e} + b\mathbf{e}'$; $\mathbf{z} = b\mathbf{v}$; and $\mathbf{z} = a\mathbf{n} + b\mathbf{v}$ for each $\mathbf{z} \in \{\mathbf{u}, \mathbf{v}\}$. These cases are equivalent to the three disjoint cases:

$$a\mathbf{n},\ a \neq 0; \quad a\mathbf{e} + b\mathbf{e}',\ a^2 + b^2 \neq 0; \quad a\mathbf{n} + b\mathbf{v},\ ab \neq 0\,. \tag{2.3}$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u} = a\mathbf{n},\ \mathbf{v} = c\mathbf{n};\ \mathbf{u} = a\mathbf{n} + b\mathbf{v},\ \mathbf{v} = c\mathbf{n} + d\mathbf{v};$$

$$\mathbf{u} = a\mathbf{e} + b\mathbf{e}', \ \mathbf{v} = c\mathbf{e} + d\mathbf{e}', \ \beta_{2m+1}(\mathbf{z}) \neq 0, \ \mathbf{z} = \mathbf{u} \text{ or } \mathbf{z} = \mathbf{v};$$

$$\mathbf{u} = a\mathbf{n}, \ \mathbf{v} = c\mathbf{n} + d\mathbf{v}; \ \mathbf{u} = a\mathbf{v}, \ \mathbf{v} = c\mathbf{n} + d\mathbf{v};$$

which violate the $D_{2m+1h}$-irreducibility condition for $(\mathbf{u}, \mathbf{v})$, we derive the following four disjoint cases for $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{v})$:

(c1) $\mathbf{u} = a\mathbf{n}, \ \mathbf{v} = b\mathbf{e} + c\mathbf{e}', \ a(b^2 + c^2) \neq 0$;

(c2) $\mathbf{u} = a\mathbf{e}, \ \mathbf{v} = b\mathbf{v}, \ \mathbf{v} \neq \mathbf{e}, \ ab \neq 0$;

(c3) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}', \ \mathbf{v} = c\mathbf{n} + d\mathbf{e}, \ bcd \neq 0$;

(c4) $\mathbf{u} = a\mathbf{n} + b\mathbf{e}, \ \mathbf{v} = c\mathbf{n} + d\mathbf{v}, \ \mathbf{v} \neq \mathbf{e}, \ abcd \neq 0$.

With cases (c1) – (c4) we show that the two presented sets $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v})$ obey the criterion (2.3) in Part I separately, and therefore they supply the desired skewsymmetric and symmetric tensor generating sets. In fact, for case (c1) we have

$$\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1h} = C_{1h}(\mathbf{n} \times \mathbf{v}) \text{ if } \beta_{2m+1}(\mathring{\mathbf{v}}) = 0,$$

$$\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \begin{cases} \mathrm{rank}\{\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, \mathbf{n} \wedge \eta_{2m}(\mathring{\mathbf{v}})\} = 3 \\ \qquad\qquad\qquad \text{if } \beta_{2m+1}(\mathring{\mathbf{v}}) \neq 0, \\ \mathrm{rank}\{\mathbf{u} \wedge \mathbf{v}\} = 1 \text{ if } \beta_{2m+1}(\mathring{\mathbf{v}}) = 0, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \begin{cases} \mathrm{rank}(\mathrm{Sym}''_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v}, \mathbf{n} \vee (\mathbf{n} \times \mathring{\mathbf{v}})\}) = 6 \\ \qquad\qquad\qquad \text{if } \beta_{2m+1}(\mathring{\mathbf{v}}) \neq 0, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}, \mathbf{u} \vee \mathbf{v}\} = 4 \\ \qquad\qquad\qquad \text{if } \beta_{2m+1}(\mathring{\mathbf{v}}) = 0. \end{cases}$$

For case (c2) we have

$$\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}\{\mathbf{u} \wedge \mathbf{v}\} = 1,$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}\} = 4;$$

For case (c3) we have

$$\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}\{\beta_{2m+1}(\mathring{\mathbf{u}})\mathbf{N}, \mathbf{n} \wedge \mathring{\mathbf{v}}, \mathbf{u} \wedge \mathbf{v}, \mathbf{n} \wedge \eta_{2m}(\mathring{\mathbf{u}})\} = 3,$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \mathrm{rank}(\mathrm{Sym}''_{2m+1}(\mathbf{v}) \cup \{\mathring{\mathbf{u}} \vee \eta_{2m}(\mathring{\mathbf{u}}), \mathbf{u} \vee \mathbf{v}\}) = 6;$$

Finally, for case (c4), by using the formula (2.4) in Part I we have

$$\text{rank}\ \text{Skw}''_{2m+1}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{n}\wedge\eta_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{n}\wedge\eta_{2m}(\overset{\circ}{\mathbf{v}}),\mathbf{u}\wedge\mathbf{v}\} = 3,$$

$$\begin{aligned}\text{rank}\ \text{Sym}''_{2m+1}(\mathbf{u},\mathbf{v}) &\geq \text{rank}(\text{Sym}''_{2m+1}(\mathbf{u})\cup\text{Sym}''_{2m+1}(\mathbf{v}))\\ &= \text{rank}(\text{Sym}(C_{1h}(\mathbf{e}'))\cup\text{Sym}(C_{1h}(\mathbf{n}\times\mathbf{v}))) = 6\,.\end{aligned}$$

In deriving the last equality, Eq. (2.4) given in Part I has been used. From the above results and Tables 2 – 3 given in Sec. 2 in Part I we deduce that the foregoing facts concerning the two sets $\text{Skw}''_{2m+1}(\mathbf{u},\mathbf{v})$ and $\text{Sym}''_{2m+1}(\mathbf{u},\mathbf{v})$ are true. Moreover, by considering the pair $\mathbf{u}_1 = \mathbf{e}'$ and $\mathbf{v}_1 = \mathbf{n}$ we infer that the last two generators in the set $\text{Skw}''_{2m+1}(\mathbf{u},\mathbf{v})$ are irreducible. Besides, by considering the pair $\mathbf{u}_2 = \mathbf{n}$ and $\mathbf{v}_2 = \mathbf{e}'$ we infer that the last two generators in the set $\text{Sym}''_{2m+1}(\mathbf{u},\mathbf{v})$ are also irreducible.

Next, consider the two presented sets $I''_{2m+1}(\mathbf{u},\mathbf{v})$ and $V''_{2m+1}(\mathbf{u},\mathbf{v})$. The former set includes as a subset the set $\{(\mathbf{u}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, (\mathbf{v}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{v}}|^2, (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}\}$. The latter is a functional basis of $(\mathbf{u},\mathbf{v})$ under the group $D_{\infty h}(\mathbf{n})$. Using this fact and following the same procedure used in Sec. 4 (**vi**) in Part I, we infer that the set $I''_{2m+1}(\mathbf{u},\mathbf{v})$ provides a desired functional basis for $(\mathbf{u},\mathbf{v})$. Next, an anisotropic vector generating set for the variables $(\mathbf{u},\mathbf{v})$ is obtainable from an isotropic vector generating set for the extended variables $(\mathbf{u},\mathbf{v},\eta_{2m}(\overset{\circ}{\mathbf{u}}),\eta_{2m}(\overset{\circ}{\mathbf{v}}),\mathbf{n}\otimes\mathbf{n})$ (see Theorem 3 in XIAO [17]). By using the related result for isotropic functions we know that the former is just given by the presented set $V''_{2m+1}(\mathbf{u},\mathbf{v})$.

(**v**) The $D_{2m+1h}$-irreducible set $(\mathbf{W},\mathbf{\Omega})$ of two skewsymmetric tensors

$$V \quad \begin{aligned}&V''_{2m+1}(\mathbf{W})\cup V''_{2m+1}(\mathbf{\Omega})\cup\{\mathbf{W}\eta_{2m}(\mathbf{\Omega n}),\mathbf{\Omega}\eta_{2m}(\mathbf{Wn}),\\ &|\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega n})(\mathbf{E}:\mathbf{W})+|\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{Wn})(\mathbf{E}:\mathbf{\Omega})\}\end{aligned}$$

$$(\equiv V''_{2m+1}(\mathbf{W},\mathbf{\Omega}))$$

$$R \quad \begin{aligned}&\mathbf{r}\cdot V''_{2m+1}(\mathbf{W}),\mathbf{r}\cdot V''_{2m+1}(\mathbf{\Omega}),\eta_{2m}(\mathbf{Wn})\cdot\mathbf{\Omega r},\eta_{2m}(\mathbf{\Omega n})\cdot\mathbf{Wr},\\ &|\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega n})\mathbf{r}\cdot(\mathbf{E}:\mathbf{W})+|\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{Wn})\mathbf{r}\cdot(\mathbf{E}:\mathbf{\Omega});.\end{aligned}$$

To prove the above result, we first work out the $D_{2m+1h}$-irreducible set $(\mathbf{W},\mathbf{\Omega})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z})\cap D_{2m+1h}\neq\Gamma(\mathbf{W},\mathbf{\Omega})\cap D_{2m+1h}$ for $\mathbf{z}=\mathbf{W},\ \mathbf{\Omega}$. Evidently, $\Gamma(\mathbf{z})\cap D_{2m+1h}\neq C_1$ for $\mathbf{z}=\mathbf{W},\ \mathbf{\Omega}$. The latter means that either $-\mathbf{R}^{\pi}_{\mathbf{n}}$ or $\mathbf{R}^{\pi}_{\mathbf{v}}$ or $-\mathbf{R}^{\pi}_{\mathbf{v}'}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for each $\mathbf{z}\in\{\mathbf{W},\mathbf{\Omega}\}$. Hence each $\mathbf{z}\in\{\mathbf{W},\mathbf{\Omega}\}$ takes one of the forms

$$(2.4) \qquad c\mathbf{En},\ c\neq 0;\quad c\mathbf{E}\mathbf{v},\ c\neq 0;\quad c\mathbf{n}\wedge\mathbf{v},\ c\neq 0\,.$$

Thus, we derive the following five disjoint cases for the $D_{2m+1h}$-irreducible set $(\mathbf{W}, \mathbf{\Omega})$:

(c1) $\mathbf{W} = a\mathbf{En}$, $\mathbf{\Omega} = b\mathbf{Ee}$, $ab \neq 0$;

(c2) $\mathbf{W} = a\mathbf{En}$, $\mathbf{\Omega} = b\mathbf{n} \wedge \mathbf{e}$, $ab \neq 0$;

(c3) $\mathbf{W} = a\mathbf{Ee}$, $\mathbf{\Omega} = b\mathbf{E}\boldsymbol{\nu}$, $\boldsymbol{\nu} \neq \mathbf{e}$, $ab \neq 0$;

(c4) $\mathbf{W} = a\mathbf{Ee}$, $\mathbf{\Omega} = b\mathbf{n} \wedge \boldsymbol{\nu}$, $ab \neq 0$;

(c5) $\mathbf{W} = a\mathbf{n} \wedge \mathbf{e}$, $\mathbf{\Omega} = b\mathbf{n} \wedge \boldsymbol{\nu}$, $\boldsymbol{\nu} \neq \mathbf{e}$, $ab \neq 0$.

Then, corresponding to the above five cases, we have

$$\mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{\Omega n}), \mathbf{W}\boldsymbol{\eta}_{2m}(\mathbf{\Omega n}), \mathbf{G}\} = 3,$$

$$\mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{\Omega n}), \alpha_{2m+1}(\mathbf{\Omega n})\mathbf{n}, \mathbf{W}\boldsymbol{\eta}_{2m}(\mathbf{\Omega n})\} = 3,$$

$$\mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \boldsymbol{\eta}_{2m}(\mathbf{\Omega n}), \mathbf{\Omega}\boldsymbol{\eta}_{2m}(\mathbf{Wn})\} = 3,$$

$$\mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \alpha_{2m+1}(\mathbf{\Omega n})\mathbf{n}, \mathbf{G}\} = 3,$$

$$\mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{\Omega}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \boldsymbol{\eta}_{2m}(\mathbf{\Omega n}), \alpha_{2m+1}(\mathbf{Wn})\mathbf{n}\} = 3,$$

where $\mathbf{G}$ is used to denote the last generator in the set $V''_{2m+1}(\mathbf{W}, \mathbf{\Omega})$. From the above results we infer that the presented set $V''_{2m+1}(\mathbf{W}, \mathbf{\Omega})$ obeys the criterion (2.3) in Part I and therefore supplies a desired vector generating set. Further, by considering the two pairs $\mathbf{W}_1 = \mathbf{En}$ and $\mathbf{\Omega}_1 = \mathbf{Ee}$, $\mathbf{W}_2 = \mathbf{Ee}$ and $\mathbf{\Omega}_2 = \mathbf{En}$, we deduce that the last three generators in the set $V''_{2m+1}(\mathbf{W}, \mathbf{\Omega})$ are irreducible.

(vi) The $D_{2m+1h}$-irreducible set $(\mathbf{W}, \mathbf{A})$ of a skewsymmetric tensor and a symmetric tensor

$$V \quad V''_{2m+1}(\mathbf{W}) \cup V''_{2m+1}(\mathbf{A}) \cup \{\mathbf{W}(\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) + (-1)^m \boldsymbol{\eta}_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})),$$
$$(\mathrm{tr}\mathbf{WN})\beta_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n} + \mathbf{W}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})),$$
$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{Wn})\,\mathring{\mathbf{A}}\,\mathbf{n} + |\mathbf{Wn}|^{2m}\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{Wn}\}$$
$$(\equiv V''_{2m+1}(\mathbf{W}, \mathbf{A}))$$

$$R \quad \mathbf{r} \cdot V''_{2m+1}(\mathbf{W}), \mathbf{r} \cdot V''_{2m+1}(\mathbf{A}), (\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) + (-1)^m \boldsymbol{\eta}_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})) \cdot \mathbf{Wr},$$
$$(\mathbf{r} \cdot \mathbf{n})(\mathrm{tr}\mathbf{WN})\beta_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n}) - \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) \cdot \mathbf{Wr},$$
$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{Wn})(\mathring{\mathbf{r}} \cdot \mathring{\mathbf{A}}\,\mathbf{n}) + |\mathbf{Wn}|^{2m}\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})(\mathring{\mathbf{r}} \cdot \mathbf{Wn}).$$

Here and henceforth, $\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))$ and $\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A}))$ are the two polynomial vector-valued functions of $\mathbf{A}$ given by (2.4) and (2.5) in Part II (see XIAO, BRUHNS and MEYERS [22]).

To prove the above result, we first work out the $D_{2m+1h}$-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{W},\ \mathbf{A}$. As has been shown in (v), $\mathbf{W}$ takes one of the forms given by (2.4). Moreover, we have $\Gamma(\mathbf{A}) \cap D_{2m+1h} \neq C_1$. This implies that either $-\mathbf{R}^{\pi}_{\mathbf{n}}$ or $-\mathbf{R}^{\pi}_{\mathbf{v}'}$ or $\mathbf{R}^{\pi}_{\mathbf{v}}$ pertains to the symmetry group $\Gamma(\mathbf{A})$ of $\mathbf{A}$. Hence, we deduce that $\overset{\circ}{\mathbf{A}}$ takes one of the forms

$$(2.5) \qquad a\mathbf{D}_1 + b\mathbf{D}_2,\ a^2 + b^2 \neq 0; \quad a(\mathbf{v} \otimes \mathbf{v} - \mathbf{v}' \otimes \mathbf{v}') + b\mathbf{n} \vee \mathbf{v},\ b \neq 0;$$
$$a(\mathbf{v} \otimes \mathbf{v} - \mathbf{v}' \otimes \mathbf{v}') + b\mathbf{n} \vee \mathbf{v}',\ b \neq 0\,.$$

Considering the combinations of the above forms for $\mathbf{A}$ and the forms (2.4) for $\mathbf{W}$ and excluding the cases

$$\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0;$$
$$\mathbf{W} = f\mathbf{E}\boldsymbol{\tau},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'),\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$
$$\mathbf{W} = c\mathbf{E}\boldsymbol{\tau},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + b\mathbf{n} \vee \boldsymbol{\tau}',\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

which violate the $D_{2m+1h}$-irreducibility condition for $(\mathbf{W}, \mathbf{A})$, we derive the following nine disjoint cases for the $D_{2m+1h}$-irreducible set $(\mathbf{W}, \mathbf{A})$:

(c1) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1,\ ac \neq 0;$

(c2) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_i,\ i = 3,\ 4,\ bc \neq 0;$

(c3) $\mathbf{W} = c\mathbf{E}\mathbf{z},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \mathbf{z} \in \{\mathbf{e}, \mathbf{e}'\},\ bc \neq 0;$

$$\text{(c4)} \begin{cases} \mathbf{W} = c\mathbf{E}\mathbf{v},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_i,\ i = 3,\ 4,\ bc \neq 0\,, \\ i = 3:\ \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m\}\,, \\ i = 4:\ \mathbf{e} \neq \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,. \end{cases}$$

$$\text{(c5)} \begin{cases} \mathbf{W} = c\mathbf{n} \wedge \mathbf{v},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_i,\ i = 3,\ 4,\ bc \neq 0\,, \\ i = 3: \quad \mathbf{e} \neq \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,, \\ i = 4: \quad \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m\}\,. \end{cases}$$

Then, for case (c1) we have

$$\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank} V''_{2m+1}(\mathbf{W}, \mathbf{A}) \geq \mathrm{rank}\{\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})),$$
$$\mathbf{W}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\} = 2.$$

For case (c2), we have

$$\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = (a^{2m} + b^{2m})\mathbf{e} \neq \mathbf{0},$$
$$\rho_m(\mathbf{q}(\mathbf{A})) = a^{2m+1}\mathbf{e},$$

for $i = 3,\ 4$, and hence with $D = \mathrm{rank} V''_{2m+1}(\mathbf{W},\mathbf{A})$ we have

$$D \geq \begin{cases} \mathrm{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) \\ \qquad\qquad + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3 \text{ if } i = 3, \\ \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})), \\ \qquad\qquad \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} = 3 \text{ if } i = 4. \end{cases}$$

For case (c3) we have (note that $b \neq 0$, i.e. $\psi(\mathbf{A}) = <\mathbf{q}(\mathbf{A}), \mathbf{e}> \neq k\pi$)

$$\begin{aligned} \mathrm{rank} V''_{2m+1}(\mathbf{W},\mathbf{A}) &\geq \mathrm{rank}\{\eta_{2m}(\mathbf{W}\mathbf{n}), \eta_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_{\mathbf{e}}\eta_m(\mathbf{q}(\mathbf{A})), \\ &\qquad \mathbf{W}\pi_m(\mathbf{q}(\mathbf{A})), \mathbf{W}\rho_m(\mathbf{q}(\mathbf{A}))\} \\ &= \mathrm{rank}\{\mathbf{e}, \mathbf{z}\sin m\psi(\mathbf{A}), \\ &\qquad \mathbf{z}\sin(m+1)\psi(\mathbf{A}) \mid \mathbf{z} = \mathbf{e}', \mathbf{n}\} = 3. \end{aligned}$$

For case (c4) we have (see the first equality for case (2))

$$D \geq \begin{cases} \mathrm{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{W}\mathbf{n}\} = 3 \text{ if } i = 3, \\ \mathrm{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \eta_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} \\ \qquad\qquad = 3 \ \text{ if } \ i = 4, \end{cases}$$

where $D = \mathrm{rank} V''_{2m+1}(\mathbf{W},\mathbf{A})$. Finally, for case (c5) we have

$$\mathrm{rank} V''_{2m+1}(\mathbf{W},\mathbf{A}) \geq \begin{cases} \mathrm{rank}\{\alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}, \eta_{2m}(\mathbf{W}\mathbf{n}), \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} \\ \qquad\qquad = 3 \ \text{ if } \ i = 3, \\ \mathrm{rank}\{\alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}, \eta_{2m}(\mathbf{W}\mathbf{n}), \\ \qquad\qquad \alpha_{2m+1}(\mathbf{W}\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 3 \ \text{ if } \ i = 4. \end{cases}$$

From the above results and Table 1 in Sec. 2 in Part I, we infer that the presented set $V''_{2m+1}(\mathbf{W}, \mathbf{A})$ obeys the criterion (2.3) in Part I and hence is a desired vector generating set. Further, by considering the two pairs $\mathbf{W}_1 = \mathbf{E}\mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n} \vee \mathbf{e}'$, $\mathbf{W}_2 = \mathbf{n} \wedge \mathbf{e}$ and $\mathbf{A}_2 = \mathbf{n} \vee \mathbf{e}'$, we deduce that the last three generators in the set $V''_{2m+1}(\mathbf{W}, \mathbf{A})$ are irreducible.

(**vii**) The $D_{2m+1h}$-irreducible set $(\mathbf{A}, \mathbf{B})$ of two symmetric tensors

$$
\begin{aligned}
V \quad & V''_{2m+1}(\mathbf{A}) \cup V''_{2m+1}(\mathbf{B}) \cup \{((\pi_m(\mathbf{q}(\mathbf{B})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \\
& ((\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}, \\
& |\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\overset{\circ}{\mathbf{B}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{B}}\,\mathbf{n}, \\
& (\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})))\mathbf{n}\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad (\equiv V''_{2m+1}(\mathbf{A}, \mathbf{B})) \\
R \quad & \mathbf{r} \cdot V''_{2m+1}(\mathbf{A}), \mathbf{r} \cdot V''_{2m+1}(\mathbf{B}), \\
& (\mathbf{r} \cdot \mathbf{n})((\pi_m(\mathbf{q}(\mathbf{B})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}), \\
& (\mathbf{r} \cdot \mathbf{n})((\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot \overset{\circ}{\mathbf{B}}\,\mathbf{n}), \\
& |\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\,\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\overset{\circ}{\mathbf{B}}\,\mathbf{n}|^{2m} \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{B}}\,\mathbf{n}, \\
& (\mathbf{r} \cdot \mathbf{n})(\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A}))).
\end{aligned}
$$

Here and henceforth, $\pi_m(\mathbf{q}(\mathbf{D}))$ is the polynomial vector-valued function of the symmetric tensor $\mathbf{D}$ given by (2.5) in Part II (see XIAO, BRUHNS and MEYERS [22]), with the replacement of $\mathbf{A}$ by $\mathbf{D}$ therein.

We proceed to work out the $D_{2m+1h}$-irreducible set $(\mathbf{A}, \mathbf{B})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{A}, \mathbf{B}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{A}$, $\mathbf{B}$. From (**vi**) we know that both $\overset{\circ}{\mathbf{A}}$ and $\overset{\circ}{\mathbf{B}}$ take the forms given by (2.5). Considering the combinations of the forms given by (2.5) and excluding the cases

$$\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ \beta_{2m+1}(\mathbf{q}(\mathbf{z})) \neq 0,\ \mathbf{z} = \mathbf{A} \text{ or } \mathbf{z} = \mathbf{B};$$

$$\overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + b\mathbf{n} \vee \boldsymbol{\tau},\ \overset{\circ}{\mathbf{B}} = c(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + d\mathbf{n} \vee \boldsymbol{\tau},\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

$$\overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}'),\ \overset{\circ}{\mathbf{B}} = c(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + d\mathbf{n} \vee \boldsymbol{\tau},\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

which violate the $D_{2m+1h}$-irreducibility condition for $(\mathbf{A}, \mathbf{B})$, we derive the following six disjoint cases for $D_{2m+1h}$-irreducible set $(\mathbf{A}, \mathbf{B})$:

(c1) $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1,\ \overset{\circ}{\mathbf{B}} = c(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}'),\ \boldsymbol{\nu}\neq\mathbf{e},\ ac\neq 0;$

(c2) $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_3,\ bd\neq 0;$

(c3) $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_4,\ bd\neq 0;$

$$(\text{c4})\ \begin{cases} \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}') + b\mathbf{n}\vee\boldsymbol{\nu},\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_i,\ i=3,\ 4,\ bd\neq 0\,, \\ i=3: \quad \mathbf{e}\neq\boldsymbol{\nu}\in\{\boldsymbol{\tau}_{2r}\mid r=1,\cdots,2r\}\,, \\ i=4: \quad \boldsymbol{\nu}\in\{\boldsymbol{\tau}_{2r}\mid r=0,1,\cdots,2r\}\,; \end{cases}$$

(c5) $\overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}') + b\mathbf{n}\vee\boldsymbol{\nu}',\ \overset{\circ}{\mathbf{B}} = c\mathbf{D}_1 + d\mathbf{D}_4,\ bd\neq 0,\ \boldsymbol{\nu}\neq\mathbf{e}.$

Then, for case (c1) we have

$$\Gamma(\mathbf{A},\mathbf{B})\cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \geq \{\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{B}))\} = 2.$$

For case (c2), by using $\overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0}$ and $b\neq 0$, i.e. $\psi(\mathbf{A}) = <\mathbf{q}(\mathbf{A}),\mathbf{e}>\neq k\pi$, we have

$$\begin{aligned} \mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) &\geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}, \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \\ &\quad \mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\} \\ &= \mathrm{rank}\{\mathbf{n},\mathbf{e},\mathbf{e}'\sin m\psi(\mathbf{A}), \mathbf{e}'\sin(m+1)\psi(\mathbf{A})\} = 3\,. \end{aligned}$$

For case (c3), by using $\overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0}$ we have

$$\mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \beta_{2m+1}(\mathbf{q}(\mathbf{A}))\beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}\} = 3,$$

when $\beta_{2m+1}(\mathbf{q}(\mathbf{A}))\neq 0$, and

$$\begin{aligned} \mathrm{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) &\geq \mathrm{rank}\{\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), (\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A}))\cdot\overset{\circ}{\mathbf{B}}\,\mathbf{n})\mathbf{n}\} \\ &= \mathrm{rank}\{\mathbf{e},\boldsymbol{\nu},\mathbf{n}\sin\frac{1}{2}(2m+1-(-1)^m)\psi(\mathbf{A})\} = 3\,, \end{aligned}$$

when $\beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0$, i.e. $\sin(2m+1)\psi(\mathbf{A}) = 0$ (note $\sin\psi(\mathbf{A})\neq 0$).

For case (c4) we have

$$\operatorname{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) \geq \begin{cases} \operatorname{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \\ \qquad\qquad \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\} = 3 \text{ if } i=3\,, \\ \operatorname{rank}\{\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \\ \qquad\qquad \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{B}}\,\mathbf{n}\} = 3 \text{ if } i=4\,. \end{cases}$$

Finally, for case (c5) we have

$$\begin{aligned} \operatorname{rank}V''_{2m+1}(\mathbf{A},\mathbf{B}) &\geq \operatorname{rank}\{\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), ((\pi_m(\mathbf{q}(\mathbf{B})) \\ &\qquad +(-1)^m\eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot\,\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} \\ &= \{\mathbf{e}, \mathbf{\nu}, b(c^{2m}+d^{2m})(\mathbf{e}\cdot\mathbf{\nu}')\mathbf{n}\} = 3\,. \end{aligned}$$

Thus, from the above results and Table 1 in Sec. 2 in Part I we infer that the presented set $V''_{2m+1}(\mathbf{A},\mathbf{B})$ obeys the criterion (2.3) in Part I, and therefore it is the desired vector generating set. Further, by considering the four pairs $(\mathbf{A}_i,\mathbf{B}_i)$ given by

$$\mathbf{A}_1 = \mathbf{n}\vee\mathbf{\tau}_1,\ \mathbf{B}_1 = \mathbf{D}_1; \qquad \mathbf{A}_2 = \mathbf{D}_1, \qquad \mathbf{B}_2 = \mathbf{n}\vee\mathbf{\tau}_1;$$

$$\mathbf{A}_3 = \mathbf{\tau}_1\otimes\mathbf{\tau}_1 - \mathbf{\tau}'_1\otimes\mathbf{\tau}'_1, \qquad \mathbf{B}_3 = \mathbf{n}\vee\mathbf{e}'\,;$$

$$\mathbf{\nu}'_1 = \mathbf{n}\vee\mathbf{e}'; \qquad \mathbf{A}_4 = \mathbf{n}\vee\mathbf{e}, \qquad \mathbf{B}_4 = \mathbf{n}\vee\mathbf{e}',$$

we deduce that the last four generators in the set $V''_{2m+1}(\mathbf{A},\mathbf{B})$ are irreducible, respectively.

**(viii)** The $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{W})$ of a vector and a skewsymmetric tensor

$V$ $\quad V''_{2m+1}(\mathbf{u})\cup V''_{2m+1}(\mathbf{W})\cup\{\mathbf{Wu}\}\ (\equiv V''_{2m+1}(\mathbf{u},\mathbf{W}))$

Skw $\quad \mathrm{Skw}''_{2m+1}(\mathbf{u})\cup\mathrm{Skw}_{4m+2}(\mathbf{W})\cup\{\mathbf{u}\wedge\mathbf{Wu}, \mathbf{u}\wedge\eta_{2m}(\mathbf{Wn}),$

$|\mathbf{u}|^{2m}\mathbf{Wu}\wedge\eta_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\,\overset{\circ}{\mathbf{u}}\wedge\mathbf{W}\eta_{2m}(\overset{\circ}{\mathbf{u}})\}$

$(\equiv \mathrm{Skw}''_{2m+1}(\mathbf{u},\mathbf{W}))$

Sym $\quad \mathrm{Sym}''_{2m+2}(\mathbf{u})\cup\mathrm{Sym}_{4m+2}(\mathbf{W})\cup\{\mathbf{u}\vee\mathbf{Wu}, \mathbf{u}\vee\eta_{2m}(\mathbf{Wn}),$

$|\mathbf{u}|^{2m}\mathbf{Wu}\vee\eta_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\,\overset{\circ}{\mathbf{u}}\vee\mathbf{W}\eta_{2m}(\overset{\circ}{\mathbf{u}})\}(\equiv \mathrm{Sym}''_{2m+1}(\mathbf{u},\mathbf{W}))$

$R \quad \mathbf{r}\cdot V''_{2m+1}(\mathbf{u}), \mathbf{r}\cdot V''_{2m+1}(\mathbf{W}), \mathbf{r}\cdot\mathbf{Wu}; \mathrm{tr}\mathbf{HW};$

$\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u}), \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\,\overset{\circ}{\mathbf{u}};$

$I''_{2m+1}(\mathbf{u}) \cup I_{4m+2}(\mathbf{W}) \cup \{\mathbf{u}\cdot\mathbf{W}^2\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})\}\ (\equiv I''_{2m+1}(\mathbf{u},\mathbf{W})).$

In the above table, the skewsymmetric tensor variable $\mathbf{H}$ is treated as having the form $\mathbf{H} = c\mathbf{W}$ with $c \neq 0$, which is derived from the condition (3.3) in Part I with $(\mathbf{z}_0, \mathbf{z}) = (\mathbf{W}, \mathbf{H})$ and $g = D_{2m+1h}$. As a result, of the invariants from the scalar products $\mathbf{H} : \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W})$, we need only to retain the invariant $\mathrm{tr}\mathbf{HW}$. Moreover, consider the symmetric tensor variable $\mathbf{C}$. The $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{W})$ is specified by cases (c1) – (c6) given below. Setting $\mathbf{z} = \mathbf{C}$, $X_0 = (\mathbf{u}, \mathbf{W})$ and $g = D_{2m+1h}$ in the condition (3.3) in Part I, we derive $\overset{\circ}{\mathbf{C}}= \mathbf{O}$ for case (c1); $\overset{\circ}{\mathbf{C}}= x\mathbf{D}_1 + y\mathbf{D}_2$ for cases (c2) and (c5); and $\mathbf{C} \in$ span $\mathrm{Sym}_{4m+2}(\mathbf{W})$ for cases (c3), (c4) and (c6). From these we deduce that, of the invariants from the scalar products $\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W})$, we need only to retain the invariants given by $\mathbf{C} : \mathrm{Sym}''_{2m+1}(\mathbf{u})$ and $\mathbf{C} : \mathrm{Sym}_{4m+2}(\mathbf{W})$, as well as $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\,\overset{\circ}{\mathbf{u}}$, as has been done in the above table. Here we would mention that for cases (c2) and (c5), the subspace $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ is generated by the three generators $\mathbf{I}$, $\mathbf{n}\otimes\mathbf{n}$ and $\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}$ in the set $\mathrm{Sym}''_{2m+1}(\mathbf{u})$ as well as the generator $\overset{\circ}{\mathbf{u}}\vee\mathbf{W}\,\overset{\circ}{\mathbf{u}}$, and hence the invariant $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\,\overset{\circ}{\mathbf{u}}$ is resulted in.

To prove the results in the table given, we first work out the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{W})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z})\cap D_{2m+1h} \neq \Gamma(\mathbf{u},\mathbf{W})\cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{W}$. Considering the combinations of the forms for $\mathbf{u}$ and $\mathbf{W}$ given by (2.3) and (2.4) and excluding the cases

$$\mathbf{u} = a\mathbf{n},\ \mathbf{W} = c\mathbf{n}\wedge\mathbf{v};\ \mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \mathbf{W} = c\mathbf{En},\ \beta_{2m+1}(\mathbf{u}) \neq 0;$$

$$\mathbf{u} = a\mathbf{v},\ \mathbf{W} = c\mathbf{Ev};\ \mathbf{u} = a\mathbf{v},\ \mathbf{W} = c\mathbf{n}\wedge\mathbf{v};$$

$$\mathbf{u} = a\mathbf{n} + b\mathbf{v},\ \mathbf{W} = c\mathbf{n}\wedge\mathbf{v};$$

which violate the just-mentioned $D_{2m+1h}$-irreducibility condition, we derive the following eight disjoint cases for the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{W})$:

(c1) $\mathbf{u} = a\mathbf{n}$, $\mathbf{W} = c\mathbf{En}$, $ac \neq 0$;

(c2) $\mathbf{u} = a\mathbf{e}$, $\mathbf{W} = c\mathbf{En}$, $ac \neq 0$;

(c3) $\mathbf{u} = a\mathbf{n}$, $\mathbf{W} = c\mathbf{Ee}$, $ac \neq 0$;

(c4) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}'$, $\mathbf{W} = c\mathbf{Ez}$, $\mathbf{z} \in \{\mathbf{e},\ \mathbf{e}'\}$, $bc \neq 0$;

(c5) $\mathbf{u} = a\mathbf{n} + b\mathbf{e}$, $\mathbf{W} = c\mathbf{En}$, $abc \neq 0$;

$$
\text{(c6)} \begin{cases} \mathbf{u} = a\mathbf{n} + b\mathbf{v}, \ \mathbf{W} = c\mathbf{E}\mathbf{z}, \ \mathbf{z} \in \{\mathbf{e}, \ \mathbf{e}'\}, \ abc \neq 0\,, \\ \mathbf{z} = \mathbf{e}: \quad \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m, \}\,, \\ \mathbf{z} = \mathbf{e}': \quad \mathbf{e} \neq \mathbf{v} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,. \end{cases}
$$

With the above cases we show that the three presented sets $V''_{2m+1}(\mathbf{u}, \mathbf{W})$, $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W})$ obey the criterion (2.3) in Part I. Case (c1) can be treated easily.

For case (c2) we have

$$
\Gamma(\mathbf{u}, \mathbf{W}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}), \quad \mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathring{\mathbf{u}}, \mathbf{W}\mathbf{u}\} = 2,
$$

$$
\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathbf{W}\} = 1,
$$

$$
\mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathbf{u} \vee \mathbf{W}\mathbf{u}\} = 4.
$$

For case (c3) we have

$$
\mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u}\} = 3,
$$

$$
\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{\mathbf{W}, \mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n})\} = 3,
$$

$$
\begin{aligned} \mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{W}^2, \mathbf{n} \vee \mathbf{W}\mathbf{n}, \mathbf{u} \vee \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \\ &\mathbf{W}\mathbf{u} \vee \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n})\} = 6. \end{aligned}
$$

For case (c4) we have

$$
\begin{aligned} \mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \ &\geq \ \mathrm{rank}\{\mathring{\mathbf{u}}, \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \alpha_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{n}, \mathbf{W}\mathbf{u}\} \\ &= \ \mathrm{rank}\{\mathbf{e}', \mathbf{e}, \alpha_{2m+1}(\mathbf{n} \times \mathbf{z})\mathbf{n}, a\mathbf{e} \times \mathbf{z} + b\mathbf{e}' \times \mathbf{z}\} = 3\,, \end{aligned}
$$

$$
\begin{aligned} \mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{W}) \ \geq \ \mathrm{rank}\{&\mathbf{W}, \mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), |\mathbf{u}|^{2m} \mathbf{W}\mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}) \\ &+ |\mathbf{W}|^{2m}\ \mathring{\mathbf{u}} \wedge \mathbf{W}\boldsymbol{\eta}_{2m}(\mathring{\mathbf{u}})\} = 3\,, \end{aligned}
$$

$$
\begin{aligned} \mathrm{rank}\ \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{W}) \ \geq \ \mathrm{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{W}^2, \mathbf{n} \vee \mathbf{W}\mathbf{n}, \mathbf{u} \vee \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \\ &\mathbf{u} \vee \mathbf{W}\mathbf{u}, |\mathbf{u}|^{2m}\ \mathring{\mathbf{u}} \vee \mathbf{W}\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}) \\ &+ |\mathbf{W}|^{2m}\ \mathring{\mathbf{u}} \vee \mathbf{W}\boldsymbol{\eta}_{2m}(\mathring{\mathbf{u}})\} = 6\,. \end{aligned}
$$

For case (c5) we have

$$
\mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{W}) \geq \mathrm{rank}\{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{W}\mathbf{u}\} = 3,
$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, \mathbf{W}, \mathbf{u}\wedge\mathbf{W}\mathbf{u}\} = 3,$$

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}},$$

$$\mathbf{u}\vee\mathbf{W}\mathbf{u}, \overset{\circ}{\mathbf{u}}\vee\mathbf{W}\eta_{2m}(\overset{\circ}{\mathbf{u}})\} = 6.$$

Finally, for case (c6) we have the first two of the last three expressions above and, moreover,

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{W}^2, \mathbf{n}\vee\mathbf{W}\mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{u}\vee\mathbf{W}\mathbf{u}\} = 6.$$

Then, from the above results and Tables 1 – 3 in Sec. 2 in Part I we infer that the three presented sets $V''_{2m+1}(\mathbf{u},\mathbf{W})$, $\text{Skw}''_{2m+1}(\mathbf{u},\mathbf{W})$ and $\text{Sym}''_{2m+1}(\mathbf{u},\mathbf{W})$ obey the criterion (2.3) in Part I, and therefore they supply desired vector, skewsymmetric tensor and symmetric tensor generating sets, respectively. Moreover, by considering the pair $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{W}_1 = \mathbf{E}\mathbf{e}$, we deduce that the generator $\mathbf{W}\mathbf{u}$ and the respective last two generators in the sets $\text{Skw}''_{2m+1}(\mathbf{u},\mathbf{W})$ and $\text{Sym}''_{2m+1}(\mathbf{u},\mathbf{W})$ are irreducible. By considering the pair $\mathbf{u}_2 = \mathbf{n}+\mathbf{e}$ and $\mathbf{W}_2 = \mathbf{E}\mathbf{n}$, we deduce that the two generators $\mathbf{u}\wedge\mathbf{W}\mathbf{u}$ and $\mathbf{u}\vee\mathbf{W}\mathbf{u}$ are also irreducible.

Finally, we show that the presented set $I''_{2m+1}(\mathbf{u},\mathbf{W})$ supplies a functional basis for the $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{W})$. Towards this goal it suffices to show that the former determines a functional basis for the $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{W})$ under the group $D_{\infty h}(\mathbf{n})$ (see the remark at the end of Sec. 4 (**vi**) in Part I). In fact, the just-mentioned functional basis is obtainable from an isotropic functional basis for $(\mathbf{u},\mathbf{W},\mathbf{n}\otimes\mathbf{n})$ (see BOEHLER [5]). The latter is formed by the invariants

$$(\mathbf{u}\cdot\mathbf{n})^2,\ |\overset{\circ}{\mathbf{u}}|^2,\ (\text{tr}\mathbf{W}\mathbf{N})^2, |\mathbf{W}\mathbf{n}|^2, \mathbf{u}\cdot\mathbf{W}^2\mathbf{u},\ (\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n},$$

$$(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})(\mathbf{u}\cdot\mathbf{W}^2\mathbf{n}).$$

The first six invariants given above are included in the set $I''_{2m+1}(\mathbf{u},\mathbf{W})$. The last invariant is of the form $(\text{tr}\mathbf{W}^2)(\mathbf{u}\cdot\mathbf{n})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}) + (\text{tr}\mathbf{W}\mathbf{N})(\text{tr}\mathbf{W}(\mathbf{E}\mathbf{u}))(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})$. The first term is redundant and the second term vanishes for each of cases (c1) – (c6), and hence the invariant at issue is redundant.

(**ix**) The $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{A})$ of a vector and a symmetric tensor

$V \quad V''_{2m+1}(\mathbf{u}) \cup V''_{2m+1}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{A}}\,\mathbf{u}\}\ (\equiv V''_{2m+1}(\mathbf{u},\mathbf{A}))$

$$\text{Skw}\quad \text{Skw}''_{2m+1}(\mathbf{u}) \cup \text{Skw}_{4m+2}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \rho_m(\mathbf{q}(\mathbf{A})),$$

$$\mathbf{u} \wedge \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + (-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \pi_m(\mathbf{q}(\mathbf{A})),$$

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u} \wedge \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} \; (\equiv \text{Skw}''_{2m+1}(\mathbf{u},\mathbf{A}))$$

$$\text{Sym}\quad \text{Sym}_{2m+1}(\mathbf{u}) \cup \text{Sym}_{4m+2}(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \rho_m(\mathbf{q}(\mathbf{A})),$$

$$\mathbf{u} \vee \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + (-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \pi_m(\mathbf{q}(\mathbf{A})),$$

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\vee \overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u} \vee \eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} \; (\equiv \text{Sym}''_{2m+1}(\mathbf{u},\mathbf{A}))$$

$$R\quad \mathbf{r}\cdot V''_{2m+1}(\mathbf{u}), \mathbf{r}\cdot V''_{2m+1}(\mathbf{A}), \mathbf{r}\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{u}; \mathbf{H} : \text{Skw}''_{2m+1}(\mathbf{u}); \overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}};$$

$$I''_{2m+1}(\mathbf{u}) \cup I_{4m+2}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}{}^2\, \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\, \overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}\}$$

$$(\equiv I''_{2m+1}(\mathbf{u},\mathbf{A})).$$

In the above table, the skewsymmetric tensor variable $\mathbf{H}$ pertains to span $\text{Skw}''_{2m+1}(\mathbf{u})$ or span $\text{Skw}_{4m+2}(\mathbf{A})$. This fact can be derived from cases (c1) – (c6) for the $D_{2m+1h}$-irreducible set $(\mathbf{A},\mathbf{B})$ that will be given below, as well as the condition (3.3) in Part I with $\mathbf{z}_0 = \mathbf{H}$ and $\mathbf{z} \in \{\mathbf{u},\mathbf{A}\}$ and $g = D_{2m+1h}$. As a result, of the invariants given by the scalar products $\mathbf{H} : \text{Skw}''_{2m+1}(\mathbf{u},\mathbf{A})$, we need only to retain those given by $\mathbf{H} : \text{Skw}''_{2m+1}(\mathbf{u})$ and $\mathbf{H} : \text{Skw}_{4m+2}(\mathbf{A})$. Moreover, by applying cases (c1) – (c6) for the $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{A})$ given below and the condition (3.3) in Part I with $X_0 = (\mathbf{u},\mathbf{A})$ and $(\mathbf{z}_0,\mathbf{z}) = (\mathbf{A},\mathbf{C})$ and $g = D_{2m+1h}$, we deduce that $\mathbf{C} \in \text{Sym}(C_{2h}(\mathbf{n}))$ for case (c5) below and $\mathbf{C} \in$ span $\text{Sym}_{4m+2}(\mathbf{A})$ for cases (c1) – (c4) and (c6) below. Accordingly, of the invariants given by the scalar products $\mathbf{C} : \text{Sym}''_{2m+1}(\mathbf{u},\mathbf{A})$, we need only to retain $\mathbf{C} : \text{Sym}_{4m+2}(\mathbf{A})$ and $\overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}$. The latter invariant results from the fact that for case (c5) below, the subspace $\text{Sym}(C_{2h}(\mathbf{n}))$ is generated by the generator $\overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}$ and the generators in the set $\text{Sym}_{4m+2}(\mathbf{A})$.

To prove the results in the table given, we first work out the $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2m+1h} \neq \Gamma(\mathbf{u},\mathbf{A}) \cap D_{2m+1h}$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{A}$. Considering the combinations of the forms for $\mathbf{u}$ and $\overset{\circ}{\mathbf{A}}$ given by (2.3) and (2.5) and excluding the case

$$\mathbf{u} = c\mathbf{n},\; \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu}\otimes\boldsymbol{\nu} - \boldsymbol{\nu}'\otimes\boldsymbol{\nu}') + b\mathbf{n}\vee\boldsymbol{\nu};$$

$$\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\; \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2,\; \beta_{2m+1}(\mathbf{u}) \neq 0 \text{ or } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0;$$

$$\mathbf{u} = c\boldsymbol{\tau},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\tau} \otimes \boldsymbol{\tau} - \boldsymbol{\tau}' \otimes \boldsymbol{\tau}') + b\mathbf{n} \vee \boldsymbol{\tau},\ \boldsymbol{\tau} = \boldsymbol{\tau}_k;$$

$$\mathbf{u} = c\mathbf{n} + d\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}');$$

$$\mathbf{u} = c\mathbf{n} + d\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = a(\boldsymbol{\nu} \otimes \boldsymbol{\nu} - \boldsymbol{\nu}' \otimes \boldsymbol{\nu}') + b\mathbf{n} \vee \boldsymbol{\nu};$$

which violate the just-mentioned $D_{2m+1h}$-irreducibility condition, we derive the following eight disjoint cases for the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{A})$:

(c1) $\mathbf{u} = a\mathbf{n}$, $\overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2$, $a(c^2 + d^2) \neq 0$;

(c2) $\mathbf{u} = a\mathbf{n}$, $\overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_4$, $ad \neq 0$;

(c3) $\mathbf{u} = a\boldsymbol{\nu}$, $\overset{\circ}{\mathbf{A}} = c\mathbf{D}_1$, $ac \neq 0$, $\mathbf{e} \neq \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}$;

(c4) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}'$, $\overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_i$, $i = 3,\ 4$, $bd \neq 0$;

(c5) $\mathbf{u} = a\mathbf{n} + b\mathbf{e}$, $\overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2$, $abd \neq 0$;

$$\text{(c6)} \begin{cases} \mathbf{u} = a\mathbf{n} + b\boldsymbol{\nu},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_i,\ i = 3,\ 4,\ abd \neq 0\,, \\ i = 3: \quad \mathbf{e} \neq \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 1, \cdots, 2m\}\,, \\ i = 4: \quad \boldsymbol{\nu} \in \{\boldsymbol{\tau}_{2r} \mid r = 0, 1, \cdots, 2m\}\,. \end{cases}$$

With the above cases we show that the three presented sets $V''_{2m+1}(\mathbf{u}, \mathbf{A})$, $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{A})$ obey the criterion (2.3) in Part I.

In fact, for case (c1) we have

$$\Gamma(\mathbf{u}, \mathbf{A}) \cap D_{2m+1h} = C_{1h}(\mathbf{n}) \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0,$$

$$\mathrm{rank} V''_{2m+1}(\mathbf{u}, \mathbf{A}) \geq \mathrm{rank}\{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\}$$

$$= \begin{cases} 3 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ 2 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{A}) \geq \mathrm{rank}\{\beta_{2m+1}(\mathbf{q}(\mathbf{A}))\mathbf{N}, \mathbf{n} \wedge \boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})),$$

$$\mathbf{n} \wedge \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\} = \begin{cases} 3 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,, \\ 1 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,, \end{cases}$$

$$\begin{aligned}
\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}},\Phi_{2m+1}(\mathbf{q}(\mathbf{A})),\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A})),\\
&\quad \mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A}))\}\\
&= \begin{cases} \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathbf{D}_1,\mathbf{D}_2,\mathbf{n}\vee\mathbf{e},\mathbf{n}\vee\mathbf{e}'\} = 6 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathbf{q}(\mathbf{A}))\neq 0\,, \\ \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}},\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A}))\} = 4 \\ \qquad\qquad \text{if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0\,. \end{cases}
\end{aligned}$$

For case (c2) we have

$$\text{rank}V''_{2m+1}(\mathbf{u}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\mathring{\mathbf{A}}\,\mathbf{u}\} = 3,$$

$$\begin{aligned}
\text{rank Skw}''_{2m+1}(\mathbf{u}) &\geq \text{rank}\{\mathbf{n}\wedge\mathring{\mathbf{A}}\,\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{u}\wedge\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\mathbf{n}\wedge\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\\
&\quad +(-1)^m\mathbf{n}\wedge\pi_m(\mathbf{q}(\mathbf{A}))\}\\
&= \text{rank}\{\mathbf{n}\wedge\mathbf{e}',\mathbf{e}\wedge\mathbf{e}',(c^{2m}+d^{2m})\mathbf{n}\wedge\mathbf{e}\} = 3\,,
\end{aligned}$$

$$\begin{aligned}
\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{n}\otimes\mathring{\mathbf{A}}\,\mathbf{n},\mathring{\mathbf{A}},\mathring{\mathbf{A}}\,\mathbf{u}\vee\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\\
&\quad \mathbf{n}\vee\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})+(-1)^m\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A}))\}\\
&= \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathbf{e}'\otimes\mathbf{e}',\mathbf{n}\vee\mathbf{e}',\mathbf{e}\vee\mathbf{e}',\\
&\qquad (d^{2m}+c^{2m})\mathbf{n}\vee\mathbf{e}\}) = 6\,.
\end{aligned}$$

For case (c3) we have

$$\Gamma(\mathbf{u},\mathbf{A})\cap D_{2m+1h} = C_{1h}(\mathbf{n}),$$

$$\text{rank}V''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathring{\mathbf{u}},\eta_m(\mathbf{q}(\mathbf{A}))\} = 2,$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{u}}\} = 1,$$

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}},\mathring{\mathbf{A}}\} = 4.$$

For case (c4) we have

$$\text{rank}V''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \begin{cases} \text{rank}\{\mathring{\mathbf{u}},\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n},\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\} = 3 \text{ if } i=3\,, \\ \text{rank}\{\mathring{\mathbf{u}},\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}),\mathring{\mathbf{A}}\,\mathbf{u}\} = 3 \text{ if } i=4\,, \end{cases}$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}\wedge \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{u}\wedge\boldsymbol{\eta}, \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u}\wedge\boldsymbol{\eta}\} = 3,$$

$$\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{u}\vee\boldsymbol{\eta}, \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}} + \overset{\circ}{\mathbf{A}}\,\mathbf{u}\vee\boldsymbol{\eta}\} = 6,$$

where $\boldsymbol{\eta} = \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$.

For case (c5) we have (note that $d \neq 0$, i.e., $\psi(\mathbf{A}) = <\mathbf{q}(\mathbf{A}), \mathbf{e}> \neq k\pi$)

$$\text{rank}V_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}\} = 3,$$

$$\begin{aligned}\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) \;\geq\; & \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},\\ & \mathbf{n}\wedge\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}\\ = \;& \text{rank}\{\mathbf{n}\wedge\mathbf{e}, \mathbf{e}\wedge\mathbf{e}', \mathbf{n}\wedge\mathbf{e}'\sin m\psi(\mathbf{A}),\\ & \mathbf{n}\wedge\mathbf{e}'\sin(m+1)\psi(\mathbf{A})\} = 3\,,\end{aligned}$$

$$\begin{aligned}\text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A}) \;\geq\; & \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}},\\ & \mathbf{n}\vee\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}\\ = \;& \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{n}\vee\mathbf{e}, \mathbf{e}\vee\mathbf{e}',\\ & \mathbf{n}\vee\mathbf{e}'\sin m\psi(\mathbf{A}), \mathbf{n}\vee\mathbf{e}'\sin(m+1)\psi(\mathbf{A})\} = 6\,.\end{aligned}$$

Finally, for case (c6) we have

$$\text{rank}V''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 3,$$

$$\text{rank Skw}''_{2m+1}(\mathbf{u},\mathbf{A}) \geq \begin{cases}\text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{u}\wedge\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\}\\ \qquad = 3 \text{ if } \boldsymbol{\nu}\neq\mathbf{e},\\ \text{rank}\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}\wedge\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\}\\ \qquad = 3 \text{ if } i=4,\ \boldsymbol{\nu}=\mathbf{e},\end{cases}$$

$$D \geq \begin{cases}\text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}\} = 6 \text{ if } \boldsymbol{\nu}\neq\mathbf{e},\\ \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}\vee\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\}\\ \qquad = 3 \text{ if } i=4,\ \boldsymbol{\nu}=\mathbf{e},\end{cases}$$

where $D = \text{rank Sym}''_{2m+1}(\mathbf{u},\mathbf{A})$.

From the above results we infer that the three presented sets $V''_{2m+1}(\mathbf{u}, \mathbf{A})$, $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{A})$ obey the criterion (2.3) in Part I, and hence they supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets, respectively. Further, by considering the pair $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n}\vee\mathbf{e}'$, we deduce that the generators $\overset{\circ}{\mathbf{A}}\,\mathbf{u}$ and the respective last two generators in the sets $\mathrm{Skw}''_{2m+1}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{A})$ are irreducible. By considering the pair $\mathbf{u}_2 = \mathbf{n}$ and $\mathbf{A}_2 = \mathbf{e} \vee \mathbf{e}'$, we infer that the two generators $(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \rho_{2m}(\mathbf{q}(\mathbf{A}))$ and $(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \rho_{2m}(\mathbf{q}(\mathbf{A}))$ are irreducible. By considering the pair $\mathbf{u}_3 = \mathbf{v}_2$ and $\mathbf{A}_3 = \mathbf{D}_1$, we infer that the generator $\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}$ is irreducible.

Finally, we show that the presented $I''_{2m+1}(\mathbf{u}, \mathbf{A})$ supplies a functional basis for the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{A})$ under the group $D_{2m+1h}$. Towards this goal it suffices to show that this set determines a functional basis for the set $(\mathbf{u}, \mathbf{A})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$ (see the remark at the end of Sec. 4 (**vi**) in Part I). In fact, the latter is obtainable from an isotropic functional basis for $(\mathbf{u}, \overset{\circ}{\mathbf{A}}, \mathbf{n}\otimes\mathbf{n})$ (see BOEHLER [5]), plus the invariants $\mathbf{n}\cdot\mathbf{A}\mathbf{n}$ and $\mathrm{tr}\mathbf{A}$. By using the related result for isotropic functions we know that the just-mentioned isotropic functional basis is formed by the invariants $\mathbf{u}\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{u}$, $\mathbf{u}\cdot \overset{\circ}{\mathbf{A}}{}^2\mathbf{u}$, $(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}$, as well as certain $D_{\infty h}$-invariants of $\mathbf{u}$ or $\overset{\circ}{\mathbf{A}}$. Each of the latter is determined by the basis $I''_{2m+1}(\mathbf{u})$ or $I_{4m+2}(\mathbf{A})$. The first three invariants yield the last three invariants in the set $I''_{2m+1}(\mathbf{u}, \mathbf{A})$.

### 2.3. $D_{2m+1h}$-irreducible sets of three variables

(x) The $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ of two vectors and a skewsymmetric tensor

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{v}}, \mathbf{W}\mathbf{u}, \mathbf{W}\mathbf{v}\}$$

$$\mathrm{Skw} \quad \{\mathbf{W}, \mathbf{u}\wedge\mathbf{v}, (\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\wedge(\mathbf{u}\times\mathbf{v})\}$$

$$\mathrm{Sym} \quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \mathbf{u}\vee\mathbf{W}\mathbf{u}, \mathbf{v}\vee\mathbf{W}\mathbf{v}, \mathbf{u}\vee\mathbf{v}, (\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\vee(\mathbf{u}\times\mathbf{v})\}$$

$$R \quad \{(\mathbf{u}\cdot\mathbf{n})^2, (\mathbf{v}\cdot\mathbf{n})^2, |\,\overset{\circ}{\mathbf{u}}\,|^2, |\,\overset{\circ}{\mathbf{v}}\,|^2, (\mathrm{tr}\mathbf{W}\mathbf{N})^2\}\,.$$

From the $D_{2m+1h}$-irreducibility condition for $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ (see (3.2) in Part I) we know that $(\mathbf{x}, \mathbf{W})$ is $D_{2m+1h}$-irreducible and $\Gamma(\mathbf{x}, \mathbf{W})\cap D_{2m+1h} \neq C_1$ for each $\mathbf{x}\in\{\mathbf{u}, \mathbf{v}\}$. Hence, case (c1) or case (c2) given in (**viii**) holds for each $\mathbf{x}\in\{\mathbf{u}, \mathbf{v}\}$. It is evident that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. From these we deduce that the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ is specified by $\mathbf{u} = a\mathbf{n}$, $\mathbf{v} = b\mathbf{e}$, $\mathbf{W} = c\mathbf{E}\mathbf{n}$ with $abc \neq 0$. With the aid of the latter, the presented results can be proved easily.

In the above table, the invariants from the scalar products have been omitted. The reason is as follows. First, the invariants $\mathbf{r}\cdot\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ runs over the presented vector generating set, have been covered before. Second, the skewsymmetric tensor variable should be of the form $\mathbf{H} = c\mathbf{W}$, and hence here we need to consider only the invariant tr$\mathbf{HW}$, which has been covered before. The other form for $\mathbf{H}$ leads to $\Gamma(\mathbf{W},\boldsymbol{\Omega}) \cap D_{2m+1h} = C_1$, which has been treated in (**iv**). Finally, the symmetric tensor variable $\mathbf{C}$ should be of the form $\overset{\circ}{\mathbf{A}}= a\mathbf{D}_1 + b\mathbf{D}_2$, and hence here we need to consider only the invariants tr$\mathbf{CG}$, where $\mathbf{G}$ is the first six symmetric tensor generators given. The six invariants have also been covered before. The other form of $\mathbf{A}$ leads to $\Gamma(\mathbf{W},\mathbf{C}) \cap D_{2m+1h} = C_1$, which has been treated in (**vi**).

(**xi**) The $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{A})$ of two vectors and a symmetric tensor

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{v}}, \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\}$$

$$\text{Skw} \quad \{\mathbf{u}\wedge\mathbf{v}, \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \overset{\circ}{\mathbf{v}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{v}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}$$

$$\text{Sym} \quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \mathbf{u}\vee\mathbf{v}, \overset{\circ}{\mathbf{A}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}$$

$$R \quad \{(\mathbf{u}\cdot\mathbf{n})^2, (\mathbf{v}\cdot\mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, |\overset{\circ}{\mathbf{v}}|^2, |\mathbf{q}(\mathbf{A})|^2, \mathbf{n}\cdot\mathbf{A}\mathbf{n}, \mathrm{tr}\mathbf{A}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}\}\,.$$

From the $D_{2m+1h}$-irreducibility condition for $(\mathbf{u},\mathbf{v},\mathbf{A})$ (see (3.2) in Part I) we know that $(\mathbf{x},\mathbf{A})$ is $D_{2m+1h}$-irreducible and $\Gamma(\mathbf{x},\mathbf{A}) \cap D_{2m+1h} \neq C_1$ for each $\mathbf{x} \in \{\mathbf{u},\mathbf{v}\}$. Hence, case (c1) with $\beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0$ or case (c3) given in (**ix**) holds for each $\mathbf{x} \in \{\mathbf{u},\mathbf{v}\}$. It is evident that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. From these we deduce that the $D_{2m+1h}$-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{A})$ is specified by $\mathbf{u} = a\mathbf{n}$, $\mathbf{v} = b\boldsymbol{\nu}$, $\overset{\circ}{\mathbf{A}}= c\mathbf{D}_1$ with $\boldsymbol{\nu} \neq \mathbf{e}$ and $abc \neq 0$. With the aid of the latter, the presented results can be proved easily.

In the above table, the invariants from the scalar products have been omitted. First, it is evident that the invariants $\mathbf{r}\cdot\boldsymbol{\psi}$ and tr$\mathbf{CG}$, where $\boldsymbol{\psi}$ and $\mathbf{G}$ run over the presented vector and symmetric tensor generating sets respectively, have been covered before. Next, the skewsymmetric tensor variable $\mathbf{H}$ should be of the form $\mathbf{H} = c\mathbf{En}$, and hence here we need to consider only the invariant $\mathbf{v}\cdot\mathbf{H}\overset{\circ}{\mathbf{A}}\mathbf{v}$, which has been given before. The other form of $\mathbf{H}$ leads to $\Gamma(\mathbf{W},\mathbf{A}) \cap D_{2m+1h} = C_1$, which has been treated in (**vi**).

(**xii**) The $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{W}, \mathbf{A})$

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \eta_m(\mathbf{q}(\mathbf{A})), \mathbf{W}\eta_m(\mathbf{q}(\mathbf{A}))\}$$

$$\mathrm{Skw} \quad \{\mathbf{W}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\wedge(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})))\}$$

$$\mathrm{Sym} \quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A})), \overset{\circ}{\mathbf{A}}\,\mathbf{W}-\mathbf{W}\,\overset{\circ}{\mathbf{A}},$$
$$(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})))\}$$

$$R \quad \{(\mathbf{u}\cdot\mathbf{n})^2, (\mathrm{tr}\mathbf{WN})^2, \mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\mathbf{A}, |\mathbf{q}(\mathbf{A})|^2\}\,.$$

From the $D_{2m+1h}$-irreducibility condition for $(\mathbf{u}, \mathbf{W}, \mathbf{A})$ we know that $(\mathbf{W}, \mathbf{A})$ is $D_{2m+1h}$-irreducible and $\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1h} \neq C_1$. Hence, case (c1) given in (**vi**), i.e. $\mathbf{W} = c\mathbf{En}$ and $\overset{\circ}{\mathbf{A}}= a\mathbf{D}_1$ with $ac \neq 0$, holds. Further, we derive $\mathbf{u} = b\mathbf{n}$ with $b \neq 0$. With the aid of these, the four presented results can be proved easily.

By virtue of the same argument used in (**x**), we have omitted the invariants from the scalar products.

(**xiii**) The $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{A}, \mathbf{B})$ of a vector and two symmetric tensors

$$V \quad V''_{2m+1}(\mathbf{u}, \mathbf{A}) \cup V''_{2m+1}(\mathbf{u}, \mathbf{B})$$

$$\mathrm{Skw} \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{B})), \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\}$$

$$\mathrm{Sym} \quad \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{A}) \cup \mathrm{Sym}''_{2m+1}(\mathbf{u}, \mathbf{B})$$

$$R \quad \{(\mathbf{u}\cdot\mathbf{n})^2, \mathrm{tr}\mathbf{A}, \mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\mathbf{B}, \mathbf{n}\cdot\mathbf{Bn}, \mathrm{tr}\mathbf{A_e B_e}\}.$$

From the $D_{2m+1h}$-irreducibility condition for $(\mathbf{u}, \mathbf{A}, \mathbf{B})$ we know that $(\mathbf{A}, \mathbf{B})$ is $D_{2m+1h}$-irreducible and $\Gamma(\mathbf{A}, \mathbf{B}) \cap D_{2m+1h} \neq C_1$. Hence, case (c1) given in (**vii**) holds. Further, we derive $\mathbf{u} = b\mathbf{n}$ with $b \neq 0$. Thus, the $D_{2m+1h}$-irreducible set $(\mathbf{u}, \mathbf{A}, \mathbf{B})$ is of the form

$$\begin{cases} \mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}}= b(\mathbf{e}\otimes\mathbf{e}-\mathbf{e}'\otimes\mathbf{e}'),\ \overset{\circ}{\mathbf{B}}= c(\mathbf{v}\otimes\mathbf{v}-\mathbf{v}'\otimes\mathbf{v}'),\ abc\neq 0\,, \\ \mathbf{v}' = \mathbf{n}\times\mathbf{v},\ \mathbf{e}\neq\mathbf{v}\in\{\tau_{2r}\mid r=1,\cdots,2m\}\,. \end{cases}$$

With the aid of the latter and the formula (2.4) in Part I, the presented results can be verified easily. All the invariants and generators given here have been covered before.

(**xiv**) The set $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$ of a vector and two skewsymmetric tensors

Any given set $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$ can not be a $g$-irreducible set for each subgroup $g \subset$Orth with $-\mathbf{I} \notin g$ and hence can be omitted. In fact, this fact is obviously true if $\mathbf{W}$ and $\mathbf{\Omega}$ are linearly dependent. If $\mathbf{W}$ and $\mathbf{\Omega}$ are linearly independent, then we have $\Gamma(\mathbf{W},\boldsymbol{\Omega})\cap g = C_1$ for each just-mentioned subgroup $g$. The foregoing fact is also true.

### 2.4. The general result

Applying Theorem 2.1 in XIAO [20] and incorporating the fact indicated at the outset of this section, from (a) – (c) we obtain the following general result.

THEOREM 7. *The four sets given by*

$$I''_{2m+1}(\mathbf{u});\ I_{4m+2}(\mathbf{W});\ I_{4m+2}(\mathbf{A});\ I_{4m+2}(\mathbf{W},\boldsymbol{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C});$$

$$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}, \overset{\circ}{\mathbf{u}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{v}}), \overset{\circ}{\mathbf{v}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{u}});$$

$$(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}^2\,\overset{\circ}{\mathbf{u}}, \beta_{2m+1}(\overset{\circ}{\mathbf{u}})(\mathrm{tr}\mathbf{W}\mathbf{N}), (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{W}\mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{W}\mathbf{n}), \overset{\circ}{\mathbf{u}}\cdot\eta_{2m}(\mathbf{W}\mathbf{n}), \eta_{2m}(\mathbf{W}\mathbf{n})\cdot\mathbf{W}\mathbf{u}, \eta_{2m}(\mathbf{W}\mathbf{n})\cdot\mathbf{W}^2\mathbf{u};$$

$$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\,\overset{\circ}{\mathbf{u}}, \eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n},$$

$$\overset{\circ}{\mathbf{u}}\cdot\eta_m(\mathbf{q}(\mathbf{A})), \eta_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{A}_e\,\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{u}}\cdot\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$

$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + J(\mathbf{A})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\mathbf{u}\cdot\mathbf{n});$$

$$\mathbf{u}\cdot\mathbf{W}\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{v}})\cdot\mathbf{W}\mathbf{n} + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{W}\mathbf{n};$$

$$\mathbf{u}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{v}})\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n};$$

$$\eta_{2m}(\mathbf{W}\mathbf{n})\cdot\boldsymbol{\Omega}\mathbf{u}, \eta_{2m}(\boldsymbol{\Omega}\mathbf{n})\cdot\mathbf{W}\mathbf{u},$$

$$|\mathbf{W}|^{2m}\beta_{2m+1}(\boldsymbol{\Omega}\mathbf{n})\mathbf{u}\cdot(\mathbf{E}:\mathbf{W}) + |\boldsymbol{\Omega}|^{2m}\beta_{2m+1}(\mathbf{W}\mathbf{n})\mathbf{u}\cdot(\mathbf{E}:\boldsymbol{\Omega});$$

$$(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot\mathbf{W}\mathbf{u},$$

$$(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) - \rho_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{W}\mathbf{u}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{W}\,\overset{\circ}{\mathbf{u}},$$

$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{W}\mathbf{n})(\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + |\mathbf{W}\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n});$$

$$(\mathbf{u}\cdot\mathbf{n})(\pi_m(\mathbf{q}(\mathbf{B})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}))\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$

$$(\mathbf{u}\cdot\mathbf{n})((\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\cdot\overset{\circ}{\mathbf{B}}\,\mathbf{n}),$$

$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{n} + |\,\overset{\circ}{\mathbf{B}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{B}}\,\mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})(\beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})));$$

*and*

$$V''_{2m+1}(\mathbf{u}); V''_{2m+1}(\mathbf{W}); V''_{2m+1}(\mathbf{A}); \mathbf{W}\mathbf{u}; \overset{\circ}{\mathbf{A}}\,\mathbf{u};$$

$$\mathbf{W}\eta_{2m}(\mathbf{\Omega n}), \mathbf{\Omega}\eta_{2m}(\mathbf{Wn}),$$

$$|\mathbf{W}|^{2m}\beta_{2m+1}(\mathbf{\Omega n})(\mathbf{E}:\mathbf{W}) + |\mathbf{\Omega}|^{2m}\beta_{2m+1}(\mathbf{Wn})(\mathbf{E}:\mathbf{\Omega});$$

$$\mathbf{W}(\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})),$$

$$(\mathrm{tr}\mathbf{WN})\beta_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n} + \mathbf{W}\rho_m(\mathbf{q}(\mathbf{A})),$$

$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathbf{Wn})\,\mathring{\mathbf{A}}\,\mathbf{n} + |\mathbf{Wn}|^{2m}\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{Wn};$$

$$((\pi_m(\mathbf{q}(\mathbf{B})) + (-1)^m\eta_{2m}(\mathring{\mathbf{B}}\,\mathbf{n}))\cdot\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n},$$

$$((\pi_m(\mathbf{q}(\mathbf{A})) + (-1)^m\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}))\cdot\mathring{\mathbf{B}}\,\mathbf{n})\mathbf{n},$$

$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathring{\mathbf{B}}\,\mathbf{n})\,\mathring{\mathbf{A}}\,\mathbf{n} + |\mathring{\mathbf{B}}\,\mathbf{n}|^{2m}\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\,\mathring{\mathbf{B}}\,\mathbf{n},$$

$$(\beta_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{B})) + \beta_{2m+1}(\mathring{\mathbf{B}}\,\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})))\mathbf{n};$$

*and*

$$\mathrm{Skw}''_{2m+1}(\mathbf{u}); \mathrm{Skw}_{4m+2}(\mathbf{W}); \mathrm{Skw}_{4m+2}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\eta_{2m}(\mathring{\mathbf{v}}) + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\wedge\eta_{2m}(\mathring{\mathbf{u}});$$

$$\mathbf{W\Omega} - \mathbf{\Omega W}; \mathring{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\mathring{\mathbf{A}}; \mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}},$$

$$\mathring{\mathbf{B}}\,\mathbf{n}\wedge\,\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}\wedge\,\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n};$$

$$\mathbf{u}\wedge\mathbf{Wu}, \mathbf{u}\wedge\eta_{2m}(\mathbf{Wn}), |\mathbf{u}|^{2m}\mathbf{Wu}\wedge\eta_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\,\mathring{\mathbf{u}}\wedge\mathbf{W}\eta_{2m}(\mathring{\mathbf{u}});$$

$$\mathring{\mathbf{u}}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\rho_m(\mathbf{q}(\mathbf{A})), \mathbf{u}\wedge\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n}) + (-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\pi_m(\mathbf{q}(\mathbf{A})),$$

$$\alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n}\wedge\,\mathring{\mathbf{u}} + \mathring{\mathbf{A}}\,\mathbf{u}\wedge\eta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$(\mathrm{tr}\mathbf{WN})\mathbf{n}\wedge(\mathbf{u}\times\mathbf{v}); (\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\wedge(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})));$$

*and*

$$\mathrm{Sym}''_{2m+1}(\mathbf{u}); \mathrm{Sym}_{4m+2}(\mathbf{W}); \mathrm{Sym}_{4m+2}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|^{2m-1}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\eta_{2m}(\mathring{\mathbf{v}}) + |\mathbf{v}\cdot\mathbf{n}|^{2m-1}(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\vee\eta_{2m}(\mathring{\mathbf{u}});$$

$$\mathbf{W\Omega} + \mathbf{\Omega W}, |\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n}\vee\mathbf{N\Omega n};$$

$$\mathring{\mathbf{A}}\,\mathbf{W} - \mathbf{W}\,\mathring{\mathbf{A}}, (\mathrm{tr}\mathbf{WN})\,\mathring{\mathbf{A}}\,\mathbf{n}\vee\mathbf{N}\,\mathring{\mathbf{A}}\,\mathbf{n}; \mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}};$$

$$\mathbf{u}\vee\mathbf{Wu}, \mathbf{u}\vee\eta_{2m}(\mathbf{Wn}),$$

$$|\mathbf{u}|^{2m}\mathbf{Wu}\vee\eta_{2m}(\mathbf{Wn}) + |\mathbf{W}|^{2m}\,\mathring{\mathbf{u}}\vee\mathbf{W}\eta_{2m}(\mathring{\mathbf{u}});$$

$$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\rho_m(\mathbf{q}(\mathbf{A})),\mathbf{u}\vee\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})+(-1)^m(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\pi_m(\mathbf{q}(\mathbf{A})),$$

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{u}}+\overset{\circ}{\mathbf{A}}\,\mathbf{u}\vee\eta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n});$$

$$(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{u}\times\mathbf{v});(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{n}\times\eta_m(\mathbf{q}(\mathbf{A})));$$

*where* $(\mathbf{u},\mathbf{v})=(\mathbf{u}_i,\mathbf{u}_j)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H})=(\mathbf{W}_\mu,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C})=(\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $j>i=1,\cdots,a$, $\theta>\tau>\mu=1,\cdots,b$, $N>M>L=1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* $D_{2m+1h}$ *for each* $m\geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold rotation axis of the group* $D_{2m+1h}$.

In the above result, $I_{4m+2}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$ is used to represent the invariants depending on two or three symmetric and/or skewsymmetric tensors given in Theorem 1 in Part 1, with the replacement of $m$ by $2m+1$ therein.

## 3. Crystal and quasicrystal classes $D_{2md}$

The classes at issue take the forms

$$D_{2md}=\{(-1)^k\mathbf{R}_{\mathbf{n}}^{\theta_k},\ (-1)^k\mathbf{R}_{\sigma_k}^{\pi}\mid\theta_k=\frac{2k\pi}{4m},\quad\sigma_k=\mathbf{R}_{\mathbf{n}}^{\theta_k/2}\mathbf{e},\quad k=1,\cdots,4m\}.\tag{3.1}$$

They include the crystal class $D_{2d}$ as the particular case when $m=1$. Each $\sigma_{2r}$ and each $\sigma_{2r-1}$ are a two-fold rotation axis vector and a reflection axis vector of the group $D_{2md}$, respectively. In particular, both $\mathbf{e}$ $(=\sigma_{4m})$ and $\mathbf{e}'$ $(=\sigma_{2m})$ are two mutually orthogonal two-fold axis vectors of $D_{2md}$. Throughout this section, $\sigma$, $\mu$ and $\nu$ will be used to represent one of the vectors $\sigma_t$, one of the reflection axis vectors $\sigma_{2r-1}$ and one of the two-fold rotation axis vectors $\sigma_{2r}$, respectively. A useful fact for the group $D_{2md}$ is: if $\sigma$ is a two-fold rotation (resp. reflection) axis vector, then $\tau'=\mathbf{n}\times\tau$ is also a two-fold rotation (resp. reflection) axis vector.

For the five sets of variables, $(\mathbf{W})$, $(\mathbf{A})$, $(\mathbf{W},\mathbf{\Omega})$, $(\mathbf{W},\mathbf{A})$ and $(\mathbf{A},\mathbf{B})$, it follows from the same argument indicated at the start of Sec. 2 that results for functional bases and skewsymmetric and symmetric tensor generating sets relative to the group $D_{2md}$, as well as related invariants from the scalar products, are obtainable from the corresponding results given in Sec. 4 in Part I by the replacement of $m$ with $2m$, and hence we shall omit them in the process of derivation to come. As a result, in what follows, for the foregoing five sets of variables, we only need to derive vector generating sets and their related invariants from the scalar products. Moreover, according to Sec. 4 (**xiii**) in Part I and Sec. 2 (**xiv**) in this part, there is no need to take the set $(\mathbf{u}, \mathbf{v}, \mathbf{r})$ of three vector variables and the set $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$ of a vector variable and two skewsymmetric tensor variables into account.

### 3.1. Single variables

(**i**) A single vector $\mathbf{u}$

$$
\begin{aligned}
V \quad & \{\mathbf{u}, \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}}),\\
& (\mathbf{u}\cdot\mathbf{n})^{4m-1}\mathbf{n} + \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})\} \ (\equiv V''_{2m}(\mathbf{u}))\\
\mathrm{Skw} \quad & \{\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}})\mathbf{N} - \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\wedge\mathring{\mathbf{u}},\\
& \beta_{4m}(\mathring{\mathbf{u}})\mathbf{N} + (\mathbf{u}\cdot\mathbf{n})^{4m-1}\mathbf{n}\wedge\mathring{\mathbf{u}}\} \ (\equiv \mathrm{Skw}''_{2m}(\mathbf{u}))\\
\mathrm{Sym} \quad & \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) + \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathbf{D}_2,\\
& (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) - \beta_{2m}(\mathring{\mathbf{u}})\mathbf{n}\vee\mathring{\mathbf{u}},\\
& \beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) + (\mathbf{u}\cdot\mathbf{n})^{4m-1}\mathbf{n}\vee\mathring{\mathbf{u}}\} \ (\equiv \mathrm{Sym}''_{2m}(\mathbf{u}))\\
R \quad & \mathbf{r}\cdot\mathbf{u}, (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}}) + (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{r}}, \eta_{2m-1}(\mathring{\mathbf{u}})],\\
& (\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n})^{4m-1} - \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n}, \mathring{\mathbf{r}}, \eta_{2m-1}(\mathring{\mathbf{u}})]; \mathrm{tr}\mathbf{H}(\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}})),\\
& (\mathrm{tr}\mathbf{HN})(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}}) + \beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\cdot\mathbf{Hn},\\
& (\mathrm{tr}\mathbf{HN})\beta_{4m}(\mathring{\mathbf{u}}) - (\mathbf{u}\cdot\mathbf{n})^{4m-1}\,\mathring{\mathbf{u}}\cdot\mathbf{Hn};\\
& \mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{u}}, [\mathbf{n}, \mathring{\mathbf{C}}\,\mathbf{n}, \eta_{2m-1}(\mathring{\mathbf{u}})] - \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathrm{tr}\mathbf{CD}_2,\\
& (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{C}}\mathring{\mathbf{u}}, \eta_{2m-1}(\mathring{\mathbf{u}})] + \beta_{2m}(\mathring{\mathbf{u}})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\,\mathbf{n},\\
& \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n}, \mathring{\mathbf{C}}\mathring{\mathbf{u}}, \eta_{2m-1}(\mathring{\mathbf{u}})] - (\mathbf{u}\cdot\mathbf{n})^{4m-1}\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\,\mathbf{n};\\
& \{(\mathbf{u}\cdot\mathbf{n})^2, |\,\mathring{\mathbf{u}}\,|^2, \alpha_{4m}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}})\} \ (\equiv I''_{2m}(\mathbf{u}))
\end{aligned}
$$

First, we prove that the presented set $I''_{2m}(\mathbf{u})$ supplies a desired functional basis. In fact, the latter is obtainable from an isotropic functional basis for the extended variables $(\mathbf{u}, \mathbf{G}, \mathbf{n}\otimes\mathbf{n})$ with $\mathbf{G} = \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\mathring{\mathbf{u}})) + \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathbf{D}_2$ (see

Theorem 4 and Theorem 6 in XIAO [17] ([1]). Here and henceforth, the Kronecker delta $\delta_{1m}$ takes values 1 and 0 when $m = 1$ and $m \geq 2$ respectively. Applying the related result for isotropic functions we know that the just-mentioned isotropic functional basis is formed by the invariants

$$\mathbf{u}\cdot\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})^2, \mathrm{tr}\mathbf{G}, \mathrm{tr}\mathbf{G}^2, \mathrm{tr}\mathbf{G}^3, \mathbf{n}\cdot\mathbf{G}^i\mathbf{u}, \mathbf{u}\cdot\mathbf{G}^i\mathbf{u}, \mathbf{u}\cdot\mathbf{G}(\mathbf{n}\otimes\mathbf{n})\mathbf{u},$$

where $i = 1, 2$. From the above invariants we derive the set $I''_{2m}(\mathbf{u})$.

Next, we prove that the three presented sets $V''_{2m}(\mathbf{u})$, $\mathrm{Skw}''_{2m}(\mathbf{u})$ and $\mathrm{Sym}''_{2m}(\mathbf{u})$ supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets, respectively. Towards this goal we show that each of them obeys the criterion (2.3) in Part I. In fact, the case when $\mathbf{u} = \mathbf{0}$ is trivial. In what follows, suppose $\mathbf{u} \neq \mathbf{0}$. The respective last two generators in the foregoing three sets produce $\mathbf{n}$ and $\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})$, $\beta_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{N}$ and $\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}$, $\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}))$ and $\mathbf{n}\vee\overset{\circ}{\mathbf{u}}$, when $\Delta = \begin{vmatrix} \beta_{2m}(\overset{\circ}{\mathbf{u}}) & -\mathbf{u}\cdot\mathbf{n} \\ (\mathbf{u}\cdot\mathbf{n})^{4m-1} & \beta_{2m}(\overset{\circ}{\mathbf{u}}) \end{vmatrix} = (\mathbf{u}\cdot\mathbf{n})^{4m} + (\beta_{2m}(\overset{\circ}{\mathbf{u}}))^2 \neq 0$. Hence, we have

$$\mathrm{rank}V''_{2m}(\mathbf{u}) \geq \begin{cases} \mathrm{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{u}\} = 3 \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}})\Delta \neq 0, \\ \mathrm{rank}\{\mathbf{n}, \overset{\circ}{\mathbf{u}}\} = 2 \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}}) = 0,\ \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ \mathrm{rank}\{\mathbf{u}\} = 1 \text{ if } \Delta = 0 \text{ or } \overset{\circ}{\mathbf{u}} = \mathbf{0}, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Skw}''_{2m}(\mathbf{u}) \geq \begin{cases} \mathrm{rank}\{\mathbf{N}, \mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}})\} = 3 \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}})\Delta \neq 0, \\ \mathrm{rank}\{\mathbf{n}\wedge\overset{\circ}{\mathbf{u}}\} = 1 \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}})\Delta = 0,\ \overset{\circ}{\mathbf{u}} \neq 0, \\ 0 \text{ if } \overset{\circ}{\mathbf{u}} = \mathbf{0}, \end{cases}$$

$$\mathrm{rank}\ \mathrm{Sym}''_{2m}(\mathbf{u}) \geq \begin{cases} \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\eta', \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\vee\eta'\} = 6 \\ \qquad \text{if } \alpha_{2m}(\overset{\circ}{\mathbf{u}})\Delta \neq 0, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}\} = 4 \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}})\Delta = 0, \\ \qquad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{D}_2\} = 2 + \delta_{1m} \text{ if } \overset{\circ}{\mathbf{u}} = \mathbf{0}, \end{cases}$$

[1]) In (3.21) and (3.30) therein, some corrigenda should be made. Here and later they are: the vector-valued functions $\eta_{2m-1}(\mathbf{z})$, $\mathbf{z} = \overset{\circ}{\mathbf{u}}$, $\mathbf{Wn}$, $\overset{\circ}{\mathbf{A}}\ \mathbf{n}$, appearing in (3.21) and (3.30) (for $m = 1$) and $\mathbf{D}_1$ in (3.31) should be changed to $\mathbf{N}\eta_{2m-1}(\mathbf{z})$ and $\mathbf{D}_2$, respectively.

where $\eta' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}})$. From these results and

$$\Gamma(\mathbf{u}) \cap D_{2md} = \begin{cases} C_{1h}(\mu) \text{ if } \alpha_{2m}(\overset{\circ}{\mathbf{u}}) = 0,\ \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_2(\nu) \text{ if } \Delta = 0,\ \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_{2mv}(\mathbf{n}) \text{ if } \overset{\circ}{\mathbf{u}} = \mathbf{0}, \text{ i.e. } \mathbf{u} = c\mathbf{n}, \end{cases}$$

as well as from Tables 1 – 3 in Sec. 2 in Part I, we deduce that the three sets at issue obey the criterion (2.3) in Part I, respectively. The three presented generating sets are minimal.

**(ii)** A single skewsymmetric tensor $\mathbf{W}$

$V$ $\{\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn})), \mathbf{W}^2(\mathbf{n} \times \eta_{2m-1}(\mathbf{n} \times \mathbf{Wn})),$
$\beta_{2m}(\mathbf{Wn})\mathbf{n}\}$ $(\equiv V''_{2m}(\mathbf{W}))$

$R$ $[\mathbf{n}, \overset{\circ}{\mathbf{r}}, \eta_{2m-1}(\mathbf{Wn})], [\mathbf{n}, \mathbf{Wr}, \eta_{2m-1}(\mathbf{Wn})], [\mathbf{n}, \mathbf{W}^2\mathbf{r}, \eta_{2m-1}(\mathbf{Wn})],$
$(\mathbf{r} \cdot \mathbf{n})\beta_{2m}(\mathbf{Wn}).$

We show that the presented set $V''_{2m}(\mathbf{A})$ supplies a desired vector generating set. In fact, an anisotropic vector generating set for $\mathbf{W}$ under the group $D_{2md}$ is obtainable from an isotropic vector generating set for the extended variables $(\mathbf{W}, \mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}), \mathbf{n} \otimes \mathbf{n})$ (see Theorem 4 and Theorem 6 in XIAO [17] and the preceding footnote). From the related result for isotropic functions we know that the latter is just given by the presented set $V''_{2m}(\mathbf{W})$. Further, by considering the two tensors $\mathbf{W}_1 = \mathbf{E}(\mathbf{n} + \mathbf{e})$ and $\mathbf{W}_2 = \mathbf{En} + \mathbf{n} \wedge \sigma_1$, we infer that each of the four generators in the set $V''_{2m}(\mathbf{W})$ is irreducible.

**(iii)** A single symmetric tensor $\mathbf{A}$

$V$ $\{\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n}), \overset{\circ}{\mathbf{A}}\, (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})), \beta_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})\, \overset{\circ}{\mathbf{A}}\, \mathbf{n}, \beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n},$
$J(\mathbf{A})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\}$ $(\equiv V''_{2m}(\mathbf{A}))$

$R$ $[\mathbf{n}, \overset{\circ}{\mathbf{r}}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})], [\mathbf{n}, \overset{\circ}{\mathbf{A}}\, \mathbf{r}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})], \beta_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})\, \overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{A}}\, \mathbf{n},$
$(\mathbf{r} \cdot \mathbf{n})\beta_m(\mathbf{q}(\mathbf{A})), (\mathbf{r} \cdot \mathbf{n})J(\mathbf{A})\alpha_m(\mathbf{q}(\mathbf{A})).$

We show that the presented set $V''_{2m}(\mathbf{A})$ supplies a desired vector generating set. To this end we show that this set obeys the criterion (2.3) in Part I. The case when $\overset{\circ}{\mathbf{A}} = \mathbf{O}$ is trivial. In what follows, suppose $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$. First, utilizing the equality

$$\overset{\circ}{\mathbf{A}}\, (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})) = -\beta_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})\mathbf{n} + \mathbf{A}_{\mathbf{e}}(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})), \tag{3.2}$$

we have

$$\operatorname{rank} V''_{2m}(\mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \eta(\mathbf{A})', \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} = 3 \text{ if } \beta_{4m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \neq 0, \\ \operatorname{rank}\{\mathbf{n}, \eta(\mathbf{A})', \mathbf{A}_{\mathbf{e}}\eta(\mathbf{A})'\} = 3 \text{ if } \beta_{4m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ \qquad\qquad J(\mathbf{A}) \neq 0, \end{cases}$$

where $\eta(\mathbf{A})' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$. Second, we have

$$\operatorname{rank} V''_{2m}(\mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}\} = 2 \text{ if } J(\mathbf{A}) = \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 1 \text{ if } J(\mathbf{A}) = \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \end{cases}$$

with $\overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0}$. Third, we have

$$\operatorname{rank} V''_{2m}(\mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 1 \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0, \\ \qquad 0 \text{ if } |\overset{\circ}{\mathbf{A}}\,\mathbf{n}| = \beta_m(\mathbf{q}(\mathbf{A})) = 0. \end{cases}$$

The cases for $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$ involved in the above results cover all possible cases. Thus, from the above results and from

$$\Gamma(\mathbf{A}) \cap D_{2md} = \begin{cases} C_{1h}(\boldsymbol{\mu}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\ J(\mathbf{A}) = \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ C_2(\boldsymbol{\nu}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} \neq \mathbf{0},\ J(\mathbf{A}) = \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) = 0, \\ C_2(\mathbf{n}) \text{ or } C_{2v}(\mathbf{n}, \boldsymbol{\mu}, \mathbf{n} \times \boldsymbol{\mu}) \text{ if } \overset{\circ}{\mathbf{A}}\,\mathbf{n} = \mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0, \\ D_2(\mathbf{n}, \boldsymbol{\nu}, \mathbf{n} \times \boldsymbol{\nu}) \text{ if } |\overset{\circ}{\mathbf{A}}\,\mathbf{n}| = \beta_m(\mathbf{q}(\mathbf{A})) = 0, \end{cases}$$

for $\overset{\circ}{\mathbf{A}} \neq \mathbf{O}$, as well as Table 1 in Sec. 2 in Part I, we infer that the presented set obeys the criterion (2.3) in Part I. Further, by considering the three tensors $\mathbf{A}_1 = \mathbf{n} \vee (\mathbf{e} + \boldsymbol{\sigma}_1)$ and $\mathbf{A}_2 = \boldsymbol{\sigma}_1 \otimes \boldsymbol{\sigma}_1$ and $\mathbf{A}_3 = \mathbf{D}_1 + \mathbf{n} \vee \boldsymbol{\sigma}_1$ we deduce that the five generators given are irreducible respectively.

### 3.2. $D_{2md}$-irreducible sets of two variables

(**iv**) The $D_{2md}$-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

$$V \quad \{\mathbf{u}, \mathbf{v}, \alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{v} \times \overset{\circ}{\mathbf{u}}, \alpha_{2m}(\overset{\circ}{\mathbf{v}})\mathbf{u} \times \overset{\circ}{\mathbf{v}}\}\ (\equiv V''_{2m}(\mathbf{u}, \mathbf{v}))$$

Skw $\mathrm{Skw}''_{2m}(\mathbf{u}) \cup \mathrm{Skw}''_{2m}(\mathbf{v}) \cup \{\mathbf{u} \wedge \mathbf{v},$

$((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{u}}))\mathbf{N}\}$

$(\equiv \mathrm{Skw}''_{2m}(\mathbf{u},\mathbf{v}))$

Sym $\mathrm{Sym}''_{2m}(\mathbf{u}) \cup \mathrm{Sym}''_{2m}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v},$

$(\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})\ \overset{\circ}{\mathbf{v}} \vee (\mathbf{n}\times \overset{\circ}{\mathbf{v}})$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})\ \overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times \overset{\circ}{\mathbf{u}})\}$ $(\equiv \mathrm{Sym}''_{2m}(\mathbf{u},\mathbf{v}))$

$R$ $\mathbf{r}\cdot\mathbf{u}, \mathbf{r}\cdot\mathbf{v}, \alpha_{2m}(\overset{\circ}{\mathbf{v}})[\mathbf{r},\mathbf{u},\overset{\circ}{\mathbf{v}}], \alpha_{2m}(\overset{\circ}{\mathbf{u}})[\mathbf{r},\mathbf{v},\overset{\circ}{\mathbf{u}}];$

$\mathbf{H} : \mathrm{Skw}''_{2m}(\mathbf{u}), \mathbf{H} : \mathrm{Skw}''_{2m}(\mathbf{v}), \mathbf{u}\cdot\mathbf{H}\mathbf{v}; \mathbf{C} : \mathrm{Sym}''_{2m}(\mathbf{u}),$

$\mathbf{C} : \mathrm{Sym}''_{2m}(\mathbf{v}), \mathbf{u}\cdot \overset{\circ}{\mathbf{C}}\ \mathbf{v}; (\mathrm{tr}\mathbf{H}\mathbf{N})((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})); (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{v}})[\mathbf{n},\overset{\circ}{\mathbf{v}},\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{v}}]$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\overset{\circ}{\mathbf{u}})[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}];$

$I''_{2m}(\mathbf{u}) \cup I''_{2m}(\mathbf{v}) \cup \{(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \mathbf{u}\cdot\mathbf{v}\}$ $(\equiv I''_{2m}(\mathbf{u},\mathbf{v}))$

To prove the above results, we first work out the $D_{2md}$-irreducible set $(\mathbf{u},\mathbf{v})$, which is specified by (see (3.1) in Part I): $\Gamma(\mathbf{u},\mathbf{v}) \cap D_{2md} \neq \Gamma(\mathbf{z}) \cap D_{2md}$, $\mathbf{z} = \mathbf{u}, \mathbf{v}$. Evidently, we have $\Gamma(\mathbf{z}) \cap D_{2md} \neq C_1$. The latter implies that $\mathbf{R}^{\pi}_{\mathbf{n}}$ or $-\mathbf{R}^{\pi}_{\boldsymbol{\mu}}$ or $\mathbf{R}^{\pi}_{\boldsymbol{\nu}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for each $\mathbf{z} = \mathbf{u}, \mathbf{v}$. Hence, we deduce that each vector $\mathbf{z} \in \{\mathbf{u},\mathbf{v}\}$ takes one of the forms:

$$c\mathbf{n},\ c \neq 0; \quad a\mathbf{n} + b\boldsymbol{\mu},\ b \neq 0; \quad c\boldsymbol{\nu},\ c \neq 0\,. \tag{3.3}$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{n};\ \mathbf{u} = a\mathbf{n},\ \mathbf{v} = c\mathbf{n} + d\boldsymbol{\mu};$$

$$\mathbf{u} = a\mathbf{n} + b\boldsymbol{\mu},\ \mathbf{v} = c\mathbf{n} + d\boldsymbol{\mu};\ \mathbf{u} = a\boldsymbol{\nu},\ \mathbf{v} = c\boldsymbol{\nu};$$

which violate the $D_{2md}$-irreducibility condition for $(\mathbf{u},\mathbf{v})$, we derive the following four disjoint cases for $D_{2md}$-irreducible sets $(\mathbf{u},\mathbf{v})$:

(c1) $\mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{e},\ ab \neq 0$;

(c2) $\mathbf{u} = a\mathbf{e},\ \mathbf{v} = b\boldsymbol{\nu},\ \boldsymbol{\nu} \neq \mathbf{e},\ ab \neq 0$;

(c3) $\mathbf{u} = c\mathbf{e},\ \mathbf{v} = a\mathbf{n} + b\boldsymbol{\mu},\ bc \neq 0$;

(c4) $\mathbf{u} = a\mathbf{n} + b\sigma_1,\ \mathbf{v} = c\mathbf{n} + d\boldsymbol{\mu},\ \boldsymbol{\mu} \neq \sigma_1,\ bd \neq 0$.

Then, for case (c1) we have

$$\mathrm{rank}V''_{2m}(\mathbf{u},\mathbf{v}) \geq \mathrm{rank}\{\mathbf{u},\mathbf{v},\mathbf{u}\times \overset{\circ}{\mathbf{v}}\} = 3\,, \tag{3.4}$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{v}}), \mathbf{u}\wedge\mathbf{v}, \mathbf{N}\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \mathbf{n}\vee(\mathbf{n}\vee\eta_{2m-1}(\overset{\circ}{\mathbf{v}})),$$
$$\mathbf{u}\vee\mathbf{v}, \overset{\circ}{\mathbf{v}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{v}})\} = 6;$$

and for cases (c2) – (c4) we have (3.4) and

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{v}}), \mathbf{u}\wedge\mathbf{v}\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{v}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},$$
$$\mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{v}})), \mathbf{u}\vee\mathbf{v}\} = 6.$$

From the above results we deduce that the three sets at issue obey the criterion (2.3) in Part I, respectively, and hence they supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets. Further, by considering the pair $\mathbf{u}_0 = \mathbf{n}$ and $\mathbf{v}_0 = \mu_1$ we infer that the generator $\beta_{2m}(\overset{\circ}{\mathbf{v}})\mathbf{u}\times\overset{\circ}{\mathbf{v}}$ and the respective last two generators in the two sets $\text{Skw}''_{2m}(\mathbf{u},\mathbf{v})$ and $\text{Sym}''_{2m}(\mathbf{u},\mathbf{v})$ are irreducible, respectively. Moreover, by exchanging $\mathbf{u}_0$ and $\mathbf{v}_0$ we know that the generator $\beta_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{v}\times\overset{\circ}{\mathbf{u}}$ is also irreducible.

By means of the relevant arguments used in (**iv**), (**viii**) and (**ix**) in Sec. 2, it can be proved that the set $I''_{2m}(\mathbf{u},\mathbf{v})$ given here and the sets $I''_{2m}(\mathbf{u},\mathbf{W})$ and $I''_{2m}(\mathbf{u},\mathbf{A})$ given later supply the desired functional bases for the $D_{2md}$-irreducible sets $(\mathbf{u},\mathbf{v})$, $(\mathbf{u},\mathbf{W})$ and $(\mathbf{u},\mathbf{A})$, respectively. Henceforth, this procedure will not be repeated.

(**v**) The $D_{2md}$-irreducible set $(\mathbf{W},\mathbf{\Omega})$ of two skewsymmetric tensors

*V* $\quad V''_{2m}(\mathbf{W})\cup V''_{2m}(\mathbf{\Omega})\cup\{\mathbf{\Omega}(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n})), \mathbf{W}(\mathbf{n}\times\eta_{2m-1}(\mathbf{\Omega}\mathbf{n})),$
$((\text{tr}\mathbf{W}\mathbf{N})|\text{tr}\mathbf{W}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{\Omega}\mathbf{n}) + (\text{tr}\mathbf{\Omega}\mathbf{N})|\text{tr}\mathbf{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{W}\mathbf{n}))\mathbf{n}\}$
$(\equiv V''_{2m}(\mathbf{W},\mathbf{\Omega}))$

*R* $\quad \mathbf{r}\cdot V''_{2m}(\mathbf{W}), \mathbf{r}\cdot V''_{2m}(\mathbf{\Omega}), [\mathbf{n}, \mathbf{\Omega}\mathbf{r}, \eta_{2m-1}(\mathbf{W}\mathbf{n})], [\mathbf{n}, \mathbf{W}\mathbf{r}, \eta_{2m-1}(\mathbf{\Omega}\mathbf{n})],$
$(\mathbf{r}\cdot\mathbf{n})((\text{tr}\mathbf{W}\mathbf{N})|\text{tr}\mathbf{W}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{\Omega}\mathbf{n}) + (\text{tr}\mathbf{\Omega}\mathbf{N})|\text{tr}\mathbf{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{W}\mathbf{n}))$.

To prove the presented results, we first work out the $D_{2md}$-irreducible set $(\mathbf{W},\mathbf{\Omega})$, specified by (see (3.1) in Part I): $\Gamma(\mathbf{W},\mathbf{\Omega})\cap D_{2md} \neq \Gamma(\mathbf{z})\cap D_{2md}$, $\mathbf{z} = \mathbf{W}, \mathbf{\Omega}$. Evidently, we have $\Gamma(\mathbf{z})\cap D_{2md} \neq C_1$. The latter implies that $\mathbf{R}^{\pi}_{\mathbf{n}}$ or $-\mathbf{R}^{\pi}_{\mu}$ or $\mathbf{R}^{\pi}_{\nu}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ of $\mathbf{z}$ for each $\mathbf{z} = \mathbf{W}, \mathbf{\Omega}$. Hence, we deduce that each skewsymmetric tensor $\mathbf{z}\in\{\mathbf{W},\mathbf{\Omega}\}$ takes one of the forms:

$$c\mathbf{E}\mathbf{n},\ c\neq 0;\quad c\mathbf{E}\mu,\ c\neq 0;\quad c\mathbf{E}\nu,\ c\neq 0. \tag{3.5}$$

Considering the combinations of the above forms and excluding the case $\mathbf{W} = c\mathbf{\Omega}$ which violates the $D_{2md}$-irreducibility condition for $(\mathbf{W}, \mathbf{\Omega})$, we derive the following five cases for $D_{2md}$-irreducible set $(\mathbf{W}, \mathbf{\Omega})$:

(c1) $\mathbf{W} = a\mathbf{E}\mathbf{n}$, $\mathbf{\Omega} = b\mathbf{E}\sigma_1$, $ab \neq 0$;

(c2) $\mathbf{W} = a\mathbf{E}\mathbf{n}$, $\mathbf{\Omega} = b\mathbf{E}\mathbf{e}$, $ab \neq 0$;

(c3) $\mathbf{W} = a\mathbf{E}\mathbf{e}$, $\mathbf{\Omega} = b\mathbf{E}\nu$, $\nu \neq \mathbf{e}$, $ab \neq 0$;

(c4) $\mathbf{W} = a\mathbf{E}\sigma_1$, $\mathbf{\Omega} = b\mathbf{E}\mu$, $\mu \neq \sigma_1$, $ab \neq 0$;

(c5) $\mathbf{W} = a\mathbf{E}\mu$, $\mathbf{\Omega} = b\mathbf{E}\mathbf{e}$, $ab \neq 0$.

With the above five cases we prove that the presented set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ obeys the criterion (2.3) in Part I. We have

$$\begin{aligned}\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{&\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n}), \mathbf{\Omega}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n})),\\ &\mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n}))\} = 3,\end{aligned}$$

$$\begin{aligned}\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{&\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n})),\\ &\alpha_{2m}(\mathbf{\Omega}\mathbf{n})\mathbf{n}\} = 3,\end{aligned}$$

$$\begin{aligned}\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{&\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n}),\\ &\mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n}))\} = 3,\end{aligned}$$

for cases (c1) – (c3), respectively, and

$$\begin{aligned}\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{\Omega}) \geq \operatorname{rank}\{&\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n})),\\ &\mathbf{n} \times \eta_{2m-1}(\mathbf{\Omega}\mathbf{n})\} = 3,\end{aligned}$$

for cases (c4) – (c5).

Thus, we infer that the presented set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ supplies a desired vector generating set. Further, from case (c1) we know that the last two generators in the set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ are irreducible. By exchanging $\mathbf{W}$ and $\mathbf{\Omega}$ in case (c1) we know that the generator $\mathbf{\Omega}\eta_{2m-1}(\mathbf{W}\mathbf{n})$ is also irreducible.

**(vi)** The $D_{2md}$-irreducible set $(\mathbf{W}, \mathbf{A})$ of a skewsymmetric and a symmetric tensors

$$\begin{aligned}V \quad & V''_{2m}(\mathbf{W}) \cup V''_{2m}(\mathbf{A}) \cup \{(\operatorname{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\mathbf{n},\\ & (\alpha_{2m}(\mathbf{W}\mathbf{n}) + (\operatorname{tr}\mathbf{W}\mathbf{N})|\operatorname{tr}\mathbf{W}\mathbf{N}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n},\\ & (\operatorname{tr}\mathbf{W}\mathbf{N})^{2m-2}\mathbf{W}(\mathbf{n} \times \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})) + |\,\mathring{\mathbf{A}}\,|^{2m-2}\,\mathring{\mathbf{A}}\,(\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}))\}\\ & \qquad\qquad (\equiv V''_{2m}(\mathbf{W}, \mathbf{A}))\end{aligned}$$

$$R \quad \mathbf{r}\cdot V_{2m}(\mathbf{W}), \mathbf{r}\cdot V_{2m}(\mathbf{A}), (\mathbf{r}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$(\mathbf{r}\cdot\mathbf{n})(\alpha_{2m}(\mathbf{Wn}) + (\mathrm{tr}\mathbf{WN})|\mathrm{tr}\mathbf{WN}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A})),$$
$$(\mathrm{tr}\mathbf{WN})^{2m-2}[\mathbf{n}, \mathbf{Wr}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})] - |\overset{\circ}{\mathbf{A}}|^{2m-2}[\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{r}, \eta_{2m}(\mathbf{Wn})].$$

To prove the above result, we first work out the $D_{2md}$-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2md} \neq \Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2md}$ for $\mathbf{z} = \mathbf{W}, \mathbf{A}$. Evidently, $\Gamma(\mathbf{z}) \cap D_{2md} \neq C_1$ for $\mathbf{z} = \mathbf{W}, \mathbf{A}$. Hence, the skewsymmetric tensor $\mathbf{W}$ takes one of the forms given by (3.5). In a similar way we deduce that the symmetric tensor $\mathbf{A}$ takes one of the forms

$$(3.6) \qquad \begin{cases} \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2, \; a^2 + b^2 \neq 0\,; \\ \overset{\circ}{\mathbf{A}} = a(\mu\otimes\mu - \mu'\otimes\mu') + b\mathbf{n}\vee\mu', \; \mu' = \mathbf{n}\times\mu, \; b\neq 0\,; \\ \overset{\circ}{\mathbf{A}} = a(\nu\otimes\nu - \nu'\otimes\nu') + b\mathbf{n}\vee\nu', \; \nu' = \mathbf{n}\times\nu, \; b\neq 0\,. \end{cases}$$

Considering the combinations of the forms given by (3.5) – (3.6) and excluding the cases

$$\mathbf{W} = c\mathbf{En}, \; \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2, \; \beta_{2m}(\mathbf{q}(\mathbf{A})) \neq 0;$$
$$\mathbf{W} = c\mathbf{E}\sigma, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma');$$
$$\mathbf{W} = c\mathbf{E}\sigma, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma;$$

which violate the $D_{2md}$-irreducibility condition for $(\mathbf{W}, \mathbf{A})$, we derive the following five disjoint cases for $D_{2md}$-irreducible set $(\mathbf{W}, \mathbf{A})$:

(c1) $\mathbf{W} = c\mathbf{En}, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma'), \; ac \neq 0;$

(c2) $\mathbf{W} = c\mathbf{En}, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma', \; bc \neq 0;$

(c3) $\mathbf{W} = c\mathbf{E}\sigma, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\sigma\vee\sigma', \; bc \neq 0;$

(c4) $\mathbf{W} = c\mathbf{E}\sigma_1, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma', \; \sigma\neq\sigma_1, \; bc \neq 0;$

(c5) $\mathbf{W} = c\mathbf{Ee}, \; \overset{\circ}{\mathbf{A}} = a(\sigma\otimes\sigma - \sigma'\otimes\sigma') + b\mathbf{n}\vee\sigma', \; \sigma\neq\mathbf{e}, \; bc \neq 0.$

With the above cases we prove that the presented set $V''_{2m}(\mathbf{W}, \mathbf{A})$ obeys the criterion (2.3) in Part I. First, using the formula (2.4) in Part I and the equalities

$$(3.7) \qquad \mathrm{rank}(V(C_{1h}(\mathbf{a})) \cup V(C_{1h}(\mathbf{b}))) = 3, \quad \mathrm{rank}(V(C_{1h}(\mathbf{a})) \cup V(C_2(\mathbf{b}))) = 3\,,$$

for any two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$, we have

$$\begin{aligned} \mathrm{rank}V''_{2m}(\mathbf{W}, \mathbf{A}) &\geq \mathrm{rank}(V''_{2m}(\mathbf{W}) \cup V''_{2m}(\mathbf{A})) \\ &= \mathrm{rank}(V(\Gamma(\mathbf{W}) \cap D_{2md}) \cup V(\Gamma(\mathbf{A}) \cap D_{2md})) = 3 \end{aligned}$$

for case (c4) and for case (c5) with $\sigma = \sigma_{2r-1}$. For case (c1) we have

$$\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2md} = C_2(\mathbf{n}),$$
$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{A}) \geq \operatorname{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 1.$$

For case (c2) we have

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\eta(\mathbf{A})', \overset{\circ}{\mathbf{A}}\, \eta(\mathbf{A})', \mathbf{W}\eta(\mathbf{A})'\} = 3 \\ \qquad \text{if } \ \sigma = \sigma_{2r-1}, \\ \operatorname{rank}\{\eta(\mathbf{A})', (\operatorname{tr}\mathbf{W}\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})\mathbf{n}, \mathbf{W}\eta(\mathbf{A})'\} = 3 \\ \qquad \text{if } \ \sigma = \sigma_{2r}, \end{cases}$$

where $\eta(\mathbf{A})' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})$. For case (c3) we have

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\eta(\mathbf{W})', \mathbf{W}\eta(\mathbf{W})', \overset{\circ}{\mathbf{A}}\, \eta(\mathbf{W})'\} = 3 \\ \qquad \text{if } \ \sigma = \sigma_{2r-1}, \\ \operatorname{rank}\{\beta_m(\mathbf{q})\mathbf{n}, \alpha_m(\mathbf{q})\alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{n}, \eta(\mathbf{W})', \overset{\circ}{\mathbf{A}}\, \eta(\mathbf{W})'\} \\ \qquad = 3 \ \text{ if } \ \sigma = \sigma_{2r}, \end{cases}$$

where $\mathbf{q} = \mathbf{q}(\mathbf{A})$, $\eta(\mathbf{W})' = \mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n})$.

Finally, for case (c5) with $\sigma = \sigma_{2r}$ we have

$$\operatorname{rank} V''_{2m}(\mathbf{W}, \mathbf{A}) \geq \ \operatorname{rank}\{\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n}),$$
$$\overset{\circ}{\mathbf{A}}\, (\mathbf{n} \times \eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 3.$$

From the above results and Table 1 in Sec. 2 in Part I, we infer that the set $V''_{2m}(\mathbf{W}, \mathbf{\Omega})$ obeys the criterion (2.3) in Part I, and hence it supplies a desired vector generating set. Further, by considering the two pairs: $\mathbf{W}_1 = \mathbf{E}\mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n} \vee \mathbf{e}$, $\mathbf{W}_2 = \mathbf{E}\mathbf{n}$ and $\mathbf{A}_2 = \mathbf{e} \otimes \mathbf{e}$, we deduce that the last three generators in the set $V''_{2m}(\mathbf{W}, \mathbf{A})$ are irreducible.

(**vii**) The $D_{2md}$-irreducible set $(\mathbf{A}, \mathbf{B})$ of two symmetric tensor variables

$$V \quad V''_{2m}(\mathbf{A}) \cup V''_{2m}(\mathbf{B}) \cup \{\overset{\circ}{\mathbf{A}}\, (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\, \mathbf{n})), \overset{\circ}{\mathbf{B}}\, (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})),$$
$$(1 - \delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\, \mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})| \cdot |\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A})))\mathbf{n},$$
$$(1 - \delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\, \mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})| \cdot |\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B})))\mathbf{n}\} \ (\equiv V''_{2m}(\mathbf{A}, \mathbf{B}))$$

$$R \quad \mathbf{r}\cdot V''_{2m}(\mathbf{A}), \mathbf{r}\cdot V''_{2m}(\mathbf{B}), [\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{r}, \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})], [\mathbf{n}, \overset{\circ}{\mathbf{B}}\,\mathbf{r}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})],$$

$$(1-\delta_{1m})(\mathbf{r}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$

$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A}))),$$

$$(1-\delta_{1m})(\mathbf{r}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})$$

$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B}))).$$

We show that the presented set $V''_{2m}(\mathbf{A},\mathbf{B})$ supplies the desired vector generating set, i.e. it obeys the criterion (2.3) in Part I. To this end, we first work out the $D_{2md}$-irreducible set $(\mathbf{A},\mathbf{B})$. From the $D_{2md}$-irreducibility condition for $(\mathbf{A},\mathbf{B})$ (see (3.1) in Part I), we know that $\Gamma(\mathbf{z})\cap D_{2md}\neq C_1$ for $\mathbf{z}=\mathbf{A}$, $\mathbf{B}$. Hence, considering the combinations of the forms given by (3.6) and excluding the cases

$$\overset{\circ}{\mathbf{A}}= a\mathbf{D}_1+b\mathbf{D}_2,\ \overset{\circ}{\mathbf{B}}= c\mathbf{D}_1+d\mathbf{D}_2,\ \beta_{2m}(\mathbf{q}(\mathbf{z}))\neq 0,\ \mathbf{z}=\mathbf{A} \text{ or } \mathbf{z}=\mathbf{B};$$

$$\overset{\circ}{\mathbf{A}}= a(\sigma\otimes\sigma-\sigma'\otimes\sigma')+b\mathbf{n}\vee\sigma',\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')+d\mathbf{n}\vee\sigma';$$

$$\overset{\circ}{\mathbf{A}}= a(\sigma\otimes\sigma-\sigma'\otimes\sigma'),\ \overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')+d\mathbf{n}\vee\sigma';$$

which violate the $D_{2md}$-irreducibility condition for $(\mathbf{A},\mathbf{B})$, we derive the following five disjoint cases for the $D_{2md}$-irreducible set $(\mathbf{A},\mathbf{B})$:

(c1) $\overset{\circ}{\mathbf{A}}= a(\sigma_1\otimes\sigma_1-\sigma_1'\otimes\sigma_1')$, $\overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')$, $\sigma\neq\sigma_1,\ \sigma_1',\ ac\neq 0$;

(c2) $\overset{\circ}{\mathbf{A}}= a(\mathbf{e}\otimes\mathbf{e}-\mathbf{e}'\otimes\mathbf{e}')$, $\overset{\circ}{\mathbf{B}}= c(\nu\otimes\nu-\nu'\otimes\nu')$, $\nu=\sigma_{2r}\neq\mathbf{e},\ \mathbf{e}',\ ac\neq 0$;

(c3) $\overset{\circ}{\mathbf{A}}= a(\sigma\otimes\sigma-\sigma'\otimes\sigma')+b\sigma\vee\sigma'$, $\overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')+d\mathbf{n}\vee\sigma'$, $bd\neq 0$;

(c4) $\overset{\circ}{\mathbf{A}}= a(\sigma_1\otimes\sigma_1-\sigma_1'\otimes\sigma_1')+b\mathbf{n}\vee\sigma_1'$, $\overset{\circ}{\mathbf{B}}= c(\sigma\otimes\sigma-\sigma'\otimes\sigma')$
$+d\mathbf{n}\vee\sigma'$, $\sigma\neq\sigma_1$, $bd\neq 0$;

(c5) $\overset{\circ}{\mathbf{A}}= a\mathbf{D}_1+b\mathbf{D}_4$, $\overset{\circ}{\mathbf{B}}= c(\nu\otimes\nu-\nu'\otimes\nu')+d\mathbf{n}\vee\nu'$, $\nu=\sigma_{2r}\neq\mathbf{e},\ ac\neq 0$.

Here and henceforth, we denote $\mathbf{u}'=\mathbf{n}\times\mathbf{u}$ for every vector $\mathbf{u}$.

Then, for cases (c1) we have

$$\Gamma(\mathbf{A},\mathbf{B})\cap D_{2md}=C_2(\mathbf{n}),\quad \mathrm{rank}V''_{2m}(\mathbf{A},\mathbf{B})\geq \mathrm{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\}=1.$$

Case (c2) does not hold for $m=1$, since the group $D_{2d}$ has only two two-fold rotation axis vectors, i.e. $\mathbf{e}$ and $\mathbf{e}'$. For case (c2) with $m\geq 2$, we have the first expression above and

$$\mathrm{rank}V''_{2m}(\mathbf{A},\mathbf{B})\geq \mathrm{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, [\mathbf{n},\mathbf{q}(\mathbf{A}),\mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A})\mathbf{n}\}=1.$$

For case (c3) we have

$$\operatorname{rank} V''_{2m}(\mathbf{A},\mathbf{B}) \geq \begin{cases} \operatorname{rank}\{\eta(\mathbf{B})', \overset{\circ}{\mathbf{B}}\,\eta(\mathbf{A})', \overset{\circ}{\mathbf{A}}\,\eta(\mathbf{B})'\} = 3 \text{ if } \sigma = \sigma_{2r-1}, \\ \operatorname{rank}\{\eta(\mathbf{B})', \overset{\circ}{\mathbf{A}}\,\eta(\mathbf{B})', \alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}\} = 3 \\ \qquad \text{if } m \geq 2,\ \sigma = \sigma_{2r}, \\ \operatorname{rank}\{\eta(\mathbf{B})', \overset{\circ}{\mathbf{A}}\,\eta(\mathbf{B})', \beta_1(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 3 \text{ if } m = 1, \\ \qquad \sigma \in \{\mathbf{e}, \mathbf{e}'\}, \end{cases}$$

where $\eta(\mathbf{D})' = \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{D}}\,\mathbf{n})$, $\mathbf{D} = \mathbf{A}$, $\mathbf{B}$. For case (c4), by using the formula (2.4) in Part I and (3.7) we have

$$\begin{aligned} \operatorname{rank} V''_{2m}(\mathbf{A},\mathbf{B}) &\geq \operatorname{rank}(V''_{2m}(\mathbf{A}) \cup V''_{2m}(\mathbf{B})) \\ &= \operatorname{rank}(V(C_{1h}(\sigma_1)) \cup V(\varGamma(\mathbf{B}) \cap D_{2md})) = 3. \end{aligned}$$

Finally, for case (c5) we have

$$\begin{aligned} \operatorname{rank} V''_{2m}(\mathbf{A},\mathbf{B}) \geq \operatorname{rank}\{&\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n}), \\ &\overset{\circ}{\mathbf{B}}\,(\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3. \end{aligned}$$

From the above results and Table 1 in Sec. 2 we deduce that the presented set $V''_{2m}(\mathbf{A},\mathbf{B})$ obeys the criterion (2.3) in Part I. Further, by considering the four pairs $(\mathbf{A}_i, \mathbf{B}_i)$ given by

$$\mathbf{A}_1 = \mathbf{e} \otimes \mathbf{e},\ \mathbf{B}_1 = \mathbf{n} \vee \sigma_1;\ \mathbf{A}_2 = \mathbf{n} \vee \sigma_1,\ \mathbf{B}_2 = \mathbf{e} \vee \mathbf{e};$$

$$\mathbf{A}_3 = \mathbf{e} \otimes \mathbf{e},\ \mathbf{B}_3 = \mathbf{n} \otimes \sigma_2,\ m \geq 2;\ \mathbf{A}_4 = \mathbf{n} \otimes \sigma_2,\ \mathbf{B}_4 = \mathbf{e} \otimes \mathbf{e},\ m \geq 2,$$

we infer that the last four generators in the set $V''_{2m}(\mathbf{A},\mathbf{B})$ are irreducible.

**(viii)** The $D_{2md}$-irreducible set $(\mathbf{u}, \mathbf{W})$ of a vector and a skewsymmetric tensor

$V \quad V''_{2m}(\mathbf{u}) \cup V''_{2m}(\mathbf{W}) \cup \{\mathbf{Wu}, (\operatorname{tr}\mathbf{WN})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}\}\ (\equiv V''_{2m}(\mathbf{u},\mathbf{W}))$

Skw $\quad \{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{W}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathbf{Wn})\mathbf{N},$

$|\mathbf{u}|^{2m-2}\mathbf{u} \wedge (\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn})) + (\operatorname{tr}\mathbf{WN})^{2m-1}\mathbf{n} \wedge \eta_{2m-1}(\overset{\circ}{\mathbf{u}})\}$

$(\equiv \operatorname{Skw}''_{2m}(\mathbf{u},\mathbf{W}))$

$$\begin{aligned}
\text{Sym} \quad & \mathrm{Sym}''_{2m}(\mathbf{u}) \cup \mathrm{Sym}_{4m}(\mathbf{W}) \cup \{\delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{D}_1, \\
& \mathbf{W}\,\overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times \eta_{2m-1}(\mathbf{W}\mathbf{n})), (\mathrm{tr}\mathbf{W}\mathbf{N})\,\overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times\,\overset{\circ}{\mathbf{u}}), \\
& |\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n}\times \eta_{2m-1}(\mathbf{W}\mathbf{n})) + (\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-1}\mathbf{n} \vee \eta_{2m-1}(\overset{\circ}{\mathbf{u}})\} \\
& \qquad\qquad\qquad\qquad (\equiv \mathrm{Sym}''_{2m}(\mathbf{u},\mathbf{W})) \\
R \quad & \mathbf{r}\cdot V''_{2m}(\mathbf{u}); \mathrm{tr}\mathbf{H}\mathbf{W}; \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\,\overset{\circ}{\mathbf{u}}; \\
& I''_{2m}(\mathbf{u}) \cup I_{4m}(\mathbf{W}) \cup \{(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}^2\,\overset{\circ}{\mathbf{u}}\}\ (\equiv I''_{2m}(\mathbf{u},\mathbf{W}))\,.
\end{aligned}$$

Here, the skewsymmetric tensor variable $\mathbf{H}$ is of the form $\mathbf{H} = c\mathbf{W}$. The other case leads to $\Gamma(\mathbf{W},\mathbf{H}) \cap D_{2md} = C_1$, which has been treated in (v) in this section. Moreover, the vector variable $\mathbf{r}$ pertains to the space span$V''_{2m}(\mathbf{u})$. In fact, for each of cases (c2) – (c6) for the $D_{2md}$-irreducible set $(\mathbf{u},\mathbf{W})$ that will be given, from the condition (3.3) in Part I with $(\mathbf{z}_0,\mathbf{z}) = (\mathbf{u},\mathbf{r})$ and $g = D_{2md}$, we derive the foregoing fact for $\mathbf{r}$. For case (c1), we derive $\mathbf{r} = a\mathbf{n} + b\sigma_{2r-1}$. The case when $b \neq 0$ is excluded, since the pair $(\mathbf{r},\mathbf{W})$ yields case (c5) that has just been covered. Finally, from the condition (3.3) in Part I with $(\mathbf{z}_0,\mathbf{z}) = (\mathbf{W},\mathbf{C})$ and $g = D_{2md}$, we derive that the symmetric tensor variable $\mathbf{C}$ pertains to the space span $\mathrm{Sym}_{4m}(\mathbf{W})$ for cases (c1) – (c4) and to the space $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ for cases (c5) – (c6).

Owing to the facts shown above, of the invariants from the scalar products, we need only to retain those listed in the above table. Of them, the two invariants $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}$ and $\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\,\overset{\circ}{\mathbf{u}}$ result from the fact that the four generators $\mathbf{I}$, $\mathbf{n}\otimes\mathbf{n}$, $\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}$ and $\overset{\circ}{\mathbf{u}} \vee\mathbf{W}\,\overset{\circ}{\mathbf{u}}$ generate the space $\mathrm{Sym}(C_{2h}(\mathbf{n}))$ for either of cases (c5) – (c6). As has been indicated earlier, here we omit the invariants $\mathbf{C} : \mathrm{Sym}_{4m}(\mathbf{W})$.

We proceed to show that the three presented sets $V''_{2m}(\mathbf{u},\mathbf{W})$, $\mathrm{Skw}''_{2m}(\mathbf{u},\mathbf{W})$ and $\mathrm{Sym}''_{2m}(\mathbf{u},\mathbf{W})$ supply the desired vector, skewsymmetric tensor and symmetric tensor generating sets, i.e. each of them obeys the criterion (2.3) in Part I. Towards this goal we first work out the $D_{2md}$-irreducible set $(\mathbf{u},\mathbf{W})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2md} \neq \Gamma(\mathbf{u},\mathbf{W}) \cap D_{2md}$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{W}$. Hence, $\mathbf{u}$ and $\mathbf{W}$ take one of the forms given by (3.3) and (3.5), respectively. Considering the combinations of the forms given by (3.3) and (3.5) and excluding the cases

$$\mathbf{W} = c\mathbf{E}\mu,\ \mathbf{u} = a\mathbf{n};\ \mathbf{W} = c\mathbf{E}\mu,\ \mathbf{u} = a\mathbf{n} + b\mu';$$

$$\mathbf{W} = c\mathbf{E}\nu,\ \mathbf{u} = a\nu;$$

which violate the $D_{2md}$-irreducibility condition for $(\mathbf{u},\mathbf{W})$, we derive the following six disjoint cases for $D_{2md}$-irreducible set $(\mathbf{u},\mathbf{W})$:

(c1) $\mathbf{W} = c\mathbf{E}\mathbf{e},\ \mathbf{u} = a\mathbf{n},\ ac \neq 0;$

(c2) $\mathbf{W} = c\mathbf{E}\sigma,\ \mathbf{u} = a\mathbf{n} + b\sigma_1,\ \sigma\cdot\sigma_1 \neq 0,\ bc \neq 0;$

(c3) $\mathbf{W} = c\mathbf{E}\sigma,\ \mathbf{u} = a\mathbf{e},\ \sigma \neq \mathbf{e},\ ac \neq 0;$

(c4) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \mathbf{u} = a\mathbf{n},\ ac \neq 0;$

(c5) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \mathbf{u} = a\mathbf{n} + b\sigma_1,\ bc \neq 0;$

(c6) $\mathbf{W} = c\mathbf{E}\mathbf{n},\ \mathbf{u} = a\mathbf{e},\ ac \neq 0;$

For case (c1) we have

$$\text{rank} V''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{u}, \mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u}\} = 3\,, \tag{3.8}$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{W}, \alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{N}, \mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{W}^2, \mathbf{n}\vee\mathbf{W}\mathbf{n}, \mathbf{W}\overset{\circ}{\mathbf{u}}\vee\eta(\mathbf{W})', \mathbf{u}\vee\eta(\mathbf{W})'\} = 6,$$

where $\eta(\mathbf{W})' = \mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n})$.

For case (c2) we have

$$\text{rank} V''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}), \mathbf{W}\mathbf{u}\} = 3\,, \tag{3.9}$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{W}, \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 3\,, \tag{3.10}$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \mathbf{n}\vee\mathbf{W}\mathbf{n}, \mathbf{u}\vee(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}))\} = 6\,. \tag{3.11}$$

For case (c3), we have (3.8), (3.10) and (3.11). For case (c4) we have

$$\Gamma(\mathbf{u},\mathbf{W}) \cap D_{2md} = C_{2m}(\mathbf{n}),\ \text{rank} V''_{2m}(\mathbf{u},\mathbf{W}) = \text{rank Skw}''_{2m}(\mathbf{u},\mathbf{W}) = 1,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{W}) = \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{D}_1, \delta_{1m}\mathbf{D}_2\} = 2(1+\delta_{1m}).$$

For case (c5) we have (3.9) and

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{W}, \mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge\eta_{2m-1}(\overset{\circ}{\mathbf{u}})\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})), \mathbf{n}\vee\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}})\} = 6.$$

Finally, for case (c6) we have the last two expressions above and

$$\text{rank} V''_{2m}(\mathbf{u},\mathbf{W}) \geq \text{rank}\{\mathbf{u}, \mathbf{W}\mathbf{u}, \alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}\} = 3.$$

From the above results and Table 1 in Sec. 2 we infer that the three presented sets of generators obey the criterion (2.3) in Part I, respectively. Further, by considering the pair $\mathbf{u}_1 = \mathbf{e}$ and $\mathbf{W}_1 = \mathbf{En}$, we infer that the respective last two generators in the two sets $V''_{2m}(\mathbf{u}, \mathbf{W})$ and $\mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{W})$ are irreducible. Moreover, by considering the pair $\mathbf{u}_2 = \mathbf{n}$ and $\mathbf{W}_2 = \mathbf{Ee}$, we deduce that the last two generators in the set $\mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{W})$ and the generator $\mathbf{W}\ \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \eta_{2m-1}(\mathbf{Wn}))$ are also irreducible, respectively. Finally, by considering case (c4) we deduce that the generator $\delta_{1m}(\mathbf{u} \cdot \mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{D}_1$ is irreducible.

(x) The $D_{2md}$-irreducible set $(\mathbf{u}, \mathbf{A})$ of a vector and a symmetric tensor

$$V \qquad V''_{2m}(\mathbf{u}) \cup V''_{2m}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{A}}\ \mathbf{u}, (1 - \delta_{1m})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} \qquad (\equiv V''_{2m}(\mathbf{u}, \mathbf{A}))$$

$$\mathrm{Skw} \qquad \mathrm{Skw}_{4m}(\mathbf{A}) \cup \{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u} \cdot \mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N},$$
$$(\mathbf{u} \cdot \mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})\mathbf{N}, |\mathbf{u}|^{2m-2}\mathbf{u} \wedge (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}))$$
$$+|\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n} \wedge \overset{\circ}{\mathbf{A}}\ (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}}))\} \ (\equiv \mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A}))$$

$$\mathrm{Sym} \qquad \mathrm{Sym}''_{2m}(\mathbf{u}) \cup \mathrm{Sym}_{4m}(\mathbf{A}) \cup \{(\mathbf{u} \cdot \mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))(\mathbf{A}_\mathbf{e}\mathbf{N} - \mathbf{N}\mathbf{A}_\mathbf{e}),$$
$$(\mathbf{u} \cdot \mathbf{n})\ \overset{\circ}{\mathbf{A}}\ \mathbf{n} \vee (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})),$$
$$|\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})) + |\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n} \vee \overset{\circ}{\mathbf{A}}\ (\mathbf{n} \times \eta_{2m-1}(\overset{\circ}{\mathbf{u}}))\} \qquad (\equiv \mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A}))$$

$$R \qquad \mathbf{r} \cdot V''_{2m}(\mathbf{u});$$
$$I''_{2m}(\mathbf{u}) \cup I_{4m}(\mathbf{A}) \cup \{(\mathbf{u} \cdot \mathbf{n})\ \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\ \mathbf{n}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}{}^2\ \overset{\circ}{\mathbf{u}}\} \qquad (\equiv I''_{2m}(\mathbf{u}, \mathbf{A})).$$

For each nonvanishing skewsymmetric tensor $\mathbf{H}$ and each $D_{2md}$-irreducible set $(\mathbf{u}, \mathbf{A})$ (see cases (c1) – (c6) given later), we have $\Gamma(\mathbf{z}_0, \mathbf{H}) \cap D_{2md} = \Gamma(\mathbf{u}, \mathbf{A}, \mathbf{H}) \cap D_{2md}$ with $\mathbf{z}_0 \in \{\mathbf{u}, \mathbf{A}\}$. From this fact and the condition $(3.3)_2$ in Part I with $\mathbf{z} = \mathbf{H}$ and $g = D_{2md}$, we derive $\mathbf{H} = \mathbf{O}$. Moreover, by means of the relevant procedure used at the start of (ix), for the vector variable $\mathbf{r}$ and the symmetric tensor variable $\mathbf{C}$ we derive $\mathbf{r} \in \mathrm{span}V''_{2m}(\mathbf{u})$ and $\mathbf{C} \in \mathrm{span}\ \mathrm{Sym}_{4m}(\mathbf{A})$ from cases (c1) – (c6) given below and the condition (3.3) in Part I with $(\mathbf{z}_0, \mathbf{z}) = (\mathbf{u}, \mathbf{r})$, $(\mathbf{A}, \mathbf{C})$, and $g = D_{2md}$. Owing to these facts, of the invariants from the scalar products, we retain the invariants $\mathbf{r} \cdot V_{2m}(\mathbf{u})$ only. Besides, as has been indicated earlier, we omit the invariants $\mathbf{C} : \mathrm{Sym}_{4m}(\mathbf{A})$.

We proceed to show that the three presented sets $V''_{2m}(\mathbf{u}, \mathbf{A})$, $\mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A})$ supply desired vector, skewsymmetric tensor and symmetric tensor generating sets, i.e. each of them obeys the criterion (2.3) in Part I. Towards this goal we first work out the $D_{2md}$-irreducible set $(\mathbf{u}, \mathbf{A})$, which is specified by (see (3.1) in Part I) $\Gamma(\mathbf{z}) \cap D_{2md} \neq \Gamma(\mathbf{u}, \mathbf{A}) \cap D_{2md}$ for $\mathbf{z} = \mathbf{u},\ \mathbf{A}$. Hence, $\mathbf{u}$ and $\mathbf{A}$ take one of the forms given by (3.3) and (3.6), respectively. Considering the combinations of the forms given by (3.3) and (3.6) and excluding the cases

$$\mathbf{u} = c\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_{2m}(\mathbf{q}(\mathbf{A})) \neq 0;$$

$$\mathbf{u} = c\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a(\mu \otimes \mu - \mu' \otimes \mu') + b\mathbf{n} \vee \mu';$$

$$\mathbf{u} = a\mathbf{n} + b\mu,\ \overset{\circ}{\mathbf{A}} = c(\mu \otimes \mu - \mu' \otimes \mu');$$

$$\mathbf{u} = a\mathbf{n} + b\mu,\ \overset{\circ}{\mathbf{A}} = c(\mu \otimes \mu - \mu' \otimes \mu') + d\mathbf{n} \vee \mu;$$

$$\mathbf{u} = c\mathbf{v},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{v} \otimes \mathbf{v} - \mathbf{v}' \otimes \mathbf{v}');$$

$$\mathbf{u} = c\mathbf{v},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{v} \otimes \mathbf{v} - \mathbf{v}' \otimes \mathbf{v}') + b\mathbf{n} \vee \mathbf{v}';$$

which violate the $D_{2md}$-irreducibility condition for $(\mathbf{u}, \mathbf{A})$ (see (3.1) in Part I), we derive the following six disjoint cases for the $D_{2md}$-irreducible set $(\mathbf{u}, \mathbf{A})$:

(c1) $\mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}} = c(\sigma \otimes \sigma - \sigma' \otimes \sigma'),\ ac \neq 0;$

(c2) $\mathbf{u} = a\mathbf{n},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_4,\ ad \neq 0;$

(c3) $\mathbf{u} = a\mathbf{n} + b\sigma_1,\ \overset{\circ}{\mathbf{A}} = c(\sigma_1 \otimes \sigma_1 - \sigma_1' \otimes \sigma_1') + d\sigma_1 \vee \sigma_1',\ bd \neq 0;$

(c4) $\mathbf{u} = a\mathbf{n} + b\sigma_1,\ \overset{\circ}{\mathbf{A}} = c(\sigma \otimes \sigma - \sigma' \otimes \sigma') + d\mathbf{n} \vee \sigma,\ \sigma \neq \sigma_1,\ bd \neq 0;$

(c5) $\mathbf{u} = b\mathbf{e},\ \overset{\circ}{\mathbf{A}} = c\mathbf{D}_1 + d\mathbf{D}_2,\ bd \neq 0;$

(c6) $\mathbf{u} = b\mathbf{e},\ \overset{\circ}{\mathbf{A}} = c(\sigma \otimes \sigma - \sigma' \otimes \sigma') + d\mathbf{n} \vee \sigma',\ \sigma \neq \mathbf{e},\ bd \neq 0.$

For case (c1) we have

$$\Gamma(\mathbf{u}, \mathbf{A}) \cap D_{2md} = \begin{cases} C_{2v}(\mathbf{n}, \sigma, \sigma') & \text{if } \sigma = \sigma_{2r-1}\,, \\ C_2(\mathbf{n}) & \text{if } \sigma = \sigma_{2r}\,, \end{cases}$$

$$\operatorname{rank} V''_{2m}(\mathbf{u}, \mathbf{A}) \geq \operatorname{rank}\{\mathbf{u}\} = 1,$$

$$\operatorname{rank}\, \mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A}) \geq \begin{cases} 0 \text{ if } \sigma = \sigma_{2r-1}\,, \\ \operatorname{rank}\{(\mathbf{v} \cdot \mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N}\} = 1 \text{ if } \sigma = \sigma_{2r}\,, \end{cases}$$

$$\operatorname{rank}\, \mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{A}}\} = 3 \text{ if } \sigma = \sigma_{2r-1}\,, \\ \operatorname{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{A}_{\mathbf{e}}\mathbf{N} - \mathbf{N}\mathbf{A}_{\mathbf{e}}\} = 4 \text{ if } \sigma = \sigma_{2r}\,. \end{cases}$$

For case (c2) we have

$$\text{rank}V''_{2m}(\mathbf{u}) \geq \text{rank}\{\mathbf{u}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{A}}\,\mathbf{u}\} = 3\,, \tag{3.12}$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{N},$$
$$\mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3,$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\vee\eta(\mathbf{A})',$$
$$\mathbf{u}\vee\eta(\mathbf{A})'\} = 6,$$

where $\eta(\mathbf{A})' = \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$.

For case (c3) we have

$$\text{rank}V''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{A}}\,\mathbf{u}\} = 3,$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}})),$$
$$\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}\} = 3\,, \tag{3.13}$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}},$$
$$\mathbf{n}\vee\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}))\} = 6\,. \tag{3.14}$$

For cases (c4) we have

$$\text{rank}V''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{n}, \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} = 3,$$

$$\text{rank Skw}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{E}\eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\,\mathbf{n},$$
$$\mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 3\,, \tag{3.15}$$

$$\text{rank Sym}''_{2m}(\mathbf{u},\mathbf{A}) \geq \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}},$$
$$\mathbf{u}\vee(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\} = 6\,. \tag{3.16}$$

For case (c5) we have (3.13) – (3.14) and (note that $\beta_1(\mathbf{q}(\mathbf{A})) \neq 0$)

$$\text{rank}V''_{2m}(\mathbf{u},\mathbf{A}) \geq \{\mathbf{u}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}, \beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, (1-\delta_{1m})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}\} = 3.$$

Finally, for case (c6) we have (3.12) and (3.15) – (3.16).

Thus, from the above results and Tables 1-3 in Sec. 2 in Part I, we deduce that the three presented sets of generators obey the criterion (2.3) in Part I. Further, by considering the pair $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{A}_1 = \mathbf{n} \vee \mathbf{e}$, we infer that the generator $\overset{\circ}{\mathbf{A}}\ \mathbf{u}$ and the respective last two generators in the two sets $\mathrm{Skw}''_{2m}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}''_{2m}(\mathbf{u}, \mathbf{A})$ are irreducible. Moreover, from case (c1) we know that the two generators $(\mathbf{u} \cdot \mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N}$ and $(\mathbf{u} \cdot \mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))(\mathbf{A_e N} - \mathbf{N A_e})$ are irreducible. By considering, respectively, the pairs $\mathbf{u}_2 = \sigma_1$ and $\mathbf{A}_2 = \mathbf{e} \otimes \mathbf{e}$, $\mathbf{u}_3 = \sigma_2$ and $\mathbf{A}_3 = \mathbf{e} \otimes \mathbf{e}$ (for $m \geq 2$), we know that the generator $\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}$ and the last generator in the set $V''_{2m}(\mathbf{u}, \mathbf{A})$ are irreducible.

### 3.3. Sets of three variables

As indicated at the outset of this section, we need only to treat the four sets $(\mathbf{u}, \mathbf{v}, \mathbf{W})$, $(\mathbf{u}, \mathbf{v}, \mathbf{A})$, $(\mathbf{u}, \mathbf{W}, \mathbf{A})$ and $(\mathbf{u}, \mathbf{A}, \mathbf{B})$. We shall demonstrate that each set $X_0$ just mentioned is $D_{2md}$-reducible, i.e there is a proper subset $S \subset X_0$ such that $\Gamma(S) \cap D_{2md} = \Gamma(X_0) \cap D_{2md}$.

First, let $X_0 = (\mathbf{u}, \mathbf{v}, \mathbf{D})$ with $\mathbf{D} \in \{\mathbf{W}, \mathbf{A}\}$ a skewsymmetric or a symmetric tensor. Suppose that $X_0$ is $D_{2md}$-irreducible. Then $(\mathbf{u}, \mathbf{v})$ is $D_{2md}$-irreducible and $\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2md} \neq C_1$. From these and cases (c1) – (c4) derived in (**iv**), we know that $\mathbf{u} = a\mathbf{n}$ and $\mathbf{v} = b\mathbf{e}$ with $ab \neq 0$ (see case (c1)). Since the group $\Gamma(\mathbf{v}) \cap D_{2md}$, i.e. $C_2(\mathbf{e})$ has only two subgroups, i.e. $C_1$ and $C_2(\mathbf{e})$, we deduce $\Gamma(\mathbf{u}, \mathbf{D}) \cap D_{2md} = C_1$ or $\Gamma(\mathbf{v}, \mathbf{D}) \cap D_{2md} = C_2(\mathbf{e}) = \Gamma(\mathbf{v}) \cap D_{2md}$. Either of the two cases mentioned above indicates that the $(\mathbf{u}, \mathbf{v}, \mathbf{D})$ is $D_{2md}$-reducible, contradicting the foregoing presupposition.

Second, let $X_0 = (\mathbf{u}, \mathbf{D}, \mathbf{A})$ with $\mathbf{D} \in \{\mathbf{W}, \mathbf{B}\}$ a skewsymmetric or a symmetric tensor. Suppose that $X_0$ is $D_{2md}$-irreducible. Then, both the set $(\mathbf{u}, \mathbf{A})$ and the set $(\mathbf{D}, \mathbf{A})$ are $D_{2md}$-irreducible and $\Gamma(\mathbf{z}, \mathbf{A}) \cap D_{2md} \neq C_1$, $\mathbf{z} = \mathbf{u}$, $\mathbf{D}$. From these and cases (c1) – (c6) derived in (**x**) and cases (c1) – (c5) derived in (**vi**) (for $\mathbf{D} = \mathbf{W}$) and cases (c1) – (c6) derived in (**vii**) (for $\mathbf{D} = \mathbf{B}$), we know $\mathbf{u} = a\mathbf{n}$, $\overset{\circ}{\mathbf{A}} = b(\sigma \otimes \sigma - \sigma' \otimes \sigma')$ (see case (c1) in (**x**)), $\mathbf{D} = \mathbf{W} = c\mathbf{En}$ (see case (c1) in (**vi**)) and $\overset{\circ}{\mathbf{D}} = \mathbf{B} = d(\bar{\sigma} \otimes \bar{\sigma} - \bar{\sigma}' \otimes \bar{\sigma}')$ (see cases (c1) – (c2) in (**vii**)), where $abcd \neq 0$ and $\sigma$, $\bar{\sigma} \in \{\sigma_1, \cdots, \sigma_{4m}\}$ and $(\sigma \cdot \bar{\sigma})\sigma \times \bar{\sigma} \neq \mathbf{0}$. Thus, we deduce $\Gamma(\mathbf{u}, \mathbf{D}, \mathbf{A}) = C_2(\mathbf{n}) = \Gamma(\mathbf{D}, \mathbf{A})$, in contradiction to the foregoing presupposition.

### 3.4. The general results

Applying Theorem 2.1 in XIAO [20] and incorporating the fact indicated at the outset of this section, from (a) – (c) we obtain the following general result.

THEOREM 8. *The four sets given by*

$I''_{2m}(\mathbf{u})$; $I_{4m}(\mathbf{W})$; $I_{4m}(\mathbf{A})$; $I_{4m}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$; $\mathbf{u}\cdot\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})$,

$(\mathbf{v}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{u}}) + (\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{v}},\eta_{2m-1}(\mathring{\mathbf{u}})]$,

$(\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{v}})]$,

$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n})^{4m-1} - \beta_{2m}(\mathring{\mathbf{v}})[\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{v}})]$,

$(\mathbf{v}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n})^{4m-1} - \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n},\mathring{\mathbf{v}},\eta_{2m-1}(\mathring{\mathbf{u}})]$;

$(\mathbf{u}\cdot\mathbf{n})\ \mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathbf{W}^2\ \mathring{\mathbf{u}}, \mathrm{tr}\mathbf{W}(\mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}}))$,

$(\mathrm{tr}\mathbf{W}\mathbf{N})(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathring{\mathbf{u}}) + \beta_{2m}(\mathring{\mathbf{u}})\ \mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}$,

$(\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{4m}(\mathring{\mathbf{u}}) - (\mathbf{u}\cdot\mathbf{n})^{4m-1}\ \mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, [\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathbf{W}\mathbf{n})]$,

$[\mathbf{n},\mathbf{W}\mathbf{u},\eta_{2m-1}(\mathbf{W}\mathbf{n})], [\mathbf{n},\mathbf{W}^2\mathbf{u},\eta_{2m-1}(\mathbf{W}\mathbf{n})]$,

$(\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathbf{W}\mathbf{n})$; $\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\ \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\ \mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^2\ \mathring{\mathbf{u}}$,

$[\mathbf{n},\mathring{\mathbf{A}}\ \mathbf{n},\eta_{2m-1}(\mathring{\mathbf{u}})] - \delta_{1m}(\mathbf{u}\cdot\mathbf{n})\mathrm{tr}\mathbf{A}\mathbf{D}_2, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{u}})]$

$+\beta_{2m}(\mathring{\mathbf{u}})\ \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\ \mathbf{n}, \beta_{2m}(\mathring{\mathbf{u}})[\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{u}})]$

$-(\mathbf{u}\cdot\mathbf{n})^{4m-1}\ \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\ \mathbf{n}, [\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{A}}\ \mathbf{n})], [\mathbf{n},\mathring{\mathbf{A}}\ \mathbf{u},\eta_{2m-1}(\mathring{\mathbf{A}}\ \mathbf{n})]$,

$\beta_{2m}(\mathring{\mathbf{A}}\ \mathbf{n})\ \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\ \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})J(\mathbf{A})\alpha_m(\mathbf{q}(\mathbf{A}))$;

$\alpha_{2m}(\mathring{\mathbf{v}})[\mathbf{r},\mathbf{u},\mathring{\mathbf{v}}], \alpha_{2m}(\mathring{\mathbf{u}})[\mathbf{r},\mathbf{v},\mathring{\mathbf{u}}]$;

$\mathbf{u}\cdot\mathbf{W}\mathbf{v}, (\mathrm{tr}\mathbf{W}\mathbf{N})((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\mathring{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\mathring{\mathbf{u}}))$;

$\mathbf{u}\cdot\ \mathring{\mathbf{A}}\ \mathbf{v}, (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\mathring{\mathbf{v}})[\mathbf{n},\mathring{\mathbf{v}},\mathring{\mathbf{A}}\mathring{\mathbf{v}}]$

$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\mathring{\mathbf{u}})[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{A}}\mathring{\mathbf{u}}]$;

$[\mathbf{n},\mathbf{\Omega}\mathbf{u},\eta_{2m-1}(\mathbf{W}\mathbf{n})], [\mathbf{n},\mathbf{W}\mathbf{u},\eta_{2m-1}(\mathbf{\Omega}\mathbf{n})]$,

$(\mathbf{u}\cdot\mathbf{n})((\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{\Omega}\mathbf{n})$

$+(\mathrm{tr}\mathbf{\Omega}\mathbf{N})|\mathrm{tr}\mathbf{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{W}\mathbf{n}))$;

$(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\mathring{\mathbf{A}}\ \mathbf{n}), (\mathbf{u}\cdot\mathbf{n})(\alpha_{2m}(\mathbf{W}\mathbf{n})$

$+(\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A}))$,

$(\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-2}[\mathbf{n},\mathbf{W}\mathbf{u},\eta_{2m-1}(\mathring{\mathbf{A}}\ \mathbf{n})]$

$-|\ \mathring{\mathbf{A}}\ |^{2m-2}[\mathbf{n},\mathring{\mathbf{A}}\ \mathbf{u},\eta_{2m}(\mathbf{W}\mathbf{n})], \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\ \mathbf{W}\ \mathring{\mathbf{u}}$;

$$[\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{u}, \eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})], [\mathbf{n}, \overset{\circ}{\mathbf{B}}\,\mathbf{u}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})],$$

$$(1-\delta_{1m})(\mathbf{u}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A}))),$$

$$(1-\delta_{1m})(\mathbf{u}\cdot\mathbf{n})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B})));$$

*and*

$$V''_{2m}(\mathbf{u}), V''_{2m}(\mathbf{W}), V''_{2m}(\mathbf{A});$$

$$\alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{v}\times\overset{\circ}{\mathbf{u}}, \alpha_{2m}(\overset{\circ}{\mathbf{v}})\mathbf{u}\times\overset{\circ}{\mathbf{v}}; \boldsymbol{\Omega}(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n})), \mathbf{W}(\mathbf{n}\times\eta_{2m-1}(\boldsymbol{\Omega}\mathbf{n})),$$

$$((\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1}\alpha_{2m}(\boldsymbol{\Omega}\mathbf{n}) + (\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})|\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|^{2m-1}\alpha_{2m}(\mathbf{W}\mathbf{n}))\mathbf{n};$$

$$(\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{n}, (\alpha_{2m}(\mathbf{W}\mathbf{n}) + (\mathrm{tr}\mathbf{W}\mathbf{N})|\mathrm{tr}\mathbf{W}\mathbf{N}|^{2m-1})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n},$$

$$(\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-2}\mathbf{W}(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) + |\,\overset{\circ}{\mathbf{A}}\,|^{2m-2}\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n}));$$

$$\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})), \overset{\circ}{\mathbf{B}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})),$$

$$(1-\delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{B}))\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{A}), \mathbf{q}(\mathbf{B})]\alpha_m(\mathbf{q}(\mathbf{A})))\mathbf{n},$$

$$(1-\delta_{1m})(\alpha_m(\mathbf{q}(\mathbf{A}))\alpha_{2m}(\overset{\circ}{\mathbf{B}}\,\mathbf{n})$$
$$+(|\mathbf{q}(\mathbf{A})|\cdot|\mathbf{q}(\mathbf{B})|)^{m-1}[\mathbf{n}, \mathbf{q}(\mathbf{B}), \mathbf{q}(\mathbf{A})]\alpha_m(\mathbf{q}(\mathbf{B})))\mathbf{n};$$

$$\mathbf{W}\mathbf{u}, (\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}; \overset{\circ}{\mathbf{A}}\,\mathbf{u}, (1-\delta_{1m})\alpha_{2m}(\overset{\circ}{\mathbf{u}})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n};$$

*and*

$$\mathrm{Skw}''_{2m}(\mathbf{u}), \mathrm{Skw}_{4m}(\mathbf{W}), \mathrm{Skw}_{4m}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}, ((\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{v}}) + (\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m-1}\alpha_{2m}(\overset{\circ}{\mathbf{u}}))\mathbf{N};$$

$$\mathbf{W}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{W}; \overset{\circ}{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\overset{\circ}{\mathbf{A}}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}},$$

$$\overset{\circ}{\mathbf{B}}\,\mathbf{n}\wedge\,\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\wedge\,\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\,\mathbf{n};$$

$$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathbf{W}\mathbf{n})\mathbf{N},$$

$$|\mathbf{u}|^{2m-2}\mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\mathbf{W}\mathbf{n})) + (\mathrm{tr}\mathbf{W}\mathbf{N})^{2m-1}\mathbf{n}\wedge\eta_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{N},$$

$$|\mathbf{u}|^{2m-2}\mathbf{u}\wedge(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})) + |\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n}\wedge\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\eta_{2m-1}(\overset{\circ}{\mathbf{u}}));$$

*and*

$$\mathrm{Sym}''_{2m}(\mathbf{u}), \mathrm{Sym}_{4m}(\mathbf{W}), \mathrm{Sym}_{4m}(\mathbf{A});$$

$$\mathbf{u} \vee \mathbf{v}, (\mathbf{u}\cdot\mathbf{n})|\mathbf{u}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\mathring{\mathbf{v}})\, \mathring{\mathbf{v}} \vee (\mathbf{n}\times \mathring{\mathbf{v}})$$

$$+(\mathbf{v}\cdot\mathbf{n})|\mathbf{v}\cdot\mathbf{n}|^{2m+1}\alpha_{2m}(\mathring{\mathbf{u}})\, \mathring{\mathbf{u}} \vee (\mathbf{n}\times \mathring{\mathbf{u}});$$

$$\mathbf{W\Omega} + \mathbf{\Omega W},$$

$$|\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn} \vee \mathbf{NWn} + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n} \vee \mathbf{N\Omega n};$$

$$\mathring{\mathbf{A}}\,\mathbf{W} - \mathbf{W}\,\mathring{\mathbf{A}}, (\mathrm{tr}\mathbf{WN})\,\mathring{\mathbf{A}}\,\mathbf{n} \vee \mathbf{N}\,\mathring{\mathbf{A}}\,\mathbf{n}; \mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}};$$

$$\delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{D}_1, \mathbf{W}\,\mathring{\mathbf{u}} \vee (\mathbf{n}\times \eta_{2m-1}(\mathbf{Wn})),$$

$$(\mathrm{tr}\mathbf{WN})\,\mathring{\mathbf{u}} \vee (\mathbf{n}\times \mathring{\mathbf{u}}),$$

$$|\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n}\times \eta_{2m-1}(\mathbf{Wn})) + (\mathrm{tr}\mathbf{WN})^{2m-1}\mathbf{n} \vee \eta_{2m-1}(\mathring{\mathbf{u}});$$

$$(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A}))(\mathbf{A_e N} - \mathbf{N A_e}), (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{A}}\,\mathbf{n} \vee (\mathbf{n}\times \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})),$$

$$|\mathbf{u}|^{2m-2}\mathbf{u} \vee (\mathbf{n}\times \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})) + |\mathbf{q}(\mathbf{A})|^{2m-2}\mathbf{n} \vee \mathring{\mathbf{A}}\,(\mathbf{n}\times \eta_{2m-1}(\mathring{\mathbf{u}}));$$

*where* $(\mathbf{u},\mathbf{v},\mathbf{r}) = (\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H}) = (\mathbf{W}_\mu,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C}) = (\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $k > j > i = 1,\cdots,a$, $\theta > \tau > \mu = 1,\cdots,b$, $N > M > L = 1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* $D_{2md}$ *for each* $m \geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold rotation axis of the group* $D_{2md}$.

In the above theorem, $I_{4m}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$ is used to represent the invariants depending on two or three symmetric and/or skewsymmetric tensors given in Theorem 1 in Part I with the replacement of $m$ by $2m$ therein.

## 4. Concluding remarks

Based upon symmetry-reduced decompositions of the domain of any finite number of vector variables and second order tensor variables in XIAO [16, 20], a simple, unified procedure for constructing both generating sets and the functional bases is designed and developed in the recent work (XIAO [18 – 19]) and this series of works. This unified procedure reduces the tough problem of determining irreducible representations for anisotropic functions of any finite number of vector variables and second order tensor variables to that of determining irreducible representations for anisotropic functions of certain sets consisting of not more than three vector and/or second order tensor variables. The $g$-irreducibility conditions for sets of two and three variables (see (3.1) – (3.2) in Part I) further provide a considerable simplification in dealing with representations for sets of two and three variables. The condition (3.3) in Part I is helpful to remove some redundant invariants in forming the scalar products of the variables $\mathbf{r}$, $\mathbf{H}$ and $\mathbf{C}$ and the presented generators. In addition, the notion of isotropic extension of anisotropic functions and the much well-known results for isotropic functions (see, e.g., SPENCER [13], WANG [14], SMITH [12], BOEHLER [3]) are essential. The former was originated earlier from LOKHIN and SEDOV [9] and independently introduced and successfully applied to derive systematic results for anisotropic functions in some cases for the first time by BOEHLER *et al.* [4 – 7] and developed later by many researchers, refer to, e.g., LIU [8], BETTEN, BOEHLER, SPENCER (see [6]), RYCHLEWSKI [10], ZHANG and RYCHLEWSKI [23], BETTEN [2], ZHENG and SPENCER [25], *et al.*; see also the reviews by BETTEN [1], RYCHLEWSKI and ZHANG [11] and ZHENG [24] for detail. A substantial generalization in this aspect has been given very recently by one of the authors (see XIAO [15, 17]).

By applying the above unified procedure, together with the results for isotropic extension of anisotropic functions and the much well-known results for isotropic functions, we have derived irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vector variables and second order tensor variables under all crystal classes and quasicrystal classes as subgroups of the cylindrical group $D_{\infty h}$.

Thus, of all kinds of material symmetry groups of solids, only the five cubic crystal classes and the two icosahedral classes have not yet been covered. According to Theorem 3.2 in XIAO [16], the unified procedure outlined in Sec. 3 in Part I also applies to these classes, except for the fact that the set $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of three symmetric tensor variables should be added.

Although the procedure used merely involves irreducible representations for one, two and three variables, many details concerning these representations need to be examined. Such a situation results from the complexity of nonlinear anisotropic tensor functions. In reality, much effort and labour should be made even for isotropic tensor functions. Nevertheless, once the results for relevant representations are derived, their correctness can be guaranteed by checking the fulfilment of Criteria 1 and 2 given in Part I, as has been done. Towards this goal, it is crucial to work out the related $g$-irreducible sets of two or three variables from the conditions (3.1) or (3.2) in Part I. This crucial aspect has been treated in a definite and rigorous manner.

This series of papers is concerned with material symmetries of solids, which are described by finite and continuous infinite subgroups of the 3-dimensional full orthogonal group. Other kinds of material symmetries, such as those of liquid crystals etc., are characterized by subgroups of the 3-dimensional unimodular group $\mathcal{U}(3)$. It is expected that the procedure used may be extended to cover the latter kinds of material symmetry groups and other groups. The main basis in this more general aspect has been laid down by RYCHLEWSKI [10]. In the latter, the existence and reality of isotropic extension is proved to be true in the most general sense that an arbitrary group acts on an arbitrary set.

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