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Laminar dispersed two-phase flows at low concentration II Disturbance Equations

J. L. ACHARD and A. CARTELLIER

Laboratoire des Écoulements Géophysiques et Industriels
CNRS-UJF-INPG, B.P. 53X, 38041 Grenoble Cedex, France

In a preceding paper (Part I), a generalised system of equations was proposed to represent multi-D flows of particle-fluid mixtures. It was based on a coupling of two sets of equations, one for each phase: the Lundgren hierarchy for the continuous phase and an adaptation of the well-known B.B.G.K.Y. hierarchy for the dispersed phase. It happens that at any order, many of the equations obtained remain intricate: several important terms are difficult either to interpret or to compute effectively such as the averaged extra-deformation tensors, the interfacial force density and finally the pseudo-turbulent tensors in the momentum equations for both phases, arising from inclusion motions alone. That can be remedied by introducing the concept of an "averaged disturbance field" based on differences between two successive conditionally averaged variables. All the equations of both hierarchies are transformed in terms of these fields, which play a central role in our theory, except for the first-order equations of both hierarchies; these correspond to conservation equations of standard two-fluid models.

1. Introduction

A truncation procedure will ultimately have to cut the two hierarchies obtained in Part I using the same perturbation method based on diluteness. Besides the small dimensionless number, Θ, related to diluteness, we can expect some additional well-known parameters characterising the relevant dynamic processes, such as the inclusion Reynolds number, to appear. These parameters may be small or large and lead to some simplification. In order to take full advantage of the order of magnitude of these parameters while completing the truncation procedure in a subsequent part, it is advisable to introduce preliminary various disturbance flow fields, in the averaged sense, for both phases. These perturbations may be considered as being set up by one, two or more test inclusions, which locally modify the pre-existing averaged fluid flow which, in the absence of these test inclusions, would be created respectively by no, one or more test inclusions. One of the main goals of this paper is to replace the second-, third- and higher-order equations of both hierarchies by equations controlling these new disturbance fields.

In both new hierarchies, the equations controlling $\overline{\mathbf{v}}^{c1}(\mathbf{x})$ and $\overline{p}^{c1}(\mathbf{x})$ as well as $\overline{\mathbf{u}}^{1}(\mathbf{x})$ and $\overline{\boldsymbol{\omega}}^{1}(\mathbf{x})$ will remain unchanged. At each other order, the equations are combined in such a way that they control a particular disturbance field. The definition of disturbance fields will satisfy two requirements. First, they will vanish far from the test inclusions considered; conditional fields which move away from them lose their influence. Second, they must be symmetrical with respect to the locations of these inclusions.

A typical first-order disturbance field is:

$$\mathbf{u}^{*}(\mathbf{x}|\mathbf{x}^{\circ}) = \overline{\mathbf{u}}^{2}(\mathbf{x}|\mathbf{x}^{\circ}) - \overline{\mathbf{u}}^{1}(\mathbf{x}). \tag{1.1}$$

Similar definitions hold for $\boldsymbol{\omega}^{*}(\mathbf{x}|\mathbf{x}^{\circ})$, $\mathbf{v}^{*}(\mathbf{x}|\mathbf{x}^{\circ})$ and $p^{*}(\mathbf{x}|\mathbf{x}^{\circ})$. Attention must be paid to the definition of the first-order disturbance field relative to the dispersed-phase volume fraction, i.e., $\alpha^{*}(\mathbf{x}|\mathbf{x}^{\circ}) = \alpha^{d2}(\mathbf{x}|\mathbf{x}^{\circ}) - \alpha^{d1}(\mathbf{x}) = \alpha^{c1}(\mathbf{x}) - \alpha^{c2}(\mathbf{x}|\mathbf{x}^{\circ})$. In accordance with the convention introduced in Sec. 3.2 – Part I, the arguments which are understood in the shortened notations $\boldsymbol{\omega}^{\circ *}$, $\mathbf{v}^{\circ}$, $p^{\circ *}$ and $\alpha^{\circ *}$ are $(\mathbf{x}^{\circ}|\mathbf{x})$. Instead of disturbance densities, we will use conditional disturbance densities:

$$\chi^{*}(\mathbf{x}|\mathbf{x}^{\circ}) = \chi_{2}(\mathbf{x}|\mathbf{x}^{\circ}) - \phi_{1}(\mathbf{x}), \tag{1.2}$$

where the first conditional density $\chi_{2}(\mathbf{x}|\mathbf{x}^{\circ})$ is that introduced in ((4.32) – Part I). In unbounded physical domains, we can observe that $\mathbf{x}^{\circ}$ given $\chi^{*} \to 0$ as $\mathbf{x} \to \infty$, since inclusions which are far apart do not influence each other's position. As before, the arguments which will be understood in the shortened notation $\chi^{\circ *}$ are $(\mathbf{x}^{\circ}|\mathbf{x})$. The notion of correlation field, which is of standard use in Statistical Mechanics, is akin to that of disturbance field. Note that definition (1.2) is consistent with the double correlation fields, i.e. $\phi^{*}(\mathbf{x}, \mathbf{x}^{\circ}) = \phi_{2}(\mathbf{x}, \mathbf{x}^{\circ}) - \phi_{1}(\mathbf{x})\phi_{1}(\mathbf{x}^{\circ})$.

A typical second-order disturbance field to be introduced is:

$$\mathbf{v}^{**}(\mathbf{x}|\mathbf{x}^{\circ}, \mathbf{x}^{\circ\circ}) = \overline{\mathbf{v}}^{c3}(\mathbf{x}|\mathbf{x}^{\circ}, \mathbf{x}^{\circ\circ}) - \overline{\mathbf{v}}^{c2}(\mathbf{x}|\mathbf{x}^{\circ}) - \overline{\mathbf{v}}^{c2}(\mathbf{x}|\mathbf{x}^{\circ\circ}) + \overline{\mathbf{v}}^{c1}(\mathbf{x}). \tag{1.3}$$

The fields $p^{**}(\mathbf{x}|\mathbf{x}^{\circ}, \mathbf{x}^{\circ\circ})$ and $\alpha^{**}(\mathbf{x}|\mathbf{x}^{\circ}, \mathbf{x}^{\circ\circ})$ are defined in the same way. Note that the locations $\mathbf{x}^{\circ}$ and $\mathbf{x}^{\circ\circ}$ of the fixed inclusions are commutable. Incidentally, an extra convention will be adopted: the arguments which are understood in the shortened notations $\mathbf{v}^{\circ\circ **}$, $p^{\circ\circ **}$ and $\alpha^{\circ\circ **}$ are $(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ}, \mathbf{x})$.

It should be recalled that for the theory being developed here, the B.B.G.K.Y. and Lundgren hierarchies were set up only up to the second and third orders respectively (cf. Part I) because the equations rapidly become unmanageable. As a consequence, a description of the dispersed phase does not imply second- (or higher-) order disturbance fields. However, it may be useful to introduce the triple number density $\phi^{(3)}(\mathbf{x}, \mathbf{x}^{\circ}, \mathbf{x}^{\circ\circ})$, the definition of which follows (see Sec. 3.2 – Part I):

$$\phi^{(3)}(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) = N(N-1)(N-2)\mathrm{E}[\varphi_1\varphi_2\varphi_3] = N(N-1)(N-2)\phi_3(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}), \tag{1.4}$$

As conditional disturbance densities are being used, it is necessary to adopt:

$$\chi^{**}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) = \chi_3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) - \chi_2(\mathbf{x}|\mathbf{x}^\circ) - \chi_2(\mathbf{x}|\mathbf{x}^{\circ\circ}) + \phi_1(\mathbf{x}) \tag{1.5}$$

where a second conditional density $\chi_3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) = \phi_3(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})/\phi_2(\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})$ has been introduced. Likewise, $\mathbf{x}^\circ$ and $\mathbf{x}^{\circ\circ}$ being given, it may be observed that $\chi^{**} \to 0$ as $\mathbf{x} \to \infty$. As before, the arguments understood in the shortened notation $\chi^{\circ\circ**}$ are $(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ,\mathbf{x})$. Such a definition is not equivalent to the triple correlation fields of the S.M.

The continuous phase description does not imply either third- (or higher-) order disturbance fields but for the sake of completeness we will give its definition

$$\begin{aligned}\mathbf{v}^{***}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ}) = {} & \overline{\mathbf{x}}^{c4}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{v}}^{c3}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) \\ & -\overline{\mathbf{v}}^{c3}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{v}}^{c3}(\mathbf{x}|\mathbf{x}^{\circ\circ},\mathbf{x}^{\circ\circ\circ}) \\ & +\overline{\mathbf{v}}^{c2}(\mathbf{x}|\mathbf{x}^\circ) + \overline{\mathbf{v}}^{c2}(\mathbf{x}|\mathbf{x}^{\circ\circ}) + \overline{\mathbf{v}}^{c2}(\mathbf{x}|\mathbf{x}^{\circ\circ\circ}) - \overline{\mathbf{v}}^{c1}(\mathbf{x}).\end{aligned} \tag{1.6}$$

The first equations of the Lundgren hierarchy (continuous phase) and of the B.B.G.K.Y. hierarchy (dispersed phase) as well as corresponding boundary conditions in terms of these new disturbance fields, i.e. in terms of α^{d1}, $\overline{\mathbf{v}}^{c1}$, $\overline{p}^{c1}$, α^*, $\mathbf{v}^*$, p^*, α^{**}, $\mathbf{v}^{**}$, p^{**}, $\overline{\mathbf{u}}^1$, $\mathbf{u}^*$, $\overline{\boldsymbol{\omega}}^1$, $\boldsymbol{\omega}^*$, ϕ_1, χ^*, ... will be rewritten in Sec. 4. First, two preliminary studies have to be conducted to split the interaction terms between phases in both hierarchies. Any kind of interfacial field has indeed to be broken down properly at each order, yielding (i) disturbance forces and torques acting on an inclusion in the dispersed-phase equations (Sec. 2) and (ii) in the continous phase equations, disturbance extra-deformation tensors and disturbance interfacial force densities (Sec. 3). An extra advantage of this break-down is actually to cancel several terms which appear at the r.h.s. of the momentum equations presented in Part I. In this way true interaction terms between phases can be displayed in a similar way where such terms are usually introduced in classical two-fluid modelling. Note that all pseudo-turbulent tensors require a similar treatment. Breaking down pseudo-turbulent tensors happens to be much less straightforward; the entire paper (Part III) is devoted to this project.

Next, the resulting "extra-deformation tensors" and "disturbance interfacial density forces" that appear in the continuous phase momentum equations receive more appropriate expressions, which are presented in Sec. 5. For instance, interfacial density forces are expanded in multipoles and in this way they can be related to the corresponding terms in the dispersed-phase equations. Finally some key results are recalled in Sec. 6.

2. Interaction terms in dispersed-phase equations

Interaction terms between phases which appear in the momentum equations obtained at the end of Part I are considered first. Beforehand, it should be pointed out that standard variables (see 3.8 and 3.9, Part I) appear naturally in these interaction terms. As they are simply considered as temporary variables in this model, they have to be expressed in terms of the moments derived from kinetic equations.

2.1. Standard dispersed-phase averaged variables

At the first-order, using (2.6), (3.3) and (3.8) of Part I, the following relations are obtained:

$$\alpha^{d1}(\mathbf{x}) = \sum_{j=1}^{N} E[H(a - |\mathbf{x} - \mathbf{x}_j|)] = NE[H(a - |\mathbf{x} - \mathbf{x}_1|)] \tag{2.1}$$

$$= NE\left[\int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\leq a} \delta(\tilde{\mathbf{x}} - \mathbf{x}_1) d\tilde{\mathbf{x}}\right] = N \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\leq a} \phi_1(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} = \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\leq a} \phi^{(1)} d\tilde{\mathbf{x}}.$$

Without going into detail, the following are obtained at the next orders

$$\begin{aligned} \alpha^{d2}(\mathbf{x}^{\circ}|\mathbf{x}) &= (N-1) \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ}|\leq a} \chi_2(\tilde{\mathbf{x}}|\mathbf{x}) d\tilde{\mathbf{x}} \\ \alpha^{d3}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ}, \mathbf{x}) &= (N-2) \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^{\circ\circ}|\leq a} \chi_3(\tilde{\mathbf{x}}|\mathbf{x}^{\circ}, \mathbf{x}) d\tilde{\mathbf{x}}. \end{aligned} \tag{2.2}$$

As for the dispersed-phase volume fractions, the standard velocities for the dispersed phase $\overline{\mathbf{v}}^{d1}$, $\overline{\mathbf{v}}^{d2}$ and $\overline{\mathbf{v}}^{d3}$ can be related to the dispersed-phase moments. Using (2.7), (3.4) and (3.8) of Part I, the first-order velocity is found to be:

$$\alpha^{d1}(\mathbf{x})\overline{\mathbf{v}}^{d1}(\mathbf{x}) = NE[H(a - |\mathbf{x} - \mathbf{x}_1|)[\mathbf{u}_1 + \boldsymbol{\omega}_1 \wedge (\mathbf{x} - \tilde{\mathbf{x}})] \tag{2.3}$$

$$= \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\leq a} \left[\overline{\mathbf{u}}^1(\tilde{\mathbf{x}}) + \overline{\boldsymbol{\omega}}^1(\tilde{\mathbf{x}}) \wedge (\mathbf{x} - \tilde{\mathbf{x}})\right] \phi^{(1)}(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}}.$$

Likewise, based on (3.9), the second-order velocity becomes:

$$\alpha^{d2}(\mathbf{x}^\circ|\mathbf{x})\overline{\mathbf{v}}^{d2}(\mathbf{x}^\circ|\mathbf{x}) = (N-1) \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^\circ|\leq a} [\overline{\mathbf{u}}^2(\tilde{\mathbf{x}}|\mathbf{x}) + \overline{\boldsymbol{\varpi}}^2(\tilde{\mathbf{x}}|\mathbf{x}) \wedge (\mathbf{x}^\circ - \tilde{\mathbf{x}})]\chi_2(\tilde{\mathbf{x}}|\mathbf{x})d\tilde{\mathbf{x}}. \tag{2.4}$$

A similar expression can be derived at the third order. Integrals in the first-order Eqs. (2.1) and (2.3) can be approximated by expanding the typical fields $\mathbf{f}$ $(\phi^{(1)}, \overline{\mathbf{u}}^1, \overline{\boldsymbol{\varpi}}^1, ...)$ appearing in their integrands. Suppose that the averaged fields for both phases have L as a macroscopic length scale, which is much larger than a, the inclusion radius. This allows the preceding integrals to be expanded in terms of $\beta = a/L$. If $\mathbf{f}$ is expanded in a Taylor series around the centre $\mathbf{x}$ of the test inclusion, it may be written symbolically as (RAYLEIGH [14]) if $\mathbf{k} = \tilde{\mathbf{x}} - \mathbf{x}$:

$$\mathbf{f}(\tilde{\mathbf{x}}) = 1/\mathrm{m}! \sum_{\mathrm{m}=0}^{\infty} \mathbf{k}^{\mathrm{m}} \boxed{\mathrm{m}} \frac{\partial^{\mathrm{m}}}{\partial \mathbf{x}^{\mathrm{m}}} [\mathbf{f}(\mathbf{x})] = \exp\left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{x}}\right) \mathbf{f}(\mathbf{x}), \tag{2.5}$$

where the m folded tensorial product of $\partial/\partial\mathbf{x}(\mathbf{k})$ is denoted by $\partial^{\mathrm{m}}/\partial\mathbf{x}^{\mathrm{m}}$ $(\mathbf{k}^{\mathrm{m}})$, while the symbol $\boxed{\mathrm{m}}$ indicates a full p-fold contraction between the tensors $\mathbf{k}^{\mathrm{m}}$ and $\partial^{\mathrm{m}}/\partial\mathbf{x}^{\mathrm{m}}$. Inserting (2.5) into (2.1) and (2.3) gives rise to two types of series:

$$\begin{aligned} &\int\limits_{|\mathbf{k}|<a} \mathbf{f}(\mathbf{x}+\mathbf{k})d\mathbf{k} = \left[\int\limits_{|\mathbf{k}|<a} \exp\left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{x}}\right) d\mathbf{k}\right] \mathbf{f}(\mathbf{x}), \\ &\int\limits_{|\mathbf{k}|<a} \mathbf{k} \wedge \mathbf{f}(\mathbf{x}+\mathbf{k})d\mathbf{k} = \int\limits_{|\mathbf{k}|<a} \varepsilon : \mathbf{k}\,\mathbf{f}(\mathbf{x}+\mathbf{k})d\mathbf{k} \\ &\qquad = \varepsilon : \left[\int\limits_{|\mathbf{k}|<a} \mathbf{k} \exp\left(\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{x}}\right) d\mathbf{k}\right] \mathbf{f}(\mathbf{x}), \end{aligned} \tag{2.6}$$

where $\partial/\partial\mathbf{x}$ is to be treated as a constant vector during the integration. Each type involves integrals which can be evaluated via a technique suggested by RAYLEIGH [14]) and GATIGNOL [7]). Then, by properly scaling lengths and velocities without changing their notations, we obtain:

$$\alpha^{d1} = \int\limits_{|\mathbf{k}|<a} \phi^{(1)}(\mathbf{x}+\mathbf{k})d\mathbf{k} = 4/3\pi a^3 \left[\phi^{(1)} + \beta^2\nabla^2\phi^{(1)}/10 + O(\beta^4)\right], \tag{2.7}$$

and following LHUILLIER [10], who assumed that if no external couples are applied to inclusions, the order of magnitude of $\overline{\boldsymbol{\varpi}}^1$ is the velocity scale over a:

$$\alpha^{d1}\overline{\mathbf{v}}^{d1} = 4/3\pi a^3[\phi^{(1)}\overline{\mathbf{u}}^1 + \beta\mathbf{curl}(\phi^{(1)}\overline{\boldsymbol{\varpi}}^1)/5 + \beta^2\nabla^2(\phi^{(1)}\overline{\mathbf{u}}^1)/10 + O(\beta^3)], \tag{2.8}$$

where $\Delta^{\mathrm{m}} = \nabla^{2\mathrm{m}}$ denote m successive applications of the Laplace operator. Equation (2.7) has already been found by BUYEVICH and SHCHELCHKOVA [3] and an equation similar to (2.8) can be found in FELDERHOF [6]. As a matter of fact, the future paper will give a more refined scaling of various velocities: in particular, a specific scale has to be introduced for the relative velocity.

Likewise higher-order dispersed-phase volume fractions and velocities (2, 3, etc.) can be expanded and give rise to the same (two) types of series involving integrals which are computed using the same technique. However, it is impossible to arrange expansion terms with respect to β^2 since the space scale of the dependence of various higher-order conditioned variables equals a and not L. Instead, it may be useful to separate each type of series into two parts: a leading term and a general operator expressions for the remainder of the series:

$$\begin{aligned} \mathbf{R}_1 &= \left[\frac{a^2}{10}\Delta + \frac{a^4}{280}\Delta^2 + \ldots + \frac{3(2n+2)}{(2n+3)!}a^{2n}\Delta^n + \ldots\right], \\ \mathbf{R}_2 &= \left[\frac{a^2}{14}\Delta + \frac{a^4}{504}\Delta^2 + \ldots + \frac{15(2n+2)(2n+4)}{(2n+5)!}a^{2n}\Delta^n + \ldots\right]\mathbf{curl}. \end{aligned} \tag{2.9}$$

These remainders terminate after only a very few terms since the coefficients decrease rapidly.

Then the dispersed-phase volume fractions and the velocities assume the compact forms:

$$\begin{aligned} \phi^{(1)}(\mathbf{x}^\circ)\alpha^{d2}(\mathbf{x}|\mathbf{x}^\circ) &= 4\pi/3a^3[\phi^{(2)}(\mathbf{x},\mathbf{x}^\circ) + \mathbf{R}_1(\phi^{(2)})], \\ \phi^{(2)}(\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\alpha^{d3}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) &= 4\pi/3a^3[\phi^{(3)}(\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) + \mathbf{R}_1(\phi^{(3)})], \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\phi^{(1)}(\mathbf{x}^\circ)\alpha^{d2}(\mathbf{x}|\mathbf{x}^\circ)\overline{\mathbf{v}}^{d2}(\mathbf{x}|\mathbf{x}^\circ) = 4\pi/3a^3[\phi^{(2)}\overline{\mathbf{u}}^2(\mathbf{x}|\mathbf{x}^\circ) \\ &\qquad + a^2/5\,\mathbf{curl}\,(\phi^{(2)}\overline{\boldsymbol{\varpi}}^2)(\mathbf{x}|\mathbf{x}^\circ) + \mathbf{R}_1(\phi^{(2)}\overline{\mathbf{u}}^2) + a^2/5\mathbf{R}_2(\phi^{(2)}\overline{\boldsymbol{\varpi}}^2)], \\ &\phi^{(2)}(\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\alpha^{d3}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ})\overline{\mathbf{v}}^{d3}(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) = 4\pi/3a^3[\phi^{(3)}\overline{\mathbf{u}}^3(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) \\ &\qquad + a^2/5\,\mathbf{curl}\,(\phi^{(3)}\overline{\boldsymbol{\varpi}}^3)(\mathbf{x}|\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}) + \mathbf{R}_1(\phi^{(3)}\overline{\mathbf{u}}^3) + a^2/5\mathbf{R}_2(\phi^{(3)}\overline{\boldsymbol{\varpi}}^3)]. \end{aligned} \tag{2.11}$$

Integral relations expressing standard dispersed-phase variables in terms of our variables at any order but the first may easily be restated as equations controlling disturbance fields. For instance:

(2.12)
$$\alpha^*(\mathbf{x}^\circ|\mathbf{x}) = (N-1) \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^\circ|\le a} \chi^*(\tilde{\mathbf{x}}|\mathbf{x})d\tilde{\mathbf{x}} + O(1/N),$$
$$\alpha^{**}(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ,\mathbf{x}) = (N-2) \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}^\circ|\le a} \chi^{**}(\tilde{\mathbf{x}}|\mathbf{x}^\circ,\mathbf{x})d\tilde{\mathbf{x}} + O(1/N).$$

Moreover, transformation of all these relations involving volume integrals can be simplified into relations similar to (2.6), (2.7), (2.9) and (2.10). For instance:

(2.13)
$$\alpha^*(\mathbf{x}|\mathbf{x}^\circ) = (4\pi a^3/3)(N-1)\{\chi^*(\mathbf{x}|\mathbf{x}^\circ) + \mathbf{R}_1[\chi^*(\mathbf{x}|\mathbf{x}^\circ)]\} + O(1/N).$$

2.2. Disturbance-averaged interfacial forces and torques

The overall force $\overline{\mathbf{F}}^1(\mathbf{x})$ experienced at time t by an inclusion known to be centred at $\mathbf{x}$ given in ((4.14) and (5.11) – Part I) may be broken down according to (1.1):

(2.14)
$$\overline{\mathbf{F}}^1 = a^2 \int\limits_{S(\mathbf{x})} \mathbf{n}\cdot\overline{X^c\mathbb{T}^c}^1 d\Omega + a^2 \int\limits_{S(\mathbf{x})} [-\mathbf{n}p^* + 2\mu^c\mathbf{n}\cdot\mathbb{D}(\mathbf{v}^*)](\mathbf{x}+a\mathbf{n}|\mathbf{x})d\Omega,$$

where the last term in the r.h.s is the force exerted by the disturbance flows on the test inclusion. This involves the viscous drag force and other forces such as the added mass force; it will be denoted by $\mathbf{F}^*(\mathbf{x})$ (for short $\mathbf{F}^*$, or $\mathbf{F}^{\circ *}$ if the inclusion is centred at $\mathbf{x}^\circ$). Likewise, using (1.3), Eqs. (4.20) – (5.12) from Part I, giving $\overline{\mathbf{F}}^2(\mathbf{x}^\circ|\mathbf{x}) = \overline{\mathbf{F}^\circ}^2$ yields:

(2.15)
$$\overline{\mathbf{F}^\circ}^2 = \mathbf{F}^{\circ *} + a^2 \int\limits_{S(\mathbf{x}^\circ)} \mathbf{n}\cdot\overline{X^c\mathbb{T}^c}^2(\mathbf{x}^\circ+a\mathbf{n}|\mathbf{x})d\Omega$$
$$+a^2 \int\limits_{S(\mathbf{x}^\circ)} [-\mathbf{n}p^{**} + 2\mu^c\mathbf{n}\cdot\mathbb{D}(\mathbf{v}^{**})](\mathbf{x}^\circ+a\mathbf{n}|\mathbf{x},\mathbf{x}^\circ)d\Omega,$$

where the last term in the r.h.s is the force exerted by the disturbance flow on the test inclusion at $\mathbf{x}^\circ$ conditionally averaged upon the presence of another inclusion at $\mathbf{x}$. It will be denoted by $\mathbf{F}^{**}(\mathbf{x}^\circ|\mathbf{x})$ (for short $\mathbf{F}^{\circ **}$).

Overall torques ((4.15) – Part I) and ((4.21) – Part I) may similarly be broken down:

(2.16)
$$\overline{\mathbf{K}}^1 = a^3\varepsilon : \int\limits_{S(\mathbf{x})} \mathbf{nn}\cdot\overline{X^c\mathbb{T}^c}^1 d\Omega + a^3\varepsilon : \int\limits_{S(\mathbf{x})} [-\mathbf{nn}p^*$$
$$+2\mu^c\mathbf{nn}\cdot\mathbb{D}(\mathbf{v}^*)](\mathbf{x}+a\mathbf{n}|\mathbf{x})d\Omega$$

and

$$\overline{\mathbf{K}^{\circ}}^{2} = \mathbf{K}^{\circ *} + a^3\varepsilon \int\limits_{S(\mathbf{x}^\circ)} \mathbf{nn}\cdot\overline{X^c\mathbb{T}^c}^2 d\Omega + a^3\varepsilon : \int\limits_{S(\mathbf{x}^\circ)} [-\mathbf{nn}\,p^{**} + 2\mu^c\mathbf{nn}\cdot\mathbb{D}(\mathbf{v}^{**})](\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x},\mathbf{x}^\circ)d\Omega, \tag{2.17}$$

where the last terms at the r.h.s are the torques exerted by the disturbance flows on the test inclusions. They will be denoted by $\mathbf{K}^*(\mathbf{x})$ and $\mathbf{K}^{**}(\mathbf{x}^\circ|\mathbf{x})$ (for short $\mathbf{K}^*$ and $\mathbf{K}^{\circ **}$ respectively).

The first-order averaged fields $\overline{X^c\mathbb{T}^c}^1(\mathbf{x}^\circ)$ in (2.14) and (2.16) are defined inside the entire sphere $S(\mathbf{x})$, and the Gauss theorem can be used to transform the corresponding surface integrals into volume integrals:

$$a^2 \int\limits_{S(\mathbf{x})} \mathbf{n}\cdot\overline{X^c\mathbb{T}^c}^1 d\Omega = \int\limits_{|\mathbf{k}|<a} \frac{\partial}{\partial\mathbf{k}}\cdot\left[-\overline{p}^{c1}\mathbb{I} + 2\mu^c\mathbb{D}(\overline{\mathbf{v}}^{c1})\right](\mathbf{x}+\mathbf{k})d\mathbf{k}, \tag{2.18}$$

$$a^3\varepsilon : \int\limits_{S(\mathbf{x})} \mathbf{nn}\cdot\overline{X^c\mathbb{T}^c}^1 d\Omega = \varepsilon : \int\limits_{|\mathbf{k}|<a} \mathbf{k}\left[-\frac{\partial\overline{p}^{c1}}{\partial\mathbf{k}} + \mu^c\Delta(\overline{\mathbf{v}}^{c1}) + \mu^c\frac{\partial}{\partial\mathbf{k}}\left(\frac{\partial}{\partial\mathbf{k}}\cdot\overline{\mathbf{v}}^{c1}\right)\right]d\mathbf{k}. \tag{2.19}$$

The same transformation between the surface and the volume integrals holds for integrals involving the second-order averaged fields $\overline{X^c\mathbb{T}^c}^2(\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x})$ in (2.15) and (2.17).

Introducing the expansion (2.6) of $\overline{X^c\mathbb{T}^c}^1$ about $\mathbf{x}$ and using the remainders of (2.9) yields:

$$a^2 \int\limits_{S(\mathbf{x})} \mathbf{n}\cdot\overline{X^c\mathbb{T}^c}^1 d\Omega = \frac{4\pi a^3}{3}\left\{-\frac{\partial\overline{p}^{c1}}{\partial\mathbf{x}} + \mu^c\Delta\overline{\mathbf{v}}^{c1} + \mu^c\frac{\partial}{\partial\mathbf{x}}\left(\frac{\partial}{\partial\mathbf{x}}\cdot\overline{\mathbf{v}}^{c1}\right) + \mathbf{R}_1\left[-\frac{\partial\overline{p}^{c1}}{\partial\mathbf{x}} + \mu^c\Delta\overline{\mathbf{v}}^{c1} + \mu^c\frac{\partial}{\partial\mathbf{x}}\left(\frac{\partial}{\partial\mathbf{x}}\cdot\overline{\mathbf{v}}^{c1}\right)\right]\right\} \tag{2.20}$$

and

$$a^3\varepsilon : \int\limits_{S(\mathbf{x})} \mathbf{nn}\cdot\overline{X^c\mathbb{T}^c}^1 d\Omega = 4\pi a^5\mu^c/15\{\mathbf{curl}\,(\Delta\overline{\mathbf{v}}^{c1}) + \mathbf{R}_2\,[\mathbf{curl}\,(\Delta\overline{\mathbf{v}}^{c1})]\} = -4\pi a^5\mu^c/15\{\mathbf{curl}^3(\overline{\mathbf{v}}^{c1}) + \mathbf{R}_2\,[\mathbf{curl}^3(\overline{\mathbf{v}}^{c1})]\}. \tag{2.21}$$

In (2.20) and (2.21), it is possible to arrange terms with respect to β as in (2.7) and (2.8) for it is clear that $\mathbf{R}_1$ and $\mathbf{R}_2$ are $O(\beta^2)$. On the other hand, $\overline{X^c \mathbb{T}^c}^2$ can be expanded about $\mathbf{x}$ and similar results are obtained where $\overline{p}^{c2}$ and $\overline{\mathbf{v}}^{c2}$ replaces $\overline{p}^{c1}$ and $\overline{\mathbf{v}}^{c1}$. However, $\mathbf{R}_1$ and $\mathbf{R}_2$ are not $O(\beta^2)$ when $\overline{p}^{c2}$ and $\overline{\mathbf{v}}^{c2}$ are involved, since they are not slowly varying.

3. Interaction terms in continuous phase equations

Both extra-deformation tensors and interfacial force densities have to be transformed according to the same procedure before being broken down.

3.1. Disturbance-averaged extra-deformation tensors

The averaged extra-deformation tensor for the field with no fixed inclusion is obtained by introducing the surface Dirac g.f. (Sec. 2 – Part I) and the generalised p.d.f. $\mathrm{f}_N(Z_N;t)$ (Sec. 3 – Part I) which allows averaging over Γ at any time:

$$(3.1)\qquad E[\mathbb{F}^c\delta_\Sigma](\mathbf{x}) = \sum_{j=1}^{N} E\{[\mathbf{n}_j^c\mathbf{v}_j^c]^s\delta(P_j)\} = \sum_{j=1}^{N}\int [\mathbf{n}_j^c\mathbf{v}_j^c]^s\delta(P_j)\mathrm{f}_N(Z_N)dZ_N$$

$$= N\int_{\mathcal{V}_x^d}\delta(a-|\mathbf{x}-\mathbf{x}_1|)\left[\int\{[\mathbf{n}_1^c X_1^c\mathbf{v}_1^c]^s\}(Z_N;\mathbf{x})\mathrm{f}_N(Z_N)d\xi_1 d\mathbf{z}_2...d\mathbf{z}_N\right]d\mathbf{x}_1$$

$$= N\int_{\mathcal{V}_x^d}\delta(a-r_1)\left[\left[\int\{[\mathbf{n}_1^c X_1^c\mathbf{v}_1^c]^s\}(Z_N;\mathbf{x})\mathrm{f}_N(\mathbf{x}_1,\xi_1,Z_{N-1})d\xi_1 d\mathbf{z}_2...d\mathbf{z}_N\right]\right.$$

$$\left.\times r_1^2 d\Omega\right]dr_1 = a^2 N\int_{S(\mathbf{x})}\left[\int\{[\mathbf{n}_1^c X_1^c\mathbf{v}_1^c]^s\}\,(Z_N;\mathbf{x})\mathrm{f}_N(\mathbf{x}_1,\xi_1,Z_{N-1})\right.$$

$$\left. d\xi_1 d\mathbf{z}_2...d\mathbf{z}_N\right]d\Omega.$$

The second line results from the symmetry of f_N with respect to $\mathbf{z}_1$ and $\mathbf{z}_j$ and the third line is a simple change of integration variables: $d\mathbf{x}_1 = r_1^2 d\Omega dr_1$ where $r_1(\mathbf{x}) = |\mathbf{x}-\mathbf{x}_1|$; finally, the volume integral can be converted into a surface integral (fourth line) taken over the surface of the sphere $S(\mathbf{x})$ i.e. such that $r_1(\mathbf{x}) = a$ or $\mathbf{x}_1 = \mathbf{x} + a\mathbf{n}_1^c$. By introducing the conditional velocity $\overline{\mathbf{v}}^{c2}$ ((3.7) – Part I), one obtains:

$$E[\mathbb{F}^c\delta_\Sigma](\mathbf{x}) = a^2N\int\limits_{S(\mathbf{x})}\left[\int\{[\mathbf{n}_1^c\delta(\mathbf{x}_1-\tilde{\mathbf{x}})X_1^c\mathbf{v}_1^c]^s\}\right.$$
$$\left.(\tilde{\mathbf{x}},\xi_1,Z_{N-1};\mathbf{x})\mathrm{f}_N(\tilde{\mathbf{x}},\xi_1,Z_{N-1})d\tilde{\mathbf{x}}d\xi_1 d\mathbf{z}_2...d\mathbf{z}_N\right]d\Omega \tag{3.2}$$
$$= a^2N\int\limits_{S(\mathbf{x})}E[\{X^c\delta(\mathbf{x}_1-\tilde{\mathbf{x}})[\mathbf{n}_1^c\mathbf{v}_1^c]^s\}]d\Omega = a^2\int\limits_{S(\mathbf{x})}[\mathbf{n}^c\overline{\mathbf{v}}^{c2}]^s(\mathbf{x}|\mathbf{x}_1)\phi^{(1)}(\mathbf{x}_1)d\Omega.$$

Now, $E[\mathbb{F}^c\delta_\Sigma]$ can be broken down by splitting $\overline{\mathbf{v}}^{c2}$ according to (1.1):

$$E[\mathbb{F}^c\delta_\Sigma](\mathbf{x}) = a^2\int\limits_{S(\mathbf{x})}[\mathbf{n}^c(\mathbf{x}|\mathbf{x}_1)\overline{\mathbf{v}}^{c1}(\mathbf{x})]^s\phi^{(1)}(\mathbf{x}_1)d\Omega$$
$$+a_2\int\limits_{S(\mathbf{x})}[\mathbf{n}^c\mathbf{v}^*]^s(\mathbf{x}|\mathbf{x}_1)\phi^{(1)}(\mathbf{x}_1)d\Omega, \tag{3.3}$$

where the second r.h.s. term is admittedly the disturbance extra-deformation tensor $E[\mathbb{F}^*\delta_\Sigma]$. In relation to fields with no fixed inclusions, this tensor appears as an average, weighted by $\phi^{(1)}$, over all inclusion centre positions, $\mathbf{x}_1 = \mathbf{x} + a\mathbf{n}^c$ these positions being such that the inclusion surface touches $\mathbf{x}$. The first term at the r.h.s. is transformed using the Gauss theorem and the relation (2.1) giving the dispersed-phase volume fraction:

$$a^2\int\limits_{S(\mathbf{x})}[\mathbf{n}^c\overline{\mathbf{v}}^{c1}]^s\phi^{(1)}(\mathbf{x}+a\mathbf{n}^c)d\Omega = a^2\left[\overline{\mathbf{v}}^{c1}\int\limits_{S(\mathbf{x})}\mathbf{n}^c\phi^{(1)}(\mathbf{x}+a\mathbf{n}^c)d\Omega\right]^s \tag{3.4}$$
$$= \left[\overline{\mathbf{v}}^{c1}\int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\le a}\frac{\partial}{\partial\tilde{\mathbf{x}}}\phi^{(1)}(\tilde{\mathbf{x}})d\tilde{\mathbf{x}}\right]^s = \left[\overline{\mathbf{v}}^{c1}\int\limits_{|\mathbf{k}|\le a}\frac{\partial}{\partial\mathbf{x}}\phi^{(1)}(\mathbf{x}+\mathbf{k})d\mathbf{k}\right]^s$$
$$= \left[\overline{\mathbf{v}}^{c1}\frac{\partial}{\partial\mathbf{x}}\alpha^{d1}\right]^s.$$

One advantage of this break-down is that one of the $O(\alpha^{d1})$ terms appearing at the r.h.s. of the momentum Eq. (5.18) derived at the end of Part I is cancelled.

Break-downs similar to (3.4) can be introduced in the next-order momentum Eqs. (5.19) and (5.20) of Part I, i.e.

$$E[\mathbb{F}^c\varphi_1\delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x}) = \phi_1(\mathbf{x})\left\{\left[\overline{\mathbf{v}}^{c2}\frac{\partial}{\partial\mathbf{x}^\circ}\alpha^{d2}\right]^s(\mathbf{x}^\circ|\mathbf{x}) + E[\mathbb{F}^{**}\delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x})\right\} \tag{3.5}$$

and:

$$E[\mathbb{F}^c\varphi_1\varphi_2\delta_\Sigma^{1,2}](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ,\mathbf{x}) = \phi_2(\mathbf{x}^\circ,\mathbf{x})\left\{\left[\overline{\mathbf{v}}^{c3}\frac{\partial}{\partial\mathbf{x}^{\circ\circ}}\alpha^{d3}\right]^s(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ,\mathbf{x}) + E[\mathbb{F}^{***}\delta_\Sigma^{1,2}](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ,\mathbf{x})\right\}, \tag{3.6}$$

where $E[\mathbb{F}^{**}\delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x})$ and $E[\mathbb{F}^{***}\delta_\Sigma^{1,2}](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ,\mathbf{x})$ are the higher-order disturbance-averaged extra-deformation tensors. For sake of brevity, only the first one will be given:

$$E[\mathbb{F}^{**}\delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x}) = a^2(N-1)\int\limits_{S(\mathbf{x}^\circ)}\{[\mathbf{n}^c\mathbf{v}^{**}]^s(\mathbf{x}^\circ|\mathbf{x}^\circ+a\mathbf{n}^c,\mathbf{x}) + [\mathbf{n}^c\mathbf{v}^*]^s(\mathbf{x}^\circ|\mathbf{x}^\circ+a\mathbf{n}^c)\}\{\phi_1(\mathbf{x}^\circ+a\mathbf{n}^c)+\chi^*(\mathbf{x}^\circ+a\mathbf{n}^c|\mathbf{x})\}d\Omega. \tag{3.7}$$

All these break-downs result in cancellations in the second and third-order momentum equations at the r.h.s.

3.2. Disturbance interfacial force density

The same arguments as those leading to (3.1) and (3.2) provide a new form of the averaged interfacial force density for the field with no fixed inclusions:

$$\begin{aligned} E[\mathbf{n}^c\cdot\mathbb{T}^c\delta_\Sigma](\mathbf{x}) &= \sum_{j=1}^N E[\mathbf{n}_j^c\cdot\mathbb{T}_j^c\delta(P_j)] = \sum_{j=1}^N\int\mathbf{n}_j^c\cdot\mathbb{T}_j^c\delta(P_j)\mathrm{f}_N(Z_N)dZ_N \\ &= a^2N\int\limits_{S(\mathbf{x})}\left[\int\{\mathbf{n}_1^c\cdot X_1^c\mathbb{T}_1^c\}(Z_N;\mathbf{x})\mathrm{f}_N(\mathbf{x}_1,\xi_1,Z_{N-1})d\xi_1 d\mathbf{z}_2...d\mathbf{z}_N\right]d\Omega \\ &= a^2\int\limits_{S(\mathbf{x})}\mathbf{n}^c\cdot\overline{X_1^c\mathbb{T}^c}^2(\mathbf{x}|\mathbf{x}_1)\phi^{(1)}(\mathbf{x}_1)d\Omega, \end{aligned} \tag{3.8}$$

where the simplified version of the local averaged stress upon an inclusion located at $\mathbf{x}_1$, $\overline{X_1^c\mathbb{T}^c}^2(\mathbf{x}|\mathbf{x}_1)$, ((5.11) – Part I) has been retained. Breaking down the conditional velocity and pressure $\overline{\mathbf{v}}^{c2}$ and $\overline{p}^{c2}$ appearing in it according to (1.1) gives:

$$E[\mathbf{n}^c\cdot\mathbb{T}^c\delta_\Sigma](\mathbf{x}) = a^2\int\limits_{S(\mathbf{x})}\mathbf{n}^c(\mathbf{x}|\mathbf{x}_1)\cdot[-\overline{p}^{c1}\mathbb{I}+2\mu^c\mathbb{D}(\overline{\mathbf{v}}^{c1})](\mathbf{x})\phi^{(1)}(\mathbf{x}_1)d\Omega + E[\mathbf{n}^c\cdot\mathbb{T}^*\delta_\Sigma], \tag{3.9}$$

where the disturbance interfacial force density $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma]$ is defined by:

$$(3.10) \qquad E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma] = a^2 \int_{S(\mathbf{x})} \mathbf{n}^c \cdot [-p^* \mathbb{I} + 2\mu^c \mathbb{D}(\mathbf{v}^*)](\mathbf{x}|\mathbf{x}_1)\phi^{(1)}(\mathbf{x}_1) d\Omega.$$

This term appears as an average of the local averaged stress weighted by $\phi^{(1)}$, over all inclusion centre positions, $\mathbf{x}_1 = \mathbf{x} + a\mathbf{n}^c$, these positions being such that the inclusion surfaces touch $\mathbf{x}$. Such a basic result was obtained previously by LUNDGREN [11]. The first term at the r.h.s. is transformed using the Gauss theorem and relation (2.1), which provides the dispersed-phase volume fraction:

$$(3.11) \qquad a^2 \int_{S(\mathbf{x})} [-\overline{p^c}^1 \mathbb{I} + 2\mu^c \mathbb{D}(\overline{\mathbf{v}^c}^1)] \cdot \mathbf{n}^c \phi^{(1)}(\mathbf{x}_1) d\Omega] = a^2 [-\overline{p^c}^1 \mathbb{I} + 2\mu^c \mathbb{D}(\overline{\mathbf{v}^c}^1)]$$

$$\cdot \int_{S(\mathbf{x})} \mathbf{n}^c \phi^{(1)}(\mathbf{x} + a\mathbf{n}^c) d\Omega] = [-\overline{p^c}^1 \mathbb{I} + 2\mu^c \mathbb{D}(\overline{\mathbf{v}^c}^1)]$$

$$\cdot \int_{|\tilde{\mathbf{x}} - \mathbf{x}| \le a} \mathbf{n}^c \frac{\partial}{\partial \tilde{\mathbf{x}}} \phi^{(1)}(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} = [-\overline{p^c}^1 \mathbb{I} + 2\mu^c \mathbb{D}(\overline{\mathbf{v}^c}^1)] \cdot \frac{\partial}{\partial \mathbf{x}} \alpha^{d1}(\mathbf{x}).$$

Our break-down exactly parallels the procedure followed in deterministic studies concerning hydrodynamic forces acting on a particle in arbitrary fields of flow. It is clearly $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma]$ and not $E[\mathbf{n}^c \cdot \mathbb{T} \delta_\Sigma]$ which contains drag and lift forces as well as virtual mass effects. This break-down is also reminiscent of that introduced in classical two-fluid modelling (ISHII, [8]), where only the first-order equations are available; specific mean values, e.g. interfacial velocity and pressure, are therefore introduced to break down the interaction terms. In our approach, a different break-down, which can be made at any order, is based on averages that are conditional upon the presence of one or more inclusions.

This break-down of the overall interfacial force density has one advantage, which is exactly the same as that mentioned previously in breaking down the overall extra-deformation tensor: the remaining $O(\alpha^{d1})$ term appearing at the r.h.s. of the momentum Eq. ((5.18) – Part I) is cancelled. A second point is worth mentioning, namely that the disturbance interfacial force density $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma]$ obtained here is not equal to the standard interfacial force density $\mathbb{M}_c^d$ (d in Ishii's book refers to "drag force" and c to "continuous phase"), for two reasons. As aforesaid, the reference fields in standard two-fluid modelling are interfacial averages and not averaged fields with one fixed inclusion; moreover in standard two-fluid modelling, there is an asymmetrical treatment of the interfacial pressure term and of the interfacial viscous stress term, which cannot be explained very well. Note that in our approach they are logically placed on an equal footing. These odd characteristics of $\mathbb{M}_c^d$ which make it different from $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma]$ have never prevented modellers from claiming that it must also contain drag and lift forces as well as virtual mass effects. Arbitrariness is usually so great in closing various unknown terms in a given application, that such statements can hardly be invalidated. Finally, the classical process somewhat artificially increases closure problems since, unlike our approach, it requires the difference between averaged interfacial pressure and bulk-averaged pressure to be expressed (STUHMILLER, [15]).

At the next order, the same process leads to:

$$(3.12)\qquad E[\mathbf{n}^c \cdot \mathbb{T}^c \varphi_1 \delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x}) = \phi_1(\mathbf{x})\{[-\overline{p}^{c2}\mathbb{I} + 2\mu^c \mathbb{D}^\circ(\overline{\mathbf{v}}^{c2})] \cdot \frac{\partial}{\partial \mathbf{x}^\circ} \alpha^{d2}(\mathbf{x}^\circ|\mathbf{x}) + E[\mathbf{n}^c \cdot \mathbb{T}^{**} \delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x})\},$$

where the disturbance interfacial force density relative to the fields with one fixed inclusion located at $\mathbf{x}$, denoted $E[\mathbf{n}^c \cdot \mathbb{T}^{**} \delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x})$, is defined by:

$$(3.13)\qquad E[\mathbf{n}^c \cdot \mathbb{T}^{**} \delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x}) = a^2(N-1) \int_{S(\mathbf{x}^\circ)} \mathbf{n}^c \cdot \{[-p^{\circ**}\mathbb{I} + 2\mu^c \mathbb{D}^\circ (\mathbf{v}^{\circ**})(\mathbf{x}^\circ|\mathbf{x}^\circ + a\mathbf{n}^c, \mathbf{x})] + [-p^{\circ*}\mathbb{I} + 2\mu^c \mathbb{D}^\circ(\mathbf{v}^{\circ*})(\mathbf{x}^\circ|\mathbf{x}^\circ + a\mathbf{n}^c)]\} \{\phi_1(\mathbf{x}^\circ + a\mathbf{n}^c) + \chi^*(\mathbf{x}^\circ + a\mathbf{n}^c|\mathbf{x})\} d\Omega.$$

At the third order, a similar result is obtained:

$$(3.14)\qquad E[\mathbf{n}^c \cdot \mathbb{T}^c \varphi_1 \varphi_2 \delta_\Sigma^{1,2}](\mathbf{x}^{\circ\circ}|\mathbf{x}, \mathbf{x}^\circ) = \phi_2(\mathbf{x}^\circ, \mathbf{x})\{[-\overline{p}^{c3} + 2\mu^c \mathbb{D}^{\circ\circ}(\overline{\mathbf{v}}^{c3})] \cdot \frac{\partial}{\partial \mathbf{x}^{\circ\circ}} \alpha^{d3}(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ, \mathbf{x}) + E[\mathbf{n}^c \cdot \mathbb{T}^{***} \delta_\Sigma^{1,2}](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ, \mathbf{x})\},$$

where $E[\mathbf{n}^c \cdot \mathbb{T}^{***} \delta_\Sigma^{1,2}](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ, \mathbf{x})$ due to the third-order disturbance flows can be similarly defined. Analogous cancellations occur at the r.h.s. of the corresponding momentum equations.

4. New system of equations

All the equations of the Lundgren and B.B.G.K.Y. hierarchies as well as the boundary conditions between these two sets of equations at any order but the first one can now be transformed into equations for averaged disturbance flows.

Equations at the first order are still expressed in terms of their initial variables as presented at the end of Part I but they also require some simplifications.

4.1. New first-order equations for the continuous phase

The first-order momentum Eq. ((5.18) – Part I) can be simplified thanks to (3.3), (3.4), (3.9), (3.10) and (3.11):

$$\frac{\partial \overline{\mathbf{v}}^{c1}}{\partial t} + \overline{\mathbf{v}}^{c1} \cdot \frac{\partial}{\partial \mathbf{x}} \overline{\mathbf{v}}^{c1} + \left(\frac{\partial}{\partial \mathbf{x}} \overline{p}^{c1} \right) / \rho^c - \nu^c \Delta(\overline{\mathbf{v}}^{c1}) - \nu^c \frac{\partial}{\partial \mathbf{x}} \left[\frac{\partial}{\partial \mathbf{x}} \cdot \overline{\mathbf{v}}^{c1} \right] \tag{4.1}$$
$$-\mathbf{g} = 2\nu^c \frac{\partial}{\partial \mathbf{x}} \cdot [E(\mathbb{F}^* \delta_\Sigma)\mathbb{A}]/\alpha^{c1} + E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma]/\alpha^{c1}\rho^c - \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^1_{vv}/\alpha^{c1}.$$

The l.h.s. comprises all the classical terms of the usual single-phase momentum equation for a compressible fluid whereas the r. h. s. displays three specific forcing functions due to the presence of the dispersed phase; note that these functions, which are multiplied by $(1-\alpha^{d1})^{-1}$, depend on first-order disturbance flow fields. This first-order momentum equation will ultimately remain in our modelling in the form of (4.1).

The first-order continuity Eq. ((5.7) – Part I) will be also kept as it is, and is repeated here for convenience

$$-\frac{\partial \alpha^{d1}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot [(1-\alpha^{d1})\overline{\mathbf{v}}^{c1}] = 0. \tag{4.2}$$

Integral relations (2.1) and (2.3) linking standard dispersed-phase variables and new moments derived from the kinetic equations must be retained.

4.2. Disturbance equations for the continuous phase

Before transforming the second-order and third-order momentum equations into equations for averaged disturbance flows, all extra-deformation tensors and force densities they contain are split, as shown in Sec. 3. Equation ((5.18) – Part I) is then extended over $\mathcal{V}^c_{x,x^\circ}$ and after having set $\mathbf{x} = \mathbf{x}^\circ$, the resulting equation is subtracted from ((5.19) – Part I):

$$\frac{\partial \mathbf{v}^{\circ *}}{\partial t} + \mathbf{v}^{\circ *} \cdot \frac{\partial}{\partial \mathbf{x}^\circ} \mathbf{v}^{\circ *} + \left(\frac{\partial}{\partial \mathbf{x}^\circ} p^{\circ *} \right) / \rho^c - \nu^c \Delta^\circ(\mathbf{v}^{\circ *}) - \nu^c \frac{\partial}{\partial \mathbf{x}^\circ} \left[\frac{\partial}{\partial \mathbf{x}^\circ} \cdot \mathbf{v}^{\circ *} \right] \tag{4.3}$$
$$= -\overline{\mathbf{v}}^{c1}(\mathbf{x}^\circ) \cdot \frac{\partial}{\partial \mathbf{x}^\circ} \mathbf{v}^{\circ *} - \mathbf{v}^{\circ *} \cdot \frac{\partial}{\partial \mathbf{x}^\circ} \overline{\mathbf{v}}^{c1}(\mathbf{x}^\circ)$$

(4.3) [cont.]
$$+\frac{\partial}{\partial \mathbf{x}^\circ}\cdot[E(\mathbb{F}^{**}\delta_\Sigma^1)](2\nu^c/\alpha^{c2})-\frac{\partial}{\partial \mathbf{x}^\circ}\cdot[E(\mathbb{F}^{*}\delta_\Sigma)](2\nu^c/\alpha^{c1})$$
$$+E[\mathbf{n}^c\cdot\mathbb{T}^{**}\delta_\Sigma^1]/\rho^c\alpha^{c2}-E[\mathbf{n}^c\cdot\mathbb{T}^{*}\delta_\Sigma]/\rho^c\alpha^{c1}+C_v^*.$$

In the l.h.s. a usual single-phase momentum equation can be observed for the unknowns, $\mathbf{v}^{\circ *}=\mathbf{v}^*(\mathbf{x}^\circ|\mathbf{x})$ and $p^{\circ *}=p^*(\mathbf{x}^\circ|\mathbf{x})$. At the r.h.s., five types of source function are exhibited. First, two convective terms appear, expressing the influence of the first-order unperturbed fields. Second, there is a difference of two disturbance extra-deformation tensors involving the first-order and second-order averaged perturbation fields. Third, a similar combination of two disturbance interfacial force densities follows. Fourth, fluctuations effects are represented at the bulk level by two continuous-phase velocity variance tensors (pseudo-turbulent tensors). The last term C_v^* represents cross-correlations between the two phases:

(4.4)
$$C^*_{v(\mathbf{x}^\circ|\mathbf{x})}=-\frac{\partial}{\partial \mathbf{x}^\circ}\cdot\mathbb{A}^2_{v^\circ v^\circ}/\alpha^{c2}\phi^{(1)}+\frac{\partial}{\partial \mathbf{x}^\circ}\cdot\mathbb{A}^1_{v^\circ v^\circ}/\alpha^{c1}-\frac{\partial}{\partial \mathbf{x}}$$
$$\cdot E[\varphi_1X_1^c\mathbf{u}_1\mathbf{v}^c]/\alpha^{c2}\phi_1+\overline{\mathbf{v}}^{c2}(\mathbf{x}^\circ|\mathbf{x})\cdot\frac{\partial}{\partial \mathbf{x}}E[\varphi_1X_1^c\mathbf{u}_1]/\alpha^{c2}\phi_1.$$

These will be analysed in the next paper.

Equation ((5.19) – Part I) is then extended over $\mathcal{V}^c_{\mathbf{x},\mathbf{x}^\circ,\mathbf{x}^{\circ\circ}}$; positing $\mathbf{x}^\circ=\mathbf{x}^{\circ\circ}$ and $\mathbf{x}=\mathbf{x}^\circ$, the resulting equation is subtracted from ((5.20) – Part I):

(4.5)
$$\frac{\partial \mathbf{v}^{\circ\circ**}}{\partial t}+\mathbf{v}^{\circ\circ**}\cdot\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\mathbf{v}^{\circ\circ**}+\left(\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}p^{\circ\circ**}\right)/\rho^c-\nu^c\Delta^{\circ\circ}(\mathbf{v}^{\circ\circ**})$$
$$-\nu^c\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\left[\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot\mathbf{v}^{\circ\circ**}\right]=-\mathbf{v}^{\circ\circ**}\cdot\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\{\overline{\mathbf{v}}^{c1}(\mathbf{x}^{\circ\circ})+\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x})$$
$$+\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)\}-\overline{\mathbf{v}}^{c1}(\mathbf{x}^{\circ\circ})\cdot\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\mathbf{v}^{\circ\circ**}-\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x})$$
$$\cdot\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\{\mathbf{v}^{\circ\circ**}+\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)\}-\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)\cdot\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\{\mathbf{v}^{\circ\circ**}+\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x})\}$$
$$+2\nu^c\left\{\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot[E(\mathbb{F}^{***}\delta_\Sigma^{1,2})]/\alpha^{c3}+\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot[E(\mathbb{F}^{*}\delta_\Sigma)](\mathbf{x}^{\circ\circ})]/\alpha^{c1}(\mathbf{x}^{\circ\circ})-\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\right.$$
$$\left.\cdot[E(\mathbb{F}^{**}\delta_\Sigma^1)](\mathbf{x}^{\circ\circ}|\mathbf{x})]/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})-\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot[E(\mathbb{F}^{**}\delta_\Sigma^1)](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)]/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)\right\}$$
$$+(1/\rho^c)\{E[\mathbf{n}^c\cdot\mathbb{T}^{***}\delta_\Sigma^{1,2}]/\alpha^{c3}+E[\mathbf{n}^c\cdot\mathbb{T}^{*}\delta_\Sigma](\mathbf{x}^{\circ\circ})/\alpha^{c1}$$
$$-E[\mathbf{n}^c\mathbb{T}^{**}\delta_\Sigma^1](\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)-E[\mathbf{n}^c\cdot\mathbb{T}^{**}\delta_\Sigma^1](\mathbf{x}^{\circ\circ}|\mathbf{x})/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})\}+C_v^{**},$$

which is a momentum equation for the unknowns $\mathbf{v}^{\circ\circ**} = \mathbf{v}^{**}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ},\mathbf{x})$ and $p^{\circ\circ**} = p^{**}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ},\mathbf{x})$. The forcing functions have exactly the same structure as in (4.3). The last term C_v^{**} represents correlation between the two phases.

$$
\begin{aligned}
(4.6)\quad C^{**}_{v(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ},\mathbf{x})} = &-\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot \mathbb{A}^3_{v^{\circ\circ}v^{\circ\circ}}/\alpha^{c3}\phi^{(2)} + \frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot \mathbb{A}^2_{v^{\circ\circ}v^{\circ\circ}}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})\\
&\phi^{(1)}(\mathbf{x}^{\circ}) + \frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot \mathbb{A}^2_{v^{\circ\circ}v^{\circ\circ}}(\mathbf{x}^{\circ\circ}|\mathbf{x})/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})\phi^{(1)}(\mathbf{x})\\
&+\frac{\partial}{\partial \mathbf{x}}\cdot E[X^c_1\varphi_1\mathbf{u}_1\mathbf{v}^c]/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})\phi_1(\mathbf{x}) - \overline{\mathbf{v}}^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})\cdot\frac{\partial}{\partial \mathbf{x}^{\circ}}E\\
&\times[X^c_2\varphi_2\mathbf{u}_2]/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})\phi_1(\mathbf{x}^{\circ}) + \frac{\partial}{\partial \mathbf{x}^{\circ}}\cdot E[X^c_2\varphi_2\mathbf{u}_2\mathbf{v}^c]/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})\phi_1(\mathbf{x}^{\circ})\\
&-\overline{v}^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})\cdot\frac{\partial}{\partial \mathbf{x}}E[X^c_1\varphi_1\mathbf{u}_1]/\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})\phi_1(\mathbf{x})\\
&-\frac{\partial}{\partial \mathbf{x}}\cdot E[X^c_{1,2}\varphi_1\varphi_2\mathbf{u}_1\mathbf{v}^c]/\alpha^{c3}\phi_2 + \overline{\mathbf{v}}^{c3}\cdot\frac{\partial}{\partial \mathbf{x}}E[X^c_{1,2}\varphi_1\varphi_2\mathbf{u}_1]/\alpha^{c3}\phi_2\\
&-\frac{\partial}{\partial \mathbf{x}^{\circ}}\cdot E[X^c_{1,2}\varphi_1\varphi_2\mathbf{u}_2\mathbf{v}^c]/\alpha^{c3}\phi_2 + \overline{\mathbf{v}}^{c3}\cdot\frac{\partial}{\partial \mathbf{x}^{\circ}}E[X^c_{1,2}\varphi_1\varphi_2\mathbf{u}_2]/\alpha^{c3}\phi_2.
\end{aligned}
$$

These convective terms, which have not yet been analysed, will also be considered in the next paper.

Finally, the second-order and third-order continuity equations of the Lundgren hierarchy can also be transformed into equations controlling disturbance fields $\alpha^{\circ*} = \alpha^*(\mathbf{x}^{\circ}|\mathbf{x})$ and $\alpha^{\circ\circ**} = \alpha^{**}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ},\mathbf{x})$. Consider the first-order disturbance flow equation obtained by subtracting (4.2), in which $\mathbf{x} = \mathbf{x}^{\circ}$, from ((5.21) – Part I):

$$
(4.7)\qquad \frac{\partial \alpha^{\circ*}}{\partial t} + \frac{\partial}{\partial \mathbf{x}^{\circ}}\cdot\left[\alpha^{\circ*}(\overline{\mathbf{v}}^{c1} + \mathbf{v}^{\circ*}) - \alpha^{c1}\mathbf{v}^{\circ*}\right] = -(\alpha^{c2}/\phi_1)\frac{\partial}{\partial \mathbf{x}}\cdot(\phi_1\overline{\mathbf{u}}^1) + C^*_\alpha,
$$

where $C^*_\alpha = (1/\phi_1)\dfrac{\partial}{\partial \mathbf{x}}\cdot E[\varphi_1 X^c_1\mathbf{u}_1]$. Setting $\mathbf{x}^{\circ} = \mathbf{x}^{\circ\circ}$ and $\mathbf{x} = \mathbf{x}^{\circ}$ in ((5.21) – Part I) and substracting the resulting equation from ((5.22) – Part I):

$$
\begin{aligned}
(4.8)\qquad \frac{\partial \alpha^{\circ\circ**}}{\partial t} &+ \frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot\{\alpha^{\circ\circ**}[\overline{\mathbf{v}}^{c1}(\mathbf{x}^{\circ\circ}) + \mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ}) + \mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x})\\
&+\mathbf{v}^{\circ\circ**}]\} + \frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot\{\alpha^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ})[\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}) + \mathbf{v}^{\circ\circ**}]\}
\end{aligned}
$$

(4.8) [cont.]

$$+\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot\{\alpha^*(\mathbf{x}^{\circ\circ}|\mathbf{x})[\mathbf{v}^*(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)+\mathbf{v}^{\circ\circ**}]\}$$

$$-\frac{\partial}{\partial \mathbf{x}^{\circ\circ}}\cdot[\alpha^{c1}(\mathbf{x}^{\circ\circ})\mathbf{v}^{\circ\circ**}]$$

$$=[\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x})/\phi_1]\frac{\partial}{\partial \mathbf{x}}\cdot(\phi_1\overline{\mathbf{u}}^1)+[\alpha^{c2}(\mathbf{x}^{\circ\circ}|\mathbf{x}^\circ)/\phi_1^\circ]\frac{\partial}{\partial \mathbf{x}^\circ}\cdot(\phi_1^\circ\overline{\mathbf{u}^\circ}^1)$$

$$-[\alpha^{c3}(\mathbf{x}^{\circ\circ}|\mathbf{x},\mathbf{x}^\circ)/\phi_2]\left[\frac{\partial}{\partial \mathbf{x}}\cdot(\phi_2\overline{\mathbf{u}}^2)+\frac{\partial}{\partial \mathbf{x}^\circ}\cdot(\phi_2\overline{\mathbf{u}^\circ}^2)\right]+C_\alpha^{**},$$

where C_α^{**}, the correlation term, comprises composite divergence-type sources:

$$C_\alpha^{**}=-(1/\phi_1)\frac{\partial}{\partial \mathbf{x}}\cdot E[\varphi_1X_1^c\mathbf{u}_1]-(1/\phi_1^\circ)\frac{\partial}{\partial \mathbf{x}^\circ}\cdot E[\varphi_2X_2^c\mathbf{u}_2] \tag{4.9}$$

$$+(1/\phi_2)\frac{\partial}{\partial \mathbf{x}}\cdot E[\varphi_1\varphi_2X_{1,2}^c\mathbf{u}_1]+(1/\phi_2)\frac{\partial}{\partial \mathbf{x}^\circ}\cdot E[\varphi_1\varphi_2X_{1,2}^c\mathbf{u}_2].$$

Integral relations linking (Sec. 2.1) standard dispersed-phase variables (basically α^{di}, $\mathbf{v}^{di}$, $i=2,3$) and new moments derived from the kinetic equations must be added in their initial integral form (see Eq. (2.12)) or their expanded form (see Eq. (2.13)).

4.3. New first-order equations for the dispersed phase

The first-order dispersed-phase linear momentum Eq. ((4.27) – Part I) becomes:

$$\rho^d\frac{\partial}{\partial t}\overline{\mathbf{u}}^1+\rho^d\overline{\mathbf{u}}^1\cdot\frac{\partial}{\partial \mathbf{x}}\overline{\mathbf{u}}^1+(1+\mathbf{R}_1)\left[\frac{\partial\overline{p}^{c1}}{\partial \mathbf{x}}-\mu^c\Delta\overline{\mathbf{v}}^{c1}\right. \tag{4.10}$$

$$\left.-\mu^c\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial}{\partial \mathbf{x}}\cdot\overline{\mathbf{v}}^{c1}\right)\right]-\rho^d\mathbf{g}=-\rho^d(\phi^{(1)})^{-1}\frac{\partial}{\partial \mathbf{x}}\cdot\mathbb{A}_{uu}^1+(3/4)\mathbf{F}^*/\pi a^3$$

considering (2.14) and (2.20). In this first-order equation, it should be recalled that $\mathbf{R}_1=O(\beta^2)$. It may appear surprising to meet a seemingly normal single-phase momentum equation written for an averaged "composite fluid" of a sort at the l.h.s. of (4.10). Its inertia is that of the inclusions while its viscosity is that of the carrying fluid. Two types of source terms are exhibited at the r.h.s. due to the presence of the continuous phase.

The first-order dispersed-phase angular momentum Eq. ((4.28) – Part I) may be similarly transformed by using (2.16) and (2.21)

$$\rho^d \frac{\partial}{\partial t}\overline{\omega}^1 + \rho^d \overline{\mathbf{u}}^1 \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\omega}^1 = -\mu^c(1+\mathbf{R}_2)\,\mathbf{curl}^3(\overline{\mathbf{v}}^{c1})/2 + (15/8)\mathbf{K}^*/\pi a^5 - (\phi^{(1)})^{-1}\rho^d \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^1_{\omega u}. \tag{4.11}$$

This first-order equation, where, similarly, $\mathbf{R}_2 = O(\beta^2)$, looks like a classical vorticity transport equation which could be written for the above "composite fluid", especially if the fluid spin is defined by $\overline{\boldsymbol{\varpi}}^{c1} = \mathbf{curl}\,(\overline{\mathbf{v}}^{c1})/2$. The viscous term at its r.h.s. is then the familiar $-\mu^c \mathbf{curl}^2(\overline{\boldsymbol{\varpi}}^{c1})$. However, several differences make this analogy irrelevant. The term $\overline{\boldsymbol{\varpi}}^1 \cdot \dfrac{\partial}{\partial \mathbf{x}}\overline{\mathbf{u}}^1$, which would give vorticity changes their distinctive character, is lacking. More basically, this is an equation in itself, juxtaposed to (4.11), and cannot be deduced from a more basic "Navier-Stokes equation".

For some applications, it is advantageous to transform the first-order dispersed-phase linear and angular momentum Eqs. (4.10) and (4.11) so that they directly provide the relative linear and angular velocities defined by:

$$\overline{\mathbf{u}}^r = \overline{\mathbf{u}}^1 - \overline{\mathbf{v}}^{c1}; \qquad \overline{\boldsymbol{\varpi}}^r = \overline{\boldsymbol{\varpi}}^1 - \overline{\boldsymbol{\varpi}}^{c1}. \tag{4.12}$$

The basic reason is the same as that leading to the introduction of disturbance field equations: estimating precisely the magnitude of various terms becomes easier. Moreover, $\mathbf{F}^*$ and $\mathbf{K}^*$ will appear as functions of these relative velocities. By introducing these velocities:

$$\begin{aligned} &\rho^d \frac{\partial}{\partial t}\overline{\mathbf{u}}^r + \rho^d \overline{\mathbf{u}}^r \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\mathbf{u}}^r - (3/4)\mathbf{F}^*/\pi a^3 + \rho^d(\phi^{(1)})^{-1}\frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^1_{uu} \\ &+\rho^d \overline{\mathbf{u}}^r \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\mathbf{v}}^{c1} + \rho^d \overline{\mathbf{v}}^{c1} \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\mathbf{u}}^r = -\rho^d \frac{\partial}{\partial t}\overline{\mathbf{v}}^{c1} - \rho^d \overline{\mathbf{v}}^{c1} \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\mathbf{v}}^{c1} \\ &-(1+\mathbf{R}_1)\left[\frac{\partial \overline{p}^{c1}}{\partial \mathbf{x}} - \mu^c \Delta \overline{\mathbf{v}}^{c1} - \mu^c \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial}{\partial \mathbf{x}} \cdot \overline{\mathbf{v}}^{c1}\right)\right] + \rho^d \mathbf{g}, \end{aligned} \tag{4.13}$$

and:

$$\begin{aligned} &\rho^d \frac{\partial}{\partial t}\overline{\boldsymbol{\varpi}}^r + \rho^d \overline{\mathbf{u}}^r \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\boldsymbol{\varpi}}^r - (15/8)\mathbf{K}^*/\pi a^5 + (\phi^{(1)})^{-1}\rho^d \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^1_{\omega u} \\ &+\rho^d \overline{\mathbf{v}}^{c1} \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\boldsymbol{\varpi}}^r + \rho^d \overline{\mathbf{u}}^r \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\boldsymbol{\varpi}}^{c1} = -\rho^d \frac{\partial}{\partial t}\overline{\boldsymbol{\varpi}}^{c1} - \rho^d \overline{\mathbf{v}}^{c1} \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\boldsymbol{\varpi}}^{c1} \\ &-\mu^c(1+\mathbf{R}_2)\mathbf{curl}^3(\overline{\mathbf{v}}^{c1})/2. \end{aligned} \tag{4.14}$$

In addition, the density dispersed-phase Eq. ((4.11) – Part I) itself becomes

$$\frac{\partial \phi_1}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \phi_1 \overline{\mathbf{v}}^{c1} = -\frac{\partial}{\partial \mathbf{x}} \cdot \phi_1 \overline{\mathbf{u}}^{r}. \tag{4.15}$$

4.4. Disturbance flow equations for the dispersed phase

Using (2.15), (2.17) and results similar to (2.20) and (2.21), the second-order dispersed-phase momentum Eqs. ((4.29) and (4.30) – Part I) become:

$$\rho^d \left[\frac{\partial}{\partial t}\overline{\mathbf{u}}^{\circ 2} + \overline{\mathbf{u}^{\circ}}^{2} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}}\overline{\mathbf{u}^{\circ}}^{2} + \overline{\mathbf{u}}^{2} \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\mathbf{u}^{\circ}}^{2}\right] \tag{4.16}$$
$$= (1+\mathbf{R}_1^{\circ})\left[-\frac{\partial \overline{p^{\circ}}^{c2}}{\partial \mathbf{x}^{\circ}} + \mu^c \Delta^{\circ}\overline{\mathbf{v}^{\circ}}^{c2} + \mu^c \frac{\partial}{\partial \mathbf{x}^{\circ}}\left(\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \overline{\mathbf{v}^{\circ}}^{c2}\right)\right]$$
$$+\rho^d\left[-\left(\phi^{(2)}\right)^{-1}\frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^2_{uu^{\circ}} - \left(\phi^{(2)}\right)^{-1}\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbb{A}^2_{u^{\circ}u^{\circ}} + \mathbf{g}\right]$$
$$+(3/4)(\mathbf{F}^{\circ **} + \mathbf{F}^{\circ *})/\pi a^3,$$

and

$$\rho^d \left[\frac{\partial}{\partial t}\overline{\boldsymbol{\omega}^{\circ}}^{2} + \overline{\mathbf{u}^{\circ}}^{2} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}}\overline{\boldsymbol{\omega}^{\circ}}^{2} + \overline{\mathbf{u}}^{2} \cdot \frac{\partial}{\partial \mathbf{x}}\overline{\boldsymbol{\omega}^{\circ}}^{2}\right] = -\mu^c(1+\mathbf{R}_2^{\circ}) \tag{4.17}$$
$$\mathrm{curl}^{\circ 3}(\overline{\mathbf{v}^{\circ}}^{c2})/2 - \rho^d(\phi^{(2)})^{-1}\left[\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbb{A}^2_{\omega^{\circ}u^{\circ}} + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^2_{\omega^{\circ}u}\right]$$
$$+(15/8)(\mathbf{K}^{\circ **} + \mathbf{K}^{\circ *})/\pi a^5.$$

To establish the equation for $\mathbf{u}^{\circ *}$, the first-order Eq. (4.7) written at $\mathbf{x}^{\circ}$ is subtracted from the above second-order Eq. (4.16):

$$\rho^d\left[\frac{\partial}{\partial t}\mathbf{u}^{\circ *} + u^{\circ *} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}}\mathbf{u}^{\circ *}\right] + \frac{\partial p^{\circ *}}{\partial \mathbf{x}^{\circ}} - \mu^c \Delta^{\circ}\mathbf{v}^{\circ *} - \mu^c\frac{\partial}{\partial \mathbf{x}^{\circ}}\left(\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbf{v}^{\circ *}\right) \tag{4.18}$$
$$= +\mathbf{R}_1^{\circ}\left[-\frac{\partial p^{\circ *}}{\partial \mathbf{x}^{\circ}} + \mu^c \Delta \mathbf{v}^{\circ *} + \mu^c \frac{\partial}{\partial \mathbf{x}^{\circ}}\left(\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbf{v}^{\circ *}\right)\right]$$
$$-\rho^d\left[\overline{\mathbf{u}^{\circ}}^{1} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}}\mathbf{u}^{\circ *} + \mathbf{u}^{\circ *} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}}\overline{\mathbf{u}^{\circ}}^{1} + \overline{\mathbf{u}}^{2} \cdot \frac{\partial}{\partial \mathbf{x}}\mathbf{u}^{\circ *}\right]$$
$$+\rho^d\left[-(\phi^{(2)})^{-1}\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbb{A}^2_{u^{\circ}u^{\circ}} - (\phi^{(2)})^{-1}\frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}^2_{uu^{\circ}}\right.$$
$$\left.+(\phi^{(1)})^{-1}\frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbb{A}^1_{u^{\circ}u^{\circ}}\right] + (3/4\pi a^3)\mathbf{F}^{\circ **}.$$

The l.h.s. of (4.18) corresponds to the first-order averaged disturbance momentum equation relative to the afore-mentioned "composite fluid". At the r.h.s., the source terms bear some similarities to those encountered in the corresponding Eq. (4.3) of the Lundgren hierarchy.

Finally, by substracting the first-order equation for the angular velocity (4.11) written at $\mathbf{x}^\circ$ from the second-order Eq. (4.17), the equation for $\omega^{\circ *}$ follows:

$$(4.19)\qquad \rho^d\left[\frac{\partial}{\partial t}\omega^{\circ *}+\mathbf{u}^{\circ *}\cdot\frac{\partial}{\partial \mathbf{x}^\circ}\omega^{\circ *}\right]+\mu^c\,\mathbf{curl}^{\circ 3}(\mathbf{v}^{\circ *})/2$$
$$=-\mu^c\mathbf{R}_2^\circ[\mathbf{curl}^{\circ 3}(\mathbf{v}^{\circ *})]/2-\rho^d\left[\overline{\mathbf{u}^\circ}^1\cdot\frac{\partial}{\partial \mathbf{x}^\circ}\omega^{\circ *}+\mathbf{u}^{\circ *}\cdot\frac{\partial}{\partial \mathbf{x}^\circ}\overline{\omega^\circ}^1\right.$$
$$\left.+\overline{\mathbf{u}}^2\cdot\frac{\partial}{\partial \mathbf{x}}\omega^{\circ *}\right]+\rho^d\left[-(\phi^{(2)})^{-1}\frac{\partial}{\partial \mathbf{x}^\circ}\cdot\mathbb{A}^2_{\omega^\circ u^\circ}-(\phi^{(2)})^{-1}\frac{\partial}{\partial \mathbf{x}}\cdot\mathbb{A}^2_{\omega^\circ u}\right.$$
$$\left.+(\phi^{(1)})^{-1}\frac{\partial}{\partial \mathbf{x}^\circ}\cdot\mathbb{A}^1_{\omega^\circ u^\circ}\right]+(15/8\pi a^5)\mathbf{K}^{\circ **}.$$

The equation for the disturbance $\chi^{\circ *}$ defined in (1.2), is obtained by substracting ((4.11) – Part I) written for $\phi_1^\circ=\phi_1(\mathbf{x}^\circ)$ from ((4.31) – Part I):

$$(4.20)\qquad \frac{\partial\chi^{\circ *}}{\partial t}+\frac{\partial}{\partial \mathbf{x}^\circ}\cdot\left[\phi_1^\circ\mathbf{u}^{\circ *}+\chi^{\circ *}\overline{\mathbf{u}^\circ}^1+\chi^{\circ *}\mathbf{u}^{\circ *}\right]$$
$$=-(\phi_1^\circ+\chi^{\circ *})\frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{u}^*-(\overline{\mathbf{u}}^1+\mathbf{u}^*)\cdot\frac{\partial\chi^{\circ *}}{\partial \mathbf{x}}-\frac{\phi_1^\circ+\chi^{\circ *}}{\phi_1}\mathbf{u}^*\cdot\frac{\partial\phi_1}{\partial \mathbf{x}},$$

where $\mathbf{u}^{\circ *}$ is the velocity disturbance at $\mathbf{x}^\circ$, knowing that an inclusion is located at $\mathbf{x}$.

4.5. Disturbance boundary continuity conditions

Boundary equations at the surfaces of test inclusions also have to be written in terms of disturbance flow. It should be recalled that these conditions express the continuity of tangential velocities (no-slip condition) and of normal velocity (no external fluid mass flows across the surface). From ((5.10) – Part I), at $\mathbf{x}^\circ=\mathbf{x}+a\mathbf{n}$ where $\mathbf{n}$ is the unit normal exterior to an inclusion, the first disturbance field satisfies:

$$(4.21)\qquad \mathbf{v}^*(\mathbf{x}+a\mathbf{n}|\mathbf{x})=\overline{\mathbf{u}}^1(\mathbf{x})-\overline{\mathbf{v}}^{c1}(\mathbf{x})+a\omega^1(\mathbf{x})\wedge\mathbf{n}+\overline{\mathbf{v}}^{c1}(\mathbf{x})-\overline{\mathbf{v}}^{c1}(\mathbf{x}+a\mathbf{n})$$
$$=\overline{\mathbf{u}}^r(\mathbf{x})+\overline{\omega}^r(\mathbf{x})\wedge\mathbf{n}-a\mathbb{D}[\overline{\mathbf{v}}^{c1}(\mathbf{x})]\cdot\mathbf{n}+(1/2)a^2\nabla\nabla\overline{\mathbf{v}}^{c1}(\mathbf{x}):\mathbf{nn}+\ldots$$

where ∇ and $\mathbb{D}$ are the gradient operator and its symmetrical part respectively. This equation may be viewed generally as relating the averaged disturbance field, $\mathbf{v}^*$, to three preponderant forcing functions:

• the relative dispersed-phase velocity, $\overline{\mathbf{u}}^r$, provided by the relative linear momentum equation for the dispersed phase,

• the relative dispersed-phase velocity, $\overline{\boldsymbol{\omega}}^r$, provided by the relative angular momentum equation for the dispersed phase,

• the simple shear flow, $a\mathbb{D}[\overline{\mathbf{v}}^{c1}(\mathbf{x}]\cdot\mathbf{n}$, provided by the first-order momentum equation for the continuous phase.

The second disturbance field satisfies two conditions, one at $\mathbf{x}^{\circ\circ} = \mathbf{x}^{\circ} + a\mathbf{n}$ and the other one at $\mathbf{x}^{\circ\circ} = \mathbf{x} + a\mathbf{n}$ derived from their mother condition ((5.10) – Part I):

$$\mathbf{v}^{**}(\mathbf{x}^{\circ} + a\mathbf{n}|\mathbf{x}^{\circ}, \mathbf{x}) = \mathbf{u}^{*}(\mathbf{x}^{\circ}|\mathbf{x}) + a\boldsymbol{\omega}^{*} \wedge \mathbf{n}(\mathbf{x}^{\circ}|\mathbf{x}) - \mathbf{v}^{*}(\mathbf{x}^{\circ} + a\mathbf{n}|\mathbf{x}) \tag{4.22}$$

and

$$\mathbf{v}^{**}(\mathbf{x} + a\mathbf{n}|\mathbf{x}^{\circ}, \mathbf{x}) = \mathbf{u}^{*}(\mathbf{x}|\mathbf{x}^{\circ}) + a\boldsymbol{\omega}^{*} \wedge \mathbf{n}(\mathbf{x}|\mathbf{x}^{\circ}) - \mathbf{v}^{*}(\mathbf{x} + a\mathbf{n}|\mathbf{x}^{\circ}). \tag{4.23}$$

These conditions, which are symmetrical with respect to $\mathbf{x}$ and $\mathbf{x}^{\circ}$, connect the third- and second-order disturbance flows.

On the external boundaries of $\mathcal{V}^c_{\mathbf{x},\mathbf{x}^{\circ}}$, $\mathcal{V}^c_{\mathbf{x},\mathbf{x}^{\circ},\mathbf{x}^{\circ\circ}}$ simple conditions result from ((5.9) – Part I):

$$\mathbf{v}^{*}(\mathbf{x}^{\circ}|\mathbf{x}) = 0, \quad \mathbf{x}^{\circ} \text{ on } \partial\mathcal{V}^c_w \quad \text{and} \quad \mathbf{v}^{**}(\mathbf{x}^{\circ\circ}|\mathbf{x}^{\circ}, \mathbf{x}) = 0, \quad \mathbf{x}^{\circ\circ} \text{ on } \partial\mathcal{V}^c_w \tag{4.24}$$

Regarding the dispersed phase, the condition in the vicinity of walls is derived from ((4.33) – Part I)

$$\mathbf{u}^{*}(\mathbf{x}^{\circ} - a\mathbf{n}^b|\mathbf{x}) + a\boldsymbol{\omega}^{*}(\mathbf{x}^{\circ} - a\mathbf{n}^b|\mathbf{x}) \wedge \mathbf{n}^b = 0, \quad \mathbf{x}^{\circ} \text{ on } \partial\mathcal{V}^c_w \tag{4.25}$$

while conditions on the fluid boundary of $\partial\mathcal{V}^d$ are obtained by combining Eqs. ((4.35) to (4.37) – Part I):

$$\chi^{*}(\mathbf{x}^{\circ}|\mathbf{x}) = \mathbf{u}^{*}(\mathbf{x}^{\circ}|\mathbf{x}) = \boldsymbol{\omega}^{*}(\mathbf{x}^{\circ}|\mathbf{x}) = 0, \quad \mathbf{x}^{\circ} \text{ on } \partial\mathcal{V}^d_f \tag{4.26}$$

Disturbance velocity fields satisfying conditions on the internal boundaries of $\mathcal{V}^d_{\mathbf{x},\mathbf{x}^{\circ}}$ result from ((4.34) – Part I)

$$\begin{aligned} &\mathbf{u}^{*}(\mathbf{x}|\mathbf{x} + 2a\mathbf{n}) - \mathbf{u}^{*}(\mathbf{x} + 2a\mathbf{n}|\mathbf{x}) + 2a[\boldsymbol{\omega}^{*}(\mathbf{x}|\mathbf{x} + 2a\mathbf{n}) \\ &\quad + \boldsymbol{\omega}^{*}(\mathbf{x} + 2a\mathbf{n}|\mathbf{x})] \wedge \mathbf{n} + \overline{\mathbf{u}}^1(\mathbf{x}) - \overline{\mathbf{u}}^1(\mathbf{x} + 2a\mathbf{n}) + 2a[\overline{\boldsymbol{\omega}}^1(\mathbf{x}) \\ &\quad + \overline{\boldsymbol{\omega}}^1(\mathbf{x} + 2a\mathbf{n})] \wedge \mathbf{n} = 0, \end{aligned} \tag{4.27}$$

where $\mathbf{n}$ is the unit normal exterior to the inclusion centred at $\mathbf{x}$.

5. More about continuous-phase interaction terms

5.1. Various expressions of the extra-deformation tensors

The extra-deformation tensor given by (3.2) can be rewritten by introducing the boundary condition ((5.10) – Part I) and then by applying Gauss theorem:

$$E[\mathbb{F}^c\delta_\Sigma](\mathbf{x}) = a^2 \int\limits_{S(\mathbf{x})} [\mathbf{n}^c\overline{\mathbf{v}}^{c2}]^s(\mathbf{x}|\mathbf{x}_1)\phi^{(1)}(\mathbf{x}_1)d\Omega \tag{5.1}$$

$$= a^2 \int\limits_{S(\mathbf{x})} \{\mathbf{n}^c[\overline{\mathbf{u}}^1(\mathbf{x}_1) + \overline{\omega}^1(\mathbf{x}_1)\wedge(\mathbf{x}-\mathbf{x}_1)]\}^s\phi^{(1)}(\mathbf{x}_1)d\Omega$$

$$= \int\limits_{|\tilde{\mathbf{x}}-\mathbf{x}|\le a} \left\{\frac{\partial}{\partial\tilde{\mathbf{x}}}[\overline{\mathbf{u}}^1(\tilde{\mathbf{x}}) + \overline{\boldsymbol{\omega}}^1(\tilde{\mathbf{x}})\wedge(\mathbf{x}-\tilde{\mathbf{x}})]\phi^{(1)}(\tilde{\mathbf{x}})\right\}^s d\tilde{\mathbf{x}}$$

$$= \mathbb{D}\left\{\int\limits_{|\mathbf{k}|\le a} [\overline{\mathbf{u}}^1(\mathbf{x}+\mathbf{k}) + \overline{\boldsymbol{\omega}}^1(\mathbf{x}+\mathbf{k})\wedge\mathbf{k}]\phi^{(1)}(\mathbf{x}+\mathbf{k})d\mathbf{k}\right\} = \mathbb{D}(\alpha^{d1}\overline{\mathbf{v}}^{d1}).$$

In the last line, the standard dispersed-phase velocity $\overline{\mathbf{v}}^{d1}$ (2.3) has been identified. Such a result has already been proposed by JOSEPH and LUNDGREN [9]. By averaging the fine-grained expression of the extra-deformation tensor $\mathbf{n}^c\cdot\mathbb{T}\delta_\Sigma = \mathbb{D}(X^d\mathbf{v}^d)$ (see (2.13) – Part I) they obtained directly the final term in Eq. (5.1). Besides several inferences were drawn.

Introducing the mixture velocity, $\mathbf{v}^m = X^c\mathbf{v}^c + X^d\mathbf{v}^d$, and taking the divergence of ((2.13) – Part I) yield:

$$\frac{\partial}{\partial\mathbf{x}}\cdot\{X^c\mathbb{T}^c\} = -\frac{\partial}{\partial\mathbf{x}}(X^cp^c) + \mu^c\Delta(\mathbf{v}^{\mathrm{m}}) + \frac{\partial}{\partial\mathbf{x}}\left[\frac{\partial}{\partial\mathbf{x}}\cdot(\mathbf{v}^{\mathrm{m}})\right] \tag{5.2}$$

$$= -\frac{\partial}{\partial\mathbf{x}}(X^cp^c) + \mu^c\Delta(\mathbf{v}^{\mathrm{m}})$$

since the continuity equations for both phases (i.e. the Eq. ((2.9) – Part I) and their dispersed-phase companion) give:

$$\frac{\partial}{\partial\mathbf{x}}\cdot\mathbf{v}^{\mathrm{m}} = 0. \tag{5.3}$$

Thus (5.2) offers an alternative expression for the fine-grained momentum Eq. ((2.15) – Part I). Once averaged, it is clear that all the averaged viscous terms except the averaged interfacial force density merge in the single term $\nu^c \Delta \overline{\mathbf{v}}^{m1}$, where the averaged mixture velocity has been introduced. Likewise, alternative expressions for the momentum equations at the second and third order can be obtained by averaging the appropriate fine-grained equations. These will not be presented for the sake of brevity.

Secondly, ISHII [8] discussed the form of the disturbance-averaged extra-deformation tensor $E[\mathbb{F}^*\delta_\Sigma]$ even if the above break-down was not explicitly given in his study. Employing (5.1), it is interesting to remark that $E[\mathbb{F}^*\delta_\Sigma]$ can be stated in the following form:

$$E[\mathbb{F}^*\delta_\Sigma](\mathbf{x}) = \alpha^{d1}\mathbb{D}(\overline{\mathbf{v}}^{c1})(\mathbf{x}) + \alpha^{d1}\mathbb{D}(\overline{\mathbf{v}}^{d1} - \overline{\mathbf{v}}^{c1})(\mathbf{x}) + \left[(\overline{\mathbf{v}}^{d1} - \overline{\mathbf{v}}^{c1})\frac{\partial \alpha^{d1}}{\partial \mathbf{x}}\right]^s(\mathbf{x}), \tag{5.4}$$

so the disturbance-averaged extra-deformation tensor has three components. In the present type of modelling, the standard dispersed-phase velocity (2.7) defined in Part I has to be expressed via (2.8) in terms of our own dispersed phase (linear and angular) velocities. At the leading order, we have:

$$E[\mathbb{F}^*\delta_\Sigma] = \alpha^{d1}\mathbb{D}(\overline{\mathbf{v}}^{c1}) + \alpha^{d1}\mathbb{D}(\overline{\mathbf{u}}^r) + \left[\overline{\mathbf{u}}^r\frac{\partial \alpha^{d1}}{\partial \mathbf{x}}\right]^s + O(\beta). \tag{5.5}$$

This result is quite general and is valid for any type of dispersed inclusion as long as both phases may be considered incompressible. The third component in (5.4) is precisely the one discussed conjecturally by ISHII [8]: here it is proved. Somewhat anticipating the scale analysis to be performed in a future paper, it may be admitted that this component is important when the particulate Reynolds number based on linear movements is high. Such a component is often neglected in two-fluid models. A last remark is worth making; as the extra-deformation tensor appears in the momentum equation inserted into a divergence operator (see Eq. (4.1)) the above component gives rise to a diffusion-type term among other terms:

$$\left[(\overline{\mathbf{v}}^{d1} - \overline{\mathbf{v}}^{c1})\Delta\alpha^{d1} + (\overline{\mathbf{v}}^{d1} - \overline{\mathbf{v}}^{c1})\cdot\frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{x}}\alpha^{d1}\right]. \tag{5.6}$$

The second component in (5.4) is also passed over by everyone. It is more difficult to ascertain but it is certainly important when the particulate Reynolds number based on rotational movements is high.

Finally, the first component which is important for low bulk flow Reynolds numbers appears as a systematic contribution to the continuous-phase viscosity.

It does not depend on inclusion flow regimes but becomes preponderant when the relative velocity between phases vanishes.

There are no similar alternative expressions for the second- and third-order extra-deformation tensors.

5.2. Expansion in multipoles of interfacial force densities

Interfacial force densities at a given observation point $\mathbf{x}$ involve continuous-phase interface fields resulting from the contributions of multiple inclusions placed at various positions all around the neighborhood of $\mathbf{x}$. This gives rise to various integrals that are difficult to handle. It is much easier to reverse the order namely fixing a single inclusion at $\mathbf{x}$ and integrating over its conditionally-averaged interface fields. Essentially, this is the multipole expansion method.

Furthermore, observe than in (4.1) the interfacial force density $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma](\mathbf{x})$ differs from its counterpart $\alpha^{d1}(3/4\pi a^3)\mathbf{F}^*(\mathbf{x})$ in (4.18). The latter is viewed by the dispersed phase and the former by the continuous phase. The difference in fact results from the asymmetrical treatment of both phases. The link between these terms can just be shown by an expansion of $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma](\mathbf{x})$ similar to the one performed by BUYEVICH and SHCHELCHKOVA [3] in terms of force multipoles. Their expansion involved all the overall stresses around the inclusions while the present expansion deals only with stresses arising from disturbance flows.

Let us introduce the $\mathrm{m}+1^{\mathrm{th}}$ tensor of order k, at $\mathrm{m} = 0, 1, 2$, which represents the monopole , dipole and other multipole moments of the local stress distribution over a test inclusion, due to the k^{th} order disturbance fields. In the calculations that follow, force multipoles of the first- and second-order disturbance flows are needed:

$$(5.7)\qquad \mathbb{M}^*_{\mathrm{m}+1}(\mathbf{x}) = a^2 \frac{(-1)^{\mathrm{m}+1}}{\mathrm{m}!} \int\limits_{S(\mathbf{x})} \mathbf{n}^{\mathrm{m}+1} \cdot \mathbb{T}^*(\mathbf{x} + a\mathbf{n}|\mathbf{x}) d\Omega,$$

where $\mathbf{n}^{\mathrm{m}+1}$ denotes a $\mathrm{m}+1$-fold tensor product of $\mathbf{n}$, and

$$(5.8)\qquad \mathbb{M}^{**}_{\mathrm{m}+1}(\mathbf{x}^\circ|\mathbf{x}) = a^2 \frac{(-1)^{\mathrm{m}+1}}{\mathrm{m}!} \int\limits_{S(\mathbf{x}^\circ)} \mathbf{n}^{\mathrm{m}+1} \cdot [\mathbb{T}^{**}(\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x}^\circ, \mathbf{x}) + \mathbb{T}^*(\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x}^\circ)]\, d\Omega.$$

These have the dimension of a force and are tensors of rank $\mathrm{m}+1$, that are symmetrical in their first m indices.

Firstly, it will be noted that $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma](\mathbf{x})$ defined in (3.10) may be rewritten as

$$(5.9)\qquad E[\mathbf{n}^c\cdot\mathbb{T}^*\delta_\Sigma](\mathbf{x}) = -E[\mathbf{n}\cdot\mathbb{T}^*\delta_\Sigma](\mathbf{x})$$
$$= -\int\delta(\tilde{\mathbf{x}}-\mathbf{x}+a\mathbf{n})\mathbf{n}\cdot\mathbb{T}^*(\tilde{\mathbf{x}}+a\mathbf{n}|\tilde{\mathbf{x}})\phi^{(1)}(\tilde{\mathbf{x}})d\tilde{\mathbf{x}}$$
$$= -\int\delta(\mathbf{k}+a\mathbf{n})\mathbf{n}\cdot\mathbb{T}^*(\mathbf{x}+\mathbf{k}+a\mathbf{n}|\mathbf{x}+\mathbf{k})\phi^{(1)}(\mathbf{x}+\mathbf{k})d\mathbf{k},$$

where $\mathbf{k}=\tilde{\mathbf{x}}-\mathbf{x}$. Then, by expanding $\mathbf{n}\cdot\mathbb{T}^*(\mathbf{x}+\mathbf{k}+a\mathbf{n}|\mathbf{x}+\mathbf{k})\phi^{(1)}(\mathbf{x}+\mathbf{k})$ in a Taylor series around $\mathbf{x}$, the integrand in (5.9)becomes according to the formula (2.5):

$$(5.10)\qquad E[\mathbf{n}^c\cdot\mathbb{T}^*\delta_\Sigma](\mathbf{x}) = -\int\delta(\mathbf{k}+a\mathbf{n})\sum_{\mathrm{m}=0}^{\infty}\frac{1}{\mathrm{m}!}\mathbf{k}^{\mathrm{m}}\boxed{\mathrm{m}}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}^{\mathrm{m}}}\mathbf{n}$$
$$\cdot\mathbb{T}^*(\mathbf{x}+a\mathbf{n}|\mathbf{x})\phi^{(1)}(\mathbf{x})d\mathbf{k} = \sum_{\mathrm{m}=0}^{\infty}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}}\boxed{\mathrm{m}}\phi^{(1)}(\mathbf{x})a^2\frac{(-1)^{\mathrm{m}+1}}{\mathrm{m}!}$$
$$\int(a\mathbf{n})^{\mathrm{m}}\mathbf{n}\cdot\mathbb{T}^*(\mathbf{x}+a\mathbf{n}|\mathbf{x})d\Omega,$$

where the second line results from a change in integration variables $d\mathbf{k}=r^2d\Omega dr$, where $r(\mathbf{x})=|\mathbf{k}|$. Finally, we succeeded in expanding:

$$(5.11)\qquad E[\mathbf{n}^c\cdot\mathbb{T}^*\delta_\Sigma](\mathbf{x}) = \sum_{\mathrm{m}=0}^{\infty}a^{\mathrm{m}}\frac{\partial^{\mathrm{m}}}{\partial\mathbf{x}}\boxed{\mathrm{m}}[\phi^{(1)}(\mathbf{x})\mathbb{M}^*_{\mathrm{m}+1}(\mathbf{x})]$$
$$= N\phi_1(\mathbf{x})\mathbb{M}^*_1(\mathbf{x}) + aN\frac{\partial}{\partial\mathbf{x}}\cdot[\phi_1(\mathbf{x})\mathbb{M}^*_2(\mathbf{x})] + \mathbb{R}^*(\mathbf{x}),$$

where the first and second terms represent the contributions of the monopole and dipole, while the remaining term $\mathbb{R}^*$ represents the total influence of all the other multipoles of higher order. As it will be shown in the future, the above expansion opens the way to various approximation schemes in which only the lowest-order multipoles are retained.

Consider $\overline{\mathbf{n}^{\mathrm{m}}}$, the irreducible tensor of rank m constructed with the vector $\mathbf{n}$, i.e. the traceless tensor that is symmetrical in any pair of its indices. If the tensor $\mathbf{n}^{\mathrm{m}}$ is replaced in (5.8) and (5.9) by $\overline{\mathbf{n}^{\mathrm{m}}}$, the resulting multipoles cannot be reduced and do not contain contributions of lower-rank multipoles. As shown by MAZUR and VAN SAARLOOS [12] in a different context, the above series can be transformed alternatively into a more convenient series of these irreducible multipoles; it should be noted that the monopole and dipole contributions remain the same. Moreover, it can be shown that the remaining contributions are still power expansions of $a\dfrac{\partial}{\partial\mathbf{x}}$ and thus $\mathbb{R}^*=O(\beta^2)$.

Furthermore, the first and second multipole in the multipole expansion (5.11), which are irreducible without transformation, can be easily related to the interaction term already found in the dispersed-phase model since comparing (2.14), (2.16) with (5.8) for $\mathrm{m} = 0, 1$ obviously yields:

$$\mathbf{F}^* = -\mathbb{M}_1^* \quad \text{and} \quad \mathbf{K}^* = a\varepsilon : \mathbb{M}_2^*. \tag{5.12}$$

In fact, the hydrodynamics torque $\mathbf{K}^*$ is only related to the antisymmetrical part $\mathbb{M}_2^{*a}$ of $\mathbb{M}_2$ and, conversely, $\mathbb{M}_2^{*a} = -(2a)^{-1}\varepsilon \cdot \mathbf{K}^*$. Similar relations were derived by MAZUR and VAN SAARLOOS [12]. Generally, the first two force multipoles have a clearer meaning and can be computed more easily than $E[\mathbf{n}^c \cdot \mathbb{T}^* \delta_\Sigma](\mathbf{x})$ as defined in (3.10).

The same arguments as before show that the averaged interfacial force density for the field with one fixed inclusion can be expanded according to:

$$\begin{aligned} E[\mathbf{n}^c \cdot \mathbb{T}^{**} \delta_\Sigma^1](\mathbf{x}^\circ|\mathbf{x}) &= (N-1) \sum_{\mathrm{m}=0}^{\infty} a^{\mathrm{m}} \frac{\partial^{\mathrm{m}}}{\partial \mathbf{x}^{\circ \mathrm{m}}} \boxed{\mathrm{m}} [\chi_2(\mathbf{x}^\circ|\mathbf{x}) \mathbb{M}_{\mathrm{m}+1}^{**}(\mathbf{x}^\circ|\mathbf{x})] \\ &= (N-1)\chi_2(\mathbf{x}^\circ|\mathbf{x})\mathbb{M}_1^{**}(\mathbf{x}^\circ|\mathbf{x}) + a(N-1)\frac{\partial}{\partial \mathbf{x}^\circ} \cdot [\chi_2(\mathbf{x}^\circ|\mathbf{x})\mathbb{M}_2^{**}(\mathbf{x}^\circ|\mathbf{x})] \\ &\quad + (N-1)\mathbb{R}^{**}(\mathbf{x}^\circ|\mathbf{x}). \end{aligned} \tag{5.13}$$

Unlike the previous case, it is impossible to arrange the expansion terms with respect to β since the space scale of the dependence of various higher-order conditioned variables equals a and not L (see Sec. 2.1). The averaged interfacial force density for the continuous field with two fixed inclusions could equally have been treated in a similar way but its expression will not be presented here. Of course, we have $\mathbb{M}_1^{**}(\mathbf{x}^\circ|\mathbf{x}) = -\mathbf{F}^*(\mathbf{x}^\circ) - \mathbf{F}^{**}(\mathbf{x}^\circ|\mathbf{x})$.

Thus interaction terms in momentum equations could be expressed for any order and any phase in terms of force multipoles. A monopole and a dipole are simply required in the dispersed-phase equations at any order. An infinite sequence of multipoles is stricly necessary for the continuous phase; at the first order, the first two monopoles are clearly enough to provide a good approximation in most cases; at higher-order approximations schemes are more difficult to devise.

6. Conclusions

At the end of Part I, a double hierarchy of equations, one for each phase, was available to lay the fundations of a general description of laminar flows carrying spherical inclusions. Specific two-phase flow models could in principle be extracted from these infinite hierarchies by proper truncation based on diluteness. This project could not in fact be undertaken readily as the form of several equations was very complicated. This is why the central concept of disturbance flow field in the averaged sense was defined for both phases. The equations for both hierarchies at any order but the first one have been replaced by equations controlling these new disturbance fields. In this way, the equations have proved to be not only simpler but easier to interpret.

The two first-order equations for each phase are not replaced but are just simplified. As they also form the basis of any usual two-fluid model approach, it is worth comparing the traditional position, where "unknown terms" make it necessary to guess the closure equations, to our new position, where these terms are naturally connected to higher-order equations, namely the afore-mentioned disturbance equations. Confining our attention to the momentum equation of the continuous phase, which contains the most difficult closure problems (DREW [4]), four specific "constitutive equations" are needed: the extra-deformation tensor, the interfacial force density, the pressure relation and the pseudo-turbulent stresses.

(i) The extra-deformation tensor is often by-passed in usual two-fluid models. Here, it is expressed in terms of the averaged dispersed-phase velocities and makes various contributions; some are important at low inclusion Reynolds numbers, and others at higher numbers. As far as solid particles are concerned, higher-order equations are not required; our expression was obtained previously by JOSEPH and LUNDGREN [9] in a different context.

(ii) Modelling the interfacial force density in a standard approach is tantamount to introducing various types of force exterted on an isolated inclusion calculated for a physical context defined in a relatively deterministic way: drag force, lift force, virtual mass force, etc. In our approach the isolated inclusion problem is perfectly defined: it is given by the first-order disturbance flow equations. Its solution leads to a sequence of force multipoles describing the interfacial stresses around a test inclusion with increasing precision; it may give some unexpected information, in particular near walls. In this case putting the interfacial force density in the same category as an overall force i.e. as a monopole, seems clearly inappropriate.

Incidentally, it should be noted that almost all the existing "theories" aimed at modelling two-phase flows rely on some typical small-scale hydrodynamics or thermal models for single-particle changes, in order to append the "constitutive relations" for the interfacial exchanges of momentum and energy in a heuristic manner. Our theory proposes a natural framework, the disturbance flow equations, for specifying the micro-problems that need to be solved to provide the missing information.

(iii) The pressure relation due to inclusion movements (STUHMILLER [15]) is simply not needed in our approach. Pressure differences between the averaged bulk pressure and the averaged interfacial pressure do not appear; in our case the extra averaged fields that are introduced are conditionally averaged upon the presence of one inclusion and lead on the contrary to consistent simplifications.

(iv) Unfortunately, a thorough analysis of pseudo-turbulent tensors and of correlation functions is still lacking, in both the continuous-phase momentum equation and in other equations. As they stand, they are just unclosed terms. It is not yet possible to commence the truncation procedure. What will be proposed in the next paper will be to derive expressions for any of them which can be effectively computed in terms of the selected main variables of the two hierarchies.

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