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Asymptotic analysis of heat propagation models

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THE SUBJECT OF THIS PAPER is the analysis of different models of heat propagation. As is well known, one of essential disadvantages of the classical model proposed by Fourier is the infinite velocity at which heat propagates. To avoid that unphysical phenomenon, Cattaneo has proposed a hyperbolic model. An essential feature of that model is the introduction of a relaxation time for thermal processes. In recent years several new models have been proposed which retain the relaxation time phenomenon but are parabolic in their character. When the relaxation time is small, all these models lead to singularly perturbed equations. We analyze some of these models and prove that the solution of the classical heat equation (Fourier model) is a bulk approximation to exact solutions of these models. We show also that the behaviour of the Fourier model depends on the way in which it is applied. Finally, we present numerical comparison of exact solutions with the bulk solution for the test problem of heat propagation in thin metal films heated by a laser beam.

1. Models of heat propagation

A GENERAL PROBLEM IN MODELING heat propagation is the choice of a correct model connecting the heat flux with the temperature gradient. This problem has been thoroughly discussed in two papers by JOSEPH and PREZIOSI [13, 14] and the review of recent literature can be found in the paper by CHANDRASEKHARAIAH [4]. As is well known, the classical assumption of Fourier

$$q = -k_1 \nabla T$$

leads to the parabolic heat equation

$$\rho C_p \frac{\partial T}{\partial t} = k_1 \nabla^2 T\,, \tag{1.1}$$

where ρ is the density, C_p the specific heat and k_1 the thermal conductivity of the material. One of essential features of this model is the infinite velocity of propagation of heat disturbances. During experiments involving very low temperatures near the absolute zero, extremely short pulse laser heating or very high heat fluxes, investigators found that the heat propagation velocity becomes finite. To account for the phenomena involving the finite propagation velocity, the classical Fourier heat flux model should be modified. CATTANEO [3] has proposed to model the heat flux as dependent on the history of the temperature gradient. This leads to the following differential relation between the heat flux and the temperature gradient:

$$\tau \frac{\partial q}{\partial t} + q = -k_2 \nabla T,$$

and the hyperbolic heat equation

$$\frac{\partial^2 T}{\partial t^2} + \frac{1}{\tau}\frac{\partial T}{\partial t} = c^2 \nabla^2 T\,, \tag{1.2}$$

where $c = \sqrt{k_2/\rho\tau C_p}$ is the velocity of propagation of thermal waves (τ is the relaxation time of thermal processes). For this model we obtain a finite velocity of propagation of heat waves but we loose a clear sense of the heat conductivity introduced in the Fourier model. In addition we encounter problems with the second law of thermodynamics (cf. [17]).

A different model has been proposed by JOSEPH and PREZIOSI [13] (called the Jeffreys-type model), which retains relaxation time phenomenon but also uses the effective Fourier conductivity k_1 explicitly

$$q(t) = -k_1 \nabla T - \frac{k_2}{\tau} \int\limits_{-\infty}^{t} \exp(-\frac{t-t'}{\tau}) \nabla T(t') dt'\,. \tag{1.3}$$

For steady flows the thermal conductivity is given by

$$k = k_1 + k_2,$$

i.e. is the sum of the effective thermal conductivity k_1 and the elastic conductivity k_2. Equation (1.3) leads to the following equation for the temperature

$$\frac{\partial^2 T}{\partial t^2} + \frac{1}{\tau}\frac{\partial T}{\partial t} = c^2 \nabla^2 T + \kappa_1 \nabla^2 \frac{\partial T}{\partial t}\,, \tag{1.4}$$

where $c = \sqrt{k/\rho\tau C_p}$ is the velocity of propagation of heat impulses and $\kappa_1 = k_1/\rho C_p$.

Let us mention finally the inertial theory obtained as a limiting case of the Cattaneo model where $\tau \to \infty$ but $k_2/\tau = k^*$ remains finite. Then, we obtain the following law of heat conduction

$$\frac{\partial q}{\partial t} = -k^* \nabla T\,,$$

and the pure wave equation for the temperature

(1.5) $$\frac{\partial^2 T}{\partial t^2} = c^2 \nabla^2 T,$$

where $c = \sqrt{k_2/\rho\tau C_p}$.

A nonlinear extension of the last model has been proposed by CIMMELLI and KOSIŃSKI [5, 6]. They have postulated the following law of thermal conductivity

$$q = -\chi \nabla \Theta,$$

where χ is a positive function representing the coefficient of thermal conductivity. Θ is the semi-empirical temperature representing a thermal history of the material and obey the evolution equation $\dot{\Theta} = F(T, \Theta)$. A particular linear version of this model was proposed earlier by IGNACZAK [11]

$$\frac{\partial q}{\partial t} = -k \nabla \frac{\partial T}{\partial t} - \frac{\kappa}{T_0} \nabla T.$$

The above model of heat conduction leads to the following equation for temperature

$$\rho C_p \frac{\partial^2 T}{\partial t^2} = k \nabla^2 \frac{\partial T}{\partial t} - \frac{\kappa}{T_0} \nabla^2 T.$$

The question which of these models is more realistic is difficult to answer. A number of papers defend the Fourier model (cf. [7, 8]) claiming that although the theoretical speed of heat impulses in this model is infinite, but the bulk of the heat energy propagates with a finite speed. The present paper can be considered as a voice in this discussion. We restrict our considerations to very short heat pulses at room temperature. This corresponds to a very short relaxation time τ. Hence we exclude from our comparison the analysis of inertial theory and its nonlinear extensions as they correspond to $\tau \to \infty$.

In what follows, we shall analyze the Fourier, Cattaneo and Jeffreys-type models. It is shown both by asymptotic analysis and numerical calculations that for the propagation time of macroscopic size there is no difference between predictions of all the three models. The situation changes for short distances of propagation and very short time. Here the predictions of the Fourier model differ form those of the Cattaneo and Jeffreys-type models. Which model is more realistic should be decided by comparison of numerical results with experimental data. For experimental results which we have found in the literature [2], the agreement is better with the Cattaneo and Jeffreys-type models (cf. Sec. 4).

2. Asymptotic analysis of the models

When the relaxation time is very small in comparison with the macroscopic time, Eqs. (1.2) and (1.4) are singularly perturbed. Hence we can apply singular perturbation methods to find the behaviour of solutions in short time scale (initial layer) and in long time scale (bulk approximation).

We begin our analysis with the Cattaneo Eq. (1.2). Writing this equation in the dimensionless form as a first-order system, we obtain

$$\begin{aligned}\partial_t\theta + \partial_x\eta &= 0,\\ \tau\partial_t\eta + D^2\partial_x\theta + \eta &= 0.\end{aligned} \tag{2.1}$$

In the above equations θ and η denote the dimensionless temperature and the heat flux, respectively, $D^2 = \dfrac{k_2 t_0}{\rho C_p x_0^2}$ is the dimensionless coefficient of thermal conductivity, where t_0 is the characteristic time and x_0 the characteristic length, and ∂_x denotes the nabla operator (*gradient* or *divergence*, depending on the context).

Now we apply the standard asymptotic procedure, i.e. we expand θ and η in power series of τ

$$\begin{aligned}\theta &= \theta_0 + \tau\theta_1 + \ldots,\\ \eta &= \eta_0 + \tau\eta_1 + \ldots.\end{aligned} \tag{2.2}$$

Inserting expansion (2.2) into Eq. (2.1) and comparing terms of the same order in τ, we obtain equations for consecutive terms of the bulk approximation. For the zeroth-order it gives

$$\begin{aligned}\partial_t\theta_0 + \partial_x\eta_0 &= 0,\\ D^2\partial_x\theta_0 + \eta_0 &= 0.\end{aligned}$$

After rearranging the terms we obtain the classical heat Eq. (1.1) for the zeroth-order approximation to the temperature, and the Fourier formula of the heat flux

$$\begin{aligned}\partial_t\theta_0 &= D^2\nabla^2\theta_0,\\ \eta_0 &= -D^2\nabla\theta_0.\end{aligned} \tag{2.3}$$

In the first-order we have

$$\begin{aligned}\partial_t\theta_1 + \partial_x\eta_1 &= 0,\\ \partial_t\eta_0 + D^2\partial_x\theta_1 + \eta_1 &= 0,\end{aligned}$$

which after rearrangement leads to the nonhomogeneous heat equation

$$\partial_t\theta_1 = D^2\nabla^2\theta_1 - D^2\nabla^2\partial_t\theta_0\,.$$

To account for the short time effects we introduce the new time variable

$$\sigma = \frac{t}{\tau}\,.$$

Then Eq. (2.1) take the form

$$\begin{aligned} &\frac{1}{\tau}\partial_\sigma\tilde{\theta} + \partial_x\tilde{\eta} = 0\,, \\ &\partial_\sigma\tilde{\eta} + D^2\partial_x\tilde{\theta} + \tilde{\eta} = 0\,, \end{aligned} \tag{2.4}$$

where *tilde* denotes the functions of the new variables (x,σ).

The above equations form again a singularly perturbed system with small parameter τ. We look for a solution of this system by expanding $\tilde{\theta}$ and $\tilde{\eta}$ in power series of τ. Then in the zeroth-order we obtain

$$\begin{aligned} &\partial_\sigma\tilde{\theta}_0 = 0\,, \\ &\partial_\sigma\tilde{\eta}_0 + \tilde{\eta}_0 = 0\,. \end{aligned} \tag{2.5}$$

Assuming that the initial layer solutions tend to zero at infinity, we get from (2.5)

$$\begin{aligned} \tilde{\theta}_0(\sigma,x) &= 0\,, \\ \tilde{\eta}_0(\sigma,x) &= \tilde{\eta}_0(0,x)e^{-\sigma}\,. \end{aligned}$$

Let us observe that the initial layer equations are necessary to fulfill the initial conditions. Let $\theta(0,x)$ and $\eta(0,x)$ be the initial conditions to (2.1). If these initial functions are independent of τ then Eqs. (2.1), (2.3) and (2.5) give

$$\begin{aligned} \theta(0,x) &= \theta_0(0,x)\,, \\ \eta(0,x) &= \eta_0(0,x) + \tilde{\eta}_0(0,x)\,. \end{aligned}$$

Assuming that all the considered functions are extended down to the initial surface $t=0$ and taking into account the equality

$$\eta_0(0) = -D^2\partial_x\theta_0(0),$$

we can fulfill the initial data taking

$$\begin{aligned} \theta_0(0,x) &= \theta(0,x)\,, \\ \tilde{\eta}_0(0,x) &= \eta(0,x) + D^2\partial_x\theta(0,x). \end{aligned}$$

An essential step in asymptotic analysis is the comparison of the approximate solution with the exact one. We shall carry on that procedure in the Hilbert space setting $H = L^2(\mathbb{R}^n)$. With some abuse of the notation we shall write $w \in H$ even if w is an n-dimensional vector, understanding in such a case that every component of w is in H and using the obvious extension of the norm in H to vector functions. Then the following result is standard in asymptotic analysis.

PROPOSITION 1. *Let us consider the Cauchy problem for Eq.* (2.1) *with initial data belonging to* $W_0^{2,1}(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n)$. *Then both the exact Eq.* (2.1) *and the zeroth-order approximation* (2.3) *possess solutions in* $C^1(\mathbb{R}^+, H)$ *and the following estimate holds in the norm of* H;

$$\|\theta(t) - \theta_0(t)\| = O(\tau)\,.$$

In addition for the heat flux we obtain

$$\|\eta(t) - \eta_0(t) - \tilde{\eta}_0(t/\tau)\| = O(\tau^{1/2})\,.$$

P r o o f: We omit the existence part of the proof as a standard one and concentrate on the error estimate. Let us define

$$y = \theta - \theta_0\,,$$
$$z = \eta - \eta_0 - \tilde{\eta}_0\,.$$

Inserting y and z into Eq. (2.1) and making use of the fact that θ_0 and η_0 solve Eq. (2.3), and $\tilde{\eta}_0$ Eq. (2.5), we obtain

$$\begin{aligned} \partial_t y + \partial_x z &= -\partial_x \tilde{\eta}_0\,, \\ \tau \partial_t z + D^2 \partial_x y + z &= \tau D^2 \partial_t \partial_x \theta_0\,. \end{aligned} \tag{2.6}$$

This equation can be written as an evolution equation in the matrix form

$$\partial_t u = \mathcal{S} u + \mathcal{C} u + g\,,$$

where

$$u = \begin{bmatrix} y \\ z \end{bmatrix},$$

$$\mathcal{S} = \begin{bmatrix} 0 & -\partial_x \\ -\frac{D^2}{\tau}\partial_x & 0 \end{bmatrix}, \quad \mathcal{C} = \frac{1}{\tau}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$g = \begin{bmatrix} -\partial_x \tilde{\eta}_0 \\ D^2 \partial_t \partial_x \theta_0 \end{bmatrix}$$

is the nonhomogeneous term.

We encounter a problem in this analysis because $\mathcal{S}$ is not a dissipative operator (cf. [1]). To avoid that difficulty we make a change of variables

$$\hat{u} = \begin{bmatrix} D & 0 \\ 0 & \sqrt{\tau} \end{bmatrix} u \,.$$

For $\hat{u}$ we get the equation

$$\partial_t \hat{u} = \mathcal{S}' \hat{u} + \mathcal{C}\hat{u} + g' \,,$$

where

$$\mathcal{S}' = \frac{D}{\sqrt{\tau}} \begin{bmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{bmatrix}, \quad g = \begin{bmatrix} -D\partial_x \tilde{\eta}_0 \\ \sqrt{\tau} D^2 \partial_t \partial_x \theta_0 \end{bmatrix}.$$

Operator $\mathcal{S}'$ is already dissipative (it is in fact a conservative operator in H) and generates a semigroup of contractions in H. By standard methods of asymptotic analysis (cf. [1] Chap. 6) we obtain the estimate

$$\|\hat{u}(t)\| = O(\tau)$$

uniformly for $t \in [0, T]$, for any $T < \infty$.

Returning to the original variables y and z we obtain the assertion of the proposition.

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Let us remark that the result of PROPOSITION 1 is well known in the literature (cf. GOLDSTEIN [9] Chap. 2 Sec. 11) and we present it only for the readers' convenience.

It is interesting to compare the above estimate with the result of JANSSEN [12] who analyzed the abstract telegraph equation. The result of Janssen in the Hilbert space $H = L^2(\mathbb{R}^n)$ setting has the form of

PROPOSITION 2. Let $\theta_0(t)$ be a bounded solution of the heat Eq. (2.3) for initial data belonging to $W_0^{2,1} \cap W^{2,2}$. Then the telegraph Eq. (1.2) with the same initial data possesses a solution $\theta(t)$ and for every $c_0 > 0$ there exists a constant C such that

$$\|\theta(t) - \theta_0(t)\| \le C \left(\frac{1}{t/\tau} + \frac{1}{(t/\tau)^{1/2} c} \right),$$

for every $c > c_0$, $t > 0$ and $\tau > 0$, where $c^2 \tau = D^2$.

PROPOSITION 2 gives a long time convergence for solutions of the telegraph equation to solutions of the heat equation whereas PROPOSITION 1 gives only estimate on a bounded time interval. The term $\tau^{1/2}$ in the estimate of PROPOSITION 2 seems to be unexpected (we expect $O(\tau)$). Let us observe however, that

$c\tau^{1/2} = D$, where D is a constant. Hence the estimate in PROPOSITION 2 is in fact of order $O(\tau)$ which is standard for asymptotic analysis.

We proceed now to the analysis of the Jeffreys-type model of Joseph and Preziosi. As in the case of Cattaneo's model, we write this equation in the dimensionless form as a first-order system

$$\begin{aligned} &\partial_t\theta + \partial_x\eta = 0\,, \\ &\tau\partial_t\eta + D^2\partial_x\theta + \eta - \kappa^*\tau\partial_{xx}\eta = 0\,, \end{aligned} \tag{2.7}$$

where $D^2 = \dfrac{kt_0}{\rho C_p x_0^2}$ and $\kappa^* = \dfrac{k_1 t_0}{\rho C_p x_0^2}$.

Carrying asymptotic analysis we obtain in the zeroth-order the result analogous to that obtained for the Cattaneo model, i.e. the classical heat equation and the Fourier formula of the heat flux

$$\begin{aligned} \partial_t\theta_0 &= D^2\nabla^2\theta_0\,, \\ \eta_0 &= -D^2\nabla\theta_0\,. \end{aligned} \tag{2.8}$$

Proceeding like previously we obtain for the initial layer in the zeroth-order

$$\begin{aligned} \partial_\sigma\tilde{\theta}_0 &= 0\,, \\ \partial_\sigma\tilde{\eta}_0 + \tilde{\eta}_0 &= 0\,, \end{aligned} \tag{2.9}$$

which are the same equations as for Cattaneo's model.

Then we can formulate our main result for the Jeffreys-type model.

PROPOSITION 3. Let us consider the Cauchy problems for Eqs. (2.7) and (2.8) with the same initial data belonging to $W_0^{2,1}(\mathbb{R}^n)\cap W^{2,2}(\mathbb{R}^n)$. Both these systems possess solutions in $C^1(\mathbb{R}^+, H)$ and the following estimate holds in the norm of H

$$\|\theta(t) - \theta_0(t)\| = O(\tau)\,.$$

In addition for the heat flux we obtain

$$\|\eta(t) - \eta_0(t) - \tilde{\eta}_0(t/\tau)\| = O(\tau^{1/2})\,.$$

Proof: The idea of the proof is analogous to the proof of PROPOSTION 1. We concentrate on an error estimate and define as previously

$$\begin{aligned} y &= \theta - \theta_0\,, \\ z &= \eta - \eta_0 - \tilde{\eta}_0\,. \end{aligned}$$

Inserting y and z into Eq. (2.7) and making use of the fact that θ_0 and η_0 solve Eq. (2.8), and $\tilde{\eta}_0$ Eq. (2.9), we obtain

$$\begin{aligned} &\partial_t y + \partial_x z = -\partial_x \tilde{\eta}_0 , \\ &\tau\partial_t z + D^2\partial_x y + z - \kappa^*\tau\partial_{xx} z = \tau(D^4 - \kappa^* D^2)\partial_{xxx}\theta_0 + \tau\kappa^*\partial_{xx}\tilde{\eta}_0 . \end{aligned} \tag{2.10}$$

With these equations we have similar problems as with (2.6). Changing variables

$$u = \begin{bmatrix} Dy \\ \sqrt{\tau} z \end{bmatrix},$$

we can write it in the matrix form

$$\partial_t u = \mathcal{S}u + \mathcal{C}u + g ,$$

where

$$\mathcal{S} = \frac{D}{\sqrt{\tau}} \begin{bmatrix} 0 & -\partial_x \\ -\partial_x & \frac{\sqrt{\tau}\kappa^*}{D}\partial_{xx} \end{bmatrix}, \quad \mathcal{C} = \frac{1}{\tau}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$g = \begin{bmatrix} -D\partial_x\tilde{\eta}_0 \\ \sqrt{\tau}(D^4 - \kappa^* D^2)\partial_{xxx}\theta_0 + \sqrt{\tau}\kappa^*\partial_{xx}\tilde{\eta}_0 \end{bmatrix}$$

is the nonhomogeneous term.

The operator $\mathcal{S}$ generates a contraction semigroup in H (cf. [16] for details). Then using standard methods of asymptotic analysis we obtain the estimate

$$\|u(t)\| = O(\tau)$$

uniformly for $t \in [0, T]$, for any $T < \infty$.

This estimate in terms of the original variables y and z is exactly the assertion of the proposition.

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3. Jeffreys-type model revisited

Because Jeffreys-type model is not related straightforwardly to experimental results, it is very difficult to make any judgement on the relation between the thermal conductivity k and the effective conductivity k_1. The only relation which follows from considerations of Sec. 1 is that $c^2 \gg \kappa_1$. But recently another approach to the derivation of the Jeffreys-type model has been proposed (cf. [10, 19]). In this model, called *the dual-phase-lag model*, the following general relation between the heat flux and the temperature gradient has been proposed:

$$q(t + \tau_q) = -k\nabla T(t + \tau_T). \tag{3.1}$$

In this relation the delay time τ_T is interpreted as caused by phonon-electron interactions or phonon scattering. The other delay time τ_q is the relaxation time due to the fast transient effects of thermal inertia.

Expanding the left-hand side of (3.1) with respect to τ_q and the right-hand side with respect to τ_T and retaining only the first-order terms, we get

$$q + \tau_q \frac{\partial q}{\partial t} = -k\nabla T - k\tau_T \frac{\partial}{\partial t}\nabla T,$$

which leads to the following equation for the temperature:

$$\frac{\partial^2 T}{\partial t^2} + \frac{1}{\tau_q}\frac{\partial T}{\partial t} = c^2\nabla^2 T + \kappa_1 \frac{\tau_T}{\tau_q}\nabla^2 \frac{\partial T}{\partial t}, \tag{3.2}$$

where $c^2 = k/\rho\tau_q C_p$ and $\kappa_1 = k/\rho C_p$. The relation between terms in the right-hand side of (3.2) depends on the ratio τ_T/τ_q. For thin gold films for which we have made comparison with experimental data in both papers [10] and [19], it is reported that $\tau_T/\tau_q \approx 120$ which is $\gg 1$ and makes the analysis performed below justifiable.

We write Eq. (3.2) in the dimensionless form as a first order system

$$\begin{aligned} \partial_t\theta + \partial_x\eta &= 0, \\ \tau\partial_t\eta + D^2\partial_x\theta + \eta - \kappa_2\partial_{xx}\eta &= 0, \end{aligned} \tag{3.3}$$

where $D^2 = \dfrac{kt_0}{\rho C_p x_0^2}$, $\kappa_2 = \dfrac{k\tau_T}{\rho C_p x_0^2}$ and τ is the dimensionless value of τ_q.

Since $\tau_T \gg \tau_q$ coefficient κ_2 is of order $O(1)$ with respect to τ. Hence the standard procedure leads to the following zeroth-order approximation for the dimensionless temperature and the heat flux

$$\begin{aligned} \partial_t\theta_0 &= D^2\partial_{xx}\theta_0 + \kappa_2\partial_t\partial_{xx}\theta_0, \\ \eta_0 &= -D^2\partial_x\theta_0 - \kappa_2\partial_t\partial_x\theta_0. \end{aligned} \tag{3.4}$$

Let us observe the difference between this system and the system obtained in the zeroth-order for the original Jeffreys model. Also the initial layer equation has a different form. In the zeroth-order we obtain

$$\partial_\sigma\tilde{\eta}_0 = \kappa_2\partial_{xx}\tilde{\eta}_0 - \eta_0.$$

Then we can write the equation for the error term. Using the same notation as in the preceding section we obtain

$$\begin{aligned} &\partial_t y + \partial_x z = -\partial_x\tilde{\eta}_0, \\ &\tau\partial_t z + D^2\partial_x y + z - \kappa_2\partial_{xx} z = \tau D^2\partial_t\partial_x\theta_0 + \tau\kappa_2\partial_{tt}\partial_x\theta_0. \end{aligned} \tag{3.5}$$

The above system is analogous to Eq. (2.10) (the difference is only in the non-homogeneous term). Hence PROPOSITION 3 remains valid also for systems (3.4) and (3.5).

We can now return to Eq. (3.4). This is an example of Sobolev's equation (cf. [18]) and its analysis is relatively simple. First, let us write this equation in the more convenient form

$$\partial_t u = D^2(I - \kappa_2\Delta)^{-1}\Delta u\,. \tag{3.6}$$

To prove that Eq. (3.6) possesses smooth solutions let us take $K(x)$, the fundamental solution of the operator $I - \kappa_2\Delta$. Then Eq. (3.6) can be written as

$$\partial_t u = D^2 K \star \Delta u\,.$$

The following result is due to KARCH [15].

LEMMA 4. Let $A = D^2K \star \Delta$, then A has a unique extension to a bounded operator on $L^2(\mathbb{R}^n)$ and is the infinitesimal generator of uniformly continuous group $\mathcal{T}(t)$.

Let us consider the heat equation

$$\partial_t u = D^2 \Delta u\,, \tag{3.7}$$

and let $\mathcal{T}_0(t)$ denote the semigroup generated by operator $D^2\Delta$. Then we have the following result.

PROPOSITION 5. Let $u_0(x) \in W_0^{2,1}(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n)$ denote initial data for Eqs. (3.6) and (3.7). Then in the Hilbert space $H = L^2(\mathbb{R}^n)$ we have the following time asymptotics for the solutions of these two equations:

$$t^{n/4}\|\mathcal{T}(t)u_0 - \mathcal{T}_0(t)u_0\| \to 0 \quad \text{as} \quad t \to \infty\,.$$

Proof: Passing to the Fourier transform we can write

$$\mathcal{T}(t)u_0 = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \exp\Big(\frac{-D^2\xi^2 t}{1+\kappa_2\xi^2} + ix\xi\Big)\hat{u}_0(\xi)d\xi\,,$$

$$\mathcal{T}_0(t)u_0 = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \exp\Big(-D^2\xi^2 t + ix\xi\Big)\hat{u}_0(\xi)d\xi\,.$$

Using the Plancherel formula we get

$$\|\mathcal{T}(t)u_0 - \mathcal{T}_0(t)u_0\|^2$$

$$= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \left|\exp\left(\tfrac{-D^2\xi^2 t}{1+\kappa_2\xi^2}\right) - \exp\left(-D^2\xi^2 t\right)\right|^2 |\hat{u}_0(\xi)|^2 d\xi$$

$$\le (2\pi)^{-n} \left(\int\limits_{\mathbb{R}^n} \left|\exp\left(\frac{-D^2\xi^2 t}{1+\kappa_2\xi^2}\right) - \exp\Big(-D^2\xi^2 t\Big)\right|^2 d\xi\right)^{1/2}$$

$$\times \left(\int\limits_{\mathbb{R}^n} \left|\exp\left(\frac{-D^2\xi^2 t}{1+\kappa_2\xi^2}\right) - \exp\left(-D^2\xi^2 t\right)\right|^2 |\hat{u}_0(\xi)|^4 d\xi\right)^{1/2} .$$

If u_0 is sufficiently smooth then the second integral is bounded. In the first integral we make a change of variables $w = t^{1/2}\xi$ to obtain

$$\int\limits_{\mathbb{R}^n} \left|\exp\Big(\frac{-D^2\xi^2 t}{1+\kappa_2\xi^2}\Big) - \exp\Big(-D^2\xi^2 t + ix\xi\Big)\right|^2 d\xi$$

$$= t^{-n/2} \int\limits_{\mathbb{R}^n} \left|\exp\Big(\frac{-D^2 w^2}{1+\kappa_2 w^2 t^{-1}}\Big) - \exp\Big(-D^2 w^2\Big)\right|^2 dw .$$

It is straightforward that the last integral tends to zero as $t \to \infty$.

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4. Numerical results

The results of asymptotic analysis on the one hand, and the experimental results presented in [2] on the other, have been confronted with numerical calculations done for Fourier, Cattaneo, Jeffreys and dual-phase-lag models. Our aim was twofold: first, we would like to see how fast is the convergence of solutions of the Cattaneo, Jeffreys and dual-phase-lag models to solutions of the heat equation (Fourier model); second, we would like to compare different models with experimental results. In [2], the heat propagation in thin metal films (namely in gold films) undergoing very short impulses of laser irradiation were investigated. In these experiments heat waves of very large but finite velocity have been observed. Since the measurements were only one-dimensional, we restricted our numerical analysis also to one space dimension.

Since the Fourier model needs no special comments, we present here some technical details concerning only the Jeffreys-type model (the dual-phase-lag model is defined by the same equations as Jeffreys model, while Cattaneo model can be treated as its special case). To account for short time effects and small sample thickness of metal films (200 to 3000 Å), we introduced the dimensionless variables in the form

$$t = \tau \hat{t}, \qquad x = v\tau \hat{x}, \tag{4.1}$$

where τ is the relaxation time and v is the sound velocity of electrons in metal. Such a change of variables is due to the assumption that heat is transported mostly by the flow of hot electrons. Following [2] we take $v = 0.8 \times 10^8\,\mathrm{cm/s}$. The same paper suggests also that the reference length should be taken as equal $74\,nm$ (the mean free path for hot electrons). This gives the relaxation time $\tau = 92\,fs$ which is almost equal to the duration of laser pulses ($96\,fs$).

In these new variables the original Jeffreys-type model (2.7) is replaced by the following system of equations

$$\begin{aligned} \partial_t T + \partial_x Q &= 0, \\ \partial_t Q + \hat{D}\partial_x T + Q - \kappa^* \partial_{xx} Q &= 0, \end{aligned} \tag{4.2}$$

where $\hat{D} = k/(\rho C_p \tau v^2)$ and $\kappa^* = k_1/(\rho C_p \tau v^2)$ (hats denoting independent variables have been omitted). T and Q are the dimensionless temperature and heat flux in the new variables, respectively. It is easy to see that if the pair $\{T, Q\}$ satisfies Eq. (4.2), then in the original independent variables Eq. (2.7) are satisfied. For the dual-phase-lag model we have the same equations but with $\kappa^* = k\tau_T/(\rho C_p \tau^2 v^2)$ (the value of κ_2 in the dimensionless variables (4.1)).

For numerical treatment of the problem, we replace system (4.2) by the following finite difference equations:

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{k} &+ \frac{A}{2}\left[\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h}\right) + \left(\frac{u_{j+1}^n - u_{j-1}^n}{2h}\right)\right] \\ &+ \frac{B}{2}\left[\left(\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2}\right) + \left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}\right)\right] \\ &+ \frac{C}{2}\left(u_j^{n+1} + u_j^n\right) = 0. \end{aligned} \tag{4.3}$$

Here h and k denote the space and time steps, respectively, n is the time level number and j the space node number of the rectangular space-time grid; u_j^n is the two-dimensional vector with the first coordinate corresponding to the approximation of T, and the second – to the approximation of Q. Matrices A, B, and

C are as follows

$$A = \begin{bmatrix} 0 & 1 \\ \hat{D} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa^* \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The heat impulse is introduced by the left-hand side boundary condition depending on time. On the right-hand side we impose Dirichlet or Neumann zero conditions for both functions T and Q. As initial data we take the zero initial values for T and Q, however there are also other possibilities. Stability and convergence problems for finite difference schemes of this kind will be discussed elsewhere.

Let us now discuss the numerical experiments. We have done computations with a rectangular impulse of duration 96 fs (as in [2]) and amplitude equal to 1. The response to the impulse was observed at four distances from the insulation surface, equal to those used in experimental measurements of [2]. These distances are 500 Å, 1000 Å, 2000 Å, and 3000 Å.

On successive graphs we present the dimensionless temperature as a function of time in picoseconds counted from the start of the heat impulse. Since all the experimental curves are normalized up to their maxima (see Fig. 1), we have performed similar scaling for the numerical results. This, however, enables us to compare only the positions of extrema on the corresponding curves.

We have observed, that for the Jeffreys-type model the existence of extrema depends on the value of κ^*. For large κ^* and at a large distance from the insulation surface there is no clearly visible extremum. Hence we have chosen $\kappa^* = 0.01$ which is sufficiently small to give extrema for all the considered distances.

The best fitting in the Jeffreys and Cattaneo models has been obtained for $\hat{D} = 0.35$ (Fig. 2 Jeffreys, Fig. 3 Cattaneo). The small oscillations on the curves for the Cattaneo model are due to the hyperbolic nature of this model and a very low level of artificial diffusion in the numerical scheme. Thanks to this low diffusion we have obtained rather sharp jumps. Introducing some extra diffusion to our scheme we can diminish the oscillations but smear the jumps.

Similar results for the Fourier model are also presented ($\hat{D} = 0.35$, Fig. 4). It is visible that in the Fourier model maxima are traveling with slightly different speed than in experiments (cf. Fig. 1).

For dual-phase-lag model, which differs from the Jeffreys model only by the values of coefficients $\hat{D}$ and κ^*, computations were done for $\hat{D} = 0.0183$ and $\kappa^* = 2.06$ (Fig. 5). These values have been calculated using the data for gold from [10]. It is worth mentioning that the relaxation time suggested in this paper $\tau_q = 0.79$ ps is almost ten times larger than that used for the Cattaneo and Jeffreys-type models. On the other hand, it is not clear which value should be taken as the reference length. Therefore we have investigated how the numerical results depend on the value of the reference length and found that the position of maxima is almost independent of this value. To obtain the above mentioned values of $\hat{D}$ and κ^* we have taken the same reference length as in other calculations ($74\,nm$). It is visible from Fig. 5 that in this model the heat waves travel too slowly with respect to experimental data of [2].

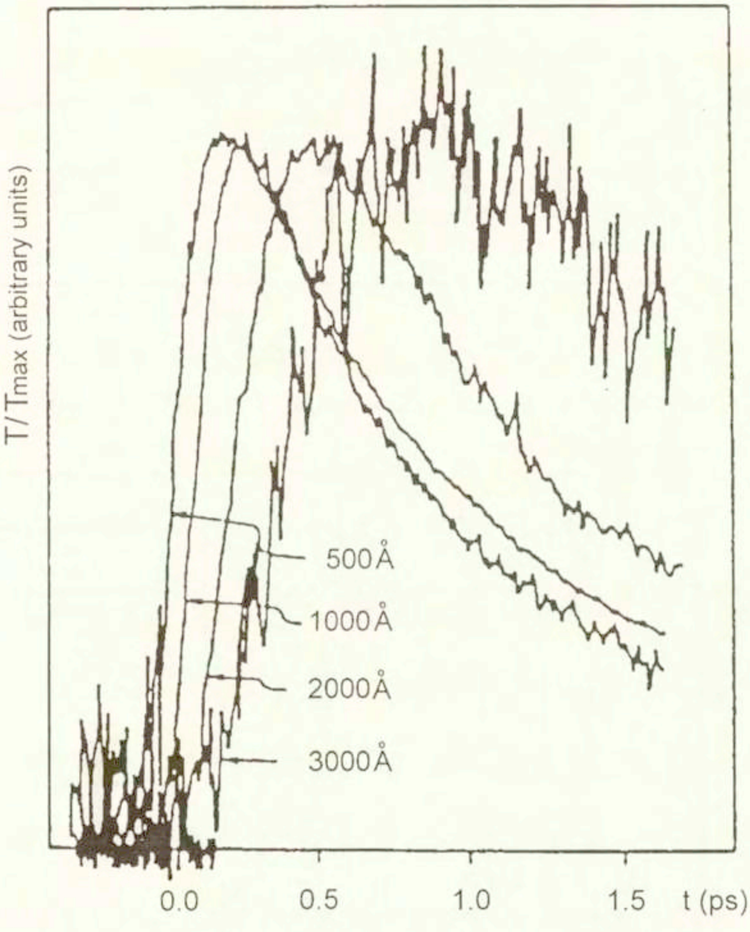

Fig. 1. Experimental data (taken from [2]).

Further, we would like to estimate the time of asymptotic confluence of the Cattaneo and Jeffreys-type models towards the Fourier model. To this end, we have done computations for the distance $x = 500$ Å and sufficiently long time (Fig. 8). It can be seen on this figure that the solutions are confluent, however, for data of the considered models, confluence is rather slow: curves stick after about 100 *ps*.

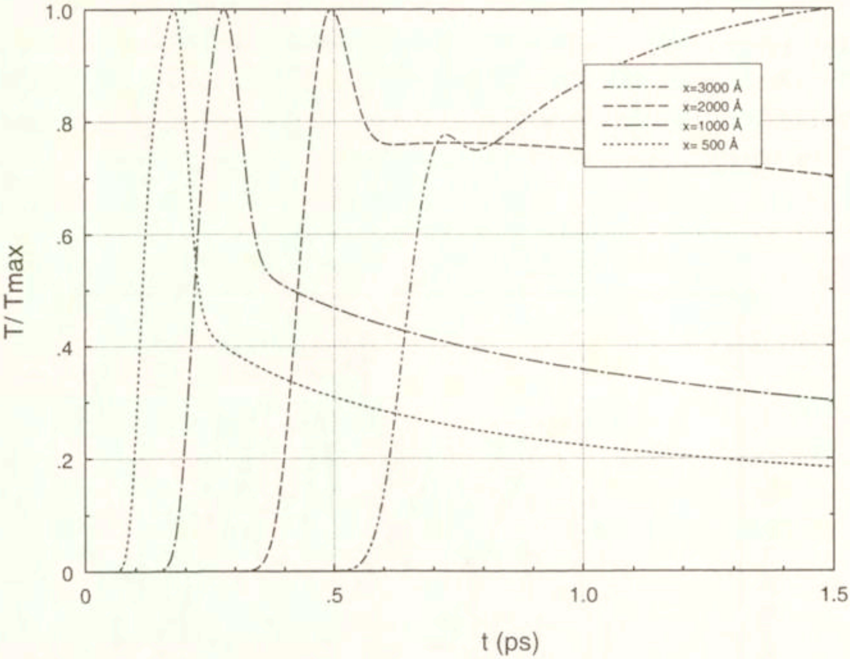

FIG. 2. Jeffreys model with rectangular impulse, $\hat{D} = 0.35$, $\kappa^* = 0.01$, $\tau = 92\,fs$.

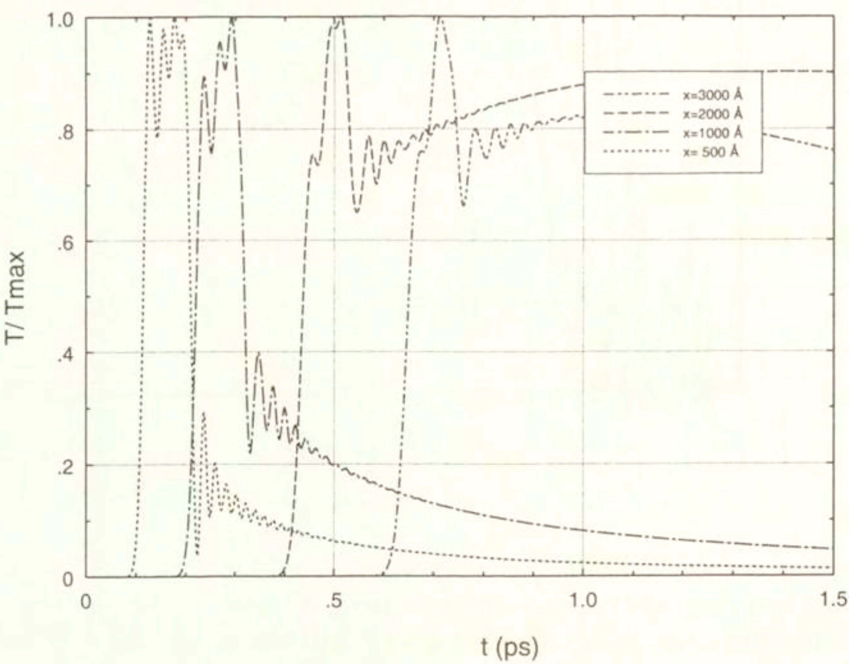

FIG. 3. Cattaneo's model with rectangular impulse, $\hat{D} = 0.35$, $\tau = 92\,fs$.

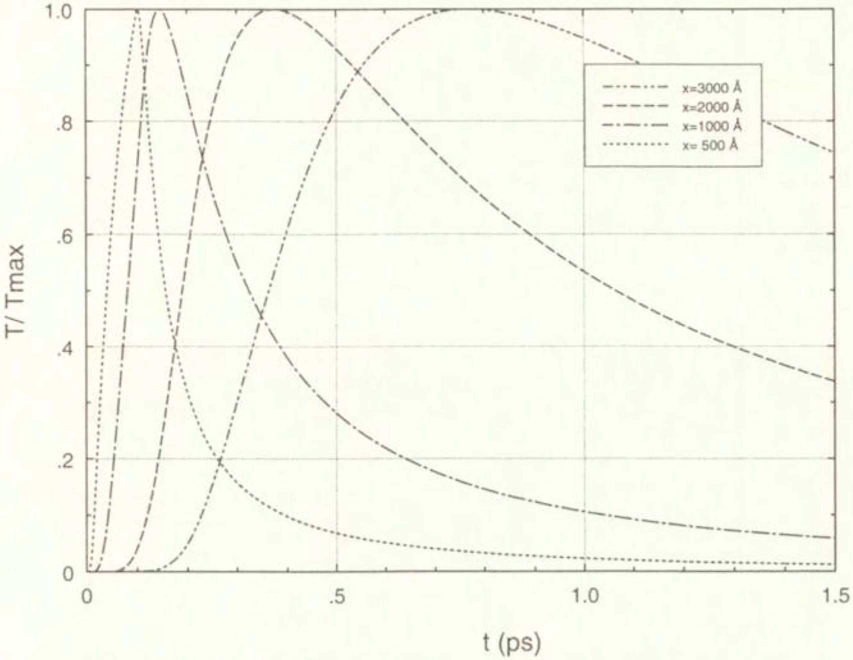

FIG. 4. Fourier model with rectangular impulse, $\hat{D} = 0.35$.

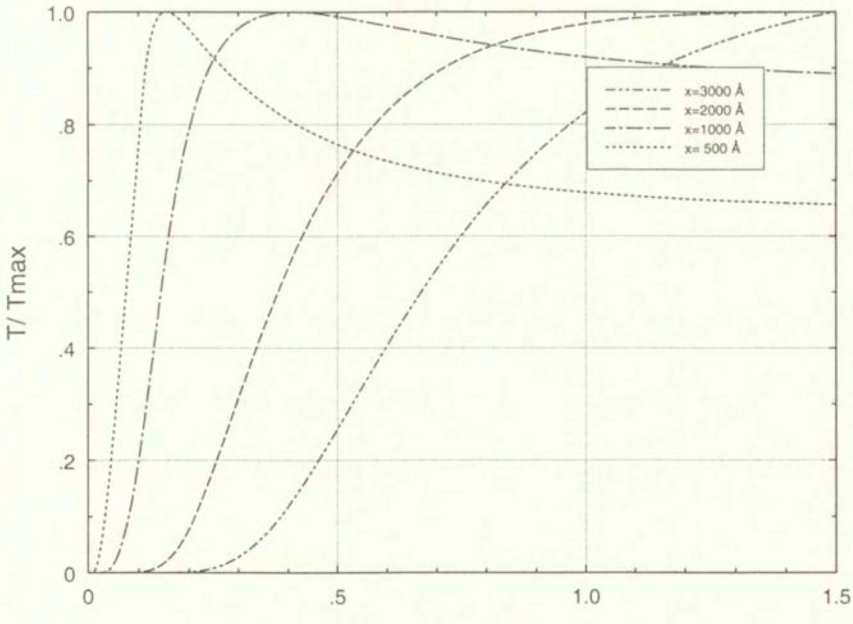

FIG. 5. Dual-phase-lag model with rectangular impulse, $\hat{D} = 0.0183$, $\kappa^* = 2.06$, $\tau = 790\, fs$.

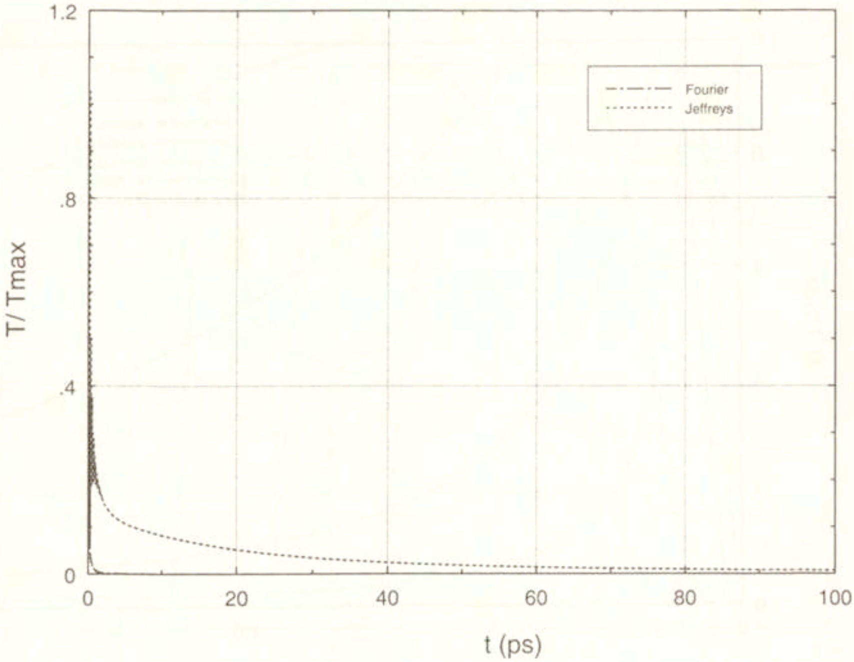

FIG. 6. Fourier and Jeffreys models confluence, temperature distribution at $x = 500$ Å, impulse via left the boundary.

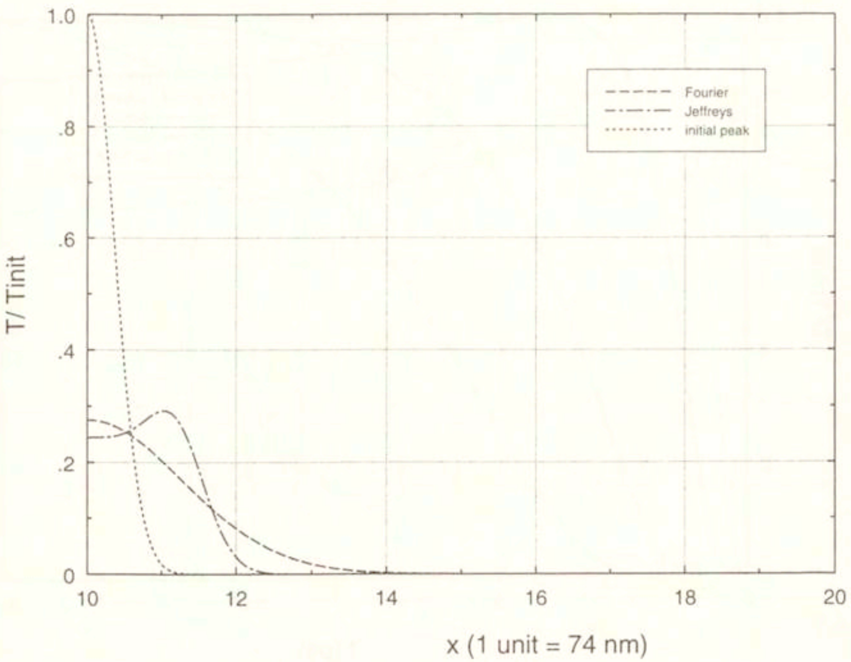

FIG. 7. Temperature distribution at $t = 0.2\,ps$ for Fourier and Jeffreys models, impulse via the initial conditions.

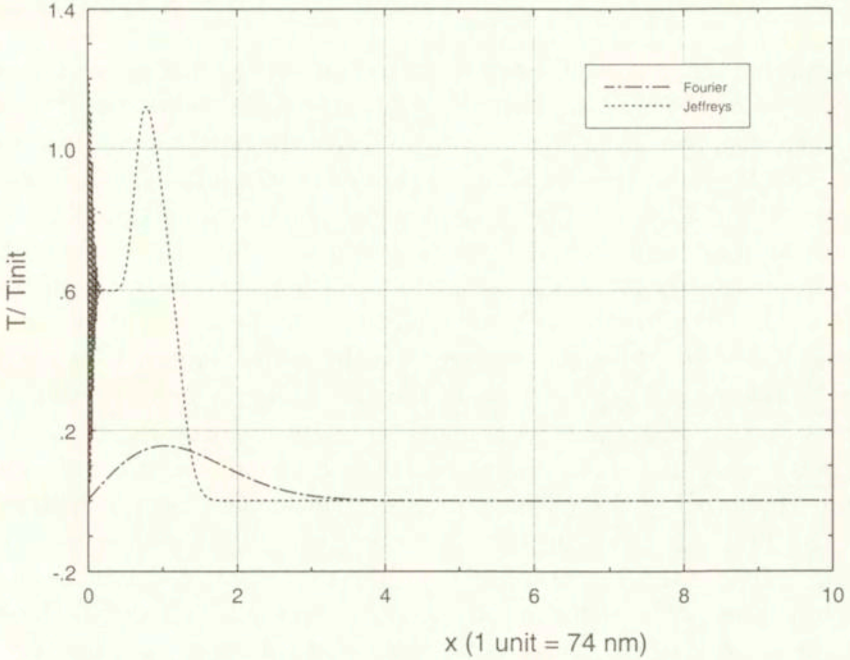

FIG. 8. Temperature distribution at $t = 0.2\,ps$ for Fourier and Jeffreys models, impulse via the left boundary.

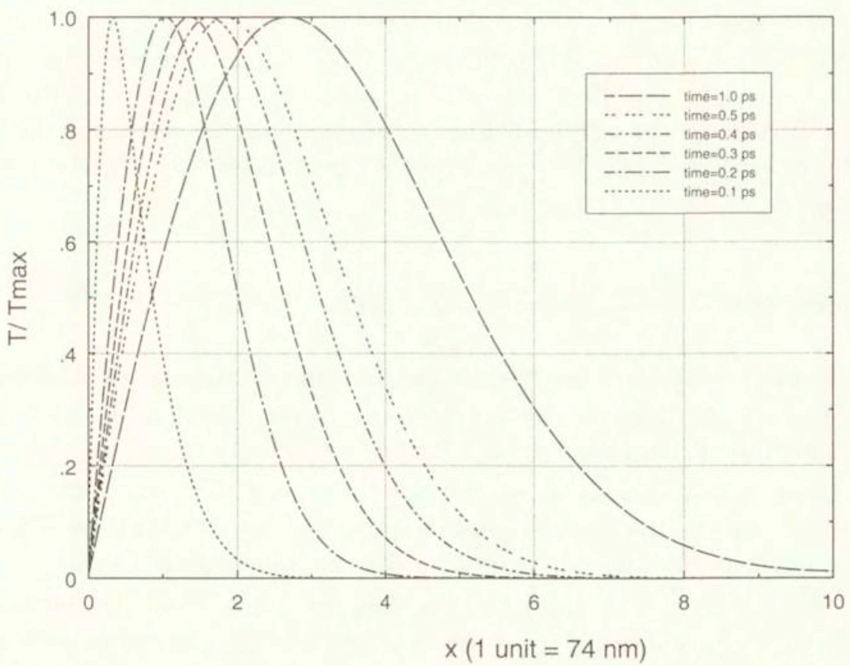

FIG. 9. Temperature distribution at different time instants for Fourier model.

Comparing the Jeffreys, Cattaneo and dual-phase-lag models with the Fourier model, it is worth to note the essential differences in the behaviour of the Fourier model from the point of view of the way in which the heat impulse is introduced: via the boundary conditions of Dirichlet type or via the initial conditions. Curves at Figures 7 and 8 present the dimensionless temperature (normalized with respect to the maximum of initial temperature) as a function of the distance x from the source of heat impulse after time $t = 0.2ps$, in both the Jeffreys and Fourier models. In Fig. 7 the heat impulse in the form of a slim and smooth peak centered at $x = 10$ was introduced via the initial condition for $t = 0$, while Figure 8 presents the same curves in the situation when the rectangular heat impulse enters through the left boundary. It is not important that the shapes of the impulses are different (rectangular impulse gives only the effect of oscillations near the left border of the graph in Fig. 8). We can see that the Jeffreys model behaves in both situations in a similar way: there is a traveling wave running from the source of the impulse. However, the behaviour of the Fourier model is completely different in each case. If the heat impulse is introduced through the initial condition (Fig. 7), there is no trace of a traveling wave for this model. With t growing, the temperature curve becomes more and more flat, its support blows up, but its maximal point is always at $x = 10$. If the heat impulse enters through the left boundary, the temperature curve in the Fourier model is completely different (see Fig. 8). It is important that its maximal point moves now to the right when t grows. When we observe the temperature curves as functions of time t this gives an effect very similar to the traveling wave of the Jeffreys model. This is clearly visible in Fig. 4, where the heat waves for the Fourier model are presented, and the positions of maxima differ only slightly from those of the Jeffreys model.

5. Conclusions

1. In all the discussed models the phenomenon of heat waves is present, but in the Fourier and dual-phase-lag models the waves travel at a speed different than that in the experiments of [2].

2. In the dual-phase-lag model not only the speed of traveling waves is lower than in experiments but, in addition this speed is decaying with time. This model gives in fact the worst approximation to the experimental results.

3. For the Fourier model (with the impulse introduced via the boundary condition), we have also observed that the speed of heat waves is lower than in experiments but the difference is not very significant. Also in this model the speed of heat waves is decaying with time (this effect is better visible in Fig. 9 where we have drawn the temperature profiles for different times).

4. There is no significant difference between the numerical results obtained by Cattaneo's and Jeffreys models. The positions of extrema in the Cattaneo and Jeffreys-type models are similar (visible instability in the Cattaneo model in the case of a rectangular impulse indicates that, perhaps the Jeffreys-type model fits better the reality).

5. The results of computations not presented here show that the positions of extrema in Jeffreys-type models depend only very weakly on the coefficient κ^*.

6. Long time computations made for the distance from the heat impulse source $x = 500$ Å show that the confluence of the Jeffreys and Fourier models occurs practically in about $100\,ps$ after the start of the initial impulse. This confirms the theoretical results we described in the preceding sections and shows that heat waves play an important role only in situations when very short impulses (of order of femtoseconds) and very thin films (of order of micrometers) are considered. The last statement is valid for experiments in room temperature and is not applicable to very low temperatures (for example in liquid helium).

7. Although the Cattaneo and Jeffreys-type models give the position of maxima of temperature profiles very close to the experimental results, the overall shape of numerical profiles is different than the experimental ones (rapid decay with time observed in experiment is not present in the numerical results for larger distances of 2000 Å and 3000 Å). This effect is maybe due to the nature of experimental data (we do not know to which temperature corresponds the bottom axis in Fig. 1 !).

8. The experimental material does not allow us to state in a decisive manner which model fits better the reality. In particular, the knowledge of the relative intensity of maxima can help in making distinction between the Fourier model, where the intensity of numerically calculated maxima is rapidly decreasing with x, and the Cattaneo and Jeffreys-type models, in which the calculated intensity of maxima is decaying much slower with x.

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