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Modelling of nonstationary heat conduction problems in micro-periodic composites using homogenisation theory with corrective terms

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HOMOGENISATION BASED on the asymptotic series expansion is used to model a nonstationary behaviour of a rigid heat conductor with micro-periodic structure. A usual first-order approximation (which cannot be assumed as a satisfactory solution for time-dependent problems) is treated as a suitable starting point for further corrections that make it admissible. An initial correction takes into account some fast processes acting on the level of the microstructure and guarantees that the initial condition is satisfied. Some higher-order correctors are intended to improve the first-order approximation far from the onset of the process, for composites with strongly different properties of components or for the case of a rough microstructure. A numerical example shows that the role of the initial corrector is prevailing in the model.

Notations

$\mathbf{x}$, $\mathbf{y}$	position vector in global and local co-ordinate system respectively,
t, τ	time,
$\Theta^k(\mathbf{x},\mathbf{y},t), T^k(\mathbf{x},t)$	temperature fields related to the k-th order of approximation,
$J^k(\mathbf{x},\mathbf{y},t)$	initial corrector to the temperature related to the k-th order of approximation,
$\mathbf{q}^k(\mathbf{x},\mathbf{y},t)$	heat flux related to k-th order of approximation,
$\chi^{(k)p}(\mathbf{y})$	p-th homogenisation function on the k-th level of approximation,
$k_{ij}(\mathbf{y}), c(\mathbf{y})$	tensor of thermal conductivity and coefficient of diffusivity,
K_{ij}	tensor of effective thermal conductivity,
$r(\mathbf{x})$	thermal sources,
$f_{,i(y)}, f_{,i(x)}, \dot{f}$	represent first derivatives of f with respect to y_i, x_i and t respectively,
$[f]_s$	represents the jump of a function f across a surface S,
$\tilde{f}(\mathbf{x},t)$ or $\langle f(\mathbf{x},t)\rangle$	represents the average value of a function $f(\mathbf{x},\mathbf{y},t)$,
$\operatorname{meas} Y$	denotes a measure of an ensemble Y.

1. Introduction

WE ANALYSE here a nonstationary heat conduction problem in a rigid, microperiodic composite. For this purpose we construct a succession of approximate models using an asymptotic theory of homogenisation with some necessary correction due to the time-dependence of the problem.

The classical theory of homogenisation, resulting in effective moduli, provides a solution that cannot satisfy exactly the local boundary conditions or the initial conditions for time-dependent problems. The theory of homogenisation yields the effective constitutive law valid on the macrolevel, starting from the microdescription of the problem (without any *a priori* hypothesis about the macro-behaviour). This is the main idea of the micromechanical approach and it is obvious that boundary condition at the macro-level are then of minor importance. In fact, boundary and initial conditions cannot influence the effective material properties. On the other hand, the behaviour of the composite material in the vicinity of the border requires a special analysis. For the nonstationary process, because of the microheterogeneity, the global solution should also be analogously corrected at the initial moment.

The reason for this is given in a recent paper by WOŹNIAK *et al.* [12]. The authors show that the usual effective modulus theory leads to important errors in the prediction of the global, nonstationary thermal behaviour of the composite. A refined theory based on the concept of macrofunctions is applied there to solve correctly the problem of nonstationary heat conduction in composites.

Here we intend to adapt a classical, asymptotic theory of homogenisation to solve the problem of nonstationary heat conduction in a body with microstructure. We show that the improvement of this theory is possible and that some corrections can give a satisfactory approximation of the exact solution. Moreover, the algorithm of this correction process is similar to that described in [2, 4, 6] and applied by us in [7] and [8]. This algorithm assumes that the first order model, resulting from the theory of homogenisation, is a convenient starting point for the construction of an improvement of this approximation at the beginning of the process. The description of the heat and temperature distribution at the level of the microstructure can be refined as well. This assumption has been verified in the problem of boundary layers of the composite and results in an efficient numerical procedure.

We consider two kinds of corrections: first, the correction including the higher-order term in the asymptotic expansion of the exact solution in the power series of the small parameter characterising the microstructure, and second - a correction of the solution near the onset of the process, called the initial correction. This second type of correction seems to be new in the literature. We analyse the influence of these two corrections. It appears that the first correction (that introduced via higher-order correctors) guarantees a weak convergence of the model to the exact solution. Hence the approximate solution far from the initial moment and inside of the body is very close to the exact one. However, the initial corrector changes qualitatively the solution at the beginning of the process and its role is still predominant, even until the process becomes steady. In this paper we present only some formal calculations. We do not analyse the regularity of the resulting family of boundary value problems because we do not expect any problem in this field. Finally, we present a simple, illustrative example which shows the efficiency of the proposed improvement.

2. Formulation of the problem and homogenization procedure

Let us consider the classical nonstationary heat conduction problem formulated for a rigid, micro-periodic composite body. Two systems of co-ordinates $\mathbf{x}$ and $\mathbf{y}$ describe the geometry of the body on the macro and micro-level, respectively. The assumed separation of scales between macro and micro phenomena is formally expressed by the following relation between two systems of co-ordinates $\mathbf{x}$ and $\mathbf{y}$ (see Fig. 1.):

$$\mathbf{y} = \varepsilon^{-1}\mathbf{x}. \tag{2.1}$$

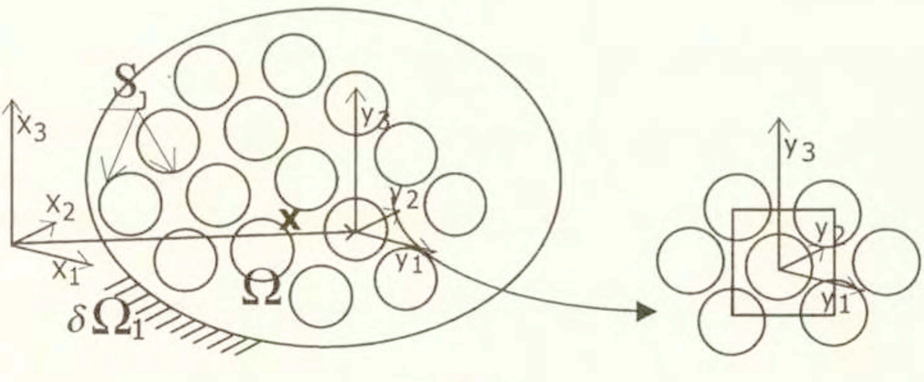


FIG. 1.

The small parameter ε, required by the homogenisation theory, is defined as the ratio between the characteristic dimension of the cell of periodicity and the diameter of Ω. This choice of the small parameter means that we are interested in the analysis of the behaviour of a solution when the microstructure is scaled down. The same symbol (used as a superscript will denote such a structure-dependent solution. Another possible choice of small parameter (such as for instance in [2]) seems to be not adequate for the analysis followed here.

The following equations define the problem under investigation:

Heat balance written for the body with diffusivity varying at the micro-level:

$$q_{i,i}(\mathbf{y},\mathbf{x},t) = \rho r(\mathbf{y},\mathbf{x},t) - c(\mathbf{y})\dot{\Theta}(\mathbf{y},\mathbf{x},t) \qquad \text{in } \Omega^{\varepsilon}. \tag{2.2}$$

Fourier law with rapidly oscillating conductivity tensor:

$$q_i(\mathbf{y},\mathbf{x},t) = -k_{ij}(\mathbf{y})\Theta_{,j}(\mathbf{y},\mathbf{x},t) \quad \text{in } \Omega^{\varepsilon}, \tag{2.3}$$

$$T(\mathbf{y},\mathbf{x},t) = T_b(\mathbf{y},\mathbf{x},t) \text{ on } \partial\Omega^{\varepsilon}_T \quad \text{and} \quad q_j(\mathbf{y},\mathbf{x},t)n_j = q_b(\mathbf{y},\mathbf{x},t) \quad \text{on} \quad \partial\Omega^{\varepsilon}_h, \tag{2.4}$$

$$T(\mathbf{y},\mathbf{x},t) = 0) = T_0(\mathbf{y},\mathbf{x}), \tag{2.5}$$

$$[T(\mathbf{y})] = 0 \; [q_j(\mathbf{y})n_j] = 0 \quad \text{on } S_J. \tag{2.6}$$

Of course, all the differentiations (appearing above and throughout the paper) are to be understood in the weak sense and will be replaced by equivalent variational formulations for detailed analysis if necessary. We assume also that all the usual requirements for the correctness of formulation hold. We suppose further that the periodicity of material characteristics imposes an analogous periodic perturbation on the studied quantities describing the thermal behaviour of the body. We will use hence the following representation for temperature:

$$\Theta^{\varepsilon}(\mathbf{x},t) \equiv \Theta^0(\mathbf{x},t) + \varepsilon^k\Theta^k(\mathbf{x},\mathbf{y},t), \tag{2.7}$$

where $k > 0$ and Θ^k are Y-periodic, i.e. takes the same values on the opposite sides of the cell of periodicity. The corresponding expansion of heat flux can be written as follows:

$$q_i^{\varepsilon}(\mathbf{x},t) = -k_{ij}(\mathbf{y})\left(\Theta^0_{,j(x)}(\mathbf{x},t) + \Theta^1_{,j(y)}(\mathbf{x},\mathbf{y},t)\right) +.. - \varepsilon^k k_{ij}(\mathbf{y})\left(\Theta^k_{,j(x)}(\mathbf{x},\mathbf{y},t) + \Theta^{k+1}_{,j(y)}(\mathbf{x},\mathbf{y},t)\right). \tag{2.8}$$

We will denote by $q_i^k(\mathbf{x},\mathbf{y})$ the flux component of order k, given in Eq. (2.8) by the expressions in parentheses, multiplied by ε^k.

We apply here and in the sequel the following notation for the chain rule of differentiation:

$$f_{,i} \equiv \frac{d}{dx_i}f = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)f \equiv f_{,i(x)} + \frac{1}{\varepsilon}f_{,(y)}. \tag{2.9}$$

We are looking for some relationships analogous to (2.7) and (2.8) but formulated for some equivalent, homogeneous body and involving averaged (or global) quantities:

$$\tilde{\Theta}(\mathbf{x},t) = \frac{1}{\mathrm{meas}Y}\int_Y \Theta(\mathbf{x},\mathbf{y},t)dY, \qquad \tilde{\mathbf{q}}(\mathbf{x},t) = \frac{1}{\mathrm{meas}Y}\int_Y \mathbf{q}(\mathbf{x},\mathbf{y},t)dY. \tag{2.10}$$

By introducing Θ and q into (2.2) and (2.3) in the forms prescribed above, we split these equations into a pair of infinite systems of equations by equating terms of the same order. The first of these systems we will call "heat balance" (hb), the second one – the "Fourier law decomposition" (Fld). The first two terms of (hb) and the first term of (Fld) are taken into account to formulate the following well-known result of the homogenisation theory (see for example [1, 10]), called here "the first order approximation":

$$\tilde{q}_{i,i}(\mathbf{x},t) = \rho r(\mathbf{x},t) - \tilde{c}\dot{\tilde{\Theta}}(\mathbf{x},t) \quad \text{in } \Omega, \tag{2.11}$$

$$\tilde{q}_i(\mathbf{x},t) = -K_{ij}\tilde{\Theta}_{,j}(\mathbf{x},t) \quad \text{in } \Theta. \tag{2.12}$$

The tensor of effective thermal conductivity $\mathbf{K}$ is defined by a vector of homogenisation functions $\chi^{(0)p}(\mathbf{y})$:

$$K_{ij} = \frac{1}{\mathrm{meas}Y}\int k_{ip}(y)\left(\delta_{pj} + \chi^{(0)p}_{,j}(\mathbf{y})\right)dY. \tag{2.13}$$

These functions satisfy the condition of local heat balance equation of order 0 ("local" means here: over the cell of periodicity):

$$\left(k_{ij}(\mathbf{y})(\delta_{ij} + \chi^{(0)p}_{,j(y)}(\mathbf{y})\right)_{,i(y)} = 0, \tag{2.14}$$

$$\left[k_{ij}(\mathbf{y})\left(\delta_{jp} + \chi^{(0)p}_{,j(y)}(\mathbf{y})\right)n_i\right] = 0, \tag{2.15}$$

(these equations express continuity of the heat flux across the interfaces of different materials inside Y),

$$\tilde{\chi}^{(0)p} = 0 \quad \text{and} \quad \chi^{(0)p}(\mathbf{y}) \quad \text{is } Y-\text{periodic}. \tag{2.16}$$

The variational counterpart of (2.14), (2.15) and (2.16) can be easily formulated (derivatives should be taken in the weak sense).

Mean form of the boundary and initial conditions complete the equivalent boundary value problem:

$$\tilde{T}(\mathbf{x},t) = \tilde{T}_b(\mathbf{x},t) \quad \text{on } \partial\Omega_T \quad \text{and} \quad \tilde{q}_j(\mathbf{x},t)n_j = \tilde{q}_b(\mathbf{x},t) \quad \text{on } \partial\Theta_h, \tag{2.17}$$

$$\tilde{T}(\mathbf{x},t=0) = \tilde{T}_0(\mathbf{x}). \tag{2.18}$$

It can be also deduced at this level of approximation that

$$\Theta^1(\mathbf{x},\mathbf{y},t) = T^1(\mathbf{x},t) + \chi^{(0)i}(\mathbf{y})T^0_{,i(x)}(\mathbf{x},t), \tag{2.19}$$

where $T^0(\mathbf{x},t) = \tilde{\Theta}(\mathbf{x},t)$ a solution of (2.11), (2.12) and $T^1(\mathbf{x},t)$ is to be defined on the next level of approximation. This fact is important since it defines a successive step of the solution of the system of equations.

In fact, the above solution, called here the "first approximation", is truncated at the term of zero order. According to the terminology adopted, all successive terms will be called "correctors", thus k-th order corrector corresponds to the temperature and heat flux with ε to the power of k in (2.7) and (2.8).

3. Higher order correctors

As a natural continuation of the approximation process initiated above, we attempt to compute the successive terms in (hb) and (Fld). We observe that the recurrent procedure initiated by (2.19) can be generalised by writing

$$\begin{aligned}\Theta^\varepsilon(\mathbf{x},\mathbf{y},t) = T^0(\mathbf{x},t) + \varepsilon(T^1(\mathbf{x},t) + \chi^{(0)i}(\mathbf{y})T^0_{,i(x)}(\mathbf{x},t)) \\ + \ldots + \varepsilon^k(T^k(\mathbf{x},t) + \chi^{(0)i}(\mathbf{y})T^{k-1}_{,i(x)}(\mathbf{x},t) \\ + \ldots + \chi^{(k-1)i..\times k..j}(\mathbf{y})T^0_{,i..\times k..j(x)}(\mathbf{x},t)).\end{aligned} \tag{3.1}$$

This form is given in [2], but our analysis differs from that which is shown in that reference. Using this representation, the equation number k in (hb) can be written in the form:

$$\begin{aligned}k_{ij}(\mathbf{y})(\delta_{jp} + \chi^{(0)p}_{,j(y)})T^k_{,pi(x)}(\mathbf{x},t) + k_{ij}(\mathbf{y})(\delta_{jp}\chi^{(0)p} + \chi^{(1)pq}_{,j(y)})T^{k-1}_{,pqi(x)}(\mathbf{x},t) \\ + \ldots + k_{ij}(\mathbf{y})(\delta_{jp}\chi^{(k-1)p..\times(k-1)..r} + \chi^{(k)p..\times(k-1)..rq}_{,j(y)})T^0_{,p..\times(k-1)..rqi(x)}(\mathbf{x},t) \\ + (k_{ij}(\mathbf{y})(\delta_{jp} + \chi^{(0)p}_{,j(y)}))_{,i(y)}T^k_{,p(x)}(\mathbf{x},t) + (k_{ij}(\mathbf{y})(\delta_{jp}\chi^{(0)p} \\ + \chi^{(1)pq}_{,j(y)}))_{,i(y)}T^{k-1}_{,pq(x)}(\mathbf{x},t) \\ + \ldots + (k_{ij}(\mathbf{y})(\delta_{jp}\chi^{(k-1)p..\times(k-1)..r} + \chi^{(k)p..\times(k-1)..q}_{,j(y)}))_{,i(y)}T^0_{,p..\times(k-1)..rq(x)}(\mathbf{x},t) \\ = c(\mathbf{y})(\dot{T}^k(\mathbf{x},t) + \chi^{(0)p}\dot{T}^{k-1}_{,p(x)}(\mathbf{x},t) + \ldots + \chi^{(k)p..\times k..q}\dot{T}^0_{,p..\times k..q(x)}(\mathbf{x},t)).\end{aligned} \tag{3.2}$$

We derive a partial differential equation defining the unknown temperature T^k by averaging over the cell of periodicity the equation number $k+1$ in (hb). We know at this step all previous T^i.

$$\text{(3.3)}\qquad K_{ij}T^k_{,ij(x)}(\mathbf{x},t) + K^1_{ipq}T^{k-1}_{,pqi(x)}(\mathbf{x},t) + \ldots + K^k_{ip..\times k..q}T^0_{,p..\times..qi(x)}(\mathbf{x},t)$$
$$= \tilde{c}\dot{T}^k(\mathbf{x},t) + C^1_p\dot{T}^{k-1}_{,p(x)}(\mathbf{x},t) + \ldots + C^{(k)}_{p..\times k..q(x)}\dot{T}^0_{,p\times k...q(x)}(\mathbf{x},t),$$

where we define the "effective coefficients of order k":

$$\text{(3.4)}\qquad K^k_{ip..\times k..q} = \int_Y k_{ij}(\mathbf{y})(\delta_{jp}\chi^{(k-1)p..\times(k-1)..r} + \chi^{(k)p..\times(k-1)..rq}_{,j(y)})dY,$$

$$\text{(3.5)}\qquad C^{(k)}_{p..\times k..q} = \int_Y c(\mathbf{y})\chi^{(k)p..\times k..q}dY.$$

The general form of the homogenisation function associated with the k-th corrector is slightly more complicated. The boundary value problem defining it is formulated over the cell of periodicity. Remembering that we know at this step all previous $\chi^{(i)}(i<k)$ and assuming a weak sense of the differentiation symbol, we can write:

$$\text{(3.6)}\qquad (k_{ij}(\mathbf{y})(\delta_{jp}\chi^{(k-1)p..\times(k-1)..r}(\mathbf{y}) + \chi^{(k)p..\times(k-1)..r}_{,j(y)}(\mathbf{y})))_{,i(y)}$$
$$= RHS^{(k)}_{p..\times(k-1)..r}(\mathbf{y}),$$

$$\text{(3.7)}\qquad \left[k_{ij}(\mathbf{y})(\delta_{ij}\chi^{(k-1)p..\times(k-1)..r}(\mathbf{y}) + \chi^{(k)p..\times(k-1)..rq}_{,j(y)}(\mathbf{y})n_i\right] = 0$$

on the interfaces S_J,

$$\text{(3.8)}\qquad \tilde{\chi}^{(k)p} = 0 \quad \text{and periodic boundary conditions are required for} \quad \chi^{(k)p}(\mathbf{y}).$$

Right-hand side in (3.6) is a function of all known at this step, values $\chi^{(i)}$ and also K^k and C^k previously computed. The complicated formula for RHS can be given in a recurrent form. Equation (3.7) expresses a continuity of the contribution to the heat flux at the k-th step, across the interfaces of different materials in Y.

We understand that the pair of functions: T^k satisfying (3.3) and $\chi^{(k)p}(\mathbf{y})$, solution of (3.6) – (3.8), defines the higher-order correction (of order k) to the first approximation Θ^0.

We note immediately that the boundary value problem defining higher-order terms is always the same, only the right-hand side has to be updated. This is of a great advantage in the computational process. It is also clear that the successive approximation depends on the higher-order of the leading term T^0, thus some problems concerning the required higher-order differentiability of T^0 can arise. In this paper, we assume everywhere a sufficient regularity. The corresponding correction to the heat flux can be easily found using (2.8) at each step.

4. Initial corrector

Let us suppose that the given initial temperature is smooth everywhere in Ω^ε (in the domain of the real composite material, not in its equivalent homogenised medium). Obviously in this case at the initial moment, the temperature gradient is also continuous and, according to our constitutive equations, the continuity of the flux (2.8) is violated. We can accept such a situation only in the presence of some heat sources (not imposed as initial loads but rather spontaneously created at the initial moment of the process) at the interfaces between materials inside each cell of periodicity. The nature of such heat sources can be explained by a quick heat exchange between layers, described in the local co-ordinate system $\mathbf{y}$ related to a single layer.

We assume in what follows that, at the beginning of heat conduction, another micro-process of heat transfer on the level of microstructure (more or less the same in each layer) takes place. It is immediately seen from the solution derived until now that the gradient of temperature is discontinuous at each step of the approximation:

$$(4.1)\qquad \Theta^\varepsilon_{,i}(\mathbf{x},\mathbf{y},t) = T^0_{,i(x)}(\mathbf{x},t) + \chi^{(0)p}_{,i(y)}(\mathbf{y})T^0_{,p(x)}(\mathbf{x},t) + \ldots + \varepsilon^k(T^k_{,i(x)}(\mathbf{x},t)$$
$$+\chi^{(0)p}_{,i(y)}(\mathbf{y})T^k_{,p(x)}(\mathbf{x},t) + \ldots + \chi^{(k-1)p..\times k..q}(\mathbf{y})T^0_{,ip..\times k..q(x)}(\mathbf{x},t)$$
$$+\chi^{(k)p..\times k..qr}_{,i(y)}(\mathbf{y})T^0_{,p..\times(k+1)..qr(x)}(\mathbf{x},t)).$$

The initial corrector will be defined to obtain the continuity of the initial temperature gradient at the moment $t = 0$. Since we intend to introduce the changes only at the beginning of the process, we define the time scaling via the following transformation:

$$(4.2)\qquad \tau = \delta^{-1}t.$$

A similar procedure can be found for example in [5] and is analogous to that employed in [7] for the spatial variables near the border. Velocities are computed now using the following expression:

$$\dot{f} \equiv \frac{d}{dt} f = \left(\frac{\partial}{\partial t} + \frac{1}{\delta} \frac{\partial}{\partial \tau} \right) f \equiv \dot{f}^t + \frac{1}{\delta} \dot{f}^\tau. \tag{4.3}$$

Instead of (2.7), we can use now the following representation for temperature, solution of (2.2) – (2.6):

$$\begin{aligned} \Theta^\varepsilon(\mathbf{x}, t) \equiv \Theta^{00}(\mathbf{x}, \mathbf{y}, t) + \delta \Theta^{01}(\mathbf{x}, \mathbf{y}, t, \tau) + .. + \delta^l \Theta^{0l}(\mathbf{x}, \mathbf{y}, t, \tau) \\ +... + \ \varepsilon^k (\Theta^{k0}(\mathbf{x}, \mathbf{y}, t) + \delta \Theta^{k1}(\mathbf{x}, \mathbf{y}, t, \tau) + .. + \delta^l \Theta^{0l}(\mathbf{x}, \mathbf{y}, t, \tau)). \end{aligned} \tag{4.4}$$

It is probably possible to restart the whole analysis with this assumption, but we find this to be tedious and not necessary. Our idea is to correct "a posteriori" the solution obtained before. According to this we identify the first term in the new development (4.4) with the already known temperature field, depending only on t. In this paper we assume that the two small parameters ε and δ are of the same order. Identifying ε with δ in the above formulae we analyse Eq. (4.4) in the following simplified form:

$$\begin{aligned} \Theta^\varepsilon(\mathbf{x}, t) \equiv \Theta^0(\mathbf{x}, \mathbf{y}, t) + \varepsilon^1(\Theta^1(\mathbf{x}, \mathbf{y}, t) + J^{01}(\mathbf{x}, \mathbf{y}, t, \tau)) \\ +... + \ \varepsilon^k(\Theta^k(\mathbf{x}, \mathbf{y}, t) + J^{0k}(\mathbf{x}, \mathbf{y}, t, \tau) + J^{1k-1}(\mathbf{x}, \mathbf{y}, t.\tau) \\ +... + J^{(k-1)1}(\mathbf{x}, \mathbf{y}, t, \tau)). \end{aligned} \tag{4.5}$$

The presence of the microstructure is at the origin of the corrective process under consideration, thus it is quite natural to deal with only one independent small parameter ε. Moreover, we do not intend to control any of the possibly time-dependent load parameters, like for example load frequency, since the only influence of the microstructure is the subject of our present analysis. It is easy to check that the following general representation of $J^k(\mathbf{x},\mathbf{y}, t)$ is suitable for our purpose:

Initial correctors of successive order to the known first order approximation

$$J^{0k}(\mathbf{x}, \mathbf{y}, t, \tau) = -\chi^{(0)p}(\mathbf{y}) I^{0k}_p(\mathbf{x}, \tau) + II^{0k}(\mathbf{x}, \tau). \tag{4.6}$$

Initial correctors of successive order to the known l-th order corrector

$$J^{lk}(\mathbf{x}, \mathbf{y}, t, \tau) = -\chi^{(l)p..q}(\mathbf{y}) I^{lk}_{p..q}(\mathbf{x}, \tau) + II^{lk}(\mathbf{x}, \tau). \tag{4.7}$$

In fact, with the initial condition:

$$I^{0k}_p(\mathbf{x}, \tau = 0) = T^0_{,p(x)}(\mathbf{x}, t = 0), \qquad I^{lk}_p(\mathbf{x}, \tau = 0) = T^l_{,p(x)}(\mathbf{x}, t = 0), \tag{4.8}$$

the initial continuity of the temperature gradient can be obtained for the first approximation and for each corrective term as well.

Using the improved representation (4.5), the equation of order 0 and k in (hb) can be written in the following form (we suppose that (3.2) is verified, therefore we write only the changes with respect to (3.2)):

$$(4.9)\qquad k_{ij}(\mathbf{y})\chi^{(0)p}_{,j(y)} I^{01}_{p,i(x)}(\mathbf{x},\tau) + (k_{ij}(\mathbf{y})\chi^{(0)p}_{(y)})_{,i(y)} I^{01}_{p,j(x)}(\mathbf{x},t) + (k_{ij}(\mathbf{y})\chi^{(0)p}_{,j(y)})_{,i(y)} I^{02}_{p(x)}(\mathbf{x},t) + (k_{ij}(\mathbf{y})\chi^{(1)pq}_{,j(y)})_{,i(y)} I^{11}_{pq,j(x)}(\mathbf{x},t) = c(\mathbf{y})(\chi^{(0)p}\dot{I}^{\tau^{01}}_{p}(\mathbf{x},\tau) + \dot{I}I^{\tau^{01}}(\mathbf{x},\tau)),$$

$$(4.10)\qquad k_{ij}(\mathbf{y})\chi^{(k)p..\times(k-1)..rq}_{,j(y)} I^{k0}_{p..\times(k-1)..rq,i(x)}(\mathbf{x},t) + \left(k_{ij}(\mathbf{y})\chi^{(k)p..\times(k-1)..rq}_{(y)}\right)_{,i(y)} T^{k0}_{p..\times(k-1)..rq,j(x)}(\mathbf{x},t) + \left(k_{ij}(\mathbf{y})\chi^{(k)p..\times(k-1)..rq}_{,j(y)}\right)_{,i(y)} T^{k1}_{p..\times(k)..rq(x)}(\mathbf{x},t) = c(\mathbf{y})\left(\chi^{(k)p..\times k..q}\dot{I}^{\tau^{k0}}_{p..\times k..q}(\mathbf{x},t) + \dot{I}I^{\tau^{k0}}\right).$$

It is easy to identify the right-hand side of the above equation as local heat sources arising at the beginning of the conduction process:

$$(4.11)\qquad r^{k}(\mathbf{x},\mathbf{y},\tau) = c(\mathbf{y})\left(\chi^{(k)p..\times k..q}\dot{I}^{\tau^{k0}}_{p..\times k..q}(\mathbf{x},t) + \dot{I}I^{\tau^{k0}}(\mathbf{x},t)\right).$$

The above equation serve us as a physical interpretation of the initial correctors. The micro-sources defined by (4.11) are associated with requirement of the smooth distribution of the initial temperature. It means that to obtain such an initial condition one should apply locally the micro-sources (4.11). It means also that if the graph of the temperature is smooth, the micro-sources of heat can be present at the level of the microstructure. The requirement of the zero mean values of these additional sources is natural in this problem and serves us as a first condition for the determination of the unknown initial corrector function,

$$(4.12)\qquad \int\limits_{Y} r^{k}\left(\dot{I}^{\tau^{k1}}_{p}, \dot{I}I^{\tau^{k+1,1}}\right) dY = 0.$$

The known jump of the heat flux across the interface between two materials should be equal to the integral of the additional heat sources inside this material domain. This is another condition for the unknown initial corrector function:

$$\left[q_j^k(I_p^{k1}, II^{k+1,1})n_j\right]_{S_J} = \int\limits_{Y_i} r^k\left(\dot{I}_p^{\tau^{k1}}, \dot{II}^{\tau^{k+1,1}}\right) dY_i. \tag{4.13}$$

Knowing I^{ik} satisfying Eqs. (4.12), (4.13), the initial correction J^{ik} is defined. It is easy to check that I^{ik} is always of the form:

$$I_p^{k1}(\mathbf{x},t) = \exp(-\lambda^{(k)}t/\varepsilon)f(\mathbf{x}), \quad II_p^{k1}(\mathbf{x},t) = 0, \tag{4.14}$$

$$\lambda^{(k)} = \frac{\int\limits_{S_J} k_{ij}(\mathbf{y})\chi^{(k)}_{,j} n_i dS_J}{\int\limits_{Y_i} c(\mathbf{y})\chi^{(k)} dY_i}. \tag{4.15}$$

The corresponding correction to the heat flux can be easily found using (2.8) and (4.5) at each step.

5. Example

The aim of this example is to illustrate the presented method by a simple calculation that can be carried out without any special software and can be presented in a relatively closed form. We consider a laminated domain Ω^ε bounded by two planes parallel to the layers. Each periodically repeated layer is made of two components. The thickness of the strip Ω^ε is $2L$, thickness of the components of the individual layer are l_1 and l_2. We assume that $l_1 + l_2 << L$. Thermal characteristics of these layers are respectively k_1, c_1 and k_2, c_2. It is natural to assume that the temperature field depends only on the co-ordinate x_1, perpendicular to the layers, with the origin in the middle of the composite strip (x_1 will be denoted simply by x). The local co-ordinate system is reduced to y, $\varepsilon = (l_1 + l_2)/L$.

We assume (as in [12]) the boundary condition in the form:

$$\Theta(L, y, t) = \Theta(L, t) = \Theta(-L, t) = \Theta_L. \tag{5.1}$$

The initial conditions are:

$$\Theta(x, y, 0) = \Theta(x, t) = \Theta_L + \Theta_0 \cos(\pi x_1/2L). \tag{5.2}$$

All computations have been performed symbolically, using the commercial code MAPLEV3.

Unfortunately, the resulting formulae defining the needed coefficients as functions of l_i, k_i, c_i are rather long, thus we present their behaviour in the form of graphs.

We start the analysis with the definition of successive homogenisation functions (the superscript p has been dropped since we deal with $\chi^{(k)1}(\mathbf{y})$ only). The local contribution to the flux vector is expressed by the formula

$$k(y)\left(\chi^{(k-1)} + \chi^{(k)}_{,y}(\mathbf{y})\right) = q^k(y)/T^0_{,k\cdot x} \equiv h^k(y). \tag{5.3}$$

The graphs of these two functions illustrate the local behaviour of the composite structure. They are to be extrapolated by periodicity to the whole domain Ω^ε, and then scaled with appropriate derivatives of the global solutions T^k.

In Fig. 2 and Fig. 3 we show the graphs of $\chi^{(k)}$and h^k for $l_1 = l_2$, or $l_1 = 5l_2$, $k_1 = 2k_2$, and $c_1 = c_2$. The influence of the ratio between the conductivities of components on the behaviour of $\chi^{(0)}(y)$ is shown in Fig. 4 for $l_1 = l_2$ and for $l_1 = 5l_2$. The influence of the ratio between conductivities and diffusivities of components on $\chi^{(1)}(y)$ is shown in Fig. 5 for $l_1 = l_2$ and for $l_1 = 5l_2$. Influence of the local geometry and material characteristics on K^0 and K^2 is shown in Fig. 7.

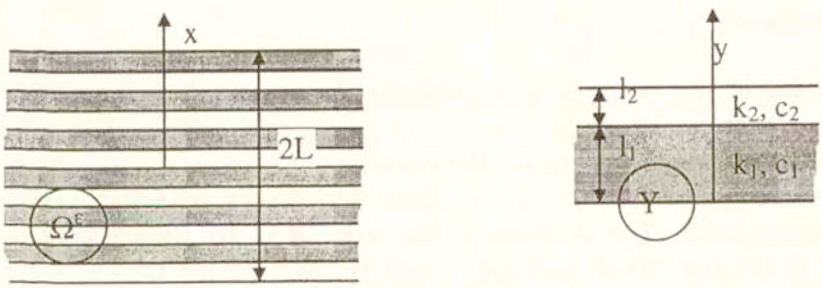


FIG. 2. Layered body and the single layer.

The next step consists of the computations of global unknowns of the model. The parabolic equations for the first approximation has the form:

$$K^0\frac{\partial^2 T^0}{\partial x^2} - \tilde{c}\frac{\partial T^0}{\partial t} = 0 \quad \text{where} \quad K^0 = \frac{(l_1+l_2)k_1k_2}{l_1k_2 + l_2k_1}. \tag{5.4}$$

The first order corrector vanishes, the second order correctors verify the following equation:

$$K^0\frac{\partial^0 T^2}{\partial x^2} - \tilde{c}\frac{\partial T^2}{\partial t} = \left(\frac{\langle c\chi^1\rangle}{\tilde{c}} - K^2\right)\frac{\partial^4 T^0}{\partial x^4} \tag{5.5}$$

where a symbolic formula for K^2 and averaging values involving c are rather long.

The following expressions stand for T^k, solutions of the above partial differential equations:

$$T^0(x,t) = \Theta_L + \Theta_0 \exp\left(-\frac{K^0}{\tilde{c}}\left(\frac{\pi}{2L}\right)^2 t\right)\cos\left(\frac{\pi}{2L}x\right), \tag{5.6}$$

$$T^2(x,t) = At\exp\left(-\frac{K^0}{\tilde{c}}\left(\frac{\pi}{2L}\right)^2 t\right)\cos\left(\frac{\pi}{2L}x\right), \tag{5.7}$$

where $A = \Theta_0\left(\frac{\pi}{2L}\right)^4\left(\tilde{c}^{-1}\langle c\chi^1\rangle K^0 - K^{(2)}\right)$.

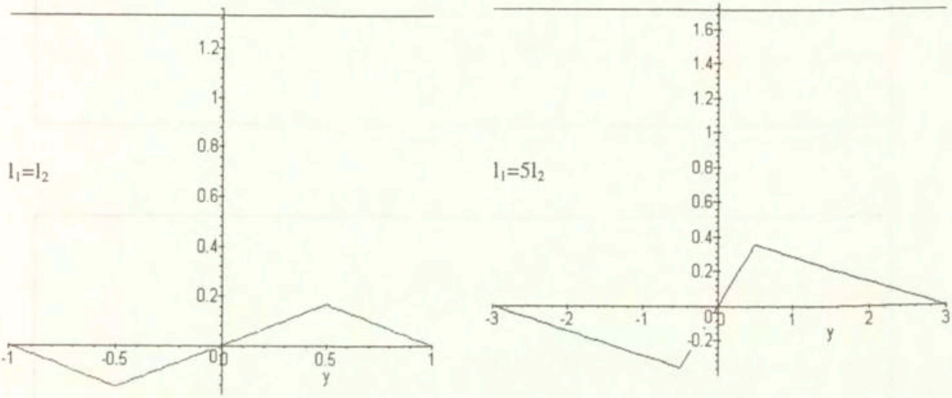


FIG. 3. Homogenisation function and corresponding heat flux for the first approximation.

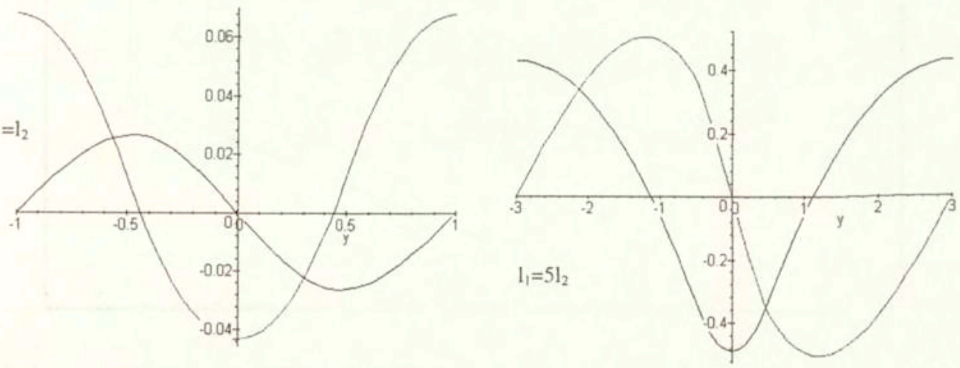


FIG. 4. Homogenisation function and corresponding heat flux for the second corrector.

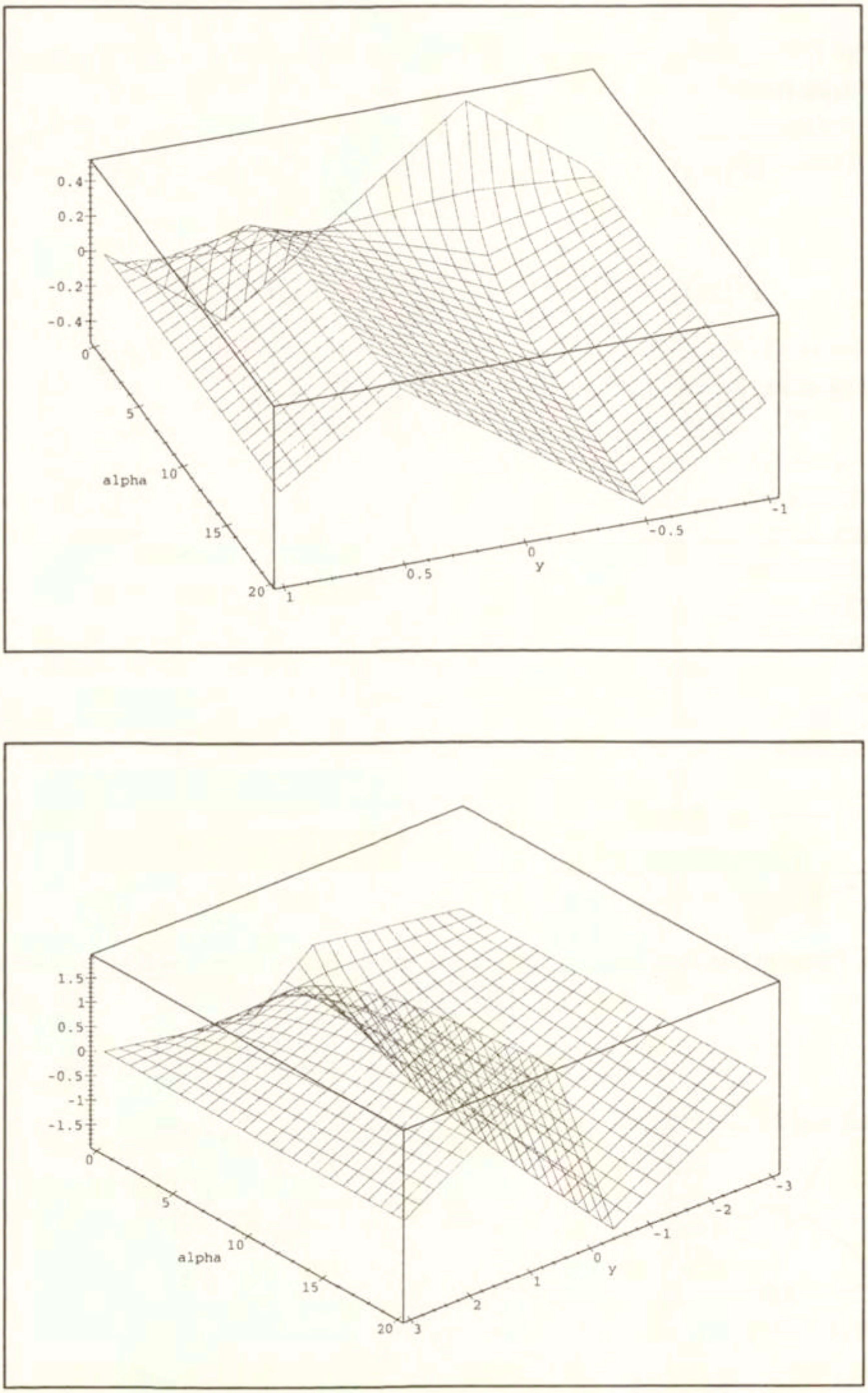
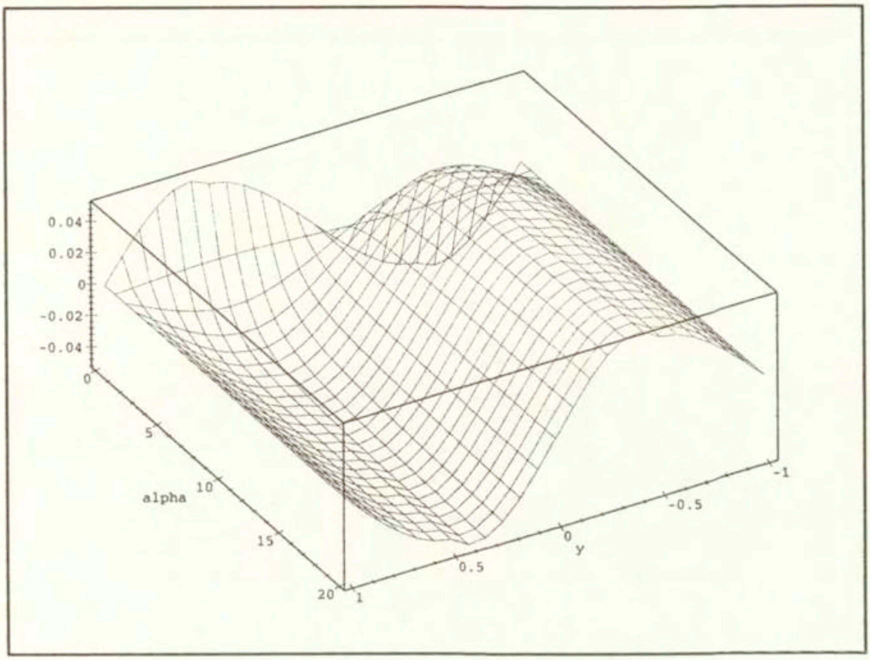


FIG. 5. Homogenisation function for the first approximation as a function of $\alpha = k_1/k_2$.



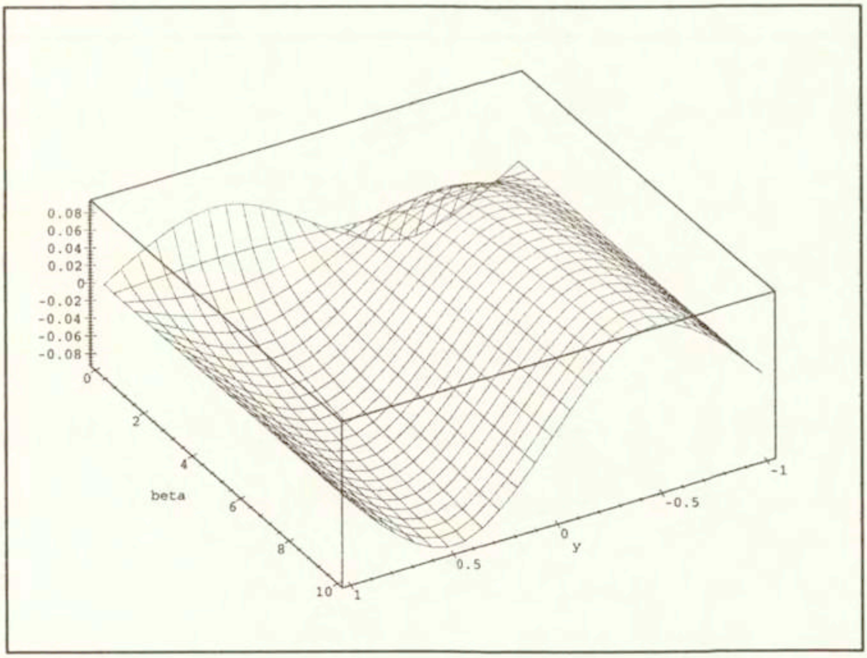


FIG. 6. Homogenisation function for the second corrector as a function of $\alpha = k_1/k_2$ and $\beta = c_1/c_2$.

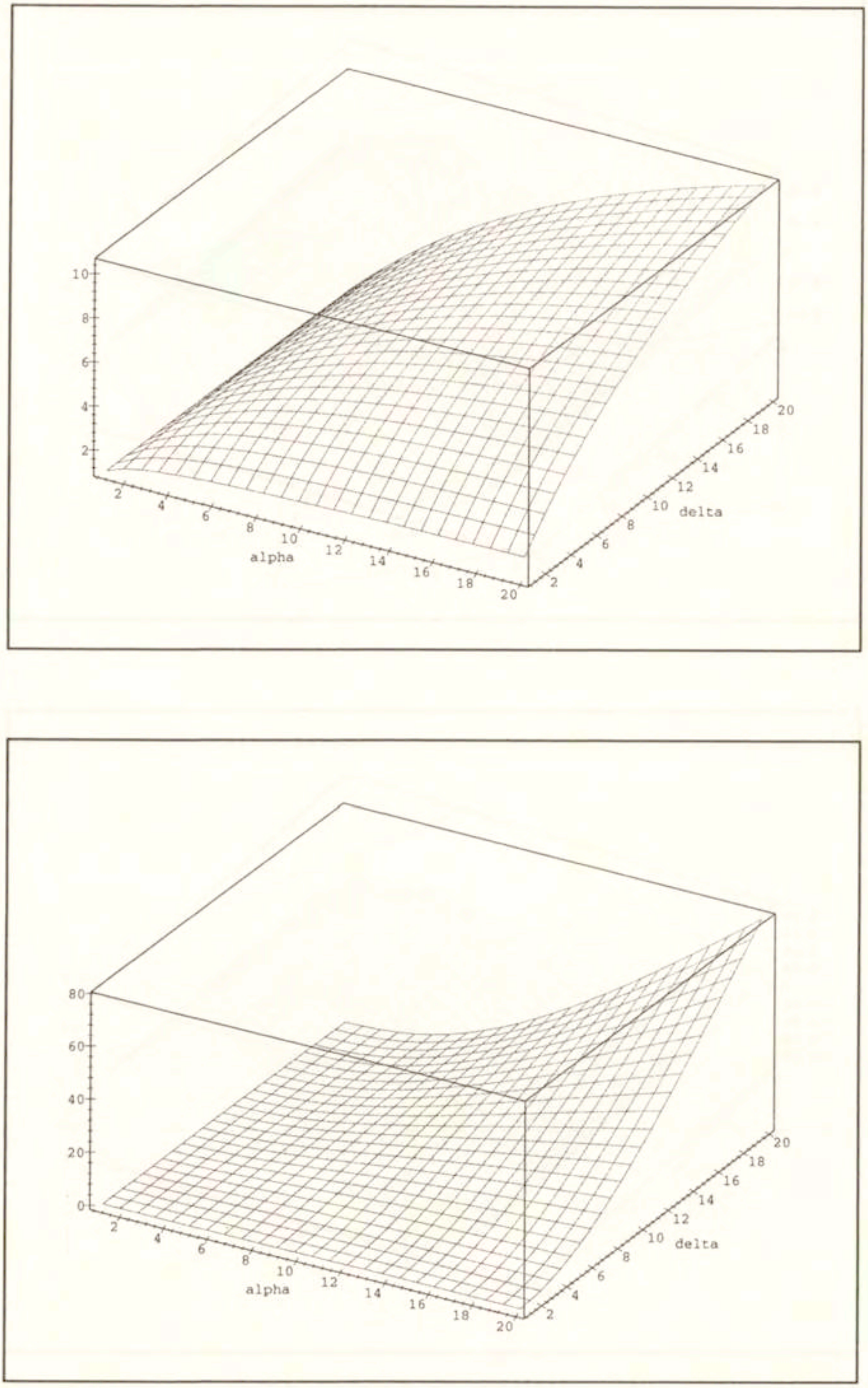


FIG. 7. Values of K^0 (first) and K^2 (second) in function of $\alpha = k_1/k_2$ and $\delta = l_1/l_2$.

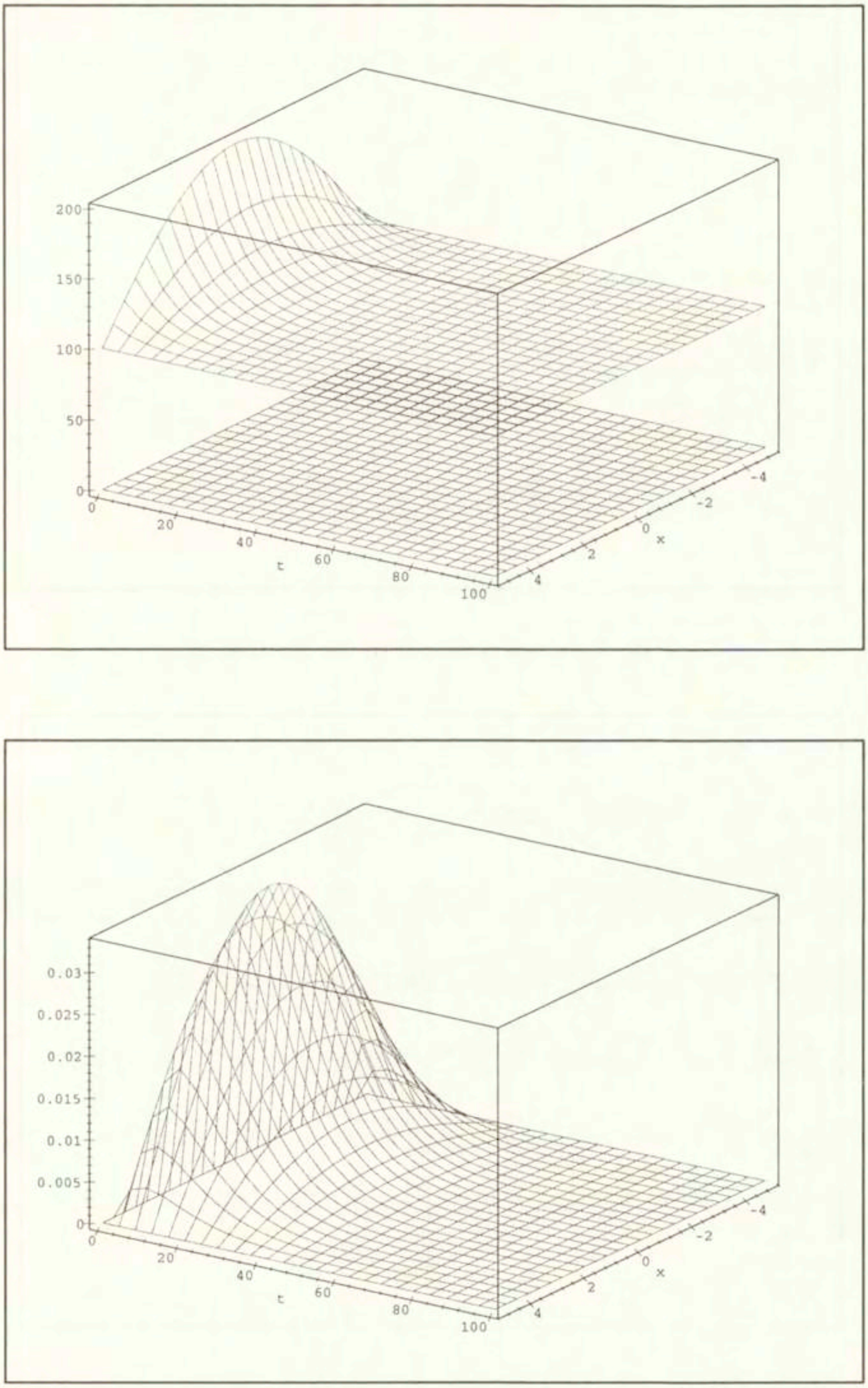


FIG. 8. $T^0(x, t)$ and the additive corrector $T^2(x, t)$ (first), the zoom of $T^2(x, t)$ (second).

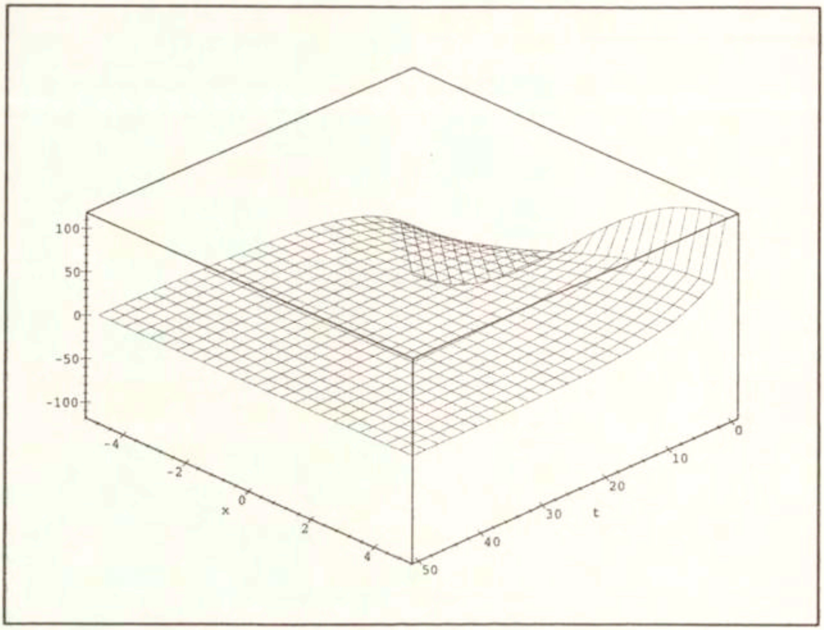


FIG. 9. Initial corrector of the first order.

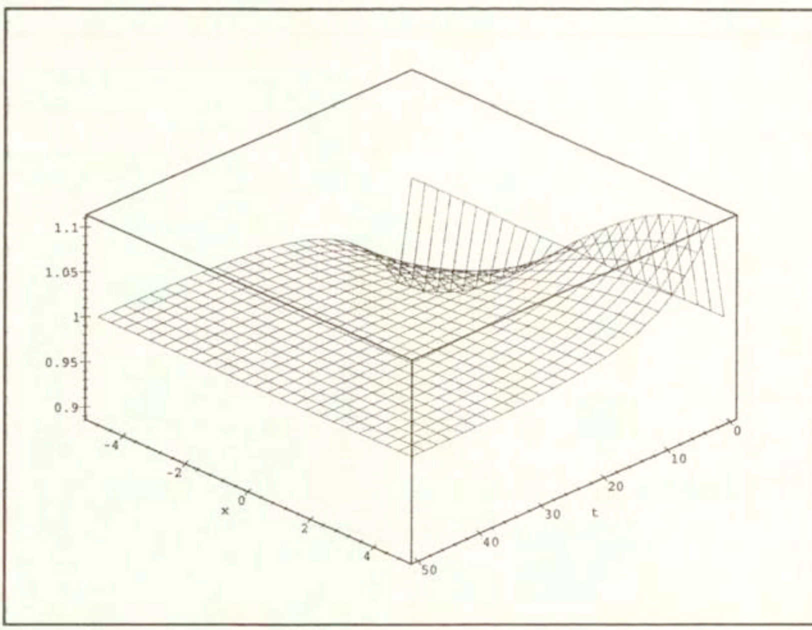


FIG. 10. Error in the average heat flux with respect to the refined solution given in [12]:

$$\text{error} = \frac{\text{our solution}}{\text{solution in[12]}}.$$

We present below (Fig. 8) both graphs: $T^0(x,t)$ and the additive corrector $T^2(x,t)$, then the zoom of $T^2(x,t)$.

In the third step of the analysis we compute the initial correctors. We limit ourselves to the first term. According to (4.14) and (4.15), the closed expression for $J^0(x,t)$ has the form:

$$J^{01}(\mathbf{x},t) = \Theta_0 \frac{\pi}{2L} \exp\left(-4\frac{L}{(l_1+l_2)}\frac{|k_2-k_1|}{(c_1l_1+c_2l_2)}t\right)\sin\left(\frac{\pi x}{2L}\right). \tag{5.8}$$

The set of Figures 9, 10, 11 below illustrates the behaviour of solution with initial corrector.

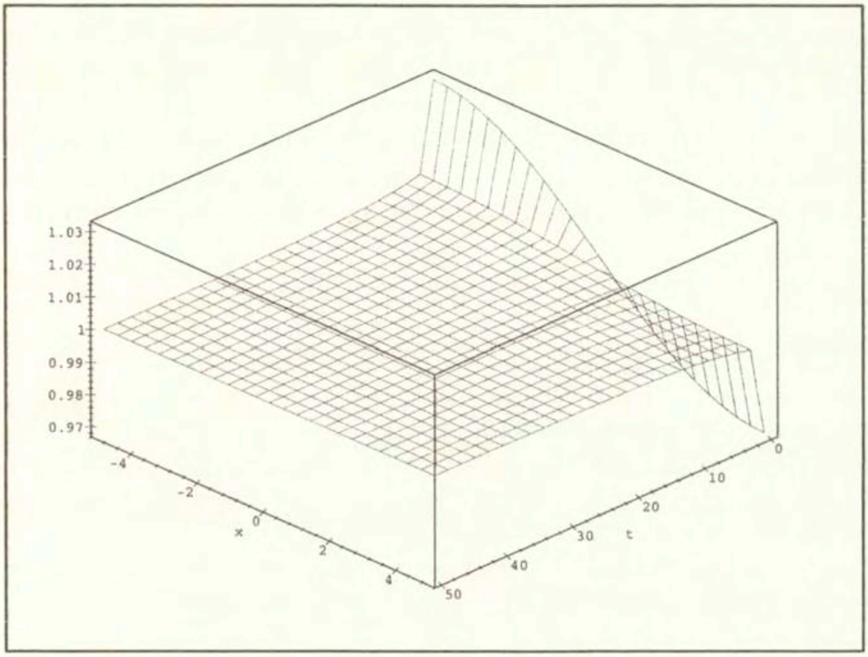


FIG. 11. Error in the average temperature gradient with respect to the refined solution given in [12]: error $= \dfrac{\text{our solution}}{\text{solution in[12]}}$.

6. Conclusions

We show in this paper that the procedure of successive corrections of the classical first order approximation resulting from the homogenisation theory is efficient. Improved approximation is qualitatively satisfactory: initial conditions are satisfied within the given precision level, the simulated evolution of temperature and heat flux is close to the exact one. A numerical example reveals that our results are close to those obtained from the refined theory presented in [12].

The most important advantage of the proposed method is its recurrent character. The algorithm of the computations is thus composed of the same procedures in each step, using as the input the flux of data resulting from the preceding step. In the studied example, the influence of the higher-order corrector on the quantitative and qualitative behaviour of the solution was negligible. These correctors may be useful only for strongly differing values of the component parameters and in the case when the scale resolution is not sharp enough (small parameter larger than 0.1).

On the other hand, the improvement due to the initial corrector is remarkable. The initial correction changed qualitatively the approximate solution at the beginning of the process. In the studied example, the initial corrector is associated via (4.11) with a field of micro-sources of heat that is needed to satisfy the assumed initial conditions. Such a correction should always be used when a nonstationary process is modelled. In this case we estimate that even for composite with strongly different properties of components, the first corrective term is sufficient.

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