

Stress distribution in the muscles of the diaphragm ¹⁾

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THE DIAPHRAGM is treated as a membrane subject to a uniform pressure. The muscle action is modeled as a uniaxial active stress in the direction of muscle bundles. In the present paper we look for the generalized stress distribution in the membrane. The shape of the active diaphragm is approximated by a surface known as a cyclide. For this simplified shape, a closed form solution of the equilibrium equations is obtained.

1. Introduction

THE DIAPHRAGM, by virtue of its curved shape, converts muscle tension to trans-diaphragmatic pressure and muscle shortening to volume displacement. During breathing, the displacement of the diaphragm is large (see [1, 2]); for example in dogs, in quiet spontaneous breathing, muscle bundles shorten on average to 75% of their original length [1].

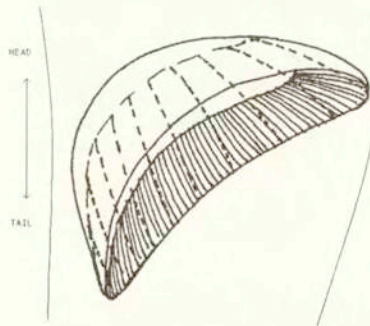


FIG. 1.a) Typical dog diaphragm displacement during breathing.

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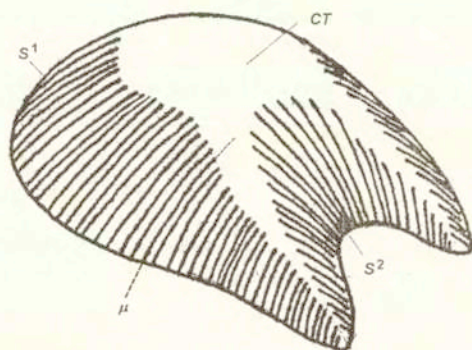


FIG. 1. b) Schematic view showing central tendon (CT), costal diaphragm (S^1) and crural diaphragm (S^2).

The diaphragm is a rather complex muscular structure. Some features of the diaphragm common to many mammals, such as humans and dogs, can be recognized:

a) The muscle bundles extend from the chest wall to a tendinous island known as the central tendon.

b) The muscle bundles and the central tendon form a curved sheet that separates the abdominal from the thoracic cavities.

c) The muscle bundles are connected transversely by fibers of connective tissue.

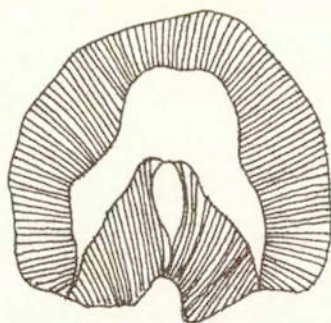


FIG. 2. a) Upper view of a typical dog diaphragm laid flat on a plane.

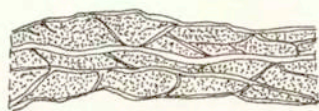


FIG. 2. b) Section of diaphragm tissue transverse to muscle fibers, showing collagen fibers.

The complex geometry and mechanics of the diaphragm, which are crucial to its function, have not been thoroughly described. As a result, the relations between muscle tension and transdiaphragmatic pressure are poorly understood.

The description of the geometry and of the state of stress of the active diaphragm, that is of equilibrium configurations at end of inspiration, is the scope of the research we are currently conducting in collaboration with scientists from the Baylor College of Medicine of Houston. In this paper the more mechanical results of this research, regarding the stress analysis of an active membrane modeling the canine diaphragm, are presented.

The central tendon consists of connective tissue and is effectively inextensible under physiological stress. The muscle layer is bounded by membranes consisting of connective tissue and contains connective tissue. At physiological level of stress, it is effectively inextensible in the direction transverse to muscle fibers. Therefore we picture the stress along the muscle bundles in the active diaphragm as being primarily the results of the active stress in the muscle and the stress in the transverse direction as being carried by connective tissue. Considering the small resistance to bending of the diaphragm, its structure is treated as a membrane with no bending stiffness. Due to its curved shape, the diaphragm converts this stress into a pressure difference across the surface, balanced by the recoil of the lungs. Each muscle bundle, when activated, exerts a force per unit length in the direction of the curvature vector of the surface. Such force is proportional to the curvature.

In recent papers BORIEK *et al.* [5] reported data on the shape of muscle bundles and the shape of the surface in the mid-costal region of the canine diaphragm. The data were obtained by attaching radio-opaque markers along muscle bundles on the peritoneal surface of the diaphragm in this region, and then determining the three-dimensional locations of the markers from biplane videofluoroscopic images recorded during passive lung inflation and spontaneous breathing. They found that, in this region, the surface of the diaphragm has approximately the shape of a right circular cylinder with the muscle bundles forming circular arcs that lie in planes orthogonal to the surface.

ANGELILLO *et al.* proposed a theory of the structure and shape of the diaphragm [6]. The theory was based on the assumptions that muscles of the diaphragm lie along lines that are both lines of maximum curvature and geodesics of the surface. These assumptions were shown to be equivalent to the assumption that muscle bundles lie in planes that are orthogonal to the surface. Thus the theory was based on the hypothesis that the relations between muscle bundles and surface geometry observed in the mid-costal region are in fact universal relations that can be obtained everywhere.

If muscle lines satisfy this assumption they are, in a sense, optimal since their energy consumption in deforming the surface is minimal: if the muscle lines are

lines of principal, maximum curvature, the contribution of the muscle force to create pressure difference is maximal; if the muscle lines are geodesics, no energy is lost in deforming the surface since the curvature vectors of these lines are orthogonal to the surface.

The class of surfaces with these properties was determined. It is a restricted class. In a surface of this class all the muscle lines must have the same shape, the orientation of the muscle lines is restricted. A special collection of surfaces belonging to this class is that of surfaces spanned by circular arcs of constant radius. Surfaces formed by circular arcs that are both principal and geodesics are known as cyclides (see HILBERT [7]). In a recent work, [8], an example of fitting of the costal and crural dog diaphragm by two cyclides having different radii and meeting smoothly at an interface is presented.

Neglecting inertia effects, we are interested in the equilibrium configuration of the diaphragm in its active state. Since the diaphragm is in contact with the lungs and the abdominal organs through a fluid interface, as a first approximation we consider the diaphragm acted on by a uniform pressure p normal to the surface, representing the transdiaphragmatic pressure. Therefore we study the equilibrium of a membrane surface having the form of a cyclide of radius ρ and subject to a given uniform pressure.

Using physical stress components $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ in the curvilinear coordinates of principal curvature of the surface (as a result of the optimality assumptions one family of these lines coincides with the muscle lines), the equilibrium equations of the membrane subject to a uniform pressure can be solved, imposing sufficient stress boundary conditions. The problem, with convenient boundary conditions, admits a closed-form solution with the shear stress $\sigma_{12} = 0$ everywhere, that is with the fibers of connective tissue orthogonal to the muscle lines.

2. Diaphragm theory

2.1. Preliminaries

The geometry of the surface in its actual configuration (*Eulerian* description), is described, in the *Euclidean* three-space, by the position vector

$$(2.1) \quad \mathbf{x} = \mathbf{x}(t^1, t^2),$$

depending on two curvilinear coordinates ranging in an open, connected domain Ω of \mathcal{R}^2 (see Fig. 3).

We assume that the surface is smooth in the sense that $\mathbf{x}(t^1, t^2)$ belongs to $C^3(\Omega)$. The natural base vectors associated to the curvilinear coordinates t^1, t^2

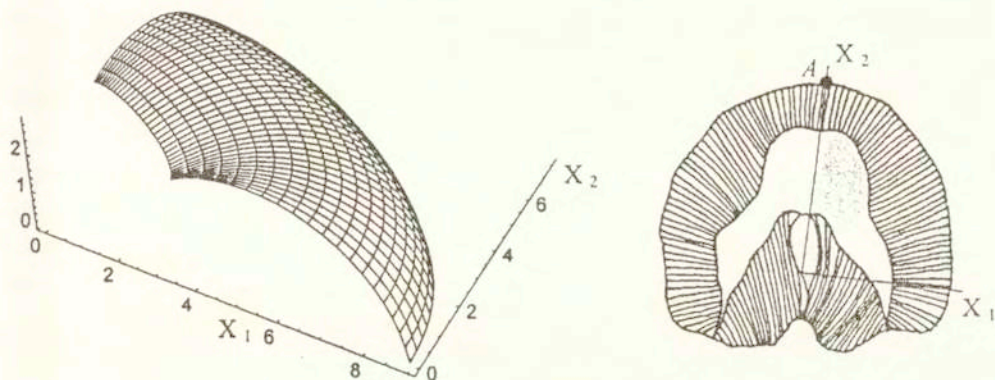


FIG. 3. Shape of diaphragm surface used for stress analysis. In the right-hand picture the top schematic view of the portion modeled is shown.

are denoted

$$(2.2) \quad \mathbf{g}_1 = \frac{\partial \mathbf{x}}{\partial t^1}, \quad \mathbf{g}_2 = \frac{\partial \mathbf{x}}{\partial t^2}.$$

The contravariant base vectors \mathbf{g}^k are defined through the relation

$$(2.3) \quad \mathbf{g}^\alpha \cdot \mathbf{g}_\beta = \delta_\beta^\alpha, \quad \alpha, \beta = 1, 2.$$

δ_β^α being the Kronecker delta. \mathbf{g}_3 is the unit normal to the surface, defined by the relation

$$(2.4) \quad \mathbf{g}_3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|}.$$

Taking into account that the element of area $da = J dt^1 dt^2$ is

$$(2.5) \quad da = |\mathbf{g}_1 \times \mathbf{g}_2| dt^1 dt^2,$$

we can write

$$(2.6) \quad \mathbf{g}_3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{J}.$$

The covariant components of the second fundamental form ρ of \mathcal{S} are denoted

$$(2.7) \quad \rho_{\alpha\beta} = \frac{\partial \mathbf{g}_\alpha}{\partial t^\beta} \cdot \mathbf{g}_3 = \frac{\partial \mathbf{x}}{\partial t^\alpha \partial t^\beta} \cdot \mathbf{g}_3.$$

2.2. Forces and equilibrium equations

The muscle fibers are arranged in bundles that span the diaphragm surface. In our continuum model we model the muscle bundles as a family of curves, say

μ , on \mathcal{S} and consider the coordinate lines t^2 directed as the μ lines (Fig. 1b, Fig. 3).

The material restriction on the generalized *Cauchy* stress \mathbf{T} (force per unit length)

$$(2.8) \quad \mathbf{T} = T^{\alpha\beta} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta,$$

define \mathcal{S} as a membrane.

Therefore the equilibrium equations of the diaphragm in its active state are

$$(2.9) \quad \frac{\partial \mathbf{T}}{\partial t^\gamma} \mathbf{g}^\gamma + p \mathbf{g}_3 = \mathbf{0},$$

and in component form

$$(2.10) \quad T^{\mu\gamma}_{,\gamma} + T^{\alpha\gamma} \mathbf{g}^\mu \cdot \mathbf{g}_{\alpha\gamma} + T^{\mu\beta} \mathbf{g}^\gamma \cdot \mathbf{g}_{\beta\gamma} = 0,$$

$$(2.11) \quad T^{\alpha\beta} \rho_{\alpha\beta} + p = 0,$$

where a comma followed by an index denotes differentiation with respect to that index.

2.3. Constitutive assumptions

We model the high number of discrete internal forces exerted by muscle fibers as a uniaxial active stress. Then it is assumed that the stress can be decomposed in the following form

$$(2.12) \quad \mathbf{T} = \mathbf{T}^a + \mathbf{T}^p,$$

where \mathbf{T}^p is the passive stress, and

$$(2.13) \quad \mathbf{T}^a = \sigma^a \mathbf{g}_2 \otimes \mathbf{g}_2,$$

is the active stress.

Considering that both the shear stiffness of the muscle tissue, and the elastic component of the muscle fiber itself is low, as a first approximation we can assume that the stress \mathbf{T} consists of a uniaxial active stress in the direction of the muscle fibers and a uniaxial passive stress in the direction of the collagen fibers

$$(2.14) \quad \mathbf{T} = \sigma^p \mathbf{b}_1 \otimes \mathbf{b}_1 + \sigma^a \mathbf{g}_2 \otimes \mathbf{g}_2,$$

\mathbf{b}_1 being a unit vector in the direction of the collagen fibers.

Therefore, introducing these simplification, the diaphragm, in its active state, behaves essentially as a net.

For what concerns the passive stress-strain behavior in the \mathbf{b}_1 direction, as pointed out in the Introduction, in the transverse direction muscle tissue exhibits a sort of locking behavior at stretches of about 1.1.

Our idea, based on the experimental observation that the diaphragm results pre-stressed in the rest physiological position, is that, in the functional state, the initial strain in the \mathbf{b}_1 direction is large enough to come up against the stop in that direction, that the transverse fibers displace at 1.1 of their natural length during normal breathing, and that the muscle tissue behaves essentially as inextensible along \mathbf{b}_1 . The passive stress σ^p becomes a reactive stress taking any value that is required for equilibrium.

3. Diaphragm shape

In [6] ANGELILLO *et al.* propose a theory on the structure of the diaphragm based on two assumptions:

1. Muscle lines are geodesics of \mathcal{S} ;
2. Muscle lines are lines of principal (maximum) curvature of \mathcal{S} .

As a consequence of these assumptions, it is shown in [6] that the shape of the μ -lines and of the surface \mathcal{S} itself are severely restricted.

In fact, it can be shown that

- i) the μ -lines are plane;
- ii) the μ -lines are universal curves on \mathcal{S} ;
- iii) the orientation of the μ -lines is restricted.

A particular simple class of surfaces satisfying all of the above implications is represented by the so-called cyclides, that is surfaces spanned by circular arcs of constant radius that are both principal and geodesics.

We model the diaphragm surface as a cyclide of radius ρ (or eventually as a composite cyclide, see [8]) with the muscle line μ directed along the circles. Such a surface can be also described as formed by circles lying in planes that are perpendicular to an arbitrary line Γ^c , with their centers on Γ^c . Since Γ^c is arbitrary, the cyclide is a flexible surface with a variety of shapes.

To describe the surface \mathcal{S} we identify t^1 as the arc-length along Γ^c :

$$(3.1) \quad \Gamma^c = \{\mathbf{x}^c = \mathbf{x}^c(t^1), \quad t^1 \in \mathbf{I}\}.$$

We assume that Γ^c is a plane curve and call $R(s)$ the radius of curvature of Γ^c . This assumption is crucial to get a simple form of the equilibrium equations.

Let us introduce the Frenè triad along Γ^c ; tangent, normal, binormal: $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$.

Since Γ^c is plane then

$$(3.2) \quad \mathbf{k}'_1 = \frac{1}{R}\mathbf{k}_2, \quad \mathbf{k}'_2 = -\frac{1}{R}\mathbf{k}_1 + \tau\mathbf{k}_3 = -\frac{1}{R}\mathbf{k}_1, \quad \mathbf{k}'_3 = -\tau\mathbf{k}_2 = 0,$$

where $(\cdot)'$ denotes differentiation with respect to t^1 .

On introducing the vector

$$(3.3) \quad \mathbf{z}(t^2) = \rho \left(\cos \frac{t^2}{\rho} \mathbf{k}_2 + \sin \frac{t^2}{\rho} \mathbf{k}_3 \right),$$

we describe the cyclide surface \mathcal{S} in the following way

$$(3.4) \quad \mathcal{S} = \{ \mathbf{x} = \mathbf{x}^c(t^1) + \mathbf{z}(t^2), \quad (t^1, t^2) \in \Omega \subset \mathcal{R}^2 \},$$

t^2 being the arc length along the muscles.

Notice that if \mathcal{S} is a cyclide, with this choice of the curvilinear coordinates (t^1, t^2) , whether or not Γ^c be plane, then the natural base vectors $\mathbf{g}_1, \mathbf{g}_2$ tangent to the curvilinear lines t^1, t^2 are orthogonal and principal.

In particular it results

$$(3.5) \quad \mathbf{g}_1 = \frac{\partial \mathbf{x}}{\partial t^1} = \left(1 + \frac{\rho}{R} \cos \frac{t^2}{\rho} \right) \mathbf{k}_1,$$

$$(3.6) \quad \mathbf{g}_2 = \frac{\partial \mathbf{x}}{\partial t^2} = \sin \frac{t^2}{\rho} \mathbf{k}_2 + \cos \frac{t^2}{\rho} \mathbf{k}_3,$$

and \mathbf{g}_2 is a unit vector. Notice that R , representing the curvature of the line of centers, is a function of t^1 .

Calling

$$(3.7) \quad f = \left(1 + \frac{\rho}{R} \cos \frac{t^2}{\rho} \right),$$

we have

$$(3.8) \quad \mathbf{g}_1 = f \mathbf{k}_1.$$

Notice that \mathbf{g}_1 is parallel to \mathbf{k}_1 .

Observing that $f > 0$, since muscle lines do not intersect each other, we also have

$$(3.9) \quad |\mathbf{g}_1| = f.$$

The unit normal to the surface is now

$$(3.10) \quad \mathbf{g}_3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1| |\mathbf{g}_2|} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{f} = \sin \frac{t^2}{\rho} \mathbf{k}_3 + \cos \frac{t^2}{\rho} \mathbf{k}_2 = \frac{\mathbf{z}}{\rho}.$$

The contravariant base vectors \mathbf{g}^i defined by the conditions (2.3) are in this case

$$(3.11) \quad \mathbf{g}^1 = \frac{1}{f} \mathbf{k}_1,$$

$$(3.12) \quad \mathbf{g}^2 = \mathbf{g}_2 = \sin \frac{t^2}{\rho} \mathbf{k}_2 + \cos \frac{t^2}{\rho} \mathbf{g}_3,$$

$$(3.13) \quad \mathbf{g}^3 = \mathbf{g}_3 = -\cos \frac{t^2}{\rho} \mathbf{k}_2 + \sin \frac{t^2}{\rho} \mathbf{k}_3.$$

4. Stress in the mid-costal diaphragm

We can write

$$(4.1) \quad p_\tau = p(\tau), \quad \tau \in [0, \bar{\tau}]$$

for the pressure at any stage of spontaneous breathing.

To simplify notations we take $p = p_{\bar{\tau}}$. Neglecting inertia effects, the Cauchy stress must be balanced by the pressure p . Locally (recall (2.9), (2.10) and (2.11)):

$$(4.2) \quad \frac{\partial \mathbf{T}}{\partial t^\alpha} \mathbf{g}^\alpha + p \mathbf{g}_3 = 0,$$

where

$$(4.3) \quad \mathbf{T} = \sigma^p \mathbf{b}_1 \otimes \mathbf{b}_1 + \sigma^a \mathbf{g}_2 \otimes \mathbf{g}_2.$$

By rewriting \mathbf{T} in physical components $\sigma_{\alpha\beta}$

$$(4.4) \quad \mathbf{T} = \sigma_{\alpha\beta} \hat{\mathbf{g}}_\alpha \otimes \hat{\mathbf{g}}_\beta, \quad \alpha, \beta = 1, 2,$$

$\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$ being the orthonormal triad directed as the natural base vectors $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$:

$$(4.5) \quad \hat{\mathbf{g}}_1 = \mathbf{g}_2/|\mathbf{g}_1|, \quad \hat{\mathbf{g}}_2 = \mathbf{g}_2, \quad \hat{\mathbf{g}}_3 = \mathbf{g}_3 = \mathbf{g}^3,$$

the following relation between the physical component of \mathbf{T} and the active and passive stresses are easily found:

$$(4.6) \quad \sigma_{11} = \sigma^p \cos^2 \omega, \quad \sigma_{12} = \sigma^p \sin \omega \cos \omega,$$

$$(4.7) \quad \sigma_{22} = \sigma^a + \sigma^p \sin^2 \omega,$$

$$(4.8) \quad \text{tg} \omega = \frac{\sigma_{12}}{\sigma_{11}}.$$

Here ω is the *net angle*, that is the angle between the direction \mathbf{b}_1 of the fibers of connective tissue and the base vector \mathbf{g}_1 , orthogonal to the muscle direction.

If the deformed shape is known, the equilibrium equations can be solved in terms of stresses if sufficient boundary conditions on the stress are given. We

consider in particular the case in which the boundary is the line $t^2 = 0$ and restrict to the case of “determined” boundary conditions. An example of such determined boundary conditions is

$$(4.9) \quad \mathbf{g}_1^0 \cdot \mathbf{Tg}_2^0 = \varphi(t^1),$$

where $\mathbf{g}_1^0, \mathbf{g}_2^0$ are the natural base vectors at $t^2 = 0$, and $\varphi(t^1)$ is a given scalar function of t^1 .

In the costal part of the diaphragm the surface is well approximated by a cyclide of constant radius ρ whose line of centers Γ^c is a plane curve, as the surface shown in Fig. 3. A particular approximation of the costal diaphragm of a dog at end of inspiration obtained by ANGELILLO *et al.* in [8] is show in Fig. 3.

Substituting (4.4) into equilibrium Eq. (4.2) and recalling (3.11 – 3.13) and (4.5), we have

$$(4.10) \quad \frac{\partial}{\partial t^1} (\sigma_{\alpha\beta} \hat{\mathbf{g}}_\alpha \otimes \hat{\mathbf{g}}_\beta) \frac{\hat{\mathbf{g}}_1}{f} + \frac{\partial}{\partial t^2} (\sigma_{\alpha\beta} \hat{\mathbf{g}}_\alpha \otimes \hat{\mathbf{g}}_\beta) \hat{\mathbf{g}}_2 + p \hat{\mathbf{g}}_3 = 0.$$

Expanding the sums in (4.10):

$$(4.11) \quad \begin{aligned} \frac{\partial}{\partial t^1} (\sigma_{11} \hat{\mathbf{g}}_1 \otimes \hat{\mathbf{g}}_1 + \sigma_{12} \hat{\mathbf{g}}_1 \otimes \hat{\mathbf{g}}_2 + \sigma_{21} \hat{\mathbf{g}}_2 \otimes \hat{\mathbf{g}}_1 + \sigma_{22} \hat{\mathbf{g}}_2 \otimes \hat{\mathbf{g}}_2) \frac{\hat{\mathbf{g}}_1}{f} \\ + \frac{\partial}{\partial t^2} (\sigma_{11} \hat{\mathbf{g}}_1 \otimes \hat{\mathbf{g}}_1 + \sigma_{12} \hat{\mathbf{g}}_1 \otimes \hat{\mathbf{g}}_2 + \sigma_{21} \hat{\mathbf{g}}_2 \otimes \hat{\mathbf{g}}_1 + \sigma_{22} \hat{\mathbf{g}}_2 \otimes \hat{\mathbf{g}}_2) \hat{\mathbf{g}}^2 \\ + p \hat{\mathbf{g}}^3 = 0, \end{aligned}$$

and taking the derivatives, we are left with

$$(4.12) \quad \begin{aligned} d \frac{1}{f} \sigma_{11,1} \hat{\mathbf{g}}_1 + \frac{1}{f} \sigma_{11} \hat{\mathbf{g}}_{1,1} + \frac{1}{f} \sigma_{12} \hat{\mathbf{g}}_1 (\hat{\mathbf{g}}_{2,1} \cdot \hat{\mathbf{g}}_1) + \frac{1}{f} \sigma_{21,1} \hat{\mathbf{g}}_2 + \frac{1}{f} \sigma_{21} \hat{\mathbf{g}}_{2,1} \\ + \frac{1}{f} \sigma_{22} \hat{\mathbf{g}}_2 (\hat{\mathbf{g}}_{2,1} \cdot \hat{\mathbf{g}}_1) + \sigma_{12,2} \hat{\mathbf{g}}_1 + \sigma_{22,2} \hat{\mathbf{g}}_2 + \sigma_{22} \hat{\mathbf{g}}_{2,2} + p \hat{\mathbf{g}}_3 = 0, \end{aligned}$$

where an index, say i , preceded by a comma stands for differentiation with respect to t^i .

Recalling (3.2), (3.5), (3.6), (3.11 – 3.13) and (4.5) we have

$$(4.13) \quad \frac{1}{f}\sigma_{11,1}\hat{\mathbf{g}}_1 + \frac{1}{Rf}\sigma_{11}\left(\sin\frac{t^2}{\rho}\hat{\mathbf{g}}_2 - \cos\frac{t^2}{\rho}\hat{\mathbf{g}}_3\right) + \frac{1}{Rf}\sigma_{12}\sin\frac{t^2}{\rho}\hat{\mathbf{g}}_1 \\ + \frac{1}{f}\sigma_{21,1}\hat{\mathbf{g}}_2 - \frac{1}{Rf}\sigma_{21}\sin\frac{t^2}{\rho}\hat{\mathbf{g}}_1 - \frac{1}{Rf}\sigma_{22}\sin\frac{t^2}{\rho}\hat{\mathbf{g}}_2 + \sigma_{12,2}\hat{\mathbf{g}}_1 + \sigma_{22,2}\hat{\mathbf{g}}_2 \\ - \frac{1}{\rho}\sigma_{22}\hat{\mathbf{g}}_3 + p\hat{\mathbf{g}}_3 = 0.$$

In components in the local Cartesian frame $\{\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3\}$

$$(4.14) \quad \frac{1}{f}\sigma_{11,1} + \sigma_{12,2} - \frac{2}{Rf}\sigma_{12}\sin\frac{t^2}{\rho} = 0, \\ \frac{1}{f}\sigma_{21,1} + \sigma_{22,2} + \frac{1}{Rf}\sigma_{11}\sin\frac{t^2}{\rho} - \frac{1}{Rf}\sigma_{22}\sin\frac{t^2}{\rho} = 0, \\ -\frac{1}{Rf}\sigma_{11}\cos\frac{t^2}{\rho} - \frac{1}{\rho}\sigma_{22} + p = 0.$$

Or, since $f = 1 + \rho/R \cos(t^2/\rho)$, $Rf = R + \rho \cos(t^2/\rho)$,

$$(4.15) \quad \frac{1}{1 + \frac{\rho}{R}\cos\frac{t^2}{\rho}}\sigma_{11,1} + \sigma_{12,2} - \frac{2}{R + \rho\cos\frac{t^2}{\rho}}\sigma_{12}\sin\frac{t^2}{\rho} = 0, \\ \frac{1}{1 + \frac{\rho}{R}\cos\frac{t^2}{\rho}}\sigma_{21,1} + \sigma_{22,2} + \frac{1}{R + \rho\cos\frac{t^2}{\rho}}(\sigma_{11} - \sigma_{22})\sin\frac{t^2}{\rho} = 0, \\ \sigma_{11}\frac{\cos\frac{t^2}{\rho}}{R + \rho\cos\frac{t^2}{\rho}} + \frac{1}{\rho}\sigma_{22} = p.$$

By multiplying the second equation by $Rf = R + \rho \cos(t^2/\rho)$, it can be written in the form

$$(4.16) \quad R\sigma_{21,1} + \frac{\partial}{\partial t^2}\left(\sigma_{22}\left(R + \rho\cos\frac{t^2}{\rho}\right)\right) + \sigma_{11}\sin\frac{t^2}{\rho} = 0.$$

The differential system (4.15) can be solved if proper conditions are given at the boundary. To any of these boundary conditions it corresponds, through (4.8), a particular distribution of the net angle ω . We look for solutions of (4.15) compatible with the conditions $\omega = 0$ (or, what is the same, $\sigma_{12} = 0$), corresponding to an orthogonal net.

Indeed, on assuming $\sigma_{12} = 0$ at the boundary (this corresponds to considering the boundary condition (4.9) with $\varphi(t^1) = 0$, the stress field

$$(4.17) \quad \begin{aligned} \sigma_{11} &= -\frac{B}{\rho \cos^2 \frac{t^2}{\rho}} + \frac{\rho p}{2}, \\ \sigma_{22} &= \frac{B}{\cos \frac{t^2}{\rho} \left(R + \rho \cos \frac{t^2}{\rho} \right)} + \rho p \frac{R + \frac{\rho}{2} \cos \frac{t^2}{\rho}}{R + \rho \cos \frac{t^2}{\rho}}, \\ \sigma_{12} &= 0, \end{aligned}$$

solve (4.34). To avoid that the normal stress σ_{11} blow up at $t^2 = \rho\pi/2$, we assume $B = 0$ and consider the solution:

$$(4.18) \quad \sigma_{11} = \frac{\rho p}{2}, \quad \sigma_{22} = \rho p \frac{R + \frac{\rho}{2} \cos \frac{t^2}{\rho}}{R + \rho \cos \frac{t^2}{\rho}}, \quad \sigma_{12} = 0.$$

This stress field appears as the generalization of the stress distribution for a toroidal surfaces to cyclides generated by plane, but otherwise arbitrary, line of centers (see, for example, [9] p. 213).

In [10], EVANS and SKALAK, in dealing with cell membranes, consider the mechanics of membranes and give the equilibrium equations for axisymmetric surfaces. A torus is axisymmetric, but here we have shown that the corresponding stress distribution can be generalized to cyclides generated by plane line of centers, a set of surfaces that are not axisymmetric.

According to (4.8), if the net angle ω vanishes, then the fibers of connective tissue are orthogonal to the muscle lines, $\sigma_{11} = \sigma^p$, $\sigma_{22} = \sigma^a$. The normal stress σ_{11} along the fibers of connective tissue is constant and the normal stress σ_{22} along the muscle fibers depends on t^1 through R .

The graph of σ_{22} for the particular cyclide \mathcal{S} , depicted in Fig. 3, approximating the costal diaphragm of a dog and generated by a plane line of centers I^c , is shown in Fig. 4. Notice that the points $(t^1, 0)$ describe the line of attachment

of the diaphragm along the chest wall and the point $(t^1, t^2) = (0, 0)$ corresponds to point *A* in Fig. 3. The value taken for the pressure is $p = 0.2$, the radius $\rho = 4$ cm and the expression for R is

$$R = 8.05 - 0.67 \cos\left(\frac{1}{4}t^1\right) + 0.22 \cos\left(\frac{1}{2}t^1\right).$$

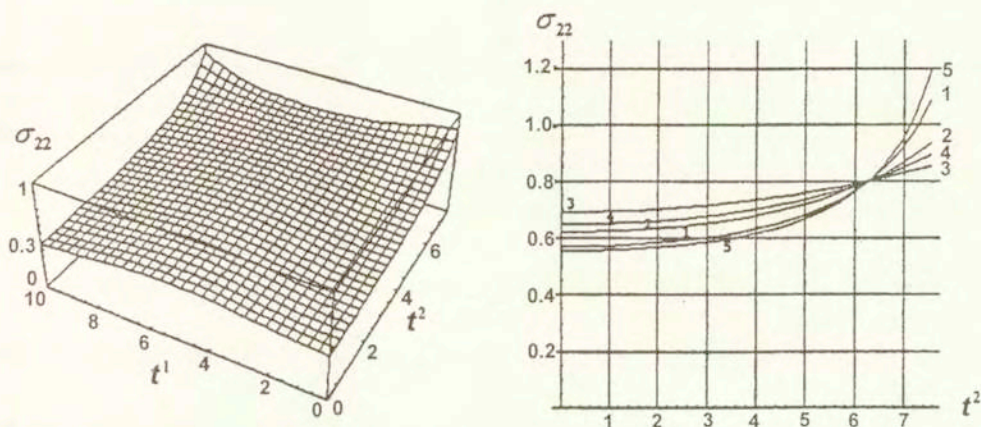


FIG. 4. 3D Plot and stress profiles at: $t^1 = 0(1)$, $t^1 = 2.5(2)$, $t^1 = 5(3)$, $t^1 = 7.5(4)$, $t^1 = 10(5)$, of the stress component σ_{22} .

Summing up, we can draw some conclusions.

Solution (4.18) generalizes to cyclides the stress distributions for a pressurized torus.

The solution we picked for the stress is compatible with the assumption that the fibers of connective tissue are orthogonal to the muscle lines in the active diaphragm configuration. As a consequence of (4.18), the active and passive stresses are principal stresses over the surface.

The stress along the fibers of connective tissue is constant.

The variation of active stress in between the lines of attachment, from chest wall to central tendon, suggest that the thickness of the diaphragm should be larger close to the central tendon. This is consistent with some physiological observations that the diaphragm is thicker at the central tendon end [11].

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