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Irreducible representations for constitutive equations of anisotropic solids II: crystal and quasicrystal classes D_{2m+1d}, D_{2m+1} and C_{2m+1v}

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A SIMPLE, UNIFIED PROCEDURE is applied to derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In this part, our concern is for all crystal classes and quasicrystal classes D_{2m+1d}, D_{2m+1} and C_{2m+1v} for all integers $m \geq 1$.

1. Introduction

IN CONTINUUM PHYSICS, complicated and varied macroscopic physical behaviours of anisotropic solids are modelled by scalar-, vector- and second order tensor-valued functions of vector variables and second order tensor variables, commonly known as material constitutive equations. Material objectivity and material symmetry place a combined invariance restriction under the material symmetry group on the tensor function forms of material constitutive equations. General reduced forms, or representations, of material constitutive equations under the just-mentioned universal invariance restriction, constitute a rational basis for consistent mathematical modelling of complex material behaviours. In the past decades, this aspect was extensively studied. Now many results for polynomial representations and some results for nonpolynomial representations are available. For detail, see, e.g., the monographs by TRUESDELL and NOLL [11], SPENCER [10], BOEHLER [4], RYCHLEWSKI [7], ERINGEN and MAUGIN [5], KIRAL and ERINGEN [6], BETTEN [2], SMITH [9], and the recent reviews by BETTEN [1], RYCHLEWSKI and ZHANG [8] and ZHENG [21], *et al.* Some references are listed in Part I of this series of paper.

Although now many results in many cases are available, general aspects of tensor function representations, especially nonpolynomial representations, are still under investigation, which are concerned with any finite number of vector variables and tensor variables and all kinds of material symmetry groups including the 32 crystal classes and all denumerably infinitely many quasicrystal classes. As compared with polynomial representations, nonpolynomial representations are not only more general both in notion and in scope, but may furnish more compact representations for constitutive equations, as noted by WANG [12] for isotropic cases and by BOEHLER [3 – 4] for anisotropic cases. There are relatively few results for irreducible nonpolynomial representations for the foregoing general cases, except for those concerning some simple material symmetry groups (see the related references in Part I of this series, i.e. XIAO, BRUHNS and MEYERS [20]. Henceforth, the just-mentioned reference will be simply referred to as Part I). In a series of work consisting of three parts, we aim to provide irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations of any finite number of vector variables and second order tensor variables relative to all crystal and quasicrystal classes as subgroups of the cylindrical group $D_{\infty h}$. In the second part, we consider the crystal and quasicrystal classes D_{2m+1d}, D_{2m+1} and C_{2m+1v} for all integers $m \geq 1$.

As it has been done in Part I, we shall apply a unified procedure based on [13 – 15] and [18] to derive the desired functional bases and generating sets. For a detailed account of such a unified procedure and for notations and preliminaries, refer to Secs. 2 – 3 in Part I and the related reference therein.

2. Crystal and quasicrystal classes D_{2m+1d}

The classes at issue are of the form

$$D_{2m+1d}(\mathbf{n},\mathbf{e}) = \{\pm\mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}, \pm\mathbf{R}_{\mathbf{a}_k}^{\pi} | \mathbf{a}_k = \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}\mathbf{e}, \quad k = 1, \cdots, 2m+1\}. \tag{2.1}$$

They include the trigonal crystal class D_{3d} as the particular case when $m = 1$. Henceforth, $\mathbf{a}$ will be used to represent one of the *two-fold axis* vectors $\mathbf{a}_1, \cdots, \mathbf{a}_{2m+1}$.

2.1. Single variables

(i) A single vector $\mathbf{u}$

Each anisotropic function of a vector variable $\mathbf{u}$ under the group D_{2m+1d} may be extended as an isotropic function of the three variables $(\mathbf{u}, \mathbf{E}\eta_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$ (see Theorem 1 in XIAO [15]). Applying the related result for isotropic functions and following the unified procedure outlined in Sec. 3 in Part I, we construct the following table.

V $\{\mathbf{u}, \mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$ $(\equiv V_{2m+1}(\mathbf{u}))$

Skw $\{\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{u} \wedge (\mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{u} \wedge \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$
$(\equiv \mathrm{Skw}_{2m+1}(\mathbf{u}))$

Sym $\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \mathbf{n} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), \overset{\circ}{\mathbf{u}} \vee (\mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})),$
$\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})(\overset{\circ}{\mathbf{u}} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) - (\mathbf{u} \cdot \mathbf{n})\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\}$ $(\equiv \mathrm{Sym}_{2m+1}(\mathbf{u}))$

R $\mathbf{r} \cdot \mathbf{u}, [\mathbf{r}, \mathbf{u}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})], \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\, \overset{\circ}{\mathbf{r}} \cdot \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}});$
$\mathrm{tr}\mathbf{H}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), [\mathbf{u}, \mathbf{H}\mathbf{u}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})], \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) \cdot \mathbf{H}\mathbf{u};$
$\mathrm{tr}\mathbf{C}, \mathbf{n} \cdot \mathbf{C}\mathbf{n}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{C}}\, \mathbf{n}], [\mathbf{u}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}],$
$\alpha_{2m}(\overset{\circ}{\mathbf{u}})(\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) \cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}} - (\mathbf{u} \cdot \mathbf{n})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) \cdot \overset{\circ}{\mathbf{C}}\, \mathbf{n});$
$\{(\mathbf{u} \cdot \mathbf{n})^2, |\overset{\circ}{\mathbf{u}}|^2, \alpha_{4m+2}(\overset{\circ}{\mathbf{u}}), (\mathbf{u} \cdot \mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{u}})\}$ $(\equiv I_{2m+1}(\mathbf{u}))$.

First, we show that the presented set $I_{2m+1}(\mathbf{u})$ of invariants is a desired functional basis. In fact, an isotropic functional basis of $(\mathbf{u}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n} \otimes \mathbf{n})$ is given by

$$|\mathbf{u}|^2, (\mathbf{u} \cdot \mathbf{n})^2, \mathbf{u} \cdot (\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))^2\mathbf{u}, \mathbf{u} \cdot (\mathbf{n} \otimes \mathbf{n})(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\mathbf{u},$$
$$\mathbf{u} \cdot (\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\ (\mathbf{n} \otimes \mathbf{n})(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))^2\mathbf{u}.$$

In deriving the above basis, many redundant invariants have been removed by using the equalities

$$(\mathbf{n} \otimes \mathbf{n})^2 = \mathbf{n} \otimes \mathbf{n}, \quad (\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))^2\mathbf{n} = -|\overset{\circ}{\mathbf{u}}|^{4m}\mathbf{n}. \tag{2.2}$$

Then, using the second equality above and the identity

$$(\mathbf{E}\mathbf{u})\mathbf{v} = \mathbf{v} \times \mathbf{u}, \tag{2.3}$$

as well as the decomposition formula (2.15) in Part I, we know that the first four invariants in the foregoing basis yield the set $I_{2m+1}(\mathbf{u})$ of invariants and, moreover, the last invariant given before is redundant. Thus, the presented set $I_{2m+1}(\mathbf{u})$ is a desired functional basis. It may readily be proved that this basis is irreducible.

Next, we prove that the two presented sets $V_{2m+1}(\mathbf{u})$ and $\mathrm{Skw}_{2m+1}(\mathbf{u})$ supply a desired vector generating set and a desired skewsymmetric tensor generating set, respectively. In fact, let $\mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) = \mathbf{0}$. Then $\mathbf{u} = x\mathbf{a}_k$ or $\mathbf{u} = x\mathbf{n}$. It can easily be shown that the foregoing two sets obey the criterion (2.3) given in

Part I. Now let $\mathbf{u}\times\boldsymbol{\eta}_{2m+1}(\overset{\circ}{\mathbf{u}})\neq\mathbf{0}$. Then we have (see (2.5), (2.6) and (2.8) in Part I)

$$\Gamma(\mathbf{u},\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\subset C_{2h}(\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\subset D_{\infty h}(\mathbf{n}),$$

and therefore we have $\Gamma(\mathbf{u},\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{n}\otimes\mathbf{n})=\Gamma(\mathbf{u},\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))$. From the latter and the criterion (2.3) given in Part I, we infer that for the case for $\mathbf{u}$ at issue, isotropic generating sets for the three variables $(\mathbf{u},\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{n}\otimes\mathbf{n})$ are obtainable from those for the two variables $(\mathbf{u},\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))$. By applying the related result for isotropic functions we know that, for the vector-valued and skewsymmetric tensor-valued cases, the latter are just given by the two presented sets $V_{2m+1}(\mathbf{u})$ and $\mathrm{Skw}_{2m+1}(\mathbf{u})$.

Finally, we show that the presented set $\mathrm{Sym}_{2m+1}(\mathbf{u})$ supplies a desired symmetric tensor generating set. To this end we prove that this set obeys the criterion (2.3) given in Part I. When $\overset{\circ}{\mathbf{u}}=\mathbf{0}$, the just-mentioned fact is evidently true. Now let $\overset{\circ}{\mathbf{u}}\neq\mathbf{0}$. Then the triplet $(\mathbf{n},\overset{\circ}{\mathbf{u}},\mathbf{n}\times\overset{\circ}{\mathbf{u}})$, denoted by $(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_2)$, is an orthogonalized basis of the vector space V, and hence the six tensors $\mathbf{e}_i\vee\mathbf{e}_j$, $i,\ j=1,2,3$, form an orthogonalized basis of the symmetric tensor space Sym. It is easily understood that the first three generators in the set $\mathrm{Sym}_{2m+1}(\mathbf{u})$ yield the three tensors $\mathbf{e}_i\otimes\mathbf{e}_i$, i=1, 2, 3. Thus, we have

$$\mathrm{rank}\ \ \mathrm{Sym}_{2m+1}(\mathbf{u})=3+\mathrm{rank}\{\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3\},$$

where the $\mathbf{G}_i$ are used to denote the last three generators in the set $\mathrm{Sym}_{2m+1}(\mathbf{u})$, i.e.

$$\mathbf{G}_1=\mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\eta}),\ \mathbf{G}_2=\overset{\circ}{\mathbf{u}}\vee(\mathbf{u}\times\boldsymbol{\eta}),\quad \mathbf{G}_3=\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})(\overset{\circ}{\mathbf{u}}\vee\boldsymbol{\eta}-(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\eta}),$$

with $\boldsymbol{\eta}=\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})$. We have

$$\Delta=\begin{vmatrix}\mathbf{G}_1:(\mathbf{e}_1\vee\mathbf{e}_2) & \mathbf{G}_1:(\mathbf{e}_2\vee\mathbf{e}_3) & \mathbf{G}_1:(\mathbf{e}_3\vee\mathbf{e}_1)\\ \mathbf{G}_2:(\mathbf{e}_1\vee\mathbf{e}_2) & \mathbf{G}_2:(\mathbf{e}_2\vee\mathbf{e}_3) & \mathbf{G}_2:(\mathbf{e}_3\vee\mathbf{e}_1)\\ \mathbf{G}_3:(\mathbf{e}_1\vee\mathbf{e}_2) & \mathbf{G}_3:(\mathbf{e}_2\vee\mathbf{e}_3) & \mathbf{G}_3:(\mathbf{e}_3\vee\mathbf{e}_1)\end{vmatrix}$$

$$=8\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\begin{vmatrix}0 & \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}) & \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\\ xy^2\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}) & 0 & -y^2\beta_{2m+1}(\overset{\circ}{\mathbf{u}})\\ -y^2\beta_{2m+1}(\overset{\circ}{\mathbf{u}}) & x\beta_{2m+1}(\overset{\circ}{\mathbf{u}}) & -x\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\end{vmatrix}$$

$$=8y^4(\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}))^2(x^2y^{4m}+(\beta_{2m+1}(\mathbf{u}))^2),$$

where $x=\mathbf{u}\cdot\mathbf{n}$ and $y=|\overset{\circ}{\mathbf{u}}|$. Hence, for $\overset{\circ}{\mathbf{u}}\neq\mathbf{0}$ we deduce

$$\text{rank Sym}_{2m+1}(\mathbf{u}) = \begin{cases} 6 \text{ if } \Delta \neq 0, \\ 4 \text{ if } \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}) = 0, \\ 4 \text{ if } x = \beta_{2m+1}(\overset{\circ}{\mathbf{u}}) = 0. \end{cases}$$

From these results and

$$\Gamma(\mathbf{u}) \cap D_{2m+1d} = \begin{cases} C_{1h}(\mathbf{a}_k) \text{ if } \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}) = 0, \\ C_2(\mathbf{a}_k) \text{ if } x = \beta_{2m+1}(\overset{\circ}{\mathbf{u}}) = 0, \end{cases}$$

for $\overset{\circ}{\mathbf{u}} \neq \mathbf{0}$, as well as Table 3 given in Sec. 2 in Part I, we infer that the set $\text{Sym}_{2m+1}(\mathbf{u})$ obeys the criterion (2.3) in Part I.

Each generating set presented is minimal and, of course, irreducible.

(ii) A skewsymmetric tensor $\mathbf{W}$

Skw $\{\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})) - (\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))\mathbf{W}\}$
$(\equiv \text{Skw}_{2m+1}(\mathbf{W}))$

Sym $\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{Wn}\otimes\mathbf{Wn}, \mathbf{n}\vee\mathbf{Wn}, \mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})) + (\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))\mathbf{W},$
$\mathbf{W}^2\mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\mathbf{Wn}))\}$ $(\equiv \text{Sym}_{2m+1}(\mathbf{W}))$

R $\text{tr}\mathbf{H}\ \mathbf{W}, \text{tr}\mathbf{H}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})), \text{tr}\mathbf{H}\ \mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))$; $\text{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn},$
$(\mathbf{Wn})\cdot \overset{\circ}{\mathbf{C}}\ \mathbf{Wn}, (\overset{\circ}{\mathbf{C}}\ \mathbf{n})\cdot\mathbf{Wn}, \text{tr}\overset{\circ}{\mathbf{C}}\ \mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})), [\mathbf{n}, \boldsymbol{\eta}_{2m}(\mathbf{Wn}), \overset{\circ}{\mathbf{C}}\ \mathbf{W}^2\mathbf{n}]$;
$\{(\text{tr}\mathbf{W}\ \mathbf{N})^2, |\mathbf{Wn}|^2, \beta_{2m+1}(\mathbf{Wn}), (\text{tr}\mathbf{W}\ \mathbf{N})\alpha_{2m+1}(\mathbf{Wn})\}$ $(\equiv I_{2m+1}(\mathbf{W}))$.

The proof for the above results is as follows. Anisotropic functions of the skewsymmetric tensor $\mathbf{W}$ under the group D_{2m+1d} may be extended as isotropic functions of the extended set $(\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$ (see Theorem 1 in XIAO [15]). Applying the related result for scalar-valued isotropic functions, we derive an isotropic functional basis of the extended variables $(\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$ as follows:

$$\text{tr}\mathbf{W}^2, |\mathbf{Wn}|^2, \text{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})), \text{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))(\mathbf{n}\otimes\mathbf{n}),$$
$$\text{tr}\mathbf{W}^2(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))(\mathbf{n}\otimes\mathbf{n}).$$

In deriving the above result, many obviously redundant invariants have been removed by using the equalities

$$(\mathbf{n}\otimes\mathbf{n})^2 = \mathbf{n}\otimes\mathbf{n},\ (\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))^2 = \boldsymbol{\eta}_{2m}(\mathbf{Wn})\otimes\boldsymbol{\eta}_{2m}(\mathbf{Wn})$$
$$-|\mathbf{Wn}|^{4m}\mathbf{I},\ \ \boldsymbol{\eta}_{2m}(\mathbf{Wn})\cdot\mathbf{n} = 0.$$

Moreover, with the aid of the decomposition formula (2.16) given in Part I, we infer that, of the foregoing basis, the third invariant is equal to the fourth invariant and hence redundant. The other four invariants form the presented set $I_{2m+1}(\mathbf{W})$ of invariants.

Next, we show that the presented set $\mathrm{Skw}_{2m+1}(\mathbf{W})$ is a desired skewsymmetric tensor generating set. Two cases are considered. First, let $\mathbf{W}$ and $\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})$ be linearly independent, i.e., either of them is nonvanishing and their axis vectors are noncollinear. Then we have (see (2.6) in Part I)

$$\Gamma(\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})) = S_2 = \Gamma(\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n}).$$

From this fact and the criterion (2.3) in Part I it follows that for the case at issue, an isotropic skewsymmetric tensor generating set for $(\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))$ supplies an isotropic skewsymmetric tensor generating set for $(\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$. By applying the related result for isotropic functions we know that the former is just given by the set $\mathrm{Skw}_{2m+1}(\mathbf{W})$. Second, let $\mathbf{W}$ and $\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})$ be linearly dependent. Then we have $(\mathrm{tr}\mathbf{W}^2)(\mathrm{tr}\mathbf{H}^2) - (\mathrm{tr}\mathbf{WH})^2 = 0$ with $\mathbf{H} = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})$. Hence we derive $\mathbf{W} = c\mathbf{Ez}$ with $\mathbf{z}\in\{\mathbf{n}, \mathbf{a}_1, \cdots, \mathbf{a}_{2m+1}\}$. Evidently, for the case at issue, a single generator $\mathbf{W}$ is enough to form a desired generating set.

Finally, we show that the presented set $\mathrm{Sym}_{2m+1}(\mathbf{W})$ is a desired symmetric tensor generating set. Towards this goal, we show that this set obeys the criterion (2.3) given in Part I. In fact, let $\mathbf{Wn}\neq\mathbf{0}$. Then the triplet $(\mathbf{n}, \mathbf{Wn}, \mathbf{n}\times\mathbf{Wn})$, denoted by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, is an orthogonalized basis of the vector space $\mathbf{V}$, and hence the six symmetric tensors $\mathbf{e}_i\vee\mathbf{e}_j$, $i, j = 1, 2, 3$ form an orthogonalized basis of the symmetric tensor space Sym. In the presented set $\mathrm{Sym}_{2m+1}(\mathbf{W})$, the first four generators yield $\mathbf{e}_3\otimes\mathbf{e}_3$ and $\mathbf{e}_i\vee\mathbf{e}_j$, $i, j = 1, 2$. For the last two generators in the set $\mathrm{Sym}_{2m+1}(\mathbf{W})$, denoted by $\mathbf{G}_1$ and $\mathbf{G}_2$, by using the formulas (2.6) and $(2.28)_{2,3}$ in Part I we have

$$\Delta = \begin{vmatrix} \mathbf{G}_1 : (\mathbf{e}_1\vee\mathbf{e}_3) & \mathbf{G}_1 : (\mathbf{e}_2\vee\mathbf{e}_3) \\ \mathbf{G}_2 : (\mathbf{e}_1\vee\mathbf{e}_3) & \mathbf{G}_2 : (\mathbf{e}_2\vee\mathbf{e}_3) \end{vmatrix}$$

$$= 4\begin{vmatrix} x\beta_{2m+1}(\mathbf{Wn}) & y^2\alpha_{2m+1}(\mathbf{Wn}) \\ -y^2\alpha_{2m+1}(\mathbf{Wn}) & xy^2\beta_{2m+1}(\mathbf{Wn}) \end{vmatrix}$$

$$= 4y^2(x^2(\beta_{2m+1}(\mathbf{Wn}))^2 + y^2(\alpha_{2m+1}(\mathbf{Wn}))^2),$$

where $x = \mathrm{tr}\mathbf{WN}$ and $y = |\mathbf{Wn}|$. Hence, for $\mathbf{Wn}\neq\mathbf{0}$ we deduce

$$\mathrm{rank}\ \mathrm{Sym}_{2m+1}(\mathbf{W}) \geq \begin{cases} \mathrm{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{Wn}\otimes\mathbf{Wn}, \mathbf{n}\vee\mathbf{Wn}\} \\ = 4 \quad \text{if} \quad \Delta = 0, \qquad \text{i.e. } \mathbf{W} = c\mathbf{Ea}_k, \\ 6 \quad \text{if} \quad \Delta\neq 0. \end{cases}$$

From the latter and $\Gamma(\mathbf{W}) \cap D_{2m+1d} = C_{2h}(\mathbf{a}_k)$ for $\mathbf{W} = c\mathbf{E}\mathbf{a}_k \neq \mathbf{O}$, as well as Table 3 given in Sec. 2 in Part I, we infer that for $\mathbf{Wn} \neq \mathbf{0}$, the set $\mathrm{Sym}_{2m+1}(\mathbf{W})$ obeys the criterion (2.3) given in Part I. Moreover, it can readily be shown that the same is true for $\mathbf{Wn} = \mathbf{0}$, i.e. $\mathbf{W} = c\mathbf{En}$.

Either of the two presented generating sets is minimal and, of course, irreducible.

(iii) A single symmetric tensor **A**

Anisotropic functions of the symmetric tensor variable **A** under the group D_{2m+1d} may be extended as isotropic functions of the four variables $(\mathbf{A}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\otimes\mathbf{n})$ (see Theorem 1 in XIAO [15]). Applying this fact and the related result for isotropic functions, one can immediately derive complete representations for the anisotropic functions at issue. However, the results thus obtained need not be irreducible. Removing the redundant elements and then following the unified procedure in Sec. 3 in Part I, we arrive at the desired irreducible representations.

$$\begin{aligned}
\mathrm{Skw}\quad & \{\beta_{2m+1}(\mathbf{q}(\mathbf{A}))\mathbf{N}, \mathbf{E}\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{A}}\,\mathbf{n}\wedge \overset{\circ}{\mathbf{A}}{}^2\mathbf{n},\\
& \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{N} + \mathbf{E}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}\ (\equiv \mathrm{Skw}_{2m+1}(\mathbf{A}))\\
\mathrm{Sym}\quad & \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}\otimes \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})),\\
& \overset{\circ}{\mathbf{A}}\,\mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})), \boldsymbol{\Phi}_{2m}(\mathbf{q}(\mathbf{A})),\\
& \mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))), \mathbf{A}_\mathbf{e}(\mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))) - (\mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})))\mathbf{A}_\mathbf{e}\}\\
& (\equiv \mathrm{Sym}_{2m+1}(\mathbf{A}))\\
R\quad & (\mathrm{tr}\mathbf{HN})\beta_{2m+1}(\mathbf{q}(\mathbf{A})), \mathrm{tr}\mathbf{H}(\mathbf{E}\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A}))), \mathrm{tr}\mathbf{H}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})),\\
& (\mathrm{tr}\mathbf{HN})J(\mathbf{A}), (\mathrm{tr}\mathbf{HN})\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + \mathrm{tr}\mathbf{H}(\mathbf{E}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})));\\
& \{\mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\mathbf{A}, |\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2, |\mathbf{q}(\mathbf{A})|^2, \mathbf{n}\cdot \overset{\circ}{\mathbf{A}}{}^3\mathbf{n}, \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),\\
& \alpha_{2m+1}(\mathbf{q}(\mathbf{A})),\\
& [\mathbf{n}, \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})]\}\ (\equiv I_{2m+1}(\mathbf{A})).
\end{aligned}$$

Here and henceforth, $\boldsymbol{\pi}_m((\mathbf{q}(\mathbf{A}))$ and $\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))$ is used to denote two vector-valued polymonial functions of the symmetric tensor **A** defined by

$$\text{(2.4)}\qquad \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) = \begin{cases} |\mathbf{q}(\mathbf{A})|^{m+1}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})) & \text{if } m \text{ is an odd number},\\ |\mathbf{q}(\mathbf{A})|^{m}\mathbf{A}_\mathbf{e}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})) & \text{if } m \text{ is an even number},\end{cases}$$

$$(2.5)\qquad \boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})) = \begin{cases} |\mathbf{q}(\mathbf{A})|^{m-1}\mathbf{A}_e\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})) \text{ if } m \text{ is an odd number}, \\ |\mathbf{q}(\mathbf{A})|^{m}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})) \text{ if } m \text{ is an even number}, \end{cases}$$

In the table given, the two sets $\mathrm{Sym}_{2m+1}(\mathbf{A})$ and $I_{2m+1}(\mathbf{A})$, cited from XIAO [16] and XIAO [17] separately, supply a desired irreducible symmetric tensor generating set and a desired irreducible functional basis, respectively. In the tables given here and in **(vi)** we omit the invariants provided by the scalar products of the symmetric tensor variable $\mathbf{C} \in \mathrm{Sym}$ and the presented symmetric tensor generators. In the final general result that will be given by Theorem 4, we intend to cite directly the related results recently established by the authors (Xiao, Bruhns & Meyers [19]), which are simpler and more compact than the foregoing invariants from the scalar product. Moreover, using the equality (2.7) below and noticing that the invariant $|\overset{\circ}{\mathbf{A}}\mathbf{n}|(\mathbf{H}\mathbf{n})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n}$ is redundant, we know the scalar product $\mathrm{tr}\mathbf{H}(\overset{\circ}{\mathbf{A}}\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}{}^2\mathbf{n})$ may be replaced by the invariant $(\mathrm{tr}\mathbf{HN})J(\mathbf{A})$, as has been done.

In what follows we prove that the presented set $\mathrm{Skw}_{2m+1}(\mathbf{A})$ is a desired irreducible skewsymmetric tensor generating set. First, we show that this set obeys the criterion (2.3) in Part I. The case when $\overset{\circ}{\mathbf{A}}= \mathbf{O}$ is trivial. Let $\overset{\circ}{\mathbf{A}}\neq \mathbf{O}$. By using the equalities

$$(2.6)\qquad \mathbf{A}_e\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})) = \alpha_{m+1}(\mathbf{q}(\mathbf{A}))\mathbf{e} + \beta_{m+1}(\mathbf{q}(\mathbf{A}))\mathbf{e}',$$

$$(2.7)\qquad \overset{\circ}{\mathbf{A}}\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}{}^2\mathbf{n} = |\overset{\circ}{\mathbf{A}}\mathbf{n}|^2\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n} + J(\mathbf{A})\mathbf{N},$$

and setting $\mathbf{q} = \mathbf{q}(\mathbf{A})$ and $D = \mathrm{rank}\ \mathrm{Skw}_{2m+1}(\mathbf{A})$, we deduce

$$D \geq \begin{cases} \mathrm{rank}\{\mathbf{N}, \mathbf{E}\boldsymbol{\pi}_m(\mathbf{q}), \mathbf{E}\boldsymbol{\rho}_m(\mathbf{q})\} = 3 \quad \text{if} \quad \beta_{2m+1}(\mathbf{q}) \neq 0, \\ \mathrm{rank}\{\mathbf{N}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}\} = 3 \quad \text{if} \quad \beta_{2m+1}(\mathbf{q}) = 0, \\ \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) \neq 0, \\ \mathrm{rank}\{\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{N}, \mathbf{E}\boldsymbol{\pi}_m(\mathbf{q})\} = 3 \quad \text{if} \quad \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) = 0, \\ J(\mathbf{A}) \neq 0, \\ \mathrm{rank}\{\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{E}\boldsymbol{\pi}_m(\mathbf{q})\} \geq 1 \quad \text{if} \quad \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) = \beta_{2m+1}(\mathbf{q}) \\ = J(\mathbf{A}) = 0. \end{cases}$$

It is clear that the above four cases for $\overset{\circ}{\mathbf{A}}\neq \mathbf{O}$ exhaust all possible cases. The last case yields

$$\overset{\circ}{\mathbf{A}}= x\mathbf{n}\vee\mathbf{a}_k + y(\mathbf{a}_k\otimes\mathbf{a}_k - \mathbf{a}'_k\otimes\mathbf{a}'_k),\ \mathbf{a}'_k = \mathbf{n}\times\mathbf{a}_k,\ x^2+y^2\neq 0.$$

This indicates that one of the two-fold axis vectors of the group D_{2m+1d} is an eigenvector of $\mathbf{A}$. Hence we have

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}) = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = J(\mathbf{A}) = 0 \Longrightarrow C_{2h}(\mathbf{a}_k) \subset \Gamma(\mathbf{A}) \cap D_{2m+1d}.$$

From the above facts and Table 2 given in Sec. 2 in Part I, we infer that the presented set $\mathrm{Skw}_{2m+1}(\mathbf{A})$ obeys the criterion (2.3) in Part I and hence is the desired skewsymmetric tensor generating set. Furthermore, let $\mathbf{A}_1 = \mathbf{e} \vee \mathbf{e}'$ and $\mathbf{A}_2 = \mathbf{n} \vee \mathbf{e}$. Then we have $\dim \mathrm{Skw}(\Gamma(\mathbf{A}_i) \cap D_{2m+1d}) = \dim \mathrm{Skw}(S_2) = 3$ and $\overset{\circ}{\mathbf{A}}_1\ \mathbf{n} = \mathbf{0}$ and $\mathbf{q}(\mathbf{A}_2) = \mathbf{0}$. From these facts we deduce that the five generators in the set $\mathrm{Skw}_{2m+1}(\mathbf{A})$ are irreducible.

2.2. D_{2m+1d}-irreducible sets of two variables

(iv) The D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

V $\quad V_{2m+1}(\mathbf{u}) \cup V_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}), \mathbf{v} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$
$(\equiv V_{2m+1}(\mathbf{u}, \mathbf{v}))$

Skw $\quad \mathrm{Skw}_{2m+1}(\mathbf{u}) \cup \mathrm{Skw}_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \wedge \mathbf{v},$
$(\mathbf{u} \cdot \mathbf{n})^{2m+1}\ \overset{\circ}{\mathbf{v}} \wedge (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})) + (\mathbf{v} \cdot \mathbf{n})^{2m+1}\ \overset{\circ}{\mathbf{u}} \wedge (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\}$
$(\equiv \mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{v}))$

Sym $\quad \mathrm{Sym}_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{2m+1}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v},$
$(\mathbf{u} \cdot \mathbf{n})^{2m+1}\ \overset{\circ}{\mathbf{v}} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})) + (\mathbf{v} \cdot \mathbf{n})^{2m+1}\ \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\}$
$(\equiv \mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{v}))$

R $\quad \mathbf{r} \cdot V_{2m+1}(\mathbf{z}), \mathbf{H} : \mathrm{Skw}_{2m+1}(\mathbf{z}), \mathbf{C} : \mathrm{Sym}_{2m+1}(\mathbf{z}),\ \mathbf{z} = \mathbf{u},\ \mathbf{v};$
$[\mathbf{r}, \mathbf{u}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})], [\mathbf{r}, \mathbf{v}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})]; \mathbf{u} \cdot \mathbf{H}\mathbf{v}; \mathbf{u} \cdot \overset{\circ}{\mathbf{C}}\ \mathbf{v};$
$(\mathbf{u} \cdot \mathbf{n})^{2m+1}[\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}), \mathbf{H}\mathbf{v}] + (\mathbf{v} \cdot \mathbf{n})^{2m+1}[\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{H}\mathbf{u}];$
$(\mathbf{u} \cdot \mathbf{n})^{2m+1}[\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}), \overset{\circ}{\mathbf{C}}\ \mathbf{v}] + (\mathbf{v} \cdot \mathbf{n})^{2m+1}[\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{C}}\ \mathbf{u}];$
$I_{2m+1}(\mathbf{u}) \cup I_{2m+1}(\mathbf{v}) \cup \{(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}), \mathbf{u} \cdot \mathbf{v}\}\ (\equiv I_{2m+1}(\mathbf{u}, \mathbf{v})).$

To prove the above results, we first work out the D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{v})$, specified by (see (3.1) in Part I)

$$\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1d} \neq \Gamma(\mathbf{z}) \cap D_{2m+1d},\ \mathbf{z} = \mathbf{u},\ \mathbf{v}.$$

It is evident that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, i.e. $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. Moreover, we have

$$\Gamma(\mathbf{z}) \cap D_{2m+1d} \neq C_1, \text{ i.e., } \operatorname{rank} V_{2m+1}(\mathbf{z}) \neq 3, \ \mathbf{z} = \mathbf{u}, \ \mathbf{v}.$$

The latter yields

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{z}})((\beta_{2m+1}(\overset{\circ}{\mathbf{z}}))^2 + (\mathbf{z} \cdot \mathbf{n})^2 | \overset{\circ}{\mathbf{z}})|^{4m}) = 0, \ \mathbf{z} = \mathbf{u}, \ \mathbf{v}.$$

Hence, each vector $\mathbf{z} \in \{\mathbf{u}, \mathbf{v}\}$ takes one of the forms

$$c\mathbf{a}, \ c \neq 0; \quad a\mathbf{n} + b\mathbf{n} \times \mathbf{a}, \ a^2 + b^2 \neq 0\,. \tag{2.8}$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u} = x\mathbf{a}, \ \mathbf{v} = y\mathbf{a}; \ \mathbf{u} = a\mathbf{n} + b\mathbf{n} \times \mathbf{a}, \ \mathbf{v} = x\mathbf{n} + y\mathbf{n} \times \mathbf{a},$$

which violate the D_{2m+1d}-irreducibility condition for $(\mathbf{u}, \mathbf{v})$, we derive the following three disjoint cases for the D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{v})$:

(c1) $\mathbf{u} = x\mathbf{e}$, $\mathbf{v} = y\mathbf{a}$, $\mathbf{a} \neq \mathbf{e}$, $xy \neq 0$;

(c2) $\mathbf{u} = x\mathbf{e}$, $\mathbf{v} = y\mathbf{n} \times \mathbf{a} + z\mathbf{n}$, $x(y^2 + z^2) \neq 0$;

(c3) $\mathbf{u} = x\mathbf{n} \times \mathbf{e} + w\mathbf{n}$, $\mathbf{v} = y\mathbf{n} \times \mathbf{a} + z\mathbf{n}$, $\mathbf{a} \neq \mathbf{e}$, $xy \neq 0$.

Then, for case (c1) we have

$$\begin{aligned}
\operatorname{rank} V_{2m+1}(\mathbf{u}, \mathbf{v}) &\geq \operatorname{rank}\{\mathbf{u}, \mathbf{v}, \mathbf{v} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\} = 3,\\
\operatorname{rank}\ \mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{v}) &\geq \operatorname{rank}\{\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}), \mathbf{u} \wedge \mathbf{v}\} = 3,\\
\operatorname{rank}\ \mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{v}) &\geq \operatorname{rank}(\mathrm{Sym}_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{2m+1}(\mathbf{v}))\\
&= \operatorname{rank}(\mathrm{Sym}(C_2(\mathbf{e})) \cup \mathrm{Sym}(C_2(\mathbf{a}))) = 6.
\end{aligned}$$

In deriving the last expression above, the formula (2.4) in Part I is used. For case (c2) we have the first expression above and

$$\begin{aligned}
\operatorname{rank}\ \mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \operatorname{rank}\{&\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{u}), \mathbf{u} \wedge \mathbf{v}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}),\\
&\overset{\circ}{\mathbf{u}} \wedge (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\} = 3,
\end{aligned}$$

$$\begin{aligned}
\operatorname{rank}\ \mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{v}) \geq \operatorname{rank}\{&\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \mathbf{n} \vee (\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\\
&\mathbf{u} \vee \mathbf{v}, \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\} = 6.
\end{aligned}$$

For case (c3) we have the second expression for case (1) and

$$\begin{aligned}
\operatorname{rank} V_{2m+1}(\mathbf{u}, \mathbf{v}) &\geq \operatorname{rank}(V_{2m+1}(\mathbf{u}) \cup V_{2m+1}(\mathbf{v}))\\
&= \operatorname{rank}(V(C_{1h}(\mathbf{e})) \cup V(C_{1h}(\mathbf{a}))) = 3,
\end{aligned}$$

$$\text{rank}\ \mathrm{Sym}_{2m+1}(\mathbf{u},\mathbf{v}) \geq \text{rank}(\mathrm{Sym}_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{2m+1}(\mathbf{v})) \\ = \text{rank}(\mathrm{Sym}(C_{1h}(\mathbf{e})) \cup \mathrm{Sym}(C_{1h}(\mathbf{a}))) = 6\,.$$

In the above, Eq. (2.4) in Part I is used again. From the above results we infer that the three sets of generators at issue obeys the criterion (2.3) in Part I, and hence they supply desired vector, skewsymmetric tensor and symmetric tensor generating sets. Further, by considering the two pairs: $\mathbf{u}_1 = \mathbf{n}$ and $\mathbf{v}_1 = \mathbf{e}$, $\mathbf{u}_2 = \mathbf{e}$ and $\mathbf{v}_2 = \mathbf{n}$, we deduce that the respective last two generators in the three presented generating sets are irreducible.

Finally, it is evident that the presented set $I_{2m+1}(\mathbf{u},\mathbf{v})$ of invariants determines a functional basis of the two variables $(\mathbf{u},\mathbf{v})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$.

(**v**) The D_{2m+1d}-irreducible set $(\mathbf{W},\mathbf{\Omega})$ of two skewsymmetric tensors

Skw $\{\mathbf{W},\mathbf{\Omega},\mathbf{W\Omega}-\mathbf{\Omega W}\}$

Sym $\mathrm{Sym}_{2m+1}(\mathbf{W}) \cup \mathrm{Sym}_{2m+1}(\mathbf{\Omega}) \cup \{\mathbf{W\Omega}+\mathbf{\Omega W},$
$|\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn} \vee \mathbf{NWn} + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n} \vee \mathbf{N\Omega n}\}$

R $\mathrm{tr}\mathbf{HW}, \mathrm{tr}\mathbf{H\Omega}; \mathbf{C} : \mathrm{Sym}_{2m+1}(\mathbf{W}), \mathbf{C} : \mathrm{Sym}_{2m+1}(\mathbf{\Omega})$
$\mathrm{tr}\mathbf{HW\Omega};\ \mathrm{tr}\overset{\circ}{\mathbf{C}}\ \mathbf{W\Omega},$
$|\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})[\mathbf{n},\mathbf{Wn},\overset{\circ}{\mathbf{C}}\ \mathbf{Wn}] + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})[\mathbf{n},\mathbf{\Omega n},\overset{\circ}{\mathbf{C}}\ \mathbf{\Omega n}];$
$I_{2m+1}(\mathbf{W}) \cup I_{2m+1}(\mathbf{\Omega}) \cup \{\mathrm{tr}\mathbf{W\Omega}\}.$

The proof for the above results is the same as that given for the corresponding case (**v**) in Sec. 4 in Part I.

(**vi**) The D_{2m+1}-irreducible set $(\mathbf{W},\mathbf{A})$ of a skewsymmetric tensor and a symmetric tensor

Skw $\mathrm{Skw}_{2m+1}(\mathbf{A}) \cup \{\mathbf{W}, \overset{\circ}{\mathbf{A}}\ \mathbf{W}+\mathbf{W}\ \overset{\circ}{\mathbf{A}}, (\mathbf{E}:\mathbf{W}) \wedge \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\}$
$(\equiv \mathrm{Skw}_{2m+1}(\mathbf{W},\mathbf{A}))$

Sym $\mathrm{Sym}_{2m+1}(\mathbf{W}) \cup \mathrm{Sym}_{2m+1}(\mathbf{A}) \cup \{(\mathrm{tr}\mathbf{WN})(\overset{\circ}{\mathbf{A}}\ \mathbf{N}-\mathbf{N}\ \overset{\circ}{\mathbf{A}}),$
$(\mathrm{tr}\mathbf{WN})((-1)^m\ \overset{\circ}{\mathbf{A}}\ \mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}) + \mathbf{n} \vee \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))\}$
$(\equiv \mathrm{Sym}_{2m+1}(\mathbf{W},\mathbf{A}))$

R $\mathrm{tr}\mathbf{HW}; \mathbf{C} : \mathrm{Sym}_{2m+1}(\mathbf{W});$
$I_{2m}(\mathbf{W}) \cup I_{2m}(\mathbf{A}) \cup \{(\mathbf{Wn})\cdot(\overset{\circ}{\mathbf{A}}\ \mathbf{n}),$
$(\mathbf{Wn})\cdot\ \overset{\circ}{\mathbf{A}}\ \mathbf{Wn}, \mathbf{n}\cdot\mathbf{W}\ \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}\}$

In the above table, the skewsymmetric tensor variable $\mathbf{H} \in \mathrm{Skw}$ is assumed to be of the form $\mathbf{H} = c\mathbf{W}$, and hence only one invariant from the scalar product concerning $\mathbf{H}$ is retained. The form for $\mathbf{H}$ just given is derived from cases (c1)–(c2) for the D_{2m+1d}-irreducible set $(\mathbf{W}, \mathbf{A})$ given later and the condition (see $(3.3)_2$ in Part I)

$$\varGamma(\mathbf{W}, \mathbf{H}) \cap D_{2m+1d} \neq \varGamma(\mathbf{W}, \mathbf{A}, \mathbf{H}) \cap D_{2m+1d} \ (= S_2).$$

For the other form of $\mathbf{H}$, we have $\varGamma(\mathbf{W}, \mathbf{H}, \mathbf{A}) \cap D_{2m+1d} = \varGamma(\mathbf{W}, \mathbf{H}) \cap D_{2m+1d} = S_2$, which has been covered by (v). Moreover, the symmetric tensor variable $\mathbf{C} \in \mathrm{Sym}$ is assmued to be subjected to the condition $\mathbf{C} \in \mathrm{span}\ \mathrm{Sym}_{2m+1}(\mathbf{A})$, and hence here appear only the invariants from the scalar products of $\mathbf{C}$ and the generators in $\mathrm{Sym}_{2m+1}(\mathbf{A})$. The form for $\mathbf{C}$ just indicated is derived from cases (c1)–(c2) for the D_{2m+1d}-irreducible set $(\mathbf{W}, \mathbf{A})$ given later and the condition (see $(3.3)_2$ in Part I)

$$\varGamma(\mathbf{A}, \mathbf{C}) \cap D_{2m+1d} \neq \varGamma(\mathbf{W}, \mathbf{A}, \mathbf{C}) \cap D_{2m+1d} \ (= S_2).$$

By using cases (c1)–(c2) for the D_{2m+1d}-irreducible set $(\mathbf{W}, \mathbf{A})$ given later, we see that $\varGamma(\mathbf{A}) \cap D_{2m+1d} = 2h(\mathbf{a})$, and hence we deduce that the other form for $\mathbf{C}$ leads to

$$\varGamma(\mathbf{W}, \mathbf{A}, \mathbf{C}) \cap D_{2m+1d} = \varGamma(\mathbf{A}, \mathbf{C}) \cap D_{2m+1d} = S_2,$$

which will be covered by the next case.

We proceed to work out the D_{2m+1d}-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1) in Part I)

$$\varGamma(\mathbf{W}, \mathbf{A}) \cap D_{2m+1d} \neq \varGamma(\mathbf{z}) \cap D_{2m+1d}, \ \mathbf{z} = \mathbf{W}, \ \mathbf{A}.$$

It is evident that $\varGamma(\mathbf{z}) \cap D_{2m+1d} \neq S_2, \ D_{2m+1d}$, should hold for $\mathbf{z} = \mathbf{W}, \ \mathbf{A}$. Hence, either $\mathbf{R}_{\mathbf{n}}^{2\pi/2m+1}$ or $\mathbf{R}_{\mathbf{a}}^{\pi}$ pertains to the symmetry group $\varGamma(\mathbf{z})$ for each $\mathbf{z} \in \{\mathbf{W}, \mathbf{A}\}$. The case when $\mathbf{R}_{\mathbf{n}}^{2\pi/2m+1} \in \varGamma(\mathbf{A})$ is excluded, since it results in $\mathbf{A} = x\mathbf{I} + y\mathbf{n} \otimes \mathbf{n}$ and hence $\varGamma(\mathbf{A}) \cap D_{2m+1d} = D_{2m+1d}$, in contradiction to the D_{2m+1d}-irreducibility condition for $(\mathbf{W}, \mathbf{A})$. From these facts we deduce that $\mathbf{W}$ and $\mathring{\mathbf{A}}$ take one of the forms

$$\mathbf{W} = c\mathbf{E}\mathbf{n}, \ c \neq 0; \quad \mathbf{W} = c\mathbf{E}\mathbf{a}, \ c \neq 0; \tag{2.9}$$

$$\mathring{\mathbf{A}} = x(\mathbf{a} \otimes \mathbf{a} - \mathbf{a}' \otimes \mathbf{a}') + y\mathbf{n} \vee \mathbf{a}', \ \mathbf{a}' = \mathbf{n} \times \mathbf{a}, \ x^2 + y^2 \neq 0. \tag{2.10}$$

Thus, combining the above forms and excluding the case

$$\mathbf{W} = f\mathbf{E}\mathbf{a}, \ \mathring{\mathbf{A}} = x(\mathbf{a} \otimes \mathbf{a} - \mathbf{a}' \otimes \mathbf{a}') + y\mathbf{n} \vee \mathbf{a}', \ \mathbf{a}' = \mathbf{n} \times \mathbf{a},$$

we derive the following two cases for D_{2m+1d}-irreducible set $(\mathbf{W}, \mathbf{A})$:

(c1) $\mathbf{W} = f\mathbf{E}\mathbf{n}$ and $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4$ with $f(a^2+b^2) \neq 0$;

(c2) $\mathbf{W} = f\mathbf{E}\mathbf{a}$ and $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4$ with $\mathbf{a} \neq \mathbf{e}$ and $f(a^2+b^2) \neq 0$.

For cases (c1)–(c2) we have

$$\text{rank } \operatorname{Skw}_{2m+1}(\mathbf{W}, \mathbf{A}) \geq \begin{cases} \operatorname{rank}\{\mathbf{W}, \mathbf{E}\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), (\mathbf{E} : \mathbf{W}) \wedge \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\} \\ = 3 \ \text{ if } \ a \neq 0, \\ \operatorname{rank}\{\mathbf{W}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\overset{\circ}{\mathbf{A}}\} \\ = 3 \ \text{ if } \ a = 0, \ b \neq 0. \end{cases}$$

Moreover, utilizing the formula (2.4) in Part I we have

$$\begin{aligned} \text{(2.11)} \qquad \text{rank } \operatorname{Sym}_{2m+1}(\mathbf{W}, \mathbf{A}) &= \operatorname{rank}(\operatorname{Sym}(C_{2h}(\mathbf{e})) \cup \{\overset{\circ}{\mathbf{A}}\,\mathbf{N} - \mathbf{N}\,\overset{\circ}{\mathbf{A}}, \\ &\qquad (-1)^m\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + \mathbf{n} \vee \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}) \\ &= \operatorname{rank}\{\mathbf{n} \otimes \mathbf{n}, \mathbf{e} \otimes \mathbf{e}, \mathbf{e}' \otimes \mathbf{e}', \mathbf{n} \vee \mathbf{e}', a\mathbf{e} \vee \mathbf{e}' - b\mathbf{n} \vee \mathbf{e}, \\ &\qquad b^{2m+1}\mathbf{e} \vee \mathbf{e}' + a^{2m+1}\mathbf{n} \vee \mathbf{e}\}) = 6, \end{aligned}$$

for case (c1). In deriving the second equality above, the following facts for case (c1) are used: $\psi(\mathbf{A}) = \dfrac{\pi}{2}$ for $b > 0$ or $\dfrac{3\pi}{2}$ for $b < 0$ and $\phi(\mathbf{A}) = 0$ for $a > 0$ or π for $a < 0$, and hence

$$\sin(2m-1)\psi(\mathbf{A}) = -(-1)^m, \ \cos(2m-1)\psi(\mathbf{A}) = 0,$$

$$\begin{aligned} \sin m\phi(\mathbf{A}) = \sin(m+1)\phi(\mathbf{A}) = 0, \ |a| \cos\frac{1}{2}(2m+1-(-1)^m)\phi(\mathbf{A}) \\ = |a|\cos\phi(\mathbf{A}) = a. \end{aligned}$$

Owing to the above facts, the last two tensors in the second equality of (2.11) are independent and hence the last equality of (2.11) holds. On the other hand, these facts explain why, in the expression of the last generator in the set $\operatorname{Sym}_{2m+1}(\mathbf{W}, \mathbf{A})$, the factor $(-1)^m$ should appear and why the vector-valued function $\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))$ (see (2.4)) should take different forms for an odd number m and an even number m. Finally, by utilizing the formula (2.4) in Part I and the equality

$$\text{(2.12)} \qquad \operatorname{span}(\operatorname{Sym}(C_{2h}(\mathbf{e})) \cup \operatorname{Sym}(C_{2h}(\mathbf{a})) = \operatorname{Sym}$$

for any two vectors $\mathbf{e}$ and $\mathbf{a}$ satifying $(\mathbf{a}\cdot\mathbf{e})\mathbf{a}\times\mathbf{e}\neq\mathbf{0}$, we have

$$\operatorname{rank}\operatorname{Sym}_{2m+1}(\mathbf{W},\mathbf{A})\geq\operatorname{rank}(\operatorname{Sym}(\Gamma(\mathbf{W})\cap D_{2m+1d})\cup\operatorname{Sym}(\Gamma(\mathbf{A})$$

$$\cup D_{2m+1d}))=\operatorname{rank}(\operatorname{Sym}(C_{2h}(\mathbf{a}))\cup\operatorname{Sym}(C_{2h}(\mathbf{e})))=6$$

for case (c2).

From the above facts and the criterion (2.3) in Part I we conclude that the two presented sets $\operatorname{Skw}_{2m+1}(\mathbf{W},\mathbf{A})$ and $\operatorname{Sym}_{2m+1}(\mathbf{W},\mathbf{A})$ supply a desired skewsymmetric tensor generating set and a desired symmetric tensor generating set respectively. Further, by considering the two pairs $(\mathbf{W}_1,\mathbf{A}_1)=(\mathbf{N},\mathbf{D}_1)$ and $(\mathbf{W}_2,\mathbf{A}_2)=(\mathbf{N},\mathbf{D}_4)$, we deduce that the the respective last two generators in the two sets $\operatorname{Skw}_{2m+1}(\mathbf{W},\mathbf{A})$ and $\operatorname{Sym}_{2m+1}(\mathbf{W},\mathbf{A})$ are irreducible.

Finally, with the aid of cases (c1)–(c2) it can readily be shown that a functional basis of the D_{2m+1}-irreducible set $(\mathbf{W},\mathbf{A})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$ is determined by the presented set $I_{2m+1}(\mathbf{W},\mathbf{A})$ of invariants, and hence the latter supplies a desired functional basis (see the remark at the end of Sec. 4 (**vi**) in Part I).

(vii) The D_{2m+1d}-irreducible set $(\mathbf{A},\mathbf{B})$ of two symmetric tensors

Skw $\operatorname{Skw}_{2m+1}(\mathbf{A})\cup\operatorname{Skw}_{2m+1}(\mathbf{B})\cup\{\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}},\mathring{\mathbf{A}}\,\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n},$
$\mathring{\mathbf{B}}\,\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n}\}$
$(\equiv\operatorname{Skw}_{2m+1}(\mathbf{A},\mathbf{B}))$

Sym $\operatorname{Sym}_{2m+1}(\mathbf{A})\cup\operatorname{Sym}_{2m+1}(\mathbf{B})\ (\equiv\operatorname{Sym}_{2m+1}(\mathbf{A},\mathbf{B})),$

R $\mathbf{H}:\operatorname{Skw}_{2m+1}(\mathbf{A}),\mathbf{H}:\operatorname{Skw}_{2m+1}(\mathbf{B});$
$\operatorname{tr}\mathbf{H}\mathring{\mathbf{A}}\mathring{\mathbf{B}},(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathbf{H}\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n},(\mathring{\mathbf{B}}\,\mathbf{n})\cdot\mathbf{H}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n};$
$I_{2m+1}(\mathbf{A})\cup I_{2m+1}(\mathbf{B})\cup\{\operatorname{tr}\mathbf{A}_{\mathbf{n}}\mathbf{B}_{\mathbf{n}},\operatorname{tr}\mathbf{A}_{\mathbf{e}}\mathbf{B}_{\mathbf{e}},\operatorname{tr}\mathring{\mathbf{A}}^2\mathring{\mathbf{B}},\operatorname{tr}\mathring{\mathbf{B}}^2\mathring{\mathbf{A}}\}$
$(\equiv I_{2m+1}(\mathbf{A},\mathbf{B})).$

To prove the above results, we first work out the D_{2m+1d}-irreducible set $(\mathbf{A},\mathbf{B})$, specified by

$$\Gamma(\mathbf{A},\mathbf{B})\cap D_{2m+1d}\neq\Gamma(\mathbf{z})\cap D_{2m+1d},\quad\mathbf{z}=\mathbf{A},\ \mathbf{B}.$$

Evidently, we have $\Gamma(\mathbf{z})\cap D_{2m+1d}\neq S_2,\ D_{2m+1d}$ for $\mathbf{z}=\mathbf{A},\ \mathbf{B}$. From the latter and the relevant argument given in (**vi**) we know that each $\mathbf{z}\in\{\mathbf{A},\mathbf{B}\}$ is of the form given by (2.10). Hence, the D_{2m+1d}-irreducible set $(\mathbf{A},\mathbf{B})$ is specified by

$$\text{(2.13)}\qquad \overset{\circ}{\mathbf{A}}= a\mathbf{D}_1 + b\mathbf{D}_4,\ \overset{\circ}{\mathbf{B}}= c(\mathbf{a}\otimes\mathbf{a} - \mathbf{a}'\otimes\mathbf{a}') + d\mathbf{n}\vee\mathbf{a}',\ \mathbf{a}\neq\mathbf{e},$$
$$(a^2+b^2)(c^2+d^2)\neq 0\,.$$

Consequently, we have

$$\Gamma(\mathbf{A})\cap D_{2m+1d} = C_{2h}(\mathbf{e}),\ \Gamma(\mathbf{B})\cap D_{2m+1d} = C_{2h}(\mathbf{a}).$$

Utilizing the above and Eq. (2.4) in Part I, we have

$$\text{(2.14)}\qquad \text{rank Skw}_{2m+1}(\mathbf{A},\mathbf{B}) = \text{rank}(\text{Skw}(C_{2h}(\mathbf{e}))\cup\text{Skw}(C_{2h}(\mathbf{a}))$$
$$\cup\{\mathbf{G}_1,\mathbf{G}_2,\mathbf{G}_3\}) = \text{rank}\{\mathbf{n}\wedge\mathbf{e}', \mathbf{n}\wedge\mathbf{a}', (bd+2ac\cos\alpha)\mathbf{N}, b^2c\sin 2\alpha\mathbf{N},$$
$$d^2a\sin 2\alpha\mathbf{N}\} = 3 \text{ if } (a^2+b^2)(c^2+d^2)\neq 0\,,$$
$$\text{rank Sym}_{2m+1}(\mathbf{A},\mathbf{B}) = \text{rank}(\text{Sym}(C_{2h}(\mathbf{e}))\cup\text{Sym}(C_{2h}(\mathbf{a}))) = 6,$$

where $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ are used to represent the last three generators in the set $\text{Skw}_{2m+1}(\mathbf{A},\mathbf{B})$, and α the angle formed by $\mathbf{e}$ and $\mathbf{a}$. Note that $\sin 2\alpha\neq 0$.

From the above results we know that the two sets $\text{Skw}_{2m+1}(\mathbf{A},\mathbf{B})$ and $\text{Sym}_{2m+1}(\mathbf{A},\mathbf{B})$ obey the criterion (2.3) in Part I, respectively. Hence they supply a desired skewsymmetric tensor generating set and a desired symmetric tensor generating set. Further, in (2.13) let $a=c=0$, $a=d=0$ and $b=c=0$, respectively. Then we have $\Gamma(\mathbf{A},\mathbf{B})\cap D_{2m+1d} = S_2$ for each of the three cases. From this and the second equality in (2.14) we deduce that each of the generators $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ is irreducible.

Finally, the presented set $I_{2m+1}(\mathbf{A},\mathbf{B})$ determines a functional basis of $(\mathbf{A},\mathbf{B})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$ and hence supplies a desired functional basis (see the comment at the end of Sec. 4 (**vi**) in Part I).

(viii) The D_{2m+1d}-irreducible set $(\mathbf{u},\mathbf{W})$ of a vector and a skewsymmetric tensor

V $\quad V_{2m+1}(\mathbf{u})\cup\{\mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, \mathbf{u}\times\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \mathbf{u}\times\mathbf{W}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$
$(\equiv V_{2m+1}(\mathbf{u},\mathbf{W}))$

Skw $\quad \text{Skw}_{2m+1}(\mathbf{u})\cup Skw_{2m+1}(\mathbf{W})\cup\{(\mathbf{E}:\mathbf{W})\wedge\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$
$(\equiv \text{Skw}_{2m+1}(\mathbf{u},\mathbf{W}))$

Sym $\quad \text{Sym}_{2m+1}(\mathbf{u})\cup\text{Sym}_{2m+1}(\mathbf{W})\cup\{(\text{tr}\mathbf{W}\mathbf{N})\ \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}}),$
$(\text{tr}\mathbf{W}\mathbf{N})\mathbf{n}\vee\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}; (\equiv \text{Sym}_{2m+1}(\mathbf{u},\mathbf{W}))$

$R \quad \mathbf{r}\cdot V_{2m+1}(\mathbf{u}); \mathbf{H} : \mathrm{Skw}_{2m+1}(\mathbf{z}), \mathbf{C} : \mathrm{Sym}_{2m+1}(\mathbf{z}),\ \mathbf{z} = \mathbf{u},\ \mathbf{W};$

$\mathrm{tr}\mathbf{HW}(\mathbf{E}\eta_{2m}(\overset{\circ}{\mathbf{u}})); (\mathrm{tr}\mathbf{WN})[\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}], (\mathrm{tr}\mathbf{WN})\eta_{2m}(\overset{\circ}{\mathbf{u}})\cdot \overset{\circ}{\mathbf{C}}\,\mathbf{n};$

$I_{2m+1}(\mathbf{u}) \cup I_{2m+1}(\mathbf{W}) \cup \{(\mathbf{u}\cdot\mathbf{n})\ \overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn}, (\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn})^2\}$

$(\equiv I_{2m+1}(\mathbf{u}, \mathbf{W})).$

In the above table and that given later, the vector variable $\mathbf{r}$ is assumed to pertain to the subspace $V(\Gamma(\mathbf{u})\cap D_{2m+1d})$, the latter being spanned by the vector generating set $V_{2m+1}(\mathbf{u})$ (see (2.4) in Part I). Owing to this fact, of the invariants from the scalar products of $\mathbf{r}$ and the vector generators in the set $V_{2m+1}(\mathbf{u}, \mathbf{D})$, the invariants except $\mathbf{r}\cdot V_{2m+1}(\mathbf{u})$ are redundant and have been and will be deleted. Here $\mathbf{D} = \mathbf{W}, \mathbf{A}$. When $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$, the foregoing condition for $\mathbf{r}$ can be derived from (see $(3.3)_2$)

$$\Gamma(\mathbf{u}, \mathbf{r}) \cap D_{2m+1d} \neq \Gamma(\mathbf{u}, \mathbf{D}, \mathbf{r}) \cap D_{2m+1d}$$

for the D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{D})$. When $\overset{\circ}{\mathbf{u}}= \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{n}$, we have $\mathbf{r} = c\mathbf{n} = x\mathbf{u}$. The case when $\overset{\circ}{\mathbf{r}}\neq \mathbf{0}$ is the same as the case when $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$.

The proof for the presented results is as follows. First, we prove that the three presented sets $I_{2m+1}(\mathbf{u}, \mathbf{W})$, $\mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{W})$ and $\mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{A})$ supply a desired functional basis, a desired skewsymmetric tensor generating set and a desired symmetric tensor generating set, respectively. In fact, since the central inversion $-\mathbf{I}$ is included in the group D_{2m+1d}, we infer that each scalar-valued or second order tensor-valued anisotropic function of the set $(\mathbf{u}, \mathbf{W})$ under D_{2m+1d} is equivalent to a scalar-valued or second tensor-valued anisotropic function of the set $(\mathbf{W}, \mathbf{u}\otimes\mathbf{u})$ under D_{2m+1d}. Evidently, the set $(\mathbf{W}, \mathbf{u}\otimes\mathbf{u})$ is the particular case of the set $(\mathbf{W}, \mathbf{A})$ considered in **(vi)** when $\mathbf{A} = \mathbf{u}\otimes\mathbf{u}$. Hence, by setting $\mathbf{A} = \mathbf{u}\otimes\mathbf{u}$ in the set $I_{2m+1}(\mathbf{W}, \mathbf{A})$ we derive a desired functional basis. After removing some obviously redundant invariants, we know that this basis is just given by the set $I_{2m+1}(\mathbf{u}, \mathbf{W})$. Similarly, a skewsymmetric tensor generating set and a symmetric tensor generating set for $(\mathbf{u}\otimes\mathbf{u}, \mathbf{W})$ can be derived. By removing some redundant generators we arrive at the desired two irreducible tensor generating sets, given by $\mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{W})$ and $\mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{W})$. The proof is as follows.

From cases (c1) and (c2) given in **(vi)**, we know that the D_{2m+1d}-irreducible set $(\mathbf{W}, \mathbf{u}\otimes\mathbf{u})$ is specified by

(c1) $\mathbf{W} = f\mathbf{En},\ \mathbf{u}\otimes\mathbf{u} = a\mathbf{D}_1 + b\mathbf{D}_4 + p\mathbf{n}\otimes\mathbf{n} + y\mathbf{I},\ f(a^2+b^2)\neq 0;$

(c2) $\mathbf{W} = f\mathbf{Ea},\ \mathbf{u}\otimes\mathbf{u} = a\mathbf{D}_1 + b\mathbf{D}_4 + p\mathbf{n}\otimes\mathbf{n} + y\mathbf{I},\ \mathbf{a}\neq\mathbf{e},\ f(a^2+b^2)\neq 0;$

Further, from the scalar products $\mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\mathbf{D}_i$ for i =1, 2, 3, 4, we derive

$$2a = (\mathbf{u}\cdot\mathbf{e})^2 - (\mathbf{u}\cdot\mathbf{e}')^2,\ b = (\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{e}'),\ (\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}') = (\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{e}) = 0.$$

These yield $\mathbf{u} = c\mathbf{e}$ or $\mathbf{u} = c\mathbf{e}' + d\mathbf{n}$ with $c \neq 0$.

With the above facts in mind we have

$$\text{rank Skw}_{2m+1}(\mathbf{u},\mathbf{W}) \geq \{\mathbf{E}\eta_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{W}, (\mathbf{E}:\mathbf{W}) \wedge \eta_{2m}(\overset{\circ}{\mathbf{u}})\} = 3,$$

for cases (c1) and (c2), and

$$\text{rank Sym}_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}(\text{Sym}_{2m+1}(\mathbf{u}) \cup \{\overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times \overset{\circ}{\mathbf{u}}),\ \mathbf{n}\vee\eta_{2m}(\overset{\circ}{\mathbf{u}})\})$$

$$= \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{n}\vee\mathbf{e}', \mathbf{e}\vee\mathbf{e}', \mathbf{n}\vee\mathbf{e}\} = 6$$

for case (c1), and by using the formula (2.4) in Part I we have

$$\text{rank Sym}_{2m+1}(\mathbf{u},\mathbf{W}) \geq \text{rank}(\text{Sym}_{2m+1}(\mathbf{u}) \cup \text{Sym}_{2m+1}(\mathbf{W}))$$

$$= \text{rank}(\text{Sym}(\varGamma(\mathbf{u}) \cap D_{2m+1d}) \cup \text{Sym}(\varGamma(\mathbf{W}) \cap D_{2m+1d}))$$

$$= \text{rank}(\text{Sym}(C_{2h}(\mathbf{a}) \cup \text{Sym}(C_{2h}(\mathbf{e}))) = 6,$$

for case (c2).

From the above results we know that the two sets $\text{Skw}_{2m+1}(\mathbf{u},\mathbf{W})$ and $\text{Sym}_{2m+1}(\mathbf{u},\mathbf{A})$ obey the criterion (2.3) in Part I, respectively, and hence they supply a desired skewsymmetric and symmetric tensor generating sets. Further, by considering the pair $\mathbf{u}_0 = \mathbf{e}$ and $\mathbf{W}_0 = \mathbf{E}\mathbf{n}$ we infer that the last generator in the set $\text{Skw}_{2m+1}(\mathbf{u},\mathbf{W})$ and the latter two generators in the set $\text{Sym}_{2m+1}(\mathbf{u},\mathbf{W})$ are irreducible.

Finally, we show that the presented set $V_{2m+1}(\mathbf{u},\mathbf{W})$ is a desired irreducible vector generating set. To this end, two cases for a nonvanishing $\mathbf{u}$ are discussed: $\overset{\circ}{\mathbf{u}}=\mathbf{0}$ and $\overset{\circ}{\mathbf{u}}\neq\mathbf{0}$. First, suppose $\overset{\circ}{\mathbf{u}}=\mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{n} \neq \mathbf{0}$. Then each form-invariant vector-valued function of $(\mathbf{u},\mathbf{W})$ under D_{2m+1d} can be extended as a vector-valued isotropic function of $(\mathbf{u}, \mathbf{W}, \mathbf{E}\eta_{2m}(\mathbf{W}\mathbf{n}), \mathbf{n}\otimes\mathbf{n})$ (see XIAO [15]). Accordingly, an anisotropic vector generating set for $(\mathbf{u},\mathbf{W})$ under D_{2m+1d} is derivable from an isotropic vector generating set for $(\mathbf{u}, \mathbf{W}, \mathbf{E}\eta_{2m}(\mathbf{W}\mathbf{n}), \mathbf{n}\otimes\mathbf{n})$. By applying the related result for isotropic functions we know that the latter is given by

$$\mathbf{u}, \mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\eta_{2m}(\mathbf{W}\mathbf{n}), (\mathbf{W}(\mathbf{E}\eta_{2m}(\mathbf{W}\mathbf{n})) - (\mathbf{E}\eta_{2m}(\mathbf{W}\mathbf{n}))\mathbf{W})\mathbf{u}.$$

In deriving the above set, some obviously redundant generators have been deleted by using the equalities $\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$. Of the five generators given above, the first four are included in the set $\mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{W})$, and the last is redundant. The latter fact is obviously true for the case when $\mathbf{Wn} = \mathbf{0}$ and the case when the first three generators above are independent. Moreover, the other case for $\mathbf{W}$ is given by $\mathbf{W} = \mathbf{Ez}$ with $\mathbf{z}$ being a vector normal to $\mathbf{n}$. Utilizing the identity $(\mathbf{Ex})\mathbf{y} = \mathbf{y} \times \mathbf{x}$ and the equality $\mathbf{u} = (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ we deduce

$$(\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})) - (\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn}))\mathbf{W})\mathbf{u} = (\mathbf{u} \times \boldsymbol{\eta}_{2m}(\mathbf{Wn})) \times \mathbf{z}$$

$$+(\mathbf{u} \times \mathbf{z}) \times \boldsymbol{\eta}_{2m}(\mathbf{Wn}) = \lambda \mathbf{n},$$

i.e. the above generator is redundant.

Second, suppose $\overset{\circ}{\mathbf{u}} \neq \mathbf{0}$. Then we have

$$\varGamma(\mathbf{u}, \mathbf{W}) \cap D_{2m+1d} = \varGamma(\mathbf{u}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{W}), \tag{2.15}$$

for the D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{W})$ with $\overset{\circ}{\mathbf{u}} \neq \mathbf{0}$. In fact, for $\overset{\circ}{\mathbf{u}} \neq \mathbf{0}$ we have

$$\varGamma(\mathbf{u}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})) = \begin{cases} C_1 & \text{if } \ \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})(\mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})) \neq \mathbf{0}, \\ C_{1h}(\mathbf{a}_k) & \text{if } \ \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}) = 0, \\ C_{\infty}(\mathbf{a}_k) & \text{if } \ \mathbf{u} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) = 0. \end{cases} \tag{2.16}$$

Of the above three cases for $\mathbf{u}$, the first means that the vector $\mathbf{u}$ is neither normal to nor collinear with any two-fold axis vector of D_{2m+1d}, and the other two yield $\mathbf{u} = c\mathbf{n} \times \mathbf{a}_k + d\mathbf{n}$ and $\mathbf{u} = c\mathbf{a}_k$ with $c \neq 0$, separately. Accordingly, we have $\varGamma(\mathbf{u}) \cap D_{2m+1d} = C_1$, $C_{1h}(\mathbf{a}_k)$, $C_2(\mathbf{a}_k)$ for the three cases at issue respectively. The first case violates the D_{2m+1d}-irreducibility condition for $(\mathbf{u}, \mathbf{A})$ and is excluded, and (2.15) holds for the second. For the last case, i.e. $\mathbf{u} = c\mathbf{a}_k$ with $c \neq 0$, the case when $\mathbf{Wa}_k = \mathbf{0}$, i.e. $\mathbf{W} = f\mathbf{Ea}_k$, violates the D_{2m+1d}-irreducibility condition for $(\mathbf{u}, \mathbf{W})$ and is excluded. Then we have $\mathbf{Wa}_k \neq \mathbf{0}$ and, moreover, $\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) = f\mathbf{a}_k$. Thus, we infer that $\varGamma(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}, \mathbf{W}) = S_2$ and $\varGamma(\mathbf{u}, \mathbf{W}) \cap D_{2m+1d} = C_1$, and hence (2.15) also holds.

Then, from (2.15) and the criterion (2.3) in Part I, for the case at issue we deduce that an isotropic vector generating set for $(\mathbf{u}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{W})$ provides the desired result. The former is just given by the generators in the set $V_{2m+1}(\mathbf{u}, \mathbf{W})$ except $(\mathbf{u} \cdot \mathbf{n})\boldsymbol{\eta}_{2m}(\mathbf{Wn})$.

Thus, we conclude that the set $V_{2m+1}(\mathbf{u}, \mathbf{W})$ is a desired vector generating set. The irreducibility of the last four vector generators in this generating set can be inferred from the pairs $(\mathbf{u}, \mathbf{W})$ given below.

$\mathbf{Wu}$ and $\mathbf{u} \times \mathbf{W}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})$: $\mathbf{u} = \mathbf{e}$, $\mathbf{W} = \mathbf{En}$;

$\mathbf{W}^2\mathbf{u}$: $\mathbf{u} = \mathbf{n}$, $\mathbf{W} = \mathbf{E}(\mathbf{n} + \mathbf{e})$;

$\mathbf{u} \times \boldsymbol{\eta}_{2m}(\mathbf{Wn})$: $\mathbf{u} = \mathbf{n}$, $\mathbf{W} = \mathbf{n} \wedge \mathbf{e}$.

(ix) The D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{A})$ of a vector and a symmetric tensor

V $\quad V_{2m+1}(\mathbf{u}) \cup \{\overset{\circ}{\mathbf{A}}\,\mathbf{u}, \mathbf{u}\times \overset{\circ}{\mathbf{A}}\,\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n} \times \boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})))\}$ $(\equiv V_{2m+1}(\mathbf{u}, \mathbf{A}))$

Skw $\quad \mathrm{Skw}_{2m+1}(\mathbf{u}) \cup \mathrm{Skw}_{2m+1}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\,\mathbf{n}\}$

$(\equiv \mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{A}))$

Sym $\quad \mathrm{Sym}_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{2m+1}(\mathbf{A})$ $(\equiv \mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{A}))$

R $\quad \mathbf{r}\cdot V_{2m+1}(\mathbf{u}); \mathbf{H} : \mathrm{Skw}_{2m+1}(\mathbf{u}), \mathbf{H} : \mathrm{Skw}_{2m+1}(\mathbf{A}); \mathbf{C} : \mathrm{Sym}_{2m+1}(\mathbf{u});$

$\overset{\circ}{\mathbf{u}}\cdot\mathbf{H}\,\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\mathbf{H}\,\overset{\circ}{\mathbf{A}}\,\mathbf{n};$

$I_{2m+1}(\mathbf{u}) \cup I_{2m+1}(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}{}^2\,\overset{\circ}{\mathbf{u}}\}$

$(\equiv I_{2m+1}(\mathbf{u}, \mathbf{A}))$

The proof for the above results is as follows. First, for each integer $r \geq 0$, each $2r$-th order tensor-valued anisotropic function of the variables $(\mathbf{u}, \mathbf{A})$ under the group D_{2m+1d} is equivalent to an anisotropic function of the variables $(\mathbf{A}, \mathbf{u}\otimes\mathbf{u})$ under the same group. As a result, setting $\mathbf{B} = \mathbf{u}\otimes\mathbf{u}$ in the corresponding results given in **(vii)**, we obtain a desired functional basis and a desired symmetric tensor generating set, as well as the related invariants from the scalar products, given by $I_{2m+1}(\mathbf{u}, \mathbf{A})$ and $\mathrm{Sym}_{2m+1}(\mathbf{u}, \mathbf{A})$ etc.

Next, we show that the presented set $\mathrm{Skw}_{2m+1}(\mathbf{u}, \mathbf{A})$ supplies a desired skewsymmetric tensor generating set. In fact, setting $\mathbf{B} = \mathbf{u} \otimes \mathbf{u}$ in $(2.13)_2$, we derive that the vector variable $\mathbf{u}$ takes one of the two forms $c\mathbf{a}$ with $c \neq 0$ and $c\mathbf{n} \times \mathbf{a} + d\mathbf{n}$ with $c \neq 0$ (see the relevant argument used in **(viii)**). Thus, combining the two forms and $(2.13)_1$, we derive the following two cases for the D_{2m+1d}-irreducible set $(\mathbf{A}, \mathbf{u}\otimes\mathbf{u})$:

$$(1)\mathbf{u} = c\mathbf{a},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4,\ \mathbf{a} \neq \mathbf{e},\ c(a^2 + b^2) \neq 0;$$

$$(2)\mathbf{u} = c\mathbf{n} \times \mathbf{a} + d\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4,\ \mathbf{a} \neq \mathbf{e},\ c(a^2 + b^2) \neq 0.$$

For the above two cases we have

$$\begin{aligned}\operatorname{rank} \mathrm{Skw}_{2m+1}(\mathbf{u},\mathbf{A}) &\geq \operatorname{rank}(\mathrm{Skw}(C_{2h}(\mathbf{a})) \cup \mathrm{Skw}(C_{2h}(\mathbf{e}))\\ &\qquad \cup\{\mathring{\mathbf{u}} \wedge \mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathring{\mathbf{u}} \cdot \mathring{\mathbf{A}}\, \mathbf{n})\, \mathring{\mathbf{u}} \wedge \mathring{\mathbf{A}}\, \mathbf{n}\})\\ &= \operatorname{rank}\{\mathbf{n}\wedge\mathbf{a}, \mathbf{n}\wedge\mathbf{e}, a\, \mathring{\mathbf{u}} \wedge \mathbf{D}_1\, \mathring{\mathbf{u}}, b^2(\mathring{\mathbf{u}} \cdot \mathbf{e})\, \mathring{\mathbf{u}} \wedge \mathbf{e}\} = 3\,.\end{aligned}$$

In deriving the first expression above, the formula (2.4) in Part I and the following equalities are used.

$$\Gamma(\mathbf{u}\otimes\mathbf{u}) \cap D_{2m+1d} = C_{2h}(\mathbf{a}),\ \ \Gamma(\mathbf{A}) \cap D_{2m+1d} = C_{2h}(\mathbf{e}).$$

From the above results we infer that the presented set $\mathrm{Skw}_{2m+1}(\mathbf{u},\mathbf{A})$ obeys the criterion (2.3) in Part I and hence supplies a desired skewsymmetric tensor generating set. Further, by considering the two pairs

$$\mathbf{u}_1 = \mathbf{a}_1,\ \mathbf{A}_1 = \mathbf{n}\vee\mathbf{e};\quad \mathbf{u}_2 = \mathbf{a}_1,\ \mathbf{A}_2 = \mathbf{e}\otimes\mathbf{e},$$

we deduce that the last two generators in the set $\mathrm{Skw}_{2m+1}(\mathbf{u},\mathbf{A})$ are irreducible.

Finally, we show that the presented set $V_{2m+1}(\mathbf{u},\mathbf{A})$ supplies the desired irreducible vector generating set. From the D_{2m+1d}-irreducibility condition for $(\mathbf{u},\mathbf{A})$ (see (3.1) in Part I) we deduce that $\Gamma(\mathbf{u}) \cap D_{2m+1d} \neq C_1$. The latter produces the three cases for $\mathbf{u}$ (see cases (c1) – (c3) derived in **(iv)**): (c1) $\mathbf{u} = c\mathbf{n}$, (c2) $\mathbf{u} = c\mathbf{e}$ and (c3) $\mathbf{u} = c\mathbf{e}' + d\mathbf{n}$, where $c \neq 0$. In what follows we prove that the set $V_{2m+1}(\mathbf{u},\mathbf{A})$ obeys the criterion (2.3) in Part I for the just-mentioned three cases, respectively. Without losing generality we set $c = 1$. First, for case (c1), i.e. $\mathbf{u} = \mathbf{n}$, we have

$$D \geq \begin{cases} \operatorname{rank}\{\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\times\boldsymbol{\eta}_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\} = 3 \text{ if } \alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n}) \neq 0\,,\\ \operatorname{rank}\{\mathbf{n}, \mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})), \mathring{\mathbf{A}}\,(\mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\} = 3 \text{ if } \beta_{2m+1}(\mathbf{q}(\mathbf{A})) \neq 0\,,\\ \operatorname{rank}\{\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\} = 3 \text{ if } \alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\\ \qquad\qquad = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = 0,\ J(\mathbf{A}) \neq 0\,,\\ \operatorname{rank}\{\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\} \geq 2 \text{ if } \alpha_{2m+1}(\mathring{\mathbf{A}}\,\mathbf{n})\\ \qquad\qquad = \beta_{2m+1}(\mathbf{q}(\mathbf{A})) = J(\mathbf{A}) = 0 \end{cases}$$

for $\mathring{\mathbf{A}} \neq \mathbf{O}$, where $D = \operatorname{rank} V_{2m+1}(\mathbf{u},\mathbf{A})$. Here and henceforth, the trivial case when $\mathring{\mathbf{A}} = \mathbf{O}$, i.e. $\mathring{\mathbf{A}}\,\mathbf{n} = \mathbf{q}(\mathbf{A}) = \mathbf{0}$, is excluded. From the above facts and

$$\Gamma(\mathbf{n},\mathbf{A})\cap D_{2m+1d}=C_{1h}(\mathbf{a}_k) \text{ if } \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \\ =\beta_{2m+1}(\mathbf{q}(\mathbf{A}))=J(\mathbf{A})=0,$$

as well as from Table 1 in Sec. 2 in Part I, for case (c1) we infer that the set $V_{2m+1}(\mathbf{u},\mathbf{A})$ obeys the criterion (2.3) in Part I.

Second, for case (c2), i.e. $\mathbf{u}=\mathbf{e}$, we have $\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})=\mathbf{e}$ and hence

$$\operatorname{rank}V_{2m+1}(\mathbf{u},\mathbf{A})\geq\begin{cases}\operatorname{rank}\{\mathbf{e},\overset{\circ}{\mathbf{A}}\,\mathbf{e},\mathbf{e}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}\} \text{ if } \mathbf{e}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}\neq\mathbf{0},\\ \operatorname{rank}\{\mathbf{e},\overset{\circ}{\mathbf{A}}\,\mathbf{e},\overset{\circ}{\mathbf{A}}\,\mathbf{n}\}\geq 2 \text{ if } \mathbf{e}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}=\mathbf{0}.\end{cases}$$

From the above facts and $\Gamma(\mathbf{e},\mathbf{A})\cap D_{2m+1d}=C_2(\mathbf{e})$ when $\mathbf{e}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}=\mathbf{0}$, as well as Table 1 in Sec. 2 in Part I, for case (c2) we infer that the set $V_{2m+1}(\mathbf{u},\mathbf{A})$ obeys the criterion (2.3) in Part I. Here and below it is helpful to note the fact: A nonvanishing vector $\mathbf{z}$ normal to $\mathbf{n}$ is an eigenvector of $\mathbf{A}$ if and only if $\mathbf{z}\times\overset{\circ}{\mathbf{A}}\,\mathbf{z}$ vanishes.

Third, for case (c3), i.e. $\mathbf{u}=\mathbf{e}'+d\mathbf{n}$, we have $\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})=(-1)^m\mathbf{e}$ and hence

$$\operatorname{rank}V_{2m+1}(\mathbf{u},\mathbf{A})\geq\begin{cases}\operatorname{rank}\{\mathbf{n},\mathbf{e}',\overset{\circ}{\mathbf{A}}\,\mathbf{u},\mathbf{u}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}\} \text{ if } \mathbf{e}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}\neq\mathbf{0},\\ \operatorname{rank}\{\mathbf{n},\mathbf{e}'\} \text{ if } \mathbf{e}\times\overset{\circ}{\mathbf{A}}\,\mathbf{e}=\mathbf{0}.\end{cases}$$

From the above facts and Table 1 in Sec. 2 and the equality : $\Gamma(\mathbf{u},\mathbf{A})\cap D_{2m+1d}=C_{1h}(\mathbf{e})$ when $\mathbf{e}$ is an eigenvector of $\mathbf{A}$, as well as Table 1 in Sec. 2 in Part I, for case (c3) we infer that the set $V_{2m+1}(\mathbf{u},\mathbf{A})$ obeys the criterion (2.3) in Part I.

From the above we conclude that the presented set $V_{2m+1}(\mathbf{u},\mathbf{A})$ supplies a desired vector generating set. Further, we infer that the last five generators in this set are irreducible by considering the paris $(\mathbf{u},\mathbf{A})$ given below:

$\overset{\circ}{\mathbf{A}}\,\mathbf{u}$ and $\mathbf{u}\times\overset{\circ}{\mathbf{A}}\,\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})$: $\mathbf{u}=\mathbf{a}_1$, $\mathbf{A}=\mathbf{e}\otimes\mathbf{e}$;

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$: $\mathbf{u}=\mathbf{n}$, $\mathbf{A}=\mathbf{n}\vee\mathbf{e}$;

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))$ and $(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})))$: $\mathbf{u}=\mathbf{n}$, $\mathbf{A}=\mathbf{e}\vee\mathbf{e}'$.

2.3. D_{2m+1d}-irreducible sets of three variables

(x) Three vector variables; two vector variables and a tensor variable

Consider any set of three variables $(\mathbf{u},\mathbf{v},\mathbf{z})$ where $\mathbf{z}$ is a vector or a skewsymmetric tensor or a symmetric tensor. From the D_{2m+1d}-irreducibility condition for $(\mathbf{u}, \mathbf{v}, \mathbf{z})$ (see (3.2) in Part I) it follows that $(\mathbf{u}, \mathbf{v})$ is a D_{2m+1d}-irreducible set, specified by cases (c1) – (c3) derived in **(iv)**. For each of cases (c1) – (c2) we have $\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1d} = C_1$. The latter leads to

$$\Gamma(\mathbf{u}, \mathbf{v}, \mathbf{z}) \cap D_{2m+1d} = (\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2m+1d}) \cap \Gamma(\mathbf{z}) = C_1.$$

The above fact shows that any set $(\mathbf{u}, \mathbf{v}, \mathbf{z})$ may be reduced to the D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{v})$, which has been covered by **(iv)**.

(xi) A vector variable and two tensor variables

Consider any set $(\mathbf{x}, \mathbf{y}, \mathbf{u})$, where $(\mathbf{x}, \mathbf{y}) \in \{(\mathbf{W}, \mathbf{\Omega}), (\mathbf{W}, \mathbf{A}), (\mathbf{A}, \mathbf{B})\}$. For each such set, tensor generating sets and functional bases have been covered before (see Theorem 3.2 in XIAO [19]). As a result, it suffices to supply a vector generating set. Towards the latter goal we work out the D_{2m+1d}-irreducible set $(\mathbf{x}, \mathbf{y}, \mathbf{u})$, specified by the condition (3.2) in Part I with $\mathbf{x}$ and $\mathbf{y}$ two tensors and $\mathbf{z} = \mathbf{u}$ and $g = D_{2m+1d}$. Let $\mathbf{D}$ represent any of the two tensors $\mathbf{x}$ and $\mathbf{y}$. Evidently, we have $\Gamma(\mathbf{D}) \cap D_{2m+1d} \neq S_2$. Hence, if the tensor $\mathbf{D}$ is skewsymmetric, we have (see (2.9)) $\mathbf{D} = c\mathbf{E}\mathbf{a}_r$ or $\mathbf{D} = c\mathbf{E}\mathbf{n}$, where $c \neq 0$. If $\mathbf{D}$ is symmetric, we have (2.10) with the replacement of $\mathbf{A}$ by $\mathbf{D}$ and hence $\Gamma(\mathbf{D}) \cap D_{2m+1d} = C_{2h}(\mathbf{a})$. From these facts and

$$\Gamma(\mathbf{u}, \mathbf{x}) \cap D_{2m+1d} \neq C_1, \qquad \Gamma(\mathbf{u}, \mathbf{y}) \cap D_{2m+1d} \neq C_1,$$

as well as $C_{2h}(\mathbf{a}_r) \cap C_{2h}(\mathbf{a}_s) = S_2$ for any two two-fold axis vectors $\mathbf{a}_r$ and $\mathbf{a}_s$ of the group D_{2m+1d}, we derive each D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ at issue taking the forms:

$(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$:

(c1) $\mathbf{u} = c\mathbf{n}$, $\mathbf{W} = a\mathbf{E}\mathbf{e}$, $\mathbf{\Omega} = b\mathbf{E}\mathbf{n}$, $abc \neq 0$;

(c2) $\mathbf{u} = c\mathbf{n}$, $\mathbf{W} = a\mathbf{E}\mathbf{e}$, $\mathbf{\Omega} = b\mathbf{E}\mathbf{a}$, $abc \neq 0$.

$(\mathbf{u}, \mathbf{W}, \mathbf{A})$:

(c1) $\mathbf{u} = c\mathbf{n}$, $\mathbf{W} = a\mathbf{e}$, $\overset{\circ}{\mathbf{A}} = b(\mathbf{a} \otimes \mathbf{a} - \mathbf{a}' \otimes \mathbf{a}') + d\mathbf{n} \vee \mathbf{a}'$, $ac(b^2 + d^2) \neq 0$;

(c2) $\mathbf{u} = c\mathbf{n}$, $\mathbf{W} = a\mathbf{n}$, $\overset{\circ}{\mathbf{A}} = b\mathbf{D}_1 + d\mathbf{D}_4$, $ac(b^2 + d^2) \neq 0$.

$(\mathbf{u}, \mathbf{A}, \mathbf{B})$:

$\mathbf{u} = f\mathbf{n}$, $\overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4$, $\overset{\circ}{\mathbf{B}} = c(\mathbf{a} \otimes \mathbf{a} - \mathbf{a}' \otimes \mathbf{a}') + d\mathbf{n} \vee \mathbf{a}'$,
$f(a^2 + b^2)(c^2 + d^2) \neq 0$.

In the above, $\mathbf{e} \neq \mathbf{a} \in \{\mathbf{a}_1, \cdots, \mathbf{a}_{2m}\}$ and $\mathbf{a}' = \mathbf{n} \times \mathbf{a}$. Then, we construct the following table for vector generating sets.

$$V \quad V_{2m+1}(\mathbf{u}, \mathbf{W}) \cup V_{2m+1}(\mathbf{u}, \mathbf{\Omega}) \cup \{(\mathbf{u} \cdot \mathbf{n})(\mathbf{W\Omega} - \mathbf{\Omega W})\mathbf{n}\}$$

$$V \quad V_{2m+1}(\mathbf{u}, \mathbf{W}) \cup V_{2m+1}(\mathbf{u}, \mathbf{A})$$

$$\cup\{(\mathbf{u} \cdot \mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\times \overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathbf{u} \cdot \mathbf{n})(\mathrm{tr}\mathbf{WN})\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))\}$$

$$V \quad V_{2m+1}(\mathbf{u}, \mathbf{A}) \cup V_{2m+1}(\mathbf{u}, \mathbf{B})$$

Applying the fact

$$\mathrm{rank}(V(C_{1h}(\mathbf{a}_r) \cup V(C_{1h}(\mathbf{a}_s)) = \mathrm{dim}V = 3$$

for any two two-fold axis vectors $\mathbf{a}_r$ and $\mathbf{a}_s$ of the group D_{2m+1d}, and the cases derived for each D_{2m+1d}-irreducible set $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ at issue, it may easily be verified that the above three presented sets supply the desired vector generating sets. The irreducibility of the three generators $(\mathbf{u} \cdot \mathbf{n})(\mathbf{W\Omega} - \mathbf{\Omega W})\mathbf{n}$, $(\mathbf{u} \cdot \mathbf{n})(\mathrm{tr}\mathbf{WN})\mathbf{n}\times \overset{\circ}{\mathbf{A}}\mathbf{n}$ and $(\mathbf{u} \cdot \mathbf{n})(\mathrm{tr}\mathbf{WN})\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}))$ can be deduced from case (c1) for $(\mathbf{u}, \mathbf{W}, \mathbf{\Omega})$, and case (c2) for $(\mathbf{u}, \mathbf{W}, \mathbf{A})$ with $d = 0$ and $b = 0$ separately.

2.4. The general result

THEOREM 4. *The four sets given by*

$$I_{2m+1}(\mathbf{u}); I_{2m+1}(\mathbf{W}); I_{2m+1}(\mathbf{A}); (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}), \mathbf{u} \cdot \mathbf{v}, [\mathbf{u}, \mathbf{v}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})],$$
$$[\mathbf{v}, \mathbf{u}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})],$$
$$\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\, \overset{\circ}{\mathbf{v}} \cdot \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \alpha_{2m+1}(\overset{\circ}{\mathbf{v}})\, \overset{\circ}{\mathbf{u}} \cdot \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}});$$
$$\mathrm{tr}\mathbf{W\Omega}, \mathrm{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{\Omega n})), \mathrm{tr}\mathbf{\Omega}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})), \mathrm{tr}\mathbf{\Omega W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})),$$
$$\mathrm{tr}\mathbf{W\Omega}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{\Omega n}));$$
$$(\overset{\circ}{\mathbf{A}}\mathbf{n})\cdot \overset{\circ}{\mathbf{B}}\mathbf{n}, \mathrm{tr}\mathbf{A}_\mathbf{e}\mathbf{B}_\mathbf{e}, \mathrm{tr}\overset{\circ}{\mathbf{A}}{}^2 \overset{\circ}{\mathbf{B}}, \mathrm{tr}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}{}^2, \mathrm{tr}\mathbf{A}_\mathbf{e}(\mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{B})))^2,$$
$$[\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\mathbf{n})], [\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n}], [\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{B}}\mathbf{n}), \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n}];$$
$$(\mathbf{Wn})\cdot \overset{\circ}{\mathbf{A}}\mathbf{Wn}, (\mathbf{Wn})\cdot \overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathbf{Wn})\cdot \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}, \mathrm{tr}\overset{\circ}{\mathbf{A}}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{Wn})),$$
$$[\mathbf{n}, \boldsymbol{\eta}_{2m}(\mathbf{Wn}), \overset{\circ}{\mathbf{A}}\mathbf{W}^2\mathbf{n}], (\mathrm{tr}\mathbf{WN})\beta_{2m+1}(\mathbf{q}(\mathbf{A})), \mathrm{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A}))),$$
$$\mathrm{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})), (\mathrm{tr}\mathbf{WN})J(\mathbf{A}), (\mathrm{tr}\mathbf{WN})\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})$$
$$+\mathrm{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})));$$

$$(\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n},\ (\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})^2,\ \mathrm{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})),\ [\mathbf{u},\mathbf{W}\mathbf{u},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})],$$

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{W}\mathbf{u};$$

$$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},\ \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\overset{\circ}{\mathbf{u}},\ (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n},\ [\mathbf{u},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}],$$

$$\alpha_{2m}(\overset{\circ}{\mathbf{u}})(\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}-(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n});\ [\mathbf{r},\mathbf{u},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})],\ [\mathbf{r},\mathbf{v},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})];$$

$$\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}\mathbf{H};\ \mathrm{tr}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{W},\ (\overset{\circ}{\mathbf{A}}\mathbf{n})\cdot\mathbf{W}\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n},\ (\overset{\circ}{\mathbf{B}}\mathbf{n})\cdot\mathbf{W}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n};$$

$$\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}\overset{\circ}{\mathbf{A}},\ |\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})[\mathbf{n},\mathbf{W}\mathbf{n},\overset{\circ}{\mathbf{A}}\mathbf{W}\mathbf{n}]$$

$$+|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})[\mathbf{n},\boldsymbol{\Omega}\mathbf{n},\overset{\circ}{\mathbf{A}}\boldsymbol{\Omega}\mathbf{n}];$$

$$\mathbf{u}\cdot\mathbf{W}\mathbf{v},\ (\mathbf{u}\cdot\mathbf{n})^{2m+1}[\mathbf{n},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}),\mathbf{W}\mathbf{v}]+(\mathbf{v}\cdot\mathbf{n})^{2m+1}[\mathbf{n},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\mathbf{W}\mathbf{u}];$$

$$\mathbf{u}\cdot\overset{\circ}{\mathbf{A}}\mathbf{v},\ (\mathbf{u}\cdot\mathbf{n})^{2m+1}[\mathbf{n},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}),\overset{\circ}{\mathbf{A}}\mathbf{v}]+(\mathbf{v}\cdot\mathbf{n})^{2m+1}[\mathbf{n},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}}\mathbf{u}];$$

$$\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$$

$$(\mathrm{tr}\mathbf{W}\mathbf{N})[\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}],\ (\mathrm{tr}\mathbf{W}\mathbf{N})\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n},\ \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},$$

$$(\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n})\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\overset{\circ}{\mathbf{A}}\mathbf{n};$$

and

$$V_{2m+1}(\mathbf{u});\ \mathbf{u}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}),\ \mathbf{v}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}});\ \mathbf{W}\mathbf{u},\ \mathbf{W}^2\mathbf{u},\ \mathbf{u}\times\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}),$$

$$\mathbf{u}\times\mathbf{W}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{A}}\mathbf{u},\ \mathbf{u}\times\overset{\circ}{\mathbf{A}}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\ (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}),$$

$$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})),\ (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{A}}(\mathbf{n}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})));$$

$$(\mathbf{u}\cdot\mathbf{n})(\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W})\mathbf{n};\ (\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})(\mathrm{tr}\mathbf{W}\mathbf{N})\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}));$$

and

$$\mathrm{Skw}_{2m+1}(\mathbf{u});\ \mathrm{Skw}_{2m+1}(\mathbf{W});\ \mathrm{Skw}_{2m+1}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v},\ (\mathbf{u}\cdot\mathbf{n})^{2m+1}\overset{\circ}{\mathbf{v}}\wedge(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}))+(\mathbf{v}\cdot\mathbf{n})^{2m+1}\overset{\circ}{\mathbf{u}}\wedge(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$$

$$\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W};\ \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}},\ \overset{\circ}{\mathbf{A}}\mathbf{n}\wedge\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n},\ \overset{\circ}{\mathbf{B}}\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n};\ \overset{\circ}{\mathbf{A}}\mathbf{W}+\mathbf{W}\overset{\circ}{\mathbf{A}},$$

$$(\mathbf{E}:\mathbf{W})\wedge\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}));\ (\mathbf{E}:\mathbf{W})\wedge\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}});\ \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},\ (\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n})\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n};$$

and

$$\mathrm{Sym}_{2m+1}(\mathbf{u}), \mathrm{Sym}_{2m+1}(\mathbf{W}), \mathrm{Sym}_{2m+1}(\mathbf{A});$$

$$\mathbf{u} \vee \mathbf{v}, (\mathbf{u}\cdot\mathbf{n})^{2m+1} \overset{\circ}{\mathbf{v}} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})) + (\mathbf{v}\cdot\mathbf{n})^{2m+1} \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$$

$$\mathbf{W\Omega} + \mathbf{\Omega W}, |\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn} \vee \mathbf{NWn} + |\mathrm{tr}\mathbf{WN}|$$

$$(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n} \vee \mathbf{N\Omega n};$$

$$(\mathrm{tr}\mathbf{WN})(\overset{\circ}{\mathbf{A}}\,\mathbf{N} - \mathbf{N}\,\overset{\circ}{\mathbf{A}}), (\mathrm{tr}\mathbf{WN})((-1)^m\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$$

$$+\mathbf{n} \vee \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})));$$

$$(\mathrm{tr}\mathbf{WN})\,\overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times\,\overset{\circ}{\mathbf{u}}), (\mathrm{tr}\mathbf{WN})\mathbf{n} \vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}});$$

where $(\mathbf{u},\mathbf{v},\mathbf{r}) = (\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H}) = (\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B}) = (\mathbf{A}_L,\mathbf{A}_M)$, $k > j > i = 1,\cdots,a$, $\theta > \tau > \sigma = 1,\cdots,b$, $M > L = 1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* D_{2m+1d} *for each* $m \geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* D_{2m+1d}.

In the above results, the invariants depending on two symmetric tensors are cited from Theorem 3 in XIAO, BRUHNS and MEYERS [19], as mentioned earlier.

3. Crystal and quasicrystal classes D_{2m+1}

The class D_{2m+1}, which includes the crystal class D_3 as the particular case when $m = 1$, is the rotation subgroup of the centrosymmetrical subgroup D_{2m+1d}, i.e. $D_{2m+1} = D_{2m+1d} \cap \mathrm{Orth}^+$. Following the procedure indicated in Sec. 5 in Part I, from the related results given in Theorem 4 we derive the results for the classes D_{2m+1} as follows.

THEOREM 5. *The four sets given by*

$$(\mathbf{u}\cdot\mathbf{n})^2, |\,\overset{\circ}{\mathbf{u}}\,|^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{u}}); I_{2m+1}(\mathbf{W}); I_{2m+1}(\mathbf{A});$$

$$I_{2m+1}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B});$$

$$\mathbf{u}\cdot\mathbf{v}, \overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}}), \overset{\circ}{\mathbf{v}}\cdot\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), [\mathbf{u},\mathbf{v},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{v}})], [\mathbf{v},\mathbf{u},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})]; [\mathbf{u},\mathbf{v},\mathbf{r}];$$

$$\mathrm{tr}\mathbf{W}(\mathbf{Eu}), \mathrm{tr}\mathbf{W}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), \overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m}(\mathbf{Wn}), \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wu}, \boldsymbol{\eta}_{2m}(\mathbf{Wn})\cdot\mathbf{Wu};$$

$$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, [\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}\,\mathbf{n}], [\mathbf{n},\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}}{}^2\mathbf{n}], \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\overset{\circ}{\mathbf{A}}\,\mathbf{u}, [\mathbf{n},\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),$$

$$\overset{\circ}{\mathbf{A}}\ ((\mathbf{n}\times\mathbf{u})\times\mathbf{u})],$$

$$(\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\mathbf{q}(\mathbf{A})), \overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}),$$

$$(\mathbf{u}\cdot\mathbf{n})J(\mathbf{A}), (\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})+\overset{\circ}{\mathbf{u}}\cdot\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}));$$

$$\mathbf{u}\cdot\mathbf{W}\mathbf{v}; \mathbf{u}\cdot\mathbf{A}\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \overset{\circ}{\mathbf{v}}, \mathbf{A}\ \overset{\circ}{\mathbf{v}}] + |\mathbf{v}\cdot\mathbf{n}|(\mathbf{v}\cdot\mathbf{n})[\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{A}\ \overset{\circ}{\mathbf{u}}];$$

$$\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}(\mathbf{E}\mathbf{u}); \mathrm{tr}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\ (\mathbf{E}\mathbf{u}), [\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}\ \mathbf{n}, \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\ \mathbf{n}], [\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{B}}\ \mathbf{n}, \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\ \mathbf{n}];$$

$$\mathrm{tr}\mathbf{W}(\mathbf{E}\mathbf{u})\ \overset{\circ}{\mathbf{A}}, |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{W}\mathbf{n}, \overset{\circ}{\mathbf{A}}\ \mathbf{W}\mathbf{n}]$$

$$-|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})[\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}];$$

and

$$\mathbf{u}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{u}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}); \mathbf{E}:\mathbf{W}, \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}), \mathbf{W}\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n});$$

$$\beta_{2m+1}(\mathbf{q}(\mathbf{A}))\mathbf{n}, \boldsymbol{\pi}_m(\mathbf{q}(\mathbf{A})), \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}), \overset{\circ}{\mathbf{A}}\ \mathbf{n}\times\ \overset{\circ}{\mathbf{A}}\ {}^2\mathbf{n},$$

$$\alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})\mathbf{n}+\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}));$$

$$\mathbf{u}\times\mathbf{v}; \mathbf{E}:(\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W}); \mathbf{E}:(\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}),$$

$$\overset{\circ}{\mathbf{A}}\ \mathbf{n}\times\ \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\ \mathbf{n}, \overset{\circ}{\mathbf{B}}\ \mathbf{n}\times\ \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\ \mathbf{n};$$

$$\overset{\circ}{\mathbf{A}}\ (\mathbf{E}:\mathbf{W}), \mathbf{W}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})); \mathbf{W}\mathbf{u}; \overset{\circ}{\mathbf{A}}\ \mathbf{u}, \mathbf{u}\times\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}));$$

and

$$\mathbf{E}\mathbf{u}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{u}\wedge\boldsymbol{\eta}_m(\overset{\circ}{\mathbf{u}}); \mathrm{Skw}_{2m+1}(\mathbf{W}); \mathrm{Skw}_{2m+1}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}; \mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\ \mathbf{n}\wedge\ \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\ \mathbf{n}, \overset{\circ}{\mathbf{B}}\ \mathbf{n}\wedge\ \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\ \mathbf{n};$$

$$\overset{\circ}{\mathbf{A}}\ \mathbf{W}+\mathbf{W}\ \overset{\circ}{\mathbf{A}}, (\mathbf{E}:\mathbf{W})\wedge\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})); \mathbf{E}(\mathbf{W}\mathbf{u}); \mathbf{E}(\overset{\circ}{\mathbf{A}}\ \mathbf{u}), \mathbf{u}\wedge\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A}));$$

and

$$\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\ \overset{\circ}{\mathbf{u}}), \mathbf{u}\vee\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\times(\mathbf{u}\times\mathbf{n}))\vee(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$$

$$\mathrm{Sym}_{2m+1}(\mathbf{W}); \mathrm{Sym}_{2m+1}(\mathbf{A}); \mathbf{u}\vee\mathbf{v}, |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})\ \overset{\circ}{\mathbf{v}}\vee(\mathbf{n}\times\ \overset{\circ}{\mathbf{v}})$$

$$+|\mathbf{v}\cdot\mathbf{n}|(\mathbf{v}\cdot\mathbf{n})\ \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\ \overset{\circ}{\mathbf{u}});$$

$$\mathbf{W}\boldsymbol{\Omega}+\boldsymbol{\Omega}\mathbf{W}, |\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n}$$

$$+|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})\boldsymbol{\Omega}\mathbf{n}\vee\mathbf{N}\boldsymbol{\Omega}\mathbf{n};$$

$$(\mathrm{tr}\mathbf{W}\mathbf{N})(\overset{\circ}{\mathbf{A}}\ \mathbf{N}-\mathbf{N}\ \overset{\circ}{\mathbf{A}}), (\mathrm{tr}\mathbf{W}\mathbf{N})((-1)^m\ \overset{\circ}{\mathbf{A}}\ \mathbf{n}\vee\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$$

$+\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})));$

$\mathbf{W}(\mathbf{E}\mathbf{u})+(\mathbf{E}\mathbf{u})\mathbf{W},|\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n}$

$+|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$

$(\mathbf{u}\cdot\mathbf{n})(\overset{\circ}{\mathbf{A}}\mathbf{N}-\mathbf{N}\overset{\circ}{\mathbf{A}}),(\mathbf{u}\cdot\mathbf{n})((-1)^m\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})$

$+\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})));$

where $(\mathbf{u},\mathbf{v},\mathbf{r})=(\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k)$, $(\mathbf{W},\boldsymbol{\Omega},\mathbf{H})=(\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B})=(\mathbf{A}_L,\mathbf{A}_M)$, $k>j>i=1,\cdots,a$, $\theta>\tau>\sigma=1,\cdots,b$, $M>L=1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* D_{2m+1} *for each* $m\geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* D_{2m+1}.

Here and henceforth, $I_{2m+1}(\mathbf{W},\boldsymbol{\Omega},\mathbf{H},\mathbf{A},\mathbf{B})$ is used to represent the invariants depending on two or three second order tensor variables given in Theorem 4.

4. Crystal and quasicrystal classes C_{2m+1v}

The classes at issue are of the form

$$C_{2m+1v}(\mathbf{n},\mathbf{e})=\{\mathbf{R}_{\mathbf{n}}^{\theta_k},-\mathbf{R}_{\mathbf{a}_k}^{\pi}\mid\mathbf{a}_k=\mathbf{R}_{\mathbf{n}}^{\theta_k}\mathbf{e},\ \theta_k=\frac{2k\pi}{2m+1},\quad k=1,\cdots,2m+1\}. \tag{4.1}$$

They include the crystal classes C_{3v} as particular case when $m=1$.

For anisotropic functions under any subgroup $g\subset C_{\infty v}$, the general cases involving any number of vector variables and tensor variables may be reduced to the cases involving not more than two variables (see Theorem 2.3 in XIAO [18]). As a result, the third step in the procedure outlined in Sec. 3 in Part I can be omitted. Further reduction is possible. Let X_0 represent any of the sets of variables, $\mathbf{W}$, $\mathbf{A}$, $(\mathbf{W},\boldsymbol{\Omega})$, $(\mathbf{W},\mathbf{A})$ and $(\mathbf{A},\mathbf{B})$. Then each scalar-valued or tensor-valued anisotropic function of X_0 under the group $C_{2mv}(\mathbf{n},\mathbf{e})$ is a scalar-valued or tensor-valued anisotropic function of X_0 under the larger group $D_{2m+1d}(\mathbf{n},\mathbf{e})$ ($\supset C_{2m+1v}(\mathbf{n},\mathbf{e})$). Thus, in the general results for the group C_{2m+1v} (Theorem 6 below), we can directly cite the invariants and the tensor generators depending on skewsymmetric and/or symmetric tensor variables in Theorem 4. Moreover, for the foregoing sets of variables, vector generating sets under the group C_{2m+1v}

and related invariants from the scalar products, can be derived by setting $\mathbf{u} = \mathbf{n}$ in the corresponding results in the tables given in Sec. 2 **(viii)**, **(ix)**, **(xi)**, since each anisotropic function of any set Y_0 of vectors and tensors under the group C_{2m+1v} is an anisotropic function of the set $(Y_0, \mathbf{n})$ under the larger group D_{2m+1d} $(\supset C_{2m+1v})$.

There remain four sets of variables that are not covered in the above, including $(\mathbf{u})$, $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}, \mathbf{W})$ and $(\mathbf{u}, \mathbf{A})$. In what follows the four sets of variables will be treated separately.

(i) A single vector $\mathbf{u}$

Results for a single vector variable $\mathbf{u}$ under the group C_{2m+1v} can be derived by setting $\mathbf{v} = \mathbf{n}$ in the table given in Sec. 2 **(iv)**. However, the results thus obtained include redundant invariants and generators. The desired irreducible functional basis and irreducible generating sets can be derived by removing some redundant invariants and generators. This can easily be done by means of the fact: when $\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}) \neq 0$, $\overset{\circ}{\mathbf{u}}$ and $\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})$ yield two independent vectors on the $\mathbf{n}$-plane and hence $\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}) = x\,\overset{\circ}{\mathbf{u}} + y\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})$. The results are as follows.

$$V \quad \{\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\} \ (\equiv V^0_{2m+1}(\mathbf{u}))$$

$$\mathrm{Skw} \quad \{\alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{N}, \mathbf{n} \wedge \overset{\circ}{\mathbf{u}}, \mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\} \ (\equiv \mathrm{Skw}^0_{2m+1}(\mathbf{u}))$$

$$\mathrm{Sym} \quad \{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \mathbf{n} \vee \overset{\circ}{\mathbf{u}}, \mathbf{n} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}))\}$$
$$(\equiv \mathrm{Sym}^0_{2m+1}(\mathbf{u}))$$

$$R \quad \mathbf{n} \cdot \mathbf{r}, \overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{r}}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})]; (\mathrm{tr}\mathbf{HN})\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \overset{\circ}{\mathbf{u}} \cdot \mathbf{Hn},$$
$$\mathrm{tr}\mathbf{H}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$$
$$\mathrm{tr}\mathbf{C}, \mathbf{n} \cdot \mathbf{Cn}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{C}}\,\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{C}}\,\mathbf{n}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})], [\mathbf{n}, \overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}}, \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})];$$
$$\{\mathbf{u} \cdot \mathbf{n}, |\overset{\circ}{\mathbf{u}}|^2, \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\} \ (\equiv I^0_{2m+1}(\mathbf{u})).$$

(ii) The C_{2m+1v}-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

The C_{2m+1v}-irreducibility condition for $(\mathbf{u}, \mathbf{v})$, i.e. (3.1) with $(\mathbf{x}, \mathbf{y})=(\mathbf{u}, \mathbf{v})$ and $g = C_{2m+1v}$, yields $\Gamma(\mathbf{z}) \cap C_{2m+1v} \neq C_1,\ C_{2m+1v},\ \mathbf{z} = \mathbf{u},\ \mathbf{v}$. The latter means that $-\mathbf{R}^{\pi}_{\mathbf{a}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for $\mathbf{z} = \mathbf{u},\ \mathbf{v}$, i.e. $\mathbf{z} = c\mathbf{n} \times \mathbf{a} + d\mathbf{n}$ with $c \neq 0$. Thus, the C_{2m+1v}-irreducible set $(\mathbf{u}, \mathbf{v})$ is specified by $\mathbf{u} = a\mathbf{n} \times \mathbf{e} + b\mathbf{n}$ and $\mathbf{v} = c\mathbf{n} \times \mathbf{a} + d\mathbf{n}$ with $\mathbf{a} \neq \mathbf{e}$ and $ac \neq 0$. Here and henceforth, $\mathbf{a}$ is used to represent a two-fold axis vector of the group C_{2m+1v}.

With the aid of the latter we construct the following table.

V	$V^0_{2m+1}(\mathbf{u}) \cup V^0_{2m+1}(\mathbf{v})$
Skw	$\{\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge \overset{\circ}{\mathbf{v}}, \overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{v}}\}$
Sym	$\mathrm{Sym}^0_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}^0_{2m+1}(\mathbf{v})$
R	$\mathbf{r}\cdot V^0_{2m+1}(\mathbf{z}), \mathbf{C} : \mathrm{Sym}^0_{2m+1}(\mathbf{z}),\ \overset{\circ}{\mathbf{z}} \cdot\mathbf{Hn},\ \mathbf{z} = \mathbf{u},\ \mathbf{v}; \overset{\circ}{\mathbf{u}} \cdot\mathbf{H}\, \overset{\circ}{\mathbf{v}};$
	$I^0_{2m+1}(\mathbf{u}) \cup I^0_{2m+1}(\mathbf{v}) \cup \{\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{v}}\}.$

(iii) The C_{2m+1v}-irreducible set $(\mathbf{u}, \mathbf{A})$ of a vector and a symmetric tensor

The C_{2m+1v}-irreducibility condition for $(\mathbf{u}, \mathbf{A})$, i.e. (3.1) with $(\mathbf{x}, \mathbf{y})=(\mathbf{u}, \mathbf{A})$ and $g = C_{2m+1v}$, produces $\Gamma(\mathbf{u}) \cap C_{2m+1v} \neq C_1$, C_{2m+1v} and $\Gamma(\mathbf{A}) \cap C_{2m+1v} \neq C_1$. These mean that $-\mathbf{R}^{\pi}_{\mathbf{a}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for $\mathbf{z} = \mathbf{u},\ \mathbf{A}$. Hence we deduce that the C_{2m+1v}-irreducible set $(\mathbf{u}, \mathbf{A})$ is specified by

$$\mathbf{u} = a\mathbf{e}' + b\mathbf{n},\ \overset{\circ}{\mathbf{A}}= c(\mathbf{a}\otimes\mathbf{a} - \mathbf{a}'\otimes\mathbf{a}') + d\mathbf{n}\vee\mathbf{a}',\ \mathbf{a}' = \mathbf{n}\times\mathbf{a},\ \mathbf{a}\neq\mathbf{e},\quad a(c^2+d^2)\neq 0\,. \tag{4.2}$$

Thus, we construct the following table.

V	$V^0_{2m+1}(\mathbf{u}) \cup V^0_{2m+1}(\mathbf{A})\ (\equiv V^0_{2m+1}(\mathbf{u}, \mathbf{A}))$
Skw	$\mathrm{Skw}_{2m+1}(\mathbf{A}) \cup \{\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}\}$
Sym	$\mathrm{Sym}^0_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{2m+1}(\mathbf{A})$
R	$\mathbf{r}\cdot V^0_{2m+1}(\mathbf{u}); \mathbf{H} : \mathrm{Skw}_{2m+1}(\mathbf{A}); \mathbf{C} : \mathrm{Sym}^0_{2m+1}(\mathbf{u});$
	$\overset{\circ}{\mathbf{u}} \cdot\mathbf{H}\, \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{u}} \cdot\mathbf{H}\, \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}};$
	$I^0_{2m+1}(\mathbf{u}) \cup I_{2m+1}(\mathbf{A}) \cup \{\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}, \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}\}$
	$(\equiv I^0_{2m+1}(\mathbf{u}, \mathbf{A})).$

Here and henceforth, $V^0_{2m+1}(\mathbf{A})$ is used to designate the vector generating set for a symmetric tensor $\mathbf{A}$ under C_{2m+1v}, given by

$$V^0_{2m+1}(\mathbf{A}) = V_{2m+1}(\mathbf{n}, \mathbf{A}) = \{\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \mathbf{n}\times\boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A})),\ \overset{\circ}{\mathbf{A}}\,(\mathbf{n}\times\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}))\}\,. \tag{4.3}$$

In the above table and the table that will be given in (**iv**), the vector variable $\mathbf{r}$ pertains to the subspace $V(\Gamma(\mathbf{u}) \cap C_{2m+1v})$, the latter being spanned by the vector generating set $V^0_{2m+1}(\mathbf{u})$. Owing to this fact, of the invariants from the scalar products of $\mathbf{r}$ and the vector generators in the set $V^0_{2m+1}(\mathbf{u},\mathbf{D})$, where $\mathbf{D}=\mathbf{A}$, $\mathbf{W}$, the invariants except $\mathbf{r}\cdot V^0_{2m+1}(\mathbf{u})$ are redundant and can be deleted, as has been and will be done. The foregoing condition for $\mathbf{r}$ is derived from (see $(3.3)_2$ in Part I)

$$\Gamma(\mathbf{u},\mathbf{r})\cap C_{2m+1v}\neq\Gamma(\mathbf{u},\mathbf{D},\mathbf{r})\cap C_{2m+1v}\ (=C_1).$$

The other case for $\mathbf{r}$, i.e. $\mathbf{r}\notin V(\Gamma(\mathbf{u})\cap C_{2m+1v})$, has been covered by **(ii)** in this section.

By virtue of (4.2), it can easily be shown that the three presented sets of generators obey the criterion (2.3) in Part I, respectively, and hence they supply the desired vector, skewsymmetric and symmetric tensor generating sets. Moreover, we prove that the presented set $I^0_{2m+1}(\mathbf{u},\mathbf{A})$ supplies a functional basis for the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{A})$ under the group C_{2m+1v}. Towards this goal it suffices to show that the set $I^0_{2m+1}(\mathbf{u},\mathbf{A})$ determines a functional basis for the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{A})$ under the transverse isotropy group $C_{\infty v}(\mathbf{n})$ (see the comment at the end of Sec. 4 **(vi)** in Part I). In fact, the latter is obtainable from an isotropic functional basis for $(\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{A}},\mathbf{n})$, plus the three $C_{\infty v}$-invariants $\mathbf{u}\cdot\mathbf{n}$, $\mathbf{n}\cdot\mathbf{A}\mathbf{n}$ and $\mathrm{tr}\mathbf{A}$. The just-mentioned isotropic functional basis is given by $I_1=\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n}$, $I_2=\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{n}$, $I_3=\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}$, $I_4=\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\overset{\circ}{\mathbf{u}}$, as well as certain invariants of a single variable $\mathbf{u}$ or $\mathbf{A}$, each of the latter being covered or determined by the basis $I^0_{2m+1}(\mathbf{u})$ or $I_{2m+1}(\mathbf{A})$. Moreover, using (4.2) we have

$$I_1=ad\cos\theta,\ I_4=a^2(c^2+d^2\cos^2\theta),$$

where $\theta=\mathbf{a}\cdot\mathbf{e}$. It is evident that I_4 is redundant, since we have $a^2=|\overset{\circ}{\mathbf{u}}|^2$, $c^2=|\mathbf{q}(\mathbf{A})|^2$ and $d^2=|\overset{\circ}{\mathbf{A}}\mathbf{n}|^2$ for the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{A})$ (see (4.2)).

(iv) The C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{W})$ of a vector and a skewsymmetric tensor

The C_{2m+1v}-irreducibility condition for $(\mathbf{u},\mathbf{W})$, i.e. (3.1) with $(\mathbf{x},\mathbf{y})=(\mathbf{u},\mathbf{W})$ and $g=C_{2m+1v}$, produces $\Gamma(\mathbf{u})\cap C_{2m+1v}\neq C_1$, C_{2m+1v} and $\Gamma(\mathbf{W})\cap C_{2m+1v}\neq C_1$. These mean that $-\mathbf{R}^{\pi}_{\mathbf{a}}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for $\mathbf{z}=\mathbf{u}$, $\mathbf{W}$. Hence we deduce that the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{W})$ is specified by the two cases

(c1) $\mathbf{u}=a\mathbf{e}'+b\mathbf{n}$ and $\mathbf{W}=c\mathbf{E}\mathbf{n}$ with $ac\neq 0$,

(c2) $\mathbf{u}=a\mathbf{e}'+b\mathbf{n}$ and $\mathbf{W}=c\mathbf{E}\mathbf{a}$ with $\mathbf{a}\neq\mathbf{e}$ and $ac\neq 0$.

Thus, we construct the following table.

V	$V^0_{2m+1}(\mathbf{u}) \cup \{\mathbf{Wn}, (\mathrm{tr}\mathbf{WN})\mathbf{n}\times \overset{\circ}{\mathbf{u}}\}\ (\equiv V^0_{2m+1}(\mathbf{u},\mathbf{W}))$
Skw	$\{\mathbf{n}\wedge \overset{\circ}{\mathbf{u}}, \mathbf{W}, (\mathbf{E}:\mathbf{W})\wedge \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}$
Sym	$\mathrm{Sym}^0_{2m+1}(\mathbf{u}) \cup \mathrm{Sym}_{2m+1}(\mathbf{W}) \cup \{(\mathrm{tr}\mathbf{WN})\ \overset{\circ}{\mathbf{u}} \vee (\mathbf{n}\times \overset{\circ}{\mathbf{u}}),$
	$(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee \boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}})\}(\equiv \mathrm{Sym}^0_{2m+1}(\mathbf{u},\mathbf{W}))$
R	$\mathbf{r}\cdot V^0_{2m+1}(\mathbf{u}); \mathbf{C}:\mathrm{Sym}^0_{2m+1}(\mathbf{u}); \overset{\circ}{\mathbf{u}}\cdot\mathbf{Hn}, \mathrm{tr}\mathbf{HW},$
	$\mathrm{tr}\mathbf{HW}(\mathbf{E}\boldsymbol{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$
	$I^0_{2m+1}(\mathbf{u}) \cup I_{2m+1}(\mathbf{W}) \cup \{\overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn}\}.$

In the above table, the symmetric tensor variable $\mathbf{C}$ pertains to the subspace $\mathrm{Sym}(\Gamma(\mathbf{u})\cap C_{2m+1v})$, the latter being spanned by the generating set $\mathrm{Sym}_{2m+1}(\mathbf{u})$. Owing to this fact, of the invariants from the scalar products of $\mathbf{C}$ and the symmetric tensor generators in the set $\mathrm{Sym}^0_{2m+1}(\mathbf{u},\mathbf{W})$, those except $\mathbf{C}\cdot V^0_{2m+1}(\mathbf{u})$ are redundant and can be deleted, as has been done. The foregoing condition for $\mathbf{C}$ is derived from the condition (see $(3.3)_2$ in Part I)

$$\Gamma(\mathbf{u},\mathbf{C})\cap C_{2m+1v} \neq \Gamma(\mathbf{u},\mathbf{W},\mathbf{C})\ (= C_1).$$

The other case for $\mathbf{C}$, i.e. $\mathbf{C}\notin \mathrm{Sym}(\Gamma(\mathbf{u})\cap C_{2m+1v})$, has been covered before by (**iii**).

With cases (c1) and (c2) for the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{W})$, it can easily be shown that the three presented sets of generators obey the criterion (2.3) in Part I, respectively, and hence they supply the desired vector, skewsymmetric and symmetric tensor generating sets. Moreover, we prove that the presented set $I^0_{2m+1}(\mathbf{u},\mathbf{W})$ supplies a functional basis for the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{W})$ under the group C_{2m+1v}. Towards this goal it suffices to show that the set $I^0_{2m+1}(\mathbf{u},\mathbf{W})$ determines a functional basis for the C_{2m+1v}-irreducible set $(\mathbf{u},\mathbf{W})$ under the transverse isotropy group $C_{\infty v}(\mathbf{n})$ (see the comment at the end of Sec. 4 (**vii**) in Part I). In fact, the latter is obtainable from an isotropic functional basis for $(\overset{\circ}{\mathbf{u}},\mathbf{W},\mathbf{n})$, plus the $C_{\infty v}$-invariant $\mathbf{u}\cdot\mathbf{n}$. The just-mentioned isotropic functional basis is given by

$$I_1 = \overset{\circ}{\mathbf{u}}\cdot\mathbf{Wn},\ I_2 = \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}^2\mathbf{n},\ I_3 = \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}^2\ \overset{\circ}{\mathbf{u}},$$

as well as certain invariants of a single variable $\mathbf{u}$ or $\mathbf{W}$, each of the latter being covered or determined by the basis $I^0_{2m+1}(\mathbf{u})$ or $I_{2m+1}(\mathbf{W})$. The two invariants I_2 and I_3 are redundant for cases (c1) and (c2) derived before. This fact is obviously true for case (c1), and it is also true for case (c2), since for case (c2) we have $I_2=0,\ I_3=-a^2c^2(\mathbf{a}\cdot\mathbf{e})^2=-(I_1)^2$.

Combining the above results, we arrive at the general result as follows.

THEOREM 6. *The four sets given by*

$$I^0_{2m+1}(\mathbf{u}); I_{2m+1}(\mathbf{W}); I_{2m+1}(\mathbf{A}); I_{2m+1}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B});$$

$$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}, [\mathbf{n},\overset{\circ}{\mathbf{u}},\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{v}})], [\mathbf{n},\overset{\circ}{\mathbf{v}},\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}})];$$

$$\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \mathrm{tr}\mathbf{W}(\mathbf{E}\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), (\mathrm{tr}\mathbf{W}\mathbf{N})\alpha_{2m+1}(\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{n}, [\mathbf{n},\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}}\mathbf{n}], [\mathbf{n},\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}];$$

$$\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\overset{\circ}{\mathbf{v}}; \mathrm{tr}\mathbf{W}\mathbf{\Omega}(\mathbf{E}\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}})); \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}};$$

and

$$V^0_{2m+1}(\mathbf{u}); V^0_{2m+1}(\mathbf{A}); \mathbf{W}\mathbf{n}, \mathbf{W}^2\mathbf{n}, \mathbf{n}\times\mathbf{\eta}_{2m}(\mathbf{W}\mathbf{n}); (\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\times\overset{\circ}{\mathbf{u}};$$
$$(\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W})\mathbf{n}; (\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{\eta}_m(\mathbf{q}(\mathbf{A}));$$

and

$$\mathrm{Skw}^0_{2m+1}(\mathbf{u}); \mathrm{Skw}_{2m+1}(\mathbf{W}); \mathrm{Skw}_{2m+1}(\mathbf{A}); \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{v}};$$

$$\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\mathbf{n}\wedge\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{B}}\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\mathbf{n};$$

$$\overset{\circ}{\mathbf{A}}\mathbf{W}+\mathbf{W}\overset{\circ}{\mathbf{A}}, (\mathbf{E}:\mathbf{W})\wedge\mathbf{\eta}_m(\mathbf{q}(\mathbf{A})); (\mathbf{E}:\mathbf{W})\wedge\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}};$$

and

$$\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}})), \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}}));$$

$$\mathrm{Sym}_{2m+1}(\mathbf{W}); \mathrm{Sym}_{2m+1}(\mathbf{A});$$

$$\mathbf{W}\mathbf{\Omega}+\mathbf{\Omega}\mathbf{W}, |\mathrm{tr}\mathbf{\Omega}\mathbf{N}|(\mathrm{tr}\mathbf{\Omega}\mathbf{N})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n}$$
$$+|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{\Omega}\mathbf{n}\vee\mathbf{N}\mathbf{\Omega}\mathbf{n};$$

$$(\mathrm{tr}\mathbf{W}\mathbf{N})(\overset{\circ}{\mathbf{A}}\mathbf{N}-\mathbf{N}\overset{\circ}{\mathbf{A}}), (\mathrm{tr}\mathbf{W}\mathbf{N})((-1)^m\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})$$
$$+\mathbf{n}\vee\mathbf{\rho}_m(\mathbf{q}(\mathbf{A})));$$

$$(\mathrm{tr}\mathbf{W}\mathbf{N})\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}}), (\mathrm{tr}\mathbf{W}\mathbf{N})\mathbf{n}\vee\mathbf{\eta}_{2m}(\overset{\circ}{\mathbf{u}});$$

where $(\mathbf{u},\mathbf{v}) = (\mathbf{u}_i,\mathbf{u}_j)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H}) = (\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B}) = (\mathbf{A}_L,\mathbf{A}_M)$, $j > i = 1,\cdots,a$, $\theta > \tau > \sigma = 1,\cdots,b$, $M > L = 1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric* *and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1, \cdots, \mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1, \cdots, \mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1, \cdots, \mathbf{A}_c$ *under the group* C_{2m+1v} *for each* $m \geq 1$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* C_{2m+1v}.

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