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Propagation and reflectivity of transient heat waves

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We consider a rigid heat conductor characterized by two relaxation times and derive a linear hyperbolic equation for the temperature which can properly describe heat waves. The wave splitting technique is applied to the propagation problem whose solution is expressed in the form of the Laplace transform of the wave propagator. The reflectivity of a heat pulse is then obtained at an interface between two different conductors. Explicit results for both the propagation and the reflection probelms are worked out under suitable conditions which allow for a second sound propagation in low temperature rigid conductors. The characteristic relaxation times of a reflecting conductor are also determined as the solution of an inverse reflection problem.

1. Introduction

A large part of the recent theories of heat conduction is modelled in such a way as to comply with the fundamental requirement of a finite propagation speed. Beside the strong physical motivation which supports these theories, a growing experimental evidence of heat waves seems to point out that the usual diffusion equation for the temperature fails in describing transient phenomena in special circumstances [1, 2]. Heat waves (second sound) have been detected in liquid helium II and in dielectric crystals in narrow ranges of very low temperatures [3, 4]. Hyperbolic heat conduction has been also observed in processed meat at room temperatures [5]. In this last case the use of a Cattaneo's type constitutive equation for the heat flux has been proved to give excellent agreement with experimental results. The heat flux in Cattaneo's constitutive model is characterized by an exponential kernel with a single relaxation time and represents the most simple generalization to the Fourier's law allowing for a hyperbolic heat equation. On the other hand, experimental results on solid heat conductors at low temperatures suggest that different relaxation times exist in connection with different mechanisms of heat conduction. This is due to the fact that any conductor possesses substructures which relax at different rates. Really, heat can be carried by free electrons or by ballistic phonons, transmitted by electron-electron,

electron-phonon or phonon-phonon collisions and by interactions of phonons and electrons with the lattice impurities. In many cases, a realistic model can be obtained accounting for only two or three different mechanisms which compete in heat transport.

In the present paper we introduce a phenomenological model for a rigid conductor adopting a constitutive equation for the heat flux characterized by an exponential kernel with two different relaxation times. We show that, along with a linearized form of the balance of energy, this assumption yields a hyperbolic heat equation which generalizes the Cattaneo model. The coefficients of the heat equation are shown to depend essentially on the relaxation times which ultimately characterize the conductor's behaviour. The remainder of the paper is devoted to the analysis of the linearized hyperbolic system for the temperature θ and the heat flux q within the phenomenological model previously outlined. In some sense, this analysis extends the results obtained in [6] for a rigid conductor governed by the Cattaneo model. Transient wave solutions are studied and expressed in terms of the Laplace transform of a wave propagator. As shown in the last section, explicit inversion of the solution can be carried out under suitable conditions on the relaxation times which allow for second sound propagation in low temperature conductors. As it occurs in all propagation phenomena, the interaction of transient heat waves with an interface between different heat conductors, plays a fundamental role in many direct and inverse problems. For this reason we have applied the usual wave splitting technique to analyze the reflectivity of heat pulses on a discontinuity surface within the conductor. The procedure parallels the known approaches on wave propagation in dissipative media and leads to a reflectivity function characterized by the relaxation times of both sides of the interface. In the last section it is also shown that reflected pulses can be exploited to determine the relaxation times of the reflecting conductor. This inverse problem is solved in the case in which the incoming wave is a second sound pulse.

2. Heat flux with two relaxation times

Let us consider a rigid isotropic heat conductor which occupies an unbounded region $\mathcal{B}$ of the physical space. The absolute temperature $\theta = \theta(\mathbf{x}, t)$ is taken as a bounded function of the position and time, defined on the vector space $V \times \mathbb{R}$ where $V \subset \mathbb{R}^3$. A constitutive model accounting for a finite speed of heat propagation in a rigid conductor was introduced by GURTIN and PIPKIN [7] within the thermodynamic theory of materials with memory. In the linear approximation they obtained the following expression for the heat flux in an isotropic conductor:

$$\mathbf{q}(\mathbf{x},t) = -\int_0^\infty a(s)\,\mathbf{g}(\mathbf{x},t-s)\,\mathrm{d}s, \tag{2.1}$$

where $\mathbf{g}(\mathbf{x},t) = \nabla\theta(\mathbf{x},t)$ is the temperature gradient. If $a(s) = \dfrac{\kappa}{\tau}\exp(-s/\tau)$, from Eq. (2.1) follows the Cattaneo's equation,

$$\tau\,\partial_t \mathbf{q} + \mathbf{q} = -\kappa\,\mathbf{g}, \tag{2.2}$$

where κ and τ are respectively the heat conductivity and the characteristic relaxation time. In [7] it is assumed that the free energy density ψ, the entropy density η and the heat flux $\mathbf{q}$ depend on the summed histories of θ and $\mathbf{g}$, i.e., $\bar\theta^t(s) = \int_0^s \theta(t-\lambda)\,d\lambda$ and $\bar{\mathbf{g}}^t(s) = \int_0^s \mathbf{g}(t-\lambda)\,d\lambda$. However, as shown by MORRO [8], an effective model, compatible with a finite speed of heat propagation can be obtained by replacing the dependence on $\bar\theta^t$ and $\bar{\mathbf{g}}^t$ with the dependence on the histories $\theta^t(s) = \theta(t-s)$ and $\mathbf{g}^t(s) = \mathbf{g}(t-s)$. The qualitative features of the model, with respect to heat waves, do not change if we restrict the constitutive functionals to the form

$$\psi = \Psi(\theta,\mathbf{g}^t), \quad \eta = N(\theta,\mathbf{g}^t), \quad \mathbf{q} = \mathbf{Q}(\theta,\mathbf{g}^t). \tag{2.3}$$

As an example, a Maxwell-Cattaneo kernel for Eq. (2.1) has been considered in [8] assuming a quadratic dependence of ψ on the heat flux. As shown in [9], this assumption is required by the compatibility of Eq. (2.2) with thermodynamics. Consistently with the analysis in [8], we generalize here the previous example assuming that the relaxation kernel $a(s)$ be characterized by two different times τ_1 and τ_2. More precisely, we assume that Eqs. $(2.3)_3$ and $(2.3)_1$ take the following form

$$\mathbf{q}(\mathbf{x},t) = -\sum_{i=1}^{2}\kappa_i[\theta(\mathbf{x},t)]\int_0^\infty \frac{\exp(-s/\tau_i)}{\tau_i}\,\mathbf{g}(\mathbf{x},t-s)\,ds, \tag{2.4}$$

$$\psi(\mathbf{x},t) = \varphi[\theta(\mathbf{x},t)] + \sum_{i,j=1}^{2}\beta_{ij}[\theta(\mathbf{x},t)]\left[\int_0^\infty \frac{\exp(-s/\tau_i)}{\tau_i}\mathbf{g}(\mathbf{x},t-s)\,ds\right] \left[\int_0^\infty \frac{\exp(-s/\tau_j)}{\tau_j}\mathbf{g}(\mathbf{x},t-s)\,ds\right]. \tag{2.5}$$

The quantities $\kappa_i(\theta)$ $(i=1,2)$ in (2.4) are supposed to be C^2 functions of θ and the relaxation times $\tau_i > 0$ $(i=1,2)$ are taken to be constant. In Eq. (2.5) $\varphi(\theta)$ is a convex function and the entries of the symmetric matrix β_{ij} are supposed to be C^2 functions of θ. The model (2.4) – (2.5) applies, in general, to composite materials in which different substructures relax at different rates. Specifically, as illustrated in the next section, it can be applied to rigid dielectrics at low temperatures where phonon normal processes and umklapp processes occur with different frequencies.

In order to obtain thermodynamic restrictions on $\kappa_i(\theta)$ and $\beta_{ij}(\theta)$, we exploit the balance equation for the energy density $e(\mathbf{x}, t)$,

$$\rho\partial_t e = -\nabla \cdot \mathbf{q} + r, \tag{2.6}$$

together with the second law in the form of the following inequality for the entropy density η,

$$\rho\partial_t \eta \geq -\nabla \cdot \frac{\mathbf{q}}{\theta} + \frac{r}{\theta}, \tag{2.7}$$

where ρ is the (constant) mass density and r is the heat supply of external sources. Accounting for the thermodynamic relation $\psi = e - \theta\eta$, Eqs. (2.6) and (2.7) yield the Clausius-Duhem inequality in the following form:

$$-\rho\partial_t\psi - \frac{\rho}{\theta}(e - \psi)\partial_t\theta \geq \frac{1}{\theta}\mathbf{q} \cdot \mathbf{g}, \tag{2.8}$$

Posing

$$\Phi_i(\mathbf{x}, t) = \int_0^\infty \frac{1}{\tau_i} \exp(-s/\tau_i)\ \mathbf{g}(\mathbf{x}, t - s)\, ds, \qquad i = 1, 2,$$

we have

$$\partial_t\Phi_i = \frac{1}{\tau_i}(\mathbf{g} - \Phi_i), \qquad i = 1, 2. \tag{2.9}$$

Exploiting this result and substituting (2.5) and (2.4) into (2.8) we obtain

$$-\frac{\rho}{\theta}\partial_t\theta \left[e - \varphi + \theta\varphi' - \sum_{i,j=1}^{2} (\beta_{ij} - \theta\beta'_{ij})\Phi_i \cdot \Phi_j\right]$$
$$-\rho \sum_{i,j=1}^{2} \beta_{ij} \left[\left(\frac{1}{\tau_i}\Phi_j + \frac{1}{\tau_j}\Phi_i\right) \cdot \mathbf{g} - \left(\frac{1}{\tau_i} + \frac{1}{\tau_j}\right) \Phi_i \cdot \Phi_j\right]$$
$$\geq -\frac{1}{\theta} \sum_{i=1}^{2} \kappa_i\Phi_i \cdot \mathbf{g}.$$

where prime denotes differentiation with respect to θ. Since β_{ij} and κ_i do not depend on $\partial_t\theta$, we obtain

$$e = \varphi - \theta\varphi' + \sum_{i,j=1}^{2} (\beta_{ij} - \theta\beta'_{ij})\Phi_i \cdot \Phi_j, \tag{2.10}$$

$$\rho\theta\sum_{i,j=1}^{2}\beta_{ij}\left[\left(\frac{1}{\tau_i}\Phi_j+\frac{1}{\tau_j}\Phi_i\right)\cdot\mathbf{g}-\left(\frac{1}{\tau_i}+\frac{1}{\tau_j}\right)\Phi_i\cdot\Phi_j\right]\le\sum_{i=1}^{2}\kappa_i\Phi_i\cdot\mathbf{g}. \tag{2.11}$$

Adapting to our purposes a lemma shown in [8] and accounting for the independence of β_{ij} and κ_i on $\mathbf{g}^t$, inequality (2.11) ultimately gives

$$\kappa_i=2\rho\theta\sum_{j=1}^{2}\beta_{ij}\frac{1}{\tau_j}, \tag{2.12}$$

$$\sum_{i,j=1}^{2}\beta_{ij}\left(\frac{1}{\tau_i}+\frac{1}{\tau_j}\right)\Phi_i\cdot\Phi_j\ge 0.$$

In view of the symmetry of β_{ij}, we conclude that the matrix $B_{ij}=\beta_{ij}/\tau_i$ is positive semidefinite and the same holds for the matrix β_{ij}. These facts imply

$$\kappa := \kappa_1+\kappa_2\ge 0, \tag{2.13}$$

$$\gamma := \frac{\kappa_1}{\tau_1}+\frac{\kappa_2}{\tau_2}\ge 0. \tag{2.14}$$

From (2.10) and (2.5) we also have

$$\eta=-\theta^2\left[\varphi'+\frac{1}{\theta}\sum_{i,j=1}^{2}\beta_{ij}\Phi_i\cdot\Phi_j\right].$$

We remark that models which are characterized by only one relaxation time yield a Cattaneo's constitutive equation for the heat flux. This fact is apparent from Eqs. (2.4) and (2.9). The usual Fourier law $\mathbf{q}=-\kappa\mathbf{g}$ follows from (2.4) in the stationary case where κ, given by (2.13), is the heat conductivity of the medium.

3. Hyperbolic heat equation

Owing to Eqs. (2.10), (2.12) and (2.4), the balance of energy density (2.6) takes the form

$$-\theta\partial_t\theta\varphi''-\sum_{i,j=1}^{2}\left[(\beta_{ij}-\theta\beta'_{ij})\left(\frac{1}{\tau_i}+\frac{1}{\tau_j}\right)+\theta\partial_t\theta\beta''_{ij}\right]\Phi_i\cdot\Phi_j$$
$$+4\theta\sum_{i,j=1}^{2}\beta'_{ij}\frac{\Phi_i}{\tau_j}\cdot\mathbf{g}-2\theta\sum_{i,j=1}^{2}\beta_{ij}\frac{1}{\tau_j}\nabla\cdot\Phi_i-\frac{r}{\rho}=0. \tag{3.1}$$

Looking at a result which bear evidence of the essential features of the model with two relaxation times, we search for a linearized form of Eq. (3.1). Since thermal equilibrium is characterized by $\theta(\mathbf{x},t) = \theta_0$ and $\Phi_i(\mathbf{x},t) = 0$ $(i = 1,2)$, we retain only linear terms in the derivatives of θ and Φ_i. In absence of external heat supplies we obtain

$$\chi\partial_t\theta(\mathbf{x},t) = 2\sum_{i,j=1}^{2}\beta^0_{ij}\frac{1}{\tau_j}\nabla\cdot\Phi_i, \tag{3.2}$$

where $\chi = -\varphi''(\theta_0)$ and $\beta^0_{ij} = \beta_{ij}(\theta_0)$. We note that, in view of the previous hypotheses we have $\chi > 0$. Eq. (3.2) allows us to arrive at a linearized heat equation in a differential form. To this aim we observe that in view of Eq. (2.9), successive differentiation of Eq. (3.2) with respect to t gives

$$\begin{aligned} 2\sum_{i,j}^{2}\beta^0_{ij}\frac{1}{\tau_i}\frac{1}{\tau_j}\nabla\cdot\Phi_i &= 2\sum_{i,j}^{2}\beta^0_{ij}\frac{1}{\tau_i}\frac{1}{\tau_j}\nabla\cdot\mathbf{g} - \chi\partial^2_t\theta,\\ 2\sum_{i,j}^{2}\beta^0_{ij}\frac{1}{\tau_i^2}\frac{1}{\tau_j}\nabla\cdot\Phi_i &= 2\sum_{i,j}^{2}\beta^0_{ij}\frac{1}{\tau_i}\frac{1}{\tau_j}\left[\frac{1}{\tau_i}\nabla\cdot\mathbf{g} - \partial_t\nabla\cdot\mathbf{g}\right] + \chi\partial^3_t\theta. \end{aligned} \tag{3.3}$$

Since $\tau_1 \neq \tau_2$ and assuming $\kappa_i \neq 0$ $(i = 1,2)$, system (3.3) can be solved for $\nabla\cdot\Phi_i$, $(i = 1,2)$. Substituting these results into (3.2) we arrive at the following heat equation

$$\begin{aligned} -\partial_t\theta = (\tau_1+\tau_2)\partial^2_t\theta + \tau_1\tau_2\partial^3_t\theta - \frac{2}{\chi}\Bigg[&\left(\frac{\beta^0_{11}}{\tau_1} + \frac{\beta^0_{22}}{\tau_2} + \beta^0_{12}\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\right)\Delta\theta \\ &+ \left(\beta^0_{11}\frac{\tau_2}{\tau_1} + \beta^0_{22}\frac{\tau_1}{\tau_2} + 2\beta^0_{12}\right)\partial_t\Delta\theta\Bigg]. \end{aligned} \tag{3.4}$$

This is an hyperbolic differential equation which, according to (2.14) admits wave propagation with the speed

$$v = \left[\frac{\gamma}{\rho\theta_0\chi}\right]^{1/2} = \left[\frac{2}{\chi}\sum_{i,j=1}^{2}\beta^0_{ij}\frac{1}{\tau_i}\frac{1}{\tau_j}\right]^{1/2}. \tag{3.5}$$

The quantities β^0_{ij} are phenomenological coefficients whose values can be assigned by experimental data. In particular, if the heat conductivity κ and the wave speed v are given, Eqs. (2.13),(2.14) and (3.5) allow us to obtain β^0_{11} and β^0_{22} to within the choice of β^0_{12}. Since β^0_{ij} is required to be positive semidefinite, we can fix β^0_{12}

under the only restriction $|\beta_{12}^0| \leq \sqrt{\beta_{11}^0 \beta_{22}^0}$. The most simple choice is $\beta_{12}^0 = 0$. In this case we obtain

$$\beta_{11}^0 = \frac{\tau_1^2}{\tau_2 - \tau_1}\left[v^2\tau_2 - \frac{\kappa}{2\rho\theta_0}\right], \qquad \beta_{22}^0 = \frac{\tau_2^2}{\tau_1 - \tau_2}\left[v^2\tau_1 - \frac{\kappa}{2\rho\theta_0}\right]. \tag{3.6}$$

The present phenomenological model is then reduced to the knowledge of the relaxation times τ_1 and τ_2.

A relevant application of the result (3.4) can be found in the problem of heat conduction in a rigid dielectric at low temperatures (see[1, 10]). In this context, a theory of the phonon gas has been developed which considers relaxation phenomena as the result of phonon's interactions. In particular, in a dielectric heat conductor two characteristic times can be introduced in connection with phonon's resistive processes which do not conserve momentum, and normal processes which conserve momentum (see [11]).We remark that in our model τ_1 and τ_2 are phenomenological quantities which are not necessarily ascribed respectivly to resistive and normal processes. However, a comparison with the 9-fields theory shows that the heat equation (3.4) has the same form of that obtained from system (3.56) in [10] where two relaxation times τ_R and τ_N account for resistive processes and normal processes, respectively. In fact Eqs. (3.56) in [10], can be rewritten as

$$\begin{aligned} \partial_t e + c^2\nabla\cdot\mathbf{p} &= 0,\\ \partial_t\mathbf{p} + \frac{1}{3}\nabla e + \nabla\cdot\mathbf{N} &= -\frac{1}{\tau_R}\mathbf{p},\\ \partial_t\mathbf{N} + \frac{2}{5}c^2\left(\nabla\mathbf{p} - \frac{1}{3}(\nabla\cdot\mathbf{p})\,\mathbf{1}\right) &= -\frac{1}{\tau}\mathbf{N}, \end{aligned} \tag{3.7}$$

where e is the energy density of phonons, $\mathbf{p}$ is the phonon momentum, $\mathbf{N}$ is the deviatoric part of the momentum flux of phonons, c is the Debye speed and $1/\tau = 1/\tau_R + 1/\tau_N$. Eliminating $\mathbf{p}$ and $\mathbf{N}$ from (3.7), we obtain the following equation for e,

$$-\frac{1}{\tau_R\tau}\partial_t e = \left(\frac{1}{\tau_R} + \frac{1}{\tau}\right)\partial_t^2 e + \partial_t^3 e - \frac{c^2}{3}\left(\frac{1}{\tau}\Delta e + \frac{9}{5}\partial_t\Delta e\right). \tag{3.8}$$

In [10] the energy density is supposed to obey the Debye law for phonons even in non-equilibrium. This fact means that $e = e(\theta)$ where θ is the absolute temperature of the conductor. In particular, the linearized form used in [10]

$$e = e_0 + \alpha(\theta - \theta_0), \tag{3.9}$$

with e_0 and α constant, turns out to be equivalent to the linearized version of (2.10). Substitution of (3.9) into (3.8) gives

$$(3.10)\qquad -\partial_t\theta = (\tau_R+\tau)\partial_t^2\theta + \tau_R\tau\partial_t s\theta - \frac{c^2}{3}\left(\tau_R\Delta\theta + \frac{9}{5}\tau_R\tau\partial_t\Delta\theta\right).$$

Equation (3.10) has the same form of (3.4) provided that $1/\tau_1$ and $1/\tau_2$ are identified respectively with the relaxation frequencies $1/\tau_R$ and $1/\tau$. We finally observe that, actually and independently on their physical interpretation, the phenomenological times τ_1 and τ_2 have to be evalued by a measure of some relaxation properties of the conductor. In the last section we turn the evaluation of $1/\tau_1$ and $1/\tau_2$ into an inverse problem for the reflection of transient heat waves.

4. Wave splitting for transient heat waves

Having introduced Cartesian coordinates (x,y,z), we suppose that the region occupied by the rigid heat conductor corresponds to the half-space $V = \{(x,y,z)\in\mathbb{R}^3 |\ x\geq 0\}$ and denote by $\mathcal{S}$ the boundary plane surface $x=0$. We assume that a uniform heat pulse is generated at $x=0$, for $t>0$ and that temperature perturbations are absent throughout V for $t\leq 0$. We restrict the analysis to the one-dimensional problem considering only the component $q_x =: q$ of $\mathbf{q}$. Looking for definite results, in the following we shall discard external heat supplies. From (2.4) and (2.6) we obtain a linear integro-differential system for θ and q in the form

$$(4.1)\qquad q(x,t) = -\int_0^\infty \left(\frac{\kappa_1}{\tau_1}\exp(-s/\tau_1) + \frac{\kappa_2}{\tau_2}\exp(-s/\tau_2)\right)\partial_x\theta(x,t-s)\,ds,$$

$$(4.2)\qquad \rho\theta_0\chi\partial_t\theta(x,t) = -\partial_x q(x,t),$$

together with the conditions

$$(4.3)\qquad \theta(0,t)=\hat{\theta}(t),\qquad \text{or}\qquad q(0,t)=\hat{q}(t),\qquad \forall t>0,$$

$$(4.4)\qquad \theta(x,t)=\theta_0,\qquad \forall x\geq 0,\quad \forall t\leq 0.$$

In Eq. (4.1) it is understood that the quantities κ_1 and κ_2 are evaluated at $\theta=\theta_0$. After differentiating (4.1) with respect to time and taking into account (4.4), we get

$$(4.5)\qquad \partial_t q(x,t) = \gamma K[\partial_x\theta(x,\cdot)](t),$$

where the linear integral operator K is defined as

$$(4.6)\qquad (Kf)(t) = -f(t) + \frac{1}{\gamma}\int_0^t \left(\frac{\kappa_1}{\tau_1}\exp(-s/\tau_1) + \frac{\kappa_2}{\tau_2}\exp(-s/\tau_2)\right) f(t-s)\,ds,$$

for any bounded $f(t)$, $(t \in \mathbb{R}^+)$. By the use of the Laplace transforms it is easy to show that K admits the following inverse

$$(4.7)\qquad (K^{-1}f)(t) = -f(t) - \frac{\gamma}{\kappa}\int_0^t \left[1 + \frac{\kappa_1\kappa_2}{\gamma^2}\left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)^2 \exp\left(-\frac{\kappa}{\tau_1\tau_2\gamma}s\right)\right] f(t-s)\,ds.$$

Then, Eqs. (4.5) and (4.2) can be rewritten as

$$(4.8)\qquad \partial_x \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} 0 & \gamma^{-1}K^{-1} \\ -\rho\theta_0\chi & 0 \end{pmatrix} \partial_t \begin{pmatrix} \theta \\ q \end{pmatrix}.$$

Equation (4.8) is analogous to the system (2.2) of WALL and OLSSON [6], where a Maxwell-Cattaneo equation is analyzed. Adopting a wave splitting technique, we parallel the analysis of [6] in the homogeneous case. Accordingly, we introduce the quantity $(\theta^+, \theta^-)^T$ in the following way:

$$(4.9)\qquad \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix} = \mathbf{D}^{-1} \begin{pmatrix} \theta - \theta_0 \\ q \end{pmatrix},$$

where

$$(4.10)\qquad \mathbf{D} = \begin{pmatrix} 1 & 1 \\ -\gamma P & \gamma P \end{pmatrix},$$

and we look for the linear operator P which diagonalizes the matrix in the right-hand side of (4.8). Substituting (4.9) and (4.10) into (4.8) we obtain

$$(4.11)\qquad \partial_x \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{v^2}P^{-1} - K^{-1}P & \frac{1}{v^2}P^{-1} + K^{-1}P \\ -\frac{1}{v^2}P^{-1} - K^{-1}P & -\frac{1}{v^2}P^{-1} + K^{-1}P \end{pmatrix} \partial_t \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix},$$

where the definition (3.5) has been used. Imposing the diagonalization condition we have

$$(4.12)\qquad \frac{1}{v^2}(P^{-1}f)(t) = -[(K^{-1}P)f](t),$$

for any bounded $f(t)$, $t \in \mathbb{R}^+$. In view of Eq. (4.7) and making use of the Laplace transforms, Eq. (4.12) yields

$$(4.13)\qquad (P^{-1}f)(t) = \pm v f(t) \pm v \int_0^t \left[F_1(\tau) + F_2(\tau) \int_0^\tau F_1(\xi) F_2(\tau - \xi)\, d\xi \right] f(t-\tau)\, d\tau,$$

where

$$F_1(t) = \frac{1}{2\tau_1} \exp\left(-\frac{t}{2\tau_1}\right) \left[I_0\left(\frac{t}{2\tau_1}\right) + I_1\left(\frac{t}{2\tau_1}\right) \right],$$

$$(4.14)\qquad F_2(t) = \frac{1}{2\tau_2}\left(1 - \frac{\kappa}{\tau_1\gamma}\right) \exp\left[-\frac{t}{2\tau_2}\left(1 + \frac{\kappa}{\tau_1\gamma}\right)\right] \times \left\{ I_0\left[\frac{t}{2\tau_2}\left(1 - \frac{\kappa}{\tau_1\gamma}\right)\right] + I_1\left[\frac{t}{2\tau_2}\left(1 - \frac{\kappa}{\tau_1\gamma}\right)\right] \right\},$$

and where I_0 and I_1 are modified Bessel functions. The sign in (4.13) can be chosen according to the meaning of θ^+ and θ^- as forward and backward propagating modes. Then, Eq. (4.11) reduces to

$$(4.15)\qquad \partial_x \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix} = \frac{1}{v} \begin{pmatrix} -1 - F_1 * - F_2 * - F_1 * F_2 * & 0 \\ 0 & 1 + F_1 * + F_2 * + F_1 * F_2 * \end{pmatrix} \partial_t \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix},$$

where the asterisk denotes time convolution, i.e.,

$$[a(\cdot) * b(\cdot)](t) = \int_0^t a(\tau) b(t-\tau)\, d\tau.$$

Since forward and backward modes propagate independently in opposite directions, for definiteness let us consider only one mode, say θ^+. We can write

$$(4.16)\qquad \partial_x \theta^+(x,t) = -\frac{1}{v}\{[\delta(\cdot) + F_1(\cdot) + F_2(\cdot) + F_1 * F_2(\cdot)] * \partial_{(\cdot)}\theta^+(x,\cdot)\}(t).$$

Equation (4.16) can be transformed into an integro-differential equation for a wave propagator $\mathcal{P}^+(x,t)$, defined as

$$\theta^+(x,t) = [\mathcal{P}^+(x,\cdot) * \theta^+(0,\cdot)](t).$$

Substituting into (4.16) we obtain

$$\partial_x[\mathcal{P}^+(x,\cdot) * \theta^+(0,\cdot)(t)] = -\frac{1}{v}\partial_t[(1+F_1+F_2+F_1*F_2) * \mathcal{P}^+(x,\cdot) * \theta^+(0,\cdot)(t)]. \tag{4.17}$$

In deriving Eq. (4.17) we have exploited the result $\theta^+(x,0) = 0, \forall x \geq 0$, which follows from (4.4), (4.9) and (4.13). The application of Laplace transforms to Eq. (4.17) allows us to obtain

$$\mathcal{L}\mathcal{P}^+(x,s) = \exp\left[-\frac{x}{v}s(1+\mathcal{L}F_1(s)+\mathcal{L}F_2(s)+\mathcal{L}F_1(s)\mathcal{L}F_2(s))\right], \tag{4.18}$$

where we have taken into account the condition $\mathcal{P}^+(0,t) = \delta(t)$. Making use of (4.14) we get

$$\mathcal{L}\mathcal{P}^+(x,s) = \exp\left[-\frac{x}{v}\sqrt{\frac{s(s+1/\tau_1)(s+1/\tau_2)}{s+\dfrac{\kappa}{\tau_1\tau_2\gamma}}}\right]. \tag{4.19}$$

We note that in a phenomenological model with a single relaxation time, the Laplace transform in Eq. (4.19) can be easily inverted to give the wave propagator for a Cattaneo type heat equation (see [6]). This can be performed by letting $\tau_2 \to \infty$ in (4.19). We obtain

$$\mathcal{P}_1^+(x,t) = \exp(-t/(2\tau_1)) \left\{\delta\left(t-\frac{x}{v}\right) + \mathcal{H}\left(t-\frac{x}{v}\right)\frac{x}{2\tau_1 v}\frac{I_1\left[\dfrac{1}{2\tau_1}\sqrt{t^2-(x/v)^2}\right]}{\sqrt{t^2-(x/v)^2}}\right\}, \tag{4.20}$$

where $\mathcal{H}(t)$ is the Heaviside unit step function. An analytical inversion of the Laplace transform in (4.19) will be accomplished under suitable approximations in the last section. To this end we give here an alternative form of the previous result. Without loss of generality we can choose $\tau_1 > \tau_2$, such that, owing to (2.12) and (2.13), the quantity

$$\sqrt{\frac{s+1/\tau_2}{s+\dfrac{\kappa}{\tau_1\tau_2\gamma}}} = \left[1-\frac{1}{\tau_2}\frac{1-\dfrac{\kappa}{\tau_1\gamma}}{s+1/\tau_2}\right]^{-1/2}$$

can be expanded into a binomial series. We obtain

$$\mathcal{L}\mathcal{P}^{+}(x,s) = \exp\left\{ -\frac{x}{v}\sqrt{s(s+1/\tau_1)} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \left[\frac{1}{\tau_2} \frac{1-\dfrac{\kappa}{\tau_1\gamma}}{s+1/\tau_2} \right]^k \right\}. \tag{4.21}$$

5. Reflectivity of heat waves

Let us consider two half-spaces $V_a = \{\mathbf{x} \in \mathbb{R}^3 |\ x \leq 0\}$ and $V_b = \{\mathbf{x} \in \mathbb{R}^3 |\ x > 0\}$ occupied respectively by two different homogeneous rigid heat conductors, modelled as in Sec. 2. They are taken to be in a thermodynamic equilibrium state at the temperature θ_0. The constitutive parameters $\rho, \kappa_1, \kappa_2, \tau_1, \tau_2$ are supposed to be constant in V_a and V_b and discontinuous at the common plane boundary $\mathcal{S}, (x = 0)$ of the two conductors. In the following we shall use the suffixes a and b to denote quantities pertaining respectively to V_a and V_b. The continuity of θ and q at the surface $\mathcal{S}$ implies

$$\begin{pmatrix} \theta \\ q \end{pmatrix}(0^+,t) = \begin{pmatrix} \theta \\ q \end{pmatrix}(0^-,t), \qquad \forall\, t \in \mathbb{R}^+, \tag{5.1}$$

where 0^+ and 0^- refer respectively to the limiting values from the right and from the left of $x = 0$. Since θ_0 is the common temperature of both conductors, from (4.9) we have

$$\left[\mathbf{D}_b \begin{pmatrix} \theta^+ \\ \theta_- \end{pmatrix}\right](0^+,t) = \left[\mathbf{D}_a \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix}\right](0^-,t), \qquad \forall\, t \in \mathbb{R}^+, \tag{5.2}$$

where

$$\mathbf{D}_b = \begin{pmatrix} 1 & 1 \\ -\gamma_b P_b & \gamma_b P_b \end{pmatrix}, \qquad \mathbf{D}_a = \begin{pmatrix} 1 & 1 \\ -\gamma_a P_a & \gamma_a P_a \end{pmatrix}.$$

Multiplicating Eq. (5.2) from the left by $\mathbf{D}_b^{-1}$ we arrive at

$$\begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix}(0^+,t) \tag{5.3}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + \dfrac{\gamma_a}{\gamma_b} P_b^{-1} * P_a & 1 - \dfrac{\gamma_a}{\gamma_b} P_b^{-1} * P_a \\ 1 - \dfrac{\gamma_a}{\gamma_b} P_b^{-1} * P_a & 1 + \dfrac{\gamma_a}{\gamma_b} P_b^{-1} * P_a \end{pmatrix} \begin{pmatrix} \theta^+ \\ \theta^- \end{pmatrix}(0^-,t).$$

Now we introduce a reflectivity function $R(t)$ by the following definition,

$$\theta^-(0^-,t) = [R(\cdot) * \theta^+(0^-,\cdot)](t),$$

and substitute into (5.3) to obtain

(5.4)
$$\theta^+(0^+,t) = \frac{1}{2}[\delta(\cdot) + \frac{\gamma_a}{\gamma_b}P_b^{-1} * P_a(\cdot) + R(\cdot) - \frac{\gamma_a}{\gamma_b}P_b^{-1} * P_a * R(\cdot)] * \theta^+(0^-,t),$$

$$\theta^-(0^+,t) = \frac{1}{2}[\delta(\cdot) - \frac{\gamma_a}{\gamma_b}P_b^{-1} * P_a(\cdot) + R(\cdot) + \frac{\gamma_a}{\gamma_b}P_b^{-1} * P_a * R(\cdot)] * \theta^+(0^-,t).$$

Owing to the causality principle we pose $\theta^-(0^+,t) = 0, \forall\, t \in \mathbb{R}^+$. Then, in view of the arbitrariness of $\theta^+(0^-,t)$, Eq. (5.4) yields

(5.5)
$$\delta(t) + R(t) - \frac{\gamma_a}{\gamma_b}P_b^{-1} * P_a(t) + \frac{\gamma_a}{\gamma_b}P_b^{-1} * P_a * R(t) = 0.$$

Applying the Laplace transforms to Eq. (5.5) and accounting for the expression of P derived in the previous section, we arrive at

(5.6)
$$\mathcal{L}R(s) = \frac{\sqrt{H(s)} - \sigma}{\sqrt{H(s)} + \sigma},$$

where

(5.7)
$$H(s) = \frac{\left(s + \frac{1}{\tau_1^b}\right)\left(s + \frac{1}{\tau_2^b}\right)\left(s + \frac{\kappa^a}{\tau_1^a\tau_2^a\gamma_a}\right)}{\left(s + \frac{1}{\tau_1^a}\right)\left(s + \frac{1}{\tau_2^a}\right)\left(s + \frac{\kappa^b}{\tau_1^b\tau_2^b\gamma_b}\right)},$$

and $\sigma = \sqrt{\rho_b\chi_b\gamma_b}/\sqrt{\rho_a\chi_a\gamma_a}$. An inversion of the Laplace transform in Eq. (5.6) can be performed by writing

$$H(s) = 1 + \frac{A_1}{s+s_1} + \frac{A_2}{s+s_2} + \frac{A_3}{s+s_3},$$

where $s_1 = 1/\tau_1^a,\ s_2 = 1/\tau_2^a,\ s_3 = \kappa_b/(\tau_1^b\tau_2^b\gamma_b)$, and

$$A_i = \left\{ -s_i^3 + \left(\frac{1}{\tau_1^b} + \frac{1}{\tau_2^b} + \frac{\kappa^a}{\tau_1^a \tau_2^a \gamma_a} \right) s_i^2 - \left(\frac{1}{\tau_1^b \tau_2^b} + \frac{\kappa^a}{\tau_1^a \tau_2^a \tau_1^b \gamma_a} + \frac{\kappa^a}{\tau_1^a \tau_2^a \tau_2^b \gamma_a} \right) s_i + \frac{\kappa^a}{\tau_1^a \tau_2^a \tau_1^b \tau_2^b \gamma_a} \right\} [(s_{i+1} - s_i)(s_{i+2} - s_i)]^{-1}$$

for $i = 1, 2, 3$. The final result is

$$R(t) = \int_0^\infty N(\xi)[G_1(\xi, \cdot) * G_2(\xi, \cdot) * G_3(\xi, \cdot)](t)\, d\xi, \tag{5.8}$$

where

$$N(\xi) = \exp(-\xi) \left[2\sigma^2 \exp(\sigma^2 \xi) \mathrm{erfc}(\sigma \sqrt{\xi}) - \frac{2\sigma}{\sqrt{\pi \xi}} + 1 \right], \tag{5.9}$$

and

$$G_i(\xi, t) = \exp(-t s_i) \left[\delta(t) - \sqrt{\frac{A_i \xi}{t}} J_1(2\sqrt{A_i \xi t}) \right], \quad i = 1, 2, 3. \tag{5.10}$$

with J_1 Bessel function of order 1. In view of Eqs. (5.8) - (5.10) we can write

$$R(t) = \nu \sum_{i=1}^{3} \mathrm{e}^{-t s_i} \delta(t) + \widetilde{R}(t),$$

where

$$\nu = \int_0^\infty N(\xi)\, d\xi = \frac{1 - \sigma}{1 + \sigma}$$

is the (instantaneous) attenuation factor and where $\widetilde{R}(t)$ yields the part of the reflected field due to convolution. Both ν and $\widetilde{R}(t)$ are characterized by the relaxation times τ_1 and τ_2 and by the partial heat conductivities κ_1 and κ_2 in a and b. As shown in the next section, if the quantities pertaining to the conductor a are known, the reflection data can be used to gain informations about the characteristic parameters in b.

6. Application to the second sound and the inverse problem

The analysis of experimental results on heat pulses in dielectric crystals has shown that second sound propagates only if θ_0 falls into a narrow range of absolute temperatures [12]. This property, called window condition, has been interpreted on the basis of phonons' model by saying that, in sufficiently pure crystals, second sound appears when the temperature is low enough to prevent resistive processes and high enough to allow for normal processes. Applying our phenomenological model to this problem we assume $1/\tau_1$ and $1/\tau_2$ to represent two characteristic frequencies accounting for resistive processes and normal processes. This hypothesis is justified by the identifications due to the comparison of the heat Eq. (3.4) with Eq. (3.10), obtained in phonon's model [10] (see the end of Sec. 3). Then the window condition holds when we have $1/\tau_1 \ll 1/\tau_2$. We exploit this condition to derive an explicit solution to the propagation and the reflection problems.

We firstly consider the wave propagator $\mathcal{P}^+(x,t)$ in the form (4.21). According to Eqs. (3.5) we can write

$$\mathcal{L}\mathcal{P}^+(x,s) = \exp\left[-\frac{x}{v}\sqrt{s(s+1/\tau_1)}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!}\left(\frac{1-\dfrac{\kappa}{2\rho\theta_0\tau_1 v^2}}{1+s\tau_2}\right)^k\right]. \tag{6.1}$$

A first approximation of Eq. (6.1) can be worked out under the window conditon assuming $\tau_2 \ll \tau_1$. Retaining only the terms for $k=0$ and $k=1$ in the binomial series, we get

$$\mathcal{L}\mathcal{P}^+(x,s) = \exp\left[-\frac{x}{v}\sqrt{s(s+1/\tau_1)}\right]\exp\left[-\frac{x}{2v\tau_2}\frac{s}{s+1/\tau_2}\right]. \tag{6.2}$$

The Laplace transform in (6.2) can be easily inverted to give

$$\mathcal{P}^+(x,t) = \exp\left(-\frac{x}{2v\tau_2}\right)\left[\mathcal{P}_1^+(x,\cdot) * \mathcal{Q}(x,\cdot)(t)\right], \tag{6.3}$$

where $\mathcal{P}_1^+(x,t)$ is given by (4.20) and where

$$\mathcal{Q}(x,t) = \exp(-t/\tau_2)\delta(t) + \exp(-t/\tau_2)\frac{1}{\tau_2}\sqrt{\frac{x}{2vt}}I_1\left(\frac{1}{\tau_2}\sqrt{\frac{2x}{v}t}\right).$$

To point out the role of normal processes in determining the wave propagator we rewrite Eq. (6.3) in terms of the nondimensional quantities $a = x/v\tau_2$, $q = \tau_2/2\tau_1$, $\lambda = t/\tau_2$. We obtain

$$\mathcal{P}^+(a,\lambda) = \mathrm{e}^{a(1/2-q)-\lambda}\delta(\lambda-a) + \mathrm{e}^{a(1/2-q)-\lambda}\frac{1}{\tau_2}\sqrt{\frac{a}{2(\lambda-a)}}\, I_1\left(\sqrt{2a(\lambda-a)}\right)$$
$$+ aq\frac{1}{\tau_2}\mathrm{e}^{-\frac{a}{2}-\lambda q}\frac{I_1(q\sqrt{\lambda^2-a^2})}{\sqrt{\lambda^2-a^2}} + \mathrm{e}^{-\frac{a}{2}-\lambda}\int_a^\lambda \mathrm{e}^{\xi(q-1)}\frac{1}{\tau_2^2}\frac{\sqrt{\frac{a}{2(\lambda-\xi)}}}{\sqrt{\xi^2-a^2}}$$
$$I_1\left[q\sqrt{\xi^2-a^2}\right]\, I_1\left[\sqrt{2a(\lambda-\xi)}\right]d\xi. \tag{6.4}$$

The first term at the right-hand side of (6.4) is the hyperbolic part of the propagator which leaves the boundary pulse undistorted but attenuated in amplitude, owing to the few resistive processes. The second term takes into account the effect of normal processes. It is the leading term and it rapidly decreases with λ. The third term is a slow decreasing function of λ, essentially due to the resistive processes. The fourth term is a mixed, quantitatively minor contribution. The effect of such propagator on a half-gaussian pulse at $x = 0$, $\theta^+(0,t) = \mathrm{e}^{-bt^2}$, $(t > 0)$, is shown in Fig. 1 for different values of τ_2. It is evident how a second sound pulse arises if τ_2 is notably smaller than τ_1. Figure 1 is in agreement with the experimental evidence of second sound in low-temperature very pure crystals (see [3]), and with the numerical results according to the 9-fields theory by [10] where a second sound pulse develops for small values of the relaxation time associated with normal processes.

Now we consider the problem of the reflectivity at the interface $\mathcal{S}$ between the half-spaces V_a and V_b occupied by two different rigid conductors. On the basis of the results obtained in Sec. 5 we suppose that the medium in V_a is a highly pure crystal which complies with the window condition at the temperature θ_0. This assumption implies that the function $H(s)$, given in (5.7), can be approximated to

$$H(s) = h\left[1 + \frac{1}{\tau s} - \frac{1}{\tau}\frac{\left(1-\frac{\tau}{\tau_1}\right)\left(1-\frac{\tau}{\tau_2}\right)}{s+\frac{\tau}{\tau_1\tau_2}}\right], \tag{6.5}$$

where τ_1 and τ_2 are the relaxation times pertinent to the conductor placed in V_b, and

$$h = \frac{\kappa^a}{\tau_1^a\gamma_a}, \qquad \tau = \frac{\kappa^b}{\gamma_b}.$$

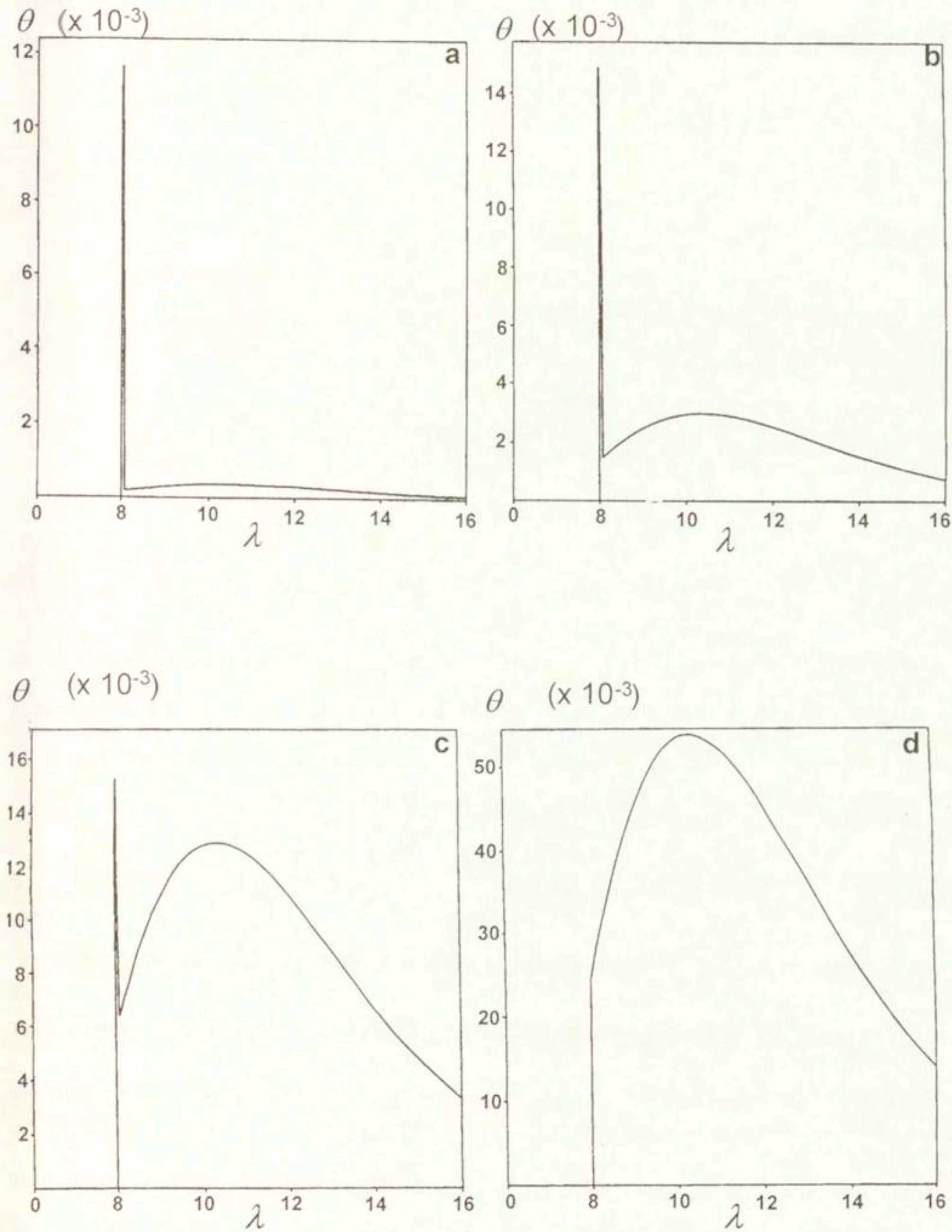

FIG. 1. Temperature at $x = av\tau_2$ as function of $\lambda = t/\tau_2$ for an half-Gaussian pulse $\theta^+(0,t) = e^{-bt^2}$, $t > 0$, for $a = 8, b = 10^6 s^{-2}$, $\tau_1 = 5$ s, and (a)$\tau_2 = 0.5$s, (b)$\tau_2 = 0.2$s, (c)$\tau_2 = 0.1$s, (d)$\tau_2 = 0.05$ s.

Substitution of Eq. (6.5) into (5.6) and application of the inverse Laplace transform yields

$$R(t) = \nu \exp\left(-\frac{\tau}{\tau_1\tau_2}t\right)\delta(t) + \widetilde{R}(t) \tag{6.6}$$

where

$$\widetilde{R}(t) = \int_0^\infty \left[-L_1(\xi,t) - L_2(\xi,t) + L_1(\xi,\cdot) * L_2(\xi,\cdot)(t)\right] N(\xi)\, d\xi, \tag{6.7}$$

with

$$\begin{aligned} L_1(\xi,t) &= \sqrt{\frac{\xi}{\tau t}} J_1\left(2\sqrt{\frac{\xi t}{\tau}}\right), \\ L_2(\xi,t) &= \exp(-\frac{\tau}{\tau_1\tau_2}t)\sqrt{\frac{1}{\tau}\left(1-\frac{\tau}{\tau_1}\right)\left(1-\frac{\tau}{\tau_2}\right)\frac{\xi}{t}} \\ &\qquad J_1\left(2\sqrt{\frac{1}{\tau}\left(1-\frac{\tau}{\tau_1}\right)\left(1-\frac{\tau}{\tau_2}\right)\xi t}\right). \end{aligned} \tag{6.8}$$

In (6.6) and (6.7) ν and $N(\xi)$ are given by Eqs. (5.11) and (5.9) where $\sigma = \sqrt{h\dfrac{\rho_b\chi_b\gamma_b}{\rho_a\chi_a\gamma_a}}$ (hereafter we shall assume $\sigma \leq 1$). From the result (6.6), which is a special case of that obtained in Sec. 5, we obtain the reflected field at $x = 0$ in the form

$$\theta^-(0^-,t) = \nu\theta^+(0^-,t) + \int_0^t \widetilde{R}(s)\theta^+(0^-,t-s)\, ds. \tag{6.9}$$

In Fig. 2 we have shown an incoming half-Gaussian pulse and its reflected profile at $x = 0$, given by Eq. (6.9) when $\sigma = 0, (\nu = 1)$ for two different values of τ_2. The reflected pulse turns out to be notably sharpened by reflection and it shows a negative tail ($\theta < \theta_0$) at $x = 0$. The effect of sharpening is more relevant for smaller values of τ_2.

Now we assume that a boundary pulse $\theta^+(0^-,t)$ be given at $\mathcal{S}$. Then, by the reflection data, we can estimate the quantities $\theta_0^- := \theta^-(0^-,0)$, $(\theta^-)'_0 := \dfrac{d}{dt}\theta^-(0^-,0)|_{t=0}$, $(\theta^-)''_0 := \dfrac{d^2}{dt^2}\theta^-(0^-,0)|_{t=0}$ at the interface $\mathcal{S}$. We show how the parameter σ and the relaxation times τ_1 and τ_2 in V_b can be derived by $\theta_0^-, (\theta^-)'_0, (\theta^-)''_0$. To this end we observe that by successive differentiation of (6.9) with respect to t and evaluating the results at $t = 0$, we obtain

$$\nu = \frac{\theta_0^-}{\theta_0^+}, \tag{6.10}$$

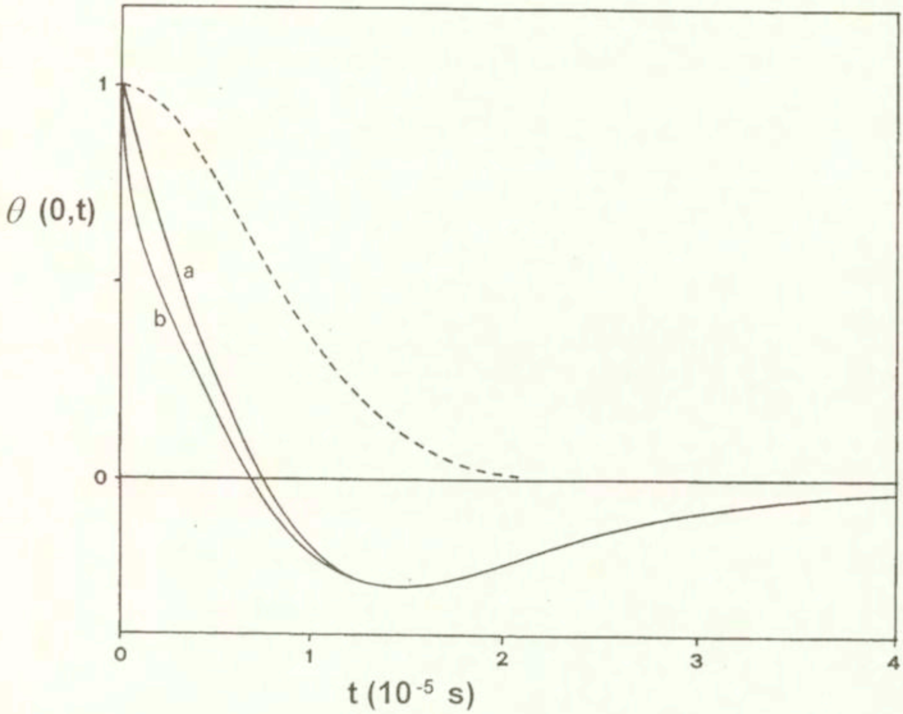

FIG. 2. Incident (dashed) and reflected (solid) temperature pulses for $\sigma = 0$ ($\nu = 1$). A half-Gaussian profile $\theta^+(0,t) = e^{-bt^2}$ is assumed with $b = 10^{10}\mathrm{s}^{-2}, \tau = 10^{-5}\mathrm{s}, \tau_1 = 5 \cdot 10^{-6}\mathrm{s}$ and (a)$\tau_2 = 5 \cdot 10^{-6}\mathrm{s}$, (b)$\tau_2 = 10^{-6}$ s.

(6.11)
$$\widetilde{R}(0) = \frac{1}{(\theta_0^+)^2}[\theta_0^+(\theta^-)_0' - \theta_0^-(\theta^+)_0'],$$
$$\widetilde{R}'(0) = \frac{1}{(\theta_0^+)^3}\{\theta_0^+[\theta_0^+(\theta^-)_0'' - \theta_0^-(\theta^+)_0''] + (\theta^+)_0'[\theta_0^-(\theta^+)_0' - \theta_0^+(\theta^-)_0']\}.$$

From Eqs. (6.10) and (5.11) we obtain the value of the parameter σ. Concerning the quantities $\widetilde{R}(0), \widetilde{R}'(0)$, they can be written in terms of the relaxation times τ_1 and τ_2. In fact, evaluating L_1, L_2 and their derivatives for $t = 0$, by Eqs. (6.8) we have

(6.12)
$$L_1(\xi,0) = \frac{\xi}{\tau}, \qquad L_2(\xi,0) = \frac{1}{\tau}\left(1-\frac{\tau}{\tau_1}\right)\left(1-\frac{\tau}{\tau_2}\right)\xi,$$
$$L_1'(\xi,0) = -\frac{\xi^2}{2\tau^2}, \qquad L_2'(\xi,0) = -\left(1-\frac{\tau}{\tau_1}\right)\left(1-\frac{\tau}{\tau_2}\right)$$
$$\left[\frac{\xi}{\tau_1\tau_2} + \frac{\xi^2}{2\tau^2}\left(1-\frac{\tau}{\tau_1}\right)\left(1-\frac{\tau}{\tau_2}\right)\right].$$

Substitution of (6.12) into (6.7) yields

$$(6.13)\qquad \begin{aligned} \widetilde{R}(0) &= -\frac{2\nu_1}{\tau} + \nu_1\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} - \frac{\tau}{\tau_1\tau_2}\right), \\ \widetilde{R}'(0) &= -\frac{\nu_1}{\tau_1\tau_2}\left[1 - \tau\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} - \frac{\tau}{\tau_1\tau_2}\right)\right] - \frac{\nu_2}{2}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} - \frac{\tau}{\tau_1\tau_2}\right)^2, \end{aligned}$$

where

$$\nu_1 = \int_0^\infty N(\xi)\xi\,\mathrm{d}\xi = \frac{\sigma^2+\sigma+1}{(\sigma+1)^2},$$

$$\nu_2 = \int_0^\infty N(\xi)\xi^2\,\mathrm{d}\xi = \frac{4\sigma^3+11\sigma^2+9\sigma+4}{2(\sigma+1)^3}.$$

Solving the system (6.13) for τ_1 and τ_2 we explicitly obtain

$$(6.14)\qquad \frac{1}{\tau_1} = a - \sqrt{a^2-4b}, \qquad \frac{1}{\tau_2} = a + \sqrt{a^2-4b},$$

with

$$(6.15)\qquad \begin{aligned} a &= \frac{\tau}{\nu_1 + \widetilde{R}(0)\tau}\left[\widetilde{R}'(0) + \frac{\nu_2}{2}\left(\frac{\widetilde{R}(0)}{\nu_1} + \frac{2}{\tau}\right)^2\right] + \frac{2}{\tau} + \frac{\widetilde{R}(0)}{\nu_1}, \\ b &= \frac{1}{\nu_1 + \widetilde{R}(0)\tau}\left[\widetilde{R}'(0) + \frac{\nu_2}{2}\left(\frac{\widetilde{R}(0)}{\nu_1} + \frac{2}{\tau}\right)^2\right]. \end{aligned}$$

Owing to (6.14), (6.15) and (6.11), the relaxation times turn out to be uniquely determined by the reflection data.

7. Conclusion

In the first part of this paper we have shown that, within the linear theory of heat conduction, a phenomenological model accounting for two relaxation times yields a hyperbolic heat equation which can be effective in describing the observed phenomena on the propagation of heat pulses. In the second part of the paper we have applied the wave splitting analysis to our phenomenological model. The solution of the propagation problem has been written in terms of the

Laplace transform of a propagator kernel. An inverse transform is explicitly obtained in the case of a low temperature conductor under the "window condition" which allows for second sound propagation. The hyperbolicity of the governing system leads to the natural question on the reflectivity of heat pulses. Beside its intrinsic intersest, we have considered the solution to this problem as a means to determine the characteristic relaxation times of a given rigid conductor when the reflection data on its boundary are available. In particular we have assumed that the incoming wave propagates in a highly pure crystal and impinges on the boundary of a unknown conducting specimen. However, the same procedure can be performed for any case in which the relaxation times of one of the two conductors are known.

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