

# Laminar dispersed two-phase flows at low concentration – I. Generalised system of equations

J. L. ACHARD and A. CARTELLIER

*Laboratoire des Écoulements Géophysiques et Industriels  
CNRS-UJF-INPG, B.P. 53X, 38041 Grenoble Cedex, France  
e-mail: Jean-Luc.Achard@hmg.inpg.fr*

TO REPRESENT MULTIDIMENSIONAL FLOWS of particle-fluid mixtures, Eulerian two-fluid models are currently used nowadays. Even if there are no pure turbulence effects in the carrier phase flow, many closure laws are required in order to supplement the conservation equations. To furnish a systematic method for deriving such closure laws that are valid at least for spherical solid inclusions, a generalised system of equations is proposed in this first paper (hereafter referred to as part I) of a sequence. It is based on the coupling of two sets of equations, one for each phase: the continuous phase is represented by an extension of a hierarchy of equations proposed by Lundgren for treating flows in porous media, and the dispersed phase by an adaptation of the well-known B.B.G.K.Y. hierarchy. The first-order equations of both hierarchies correspond to the conservation equations of standard two-fluid models; they contain the usual unknown terms. In our approach these terms appear to be provided by the second-order equations. Unfortunately, as is usual in other similar statistical theories, the second-order equations contain extra unknown terms which figure in the third-order equations and so on. Formulating closure equations is replaced by the broader problem of truncating the generalised system of equations via a perturbation method based on diluteness.

## 1. Introduction

IN ORDER TO OBTAIN multidimensional equations which relate the macroscopic phase properties of two-phase flows, several averaging processes have been proposed by (among others) DELHAYE and ACHARD [6]. They consist in applying time or space-averaging operators to a given system composed of local instant field equations that are valid each phase, supplemented by jump conditions at the interface. As it has been pointed out (HINCH [9]), ensemble-averaging operators applied over a set of “macroscopically equivalent” realisations have to be introduced in preference to other averaging processes. However, all these processes lead to formally identical mass, momentum and energy conservation equations. These provide a rational basis for many models used in engineering and especially for the most sophisticated ones, i.e., two-fluid and multidimensional models, which are starting to be used widely for numerical simulation purposes.

A consistent model for any two-phase flow phenomenon must therefore be based upon the above equations; furthermore it must include appropriate closure laws (improperly referred to as constitutive equations, or rheological relationships). In fact, these laws constitute the essential part of a given model since they express, *inter alia*, how the fluids are coupled. There is still a lack of knowledge about these closure laws, the derivation of which generally appears in a separate second step, as intuitive grafts upon well-established conservation equations. Progress may be made by settling for something less than complete generality. Indeed, even if we restrict our interest, as in this paper, to dispersed two-phase flows (i) carrying spherical inclusions (ii) which are small compared to the length scale of the averaged fields, (iii) are composed of incompressible and Newtonian fluids, (iv) where no heat transfer and no phase change occur, (v) under negligible interparticle collision conditions (vi) which remain "laminar" at all times, with possibly significant inertia effects, averaged equations nevertheless require many laws, the rational derivation and indeed the validity of which are not guaranteed.

The most promising way to improve our ability to model dispersed two-phase flows lies in an approach which mixes kinetic theory concepts and classical continuum mechanics. VAN and WIJNGAARDEN [16] was one of the first authors to introduce an equation for the bubble number density in order to handle variations in their radius. Many of these studies proceed intuitively rather than rigorously (ACHARD [1]), even if the modelling of pure turbulence effects is left aside. Fortunately, exceptions exist (BIESHEUVEL VAN and WIJNGAARDEN [2]) and among these, two series of careful studies attempt to be as systematic as possible as we shall. They were proposed by ZHANG and PROSPERETTI [17, 19] and by KOCH and his collaborators (KUMARAN and KOCH [13]; KOCH [12]). In order to represent the dispersed phases, both authors start from a sort of Liouville equation (named  $N$ -particle Smoluchowski equation by the latter and equation for the number of realisations by the former). The continuous phase is governed by creeping flow equations for the latter and by potential flow equations for the former. Due to their general formalism, several new results were obtained in their specific applications.

In the forthcoming sequence of papers, a new method pertaining to this school of thought is presented in detail; emphasis will be put on methodology. In the first place, our work will concentrate on extending and then connecting in a consistent way two hierarchies of equations existing in the literature: the well-known B.B.G.K.Y. hierarchy for the dispersed phase (GRAD [8]) and the Lundgren hierarchy for the continuous phase (LUNDGREN [15]), which was introduced initially to treat flows in porous media. Its essential feature is to produce, as a rule, all the standard conservation equations and most of the required constitutive equations at the same time, and in an unified manner.

The rest of the paper is organised as follows. The class of two-phase flows to which our theory applies will be defined in Sec. 2. In Sec. 3, we present the basic statistical tools and the main averaged variables which will be used. Most of them are conditional upon the presence of several inclusions and require specific equations which are given precisely by the revisited B.B.G.K.Y. and Lundgren hierarchies. Complete derivation of these hierarchies is a long and difficult process which will be discussed in the following two Secs. 4 and 5. The main limitations of our theory are recalled in Sec. 6 while further developments are briefly outlined.

## 2. Deterministic equations

### 2.1. Inclusion equations

The first step here is to see how far the usual starting particle equations in Statistical Mechanics (S.M.) differ from the starting inclusion equations in our method. To begin with, there is a striking difference: both phases will be modelled by equations provided by Continuum Mechanics and any molecular effects will be neglected. With regard to the continuous phase, the usual field equations are inevitably used. These consist of conservation equations (mass, linear and angular momentum and energy) supplemented by constitutive and state laws appropriate to the continuous phase material. All these equations obviously have no counterpart in S.M. With regard to the dispersed phase, we have to adopt a lumped formulation for the thermomechanical fields inside a typical inclusion. Such a lumped model is essential in order to deal with a finite number of overall particle characteristics whose values at time  $t$  correspond to  $n$  generalised coordinates (g.c.), written for short,  $\mathbf{z} = [z^1, z^2, \dots, z^n]$ . Among these, we may distinguish an external part giving the location of an inclusion centre  $\mathbf{x}$  and its unit orientation vectors, from an internal part  $\xi$  giving its intrinsic condition and specifying for instance its linear and angular velocities, its size, its temperature or even certain spherical modes in the case of deformable inclusions. To obtain such a formulation, if it exists, it is necessary to make certain approximations that exploit the physical peculiarities of the micro problem under consideration, and among others the smallness of the inclusion.

Turning now to the simplified case treated in this paper, where the inclusions are identical, spherical, rigid, with a radius  $a$ : they may represent solid particles, highly viscous droplets or small bubbles with surfactants; the equations of rigid body motion will be applied. Collisions will also be excluded. For both phases gravity may act and affect the motion of the mixture. The physical 3-D domain occupied by the mixture is denoted by  $\mathcal{V}_x^c$ . The boundaries  $\partial\mathcal{V}^c$  of  $\mathcal{V}_x^c$  may consist of rigid walls  $\partial\mathcal{V}_w^c$  as well as fluid surfaces  $\partial\mathcal{V}_f^c$  through which inclusions

(and carrying fluid) are injected or removed at a prescribed rate. Note that the physical 3-D domain allowed for inclusions is not  $\mathcal{V}_x^c$  but more precisely,  $\mathcal{V}_x^d$ , the reduced domain which is obtained by excluding all positions  $\mathbf{x}$  such that  $|\mathbf{x} - \mathbf{x}_\partial \mathcal{V}_w^c| < a$  from  $\mathcal{V}_x^c$  (*impermeability* of solid walls). Its boundaries are denoted  $\partial \mathcal{V}^d$ .

Supposing provisionally distinguishable inclusions, let  $\mathbf{z}_j = (\mathbf{x}_j, \mathbf{u}_j, \boldsymbol{\omega}_j)$  ( $j = 1, 2, \dots, N$ ) the motion of the  $j$ th inclusion in its generalized own space  $\mu_{z_j}$  ( $n = 9$ ) where  $\mathbf{x}_j$  or  $x_j^i$  ( $i = 1, 2, 3$ ) denotes its position,  $\mathbf{u}_j$  or  $u_j^i$  its translational velocity and  $\boldsymbol{\omega}_j$  or  $\omega_j^i$  its angular velocity. The density  $\rho^d$  within each inclusion is homogeneous so that the mass of the  $j$ th inclusion is  $m = 4\pi\rho^d a^3/3$  and its moment of inertia is  $I = 2ma^2/5$ . Such a motion  $\mathbf{z}_j(t)$  obeys the following equations:

$$(2.1) \quad \frac{d\mathbf{x}_j(t)}{dt} = \mathbf{u}_j(t),$$

$$(2.2) \quad m \frac{d\mathbf{u}_j(t)}{dt} = \mathbf{F}_j(t) + \mathbf{F}_j^e(t),$$

$$(2.3) \quad I \frac{d\boldsymbol{\omega}_j(t)}{dt} = \mathbf{K}_j(t),$$

where  $\mathbf{F}_j$ ,  $\mathbf{K}_j$  and  $\mathbf{F}_j^e$  are, respectively, the force and torque exerted by the fluid on inclusion  $j$ , and the external force acting on this inclusion. No external couple acts on it. Here the boundary terms are:

$$\mathbf{F}_j = \int_{S(\mathbf{x}_j)} \mathbf{n}_j^d \cdot \mathbb{T}^c dS \quad \text{and} \quad \mathbf{K}_j = \int_{S(\mathbf{x}_j)} (\mathbf{x} - \mathbf{x}_j(t)) \wedge \mathbf{n}_j^d \cdot \mathbb{T}^c dS,$$

where  $S(\mathbf{x}_j)$  is the surface of the inclusion centered at  $\mathbf{x}_j(t)$  and  $\mathbf{n}_j^d$  a unit normal to it, exterior to phase  $d$ . For the sake of simplicity the superscript  $d$  will usually be omitted and this unit vector will be defined as  $\mathbf{n}_j = (\mathbf{x} - \mathbf{x}_j(t))/a$ .

Now, consider the whole population of inclusions. Its state, at any time  $t$ , is specified by the location of one point, called a phase point, or a *configuration* of the system  $Z_N(t) = [\mathbf{z}_1(t), \mathbf{z}_2(t), \dots, \mathbf{z}_N(t)]$ . The corresponding Euclidean space of  $n\mathbf{N}$  dimensions is the phase space, denoted by  $\Gamma$ ; it is equal to the Cartesian product of all  $\mu_{z_i}$ . Of course the particle coordinates are arbitrary within each domain, except for the restrictions  $|\mathbf{x}_i - \mathbf{x}_j| > 2a$  for any couple of  $(i, j)$  having distinct values (*non-overlapping* of nondeformable particles).

The continuous fluid is Newtonian and therefore the stress tensor  $\mathbb{T}^c$  is defined by  $\mathbb{T}^c = -p^c \mathbb{I} + 2\mu^c \mathbb{D}(\mathbf{v}^c)$ , where  $\mathbb{D}(\mathbf{v}^c)$ , the rate of deformation tensor, is the

symmetric part of the velocity gradient  $\mathbb{D}() = \left[ \frac{\partial}{\partial \mathbf{x}} () \right]^s$ . Of course  $\mathbf{F}_j^e = m\mathbf{g}$  as the body force is just gravity. The fluid velocity field satisfies the classical equation

$$(2.4) \quad \mathbf{v}^c(\mathbf{x}, t) = \mathbf{v}_j^d(\mathbf{x}, t) \quad \text{for } |\mathbf{x} - \mathbf{x}_j(t)| = a$$

at the surface of the inclusions and  $\mathbf{v}_j^d$  is the velocity field inside the  $j$ th inclusion. Each inclusion is subject to a rigid motion, thus:

$$(2.5) \quad \mathbf{v}_j^d(\mathbf{x}, t) = \mathbf{u}_j(t) + \boldsymbol{\omega}_j(t) \wedge [\mathbf{x} - \mathbf{x}_j(t)] \quad \text{for } |\mathbf{x} - \mathbf{x}_j(t)| \leq a.$$

It will be assumed that collisions between inclusions are unlikely to occur or are soft enough not to give rise to significant pressure impulses. Taking into account collisions would mean supplementing motion Eqs. (2.1), (2.2) and (2.3) with jump conditions relating translational and angular velocities before and after each interaction.

Several fine-grained functions can be defined describing the complete structure of the dispersed phase in  $\mathbf{V}_x^d$ . These are the dispersed phase indicator (or structural, or characteristic) function  $X^d$  equalling unity inside the inclusion and zero otherwise, and the velocity field of the dispersed phase. In the case under consideration, they are:

$$(2.6) \quad X^d(Z_N; \mathbf{x}) = \sum_{j=1}^N H(a - |\mathbf{x} - \mathbf{x}_j|) = 1 - X^c,$$

$$(2.7) \quad \begin{aligned} X^d(Z_N; \mathbf{x})\mathbf{v}^d(Z_N; \mathbf{x}) &= \sum_{j=1}^N H(a - |\mathbf{x} - \mathbf{x}_j|)\mathbf{v}_j^d(\mathbf{x}, t) \\ &= \sum_{j=1}^N H(a - |\mathbf{x} - \mathbf{x}_j|)(\mathbf{u}_j + \omega_j \wedge (\mathbf{x} - \mathbf{x}_j)), \end{aligned}$$

where  $H$  is the Heaviside step function defined as zero when its argument is negative and one otherwise.

## 2.2. Ambient fluid equations

Formulating local instant field equations in the sense of generalised functions (g.f.) before averaging them, offers several advantages which are well-known today in two-phase flows as well as in intermittent turbulent flows after the original LEWIS thesis [14] and DOPAZO paper [7], respectively. First, time and space differential operators in the sense of g.f. commute with averaging operators;

second, applying averaging operators to equations in their ordinary sense requires a different set of calculations concerning the interfacial source terms, whereas equations in the sense of g.f. require a single step operation; finally, these source terms have a mathematical form which will help in finding the closure models. The first step in obtaining continuous phase equations in sense of g.f. is to make these equations valid throughout space-time  $\mathbb{R}^3 \times \mathbb{R}$ , i.e. even outside the mixture domain  $\mathbf{V}_x^c$  and inside each inclusion where  $|\mathbf{x} - \mathbf{x}_j| < a$ . The velocity  $\mathbf{v}^c(\mathbf{x}, t)$  and the pressure  $p^c(\mathbf{x}, t)$  which enter the Navier-Stokes and continuity equations are thus extended outside phase  $c$ ; the precise value of these extended fields is irrelevant with respect to the fields  $X^c \mathbf{v}^c(\mathbf{x}, t)$  and  $X^c p^c(\mathbf{x}, t)$ , which vanish inside the inclusions. Thus,  $X^c \mathbf{v}^c(\mathbf{x}, t)$  and  $X^c p^c(\mathbf{x}, t)$  will be treated as regular g.f. We shall also introduce a singular g.f., the surface Dirac g.f.  $\delta_\Sigma$ , which restricts a volume integration to one over the surface  $\Sigma (\Sigma = \cup S(\mathbf{x}_j), j = 1, \dots, N.)$ , representing the interface. Collecting all inclusion effects, we indeed

find  $\delta_\Sigma = \sum_{j=1}^N \delta(P_j) |\text{grad } P_j|$ . An obvious choice for  $P_j$  is the radial coordinate of

a spherical co-ordinate system having its origin at the centre of the  $j$ th sphere i.e.  $P_j(\mathbf{x}, t) = a - |\mathbf{x} - \mathbf{x}_j(t)| = a - r_j$  for which we observe  $|\text{grad } P_j| = 1$ .

The following extended equations are then valid for each member of the ensemble or realisations:

$$(2.8) \quad \left\{ X^c \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v}^c \right\} = 0 \quad \text{and} \quad \left\{ X^c \frac{\partial}{\partial t} \mathbf{v}^c \right\} + \left\{ X^c \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v}^c \mathbf{v}^c \right\} \\ - \left\{ X^c \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{T}^c \right\} / \rho^c - \{X^c \mathbf{g}\} = 0,$$

where  $\mathbb{T}^c$  denotes the stress tensor defined above and  $\mathbf{g}$  the body force density;  $\{f\}$  is the regular g.f. associated with the usual function  $f$  which here is piecewise continuous. The above equations are coupled by the non-slip condition (2.4) at the interface  $S(\mathbf{x}_j)$  of each particle.

Several formulas are available (BOUIX [3]) for transforming each g.f. associated with derivatives in the usual sense of functions (i.e.  $\{X^c \partial \mathbf{v}^c / \partial t\}$  or equivalently  $\{\partial(X^c \mathbf{v}^c) / \partial t\}$ ) into derivatives in the sense of g.f. of the same functions (i.e.  $\partial X^c \mathbf{v}^c / \partial t$ ). They are basically similar to integration by parts formulas. The continuity equation in the g.f. sense becomes:

$$(2.9) \quad \frac{\partial}{\partial t} X^c + \frac{\partial}{\partial \mathbf{x}} \cdot (X^c \mathbf{v}^c) = 0.$$

Note that a similar continuity equation for the dispersed phase can be obtained by replacing the superscript  $c$  by  $d$ . On the other hand, the momentum equation

calls for the following relation to be satisfied:

$$(2.10) \quad \left\{ X^c \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{T}^c \right\} = \frac{\partial}{\partial \mathbf{x}} \cdot \{ X^c \mathbb{T}^c \} + \mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma$$

in which the divergence argument can be written in turn:

$$(2.11) \quad \{ X^c \mathbb{T}^c \} = -X^c p^c \mathbb{I} + 2\mu^c \mathbb{D}(X^c \mathbf{v}^c) + 2\mu^c \mathbb{F}^c \delta_\Sigma.$$

Taking the divergence of (2.11):

$$(2.12) \quad \frac{\partial}{\partial \mathbf{x}} \cdot \{ X^c \mathbb{T}^c \} = -\frac{\partial}{\partial \mathbf{x}} (X^c p^c) + \mu^c \Delta (X^c \mathbf{v}^c) \\ + \mu^c \frac{\partial}{\partial \mathbf{x}} \left[ \frac{\partial}{\partial \mathbf{x}} \cdot (X^c \mathbf{v}^c) \right] + 2\mu^c \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbb{F}^c \delta_\Sigma),$$

where  $\Delta$  represents the Laplace operator; the singular g.f.  $\mathbb{F}^c \delta_\Sigma = [\mathbf{v}^c \mathbf{n}^c]^s \delta_\Sigma$  is the fine-grained extra deformation tensor, and the singular g.f.  $\mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma = (-\mathbf{n}^c p^c + 2\mu^c \mathbf{n}^c \cdot \mathbb{D}(\mathbf{v}^c)) \delta_\Sigma$  is the fine grained interfacial stress exerted by the continuous phase upon the inclusions;  $\mathbf{n}^c (= -\mathbf{n}^d = -\mathbf{n})$  is the unit normal exterior to phase  $c$ . Another expression of  $\{ X^c \mathbb{T}^c \}$  is proposed by JOSEPH and LUNDGREN [10], extending  $\mathbf{v}^c$  and  $p^c$  inside the inclusions: the pressure is made to vanish and the velocity is assumed to be given by (2.5); they obtain:

$$(2.13) \quad \{ X^c \mathbb{T}^c \} = -X^c p^c \mathbb{I} + 2\mu^c \mathbb{D}(X^c \mathbf{v}^c + X^d \mathbf{v}^d).$$

Comparing (2.11) and (2.13) gives  $\mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma = \mathbb{D}(X^d \mathbf{v}^d)$ . The momentum equations in the g.f. sense, where  $\nu^c = \mu^c / \rho^c$ , can be easily obtained once (2.11) and (2.13) are known. The former expression yields:

$$(2.14) \quad \frac{\partial X^c \mathbf{v}^c}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (X^c \mathbf{v}^c \mathbf{v}^c) = -\frac{1}{\rho^c} \frac{\partial}{\partial \mathbf{x}} (X^c p^c) + \nu^c \Delta (X^c \mathbf{v}^c) \\ + \nu^c \frac{\partial}{\partial \mathbf{x}} \left[ \frac{\partial}{\partial \mathbf{x}} \cdot (X^c \mathbf{v}^c) \right] + 2\nu^c \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbb{F}^c \delta_\Sigma) + \frac{1}{\rho^c} \mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma + X^c \mathbf{g}.$$

Strictly speaking all the above fine-grained equations are incomplete since they do not involve a singular g.f. relative to the walls, as for instance  $\mathbb{F}^c \delta_\partial \mathbf{V}^c = -[\mathbf{v}^c \mathbf{n}^b]^s \delta_\partial \mathbf{V}^c$ , where  $\mathbf{n}^b$  is the unit normal interior to  $\mathbf{V}_x^c$ . Presenting all these g.f.'s would be of no interest, since in contrast to the interfacial g.f.'s, which evolve randomly, they still remain g.f.'s after averaging. Once this process is completed, it is more convenient to rule out the g.f. approach to the equations.

At the boundary of the  $\mathcal{V}_x^c$  domain, it should simply be recalled that an equation similar to (2.4) holds, namely:

$$(2.15) \quad \mathbf{v}^c(\mathbf{x}, t) = \mathbf{v}^b(\mathbf{x}, t) \quad \text{for } \mathbf{x} \text{ on } \partial\mathcal{V}^c,$$

where  $\mathbf{v}^b$  is a prescribed velocity on the boundaries.

### 3. Probability concepts

#### 3.1. Reduced densities and observation or $\mu_z$ -space

Thus far our considerations have only been dynamic, and probability considerations have not been introduced. Once initial conditions for the above equations have been specified, the inclusions and the carrying phase behave in a deterministic fashion. However, from a practical point of view, these initial conditions are random. In our case, where the continuous phase flow is assumed to remain laminar each time (it might be laminar at the beginning and then turn turbulent, i.e. unstable, by the very presence of the inclusions), one is tempted to consider the hydrodynamic fields as continuous functionals of the dispersed phase initial data alone; under this assumption, any field (say  $\mathbf{v}^c(s; \mathbf{x}, 0)$ ,  $s$  representing a sample point) would not be an independent random variable but rather it might be introduced as an implicit and regular functional of an initial configuration  $Z_N(s; 0)$ , which would be compatible with the laws of hydrodynamics and would continue to be so at any point  $(\mathbf{x}, t)$ ; this view is incorrect in general since the fluid has its own degrees of freedom which are independent of those of the inclusions, but it is correct in the limit of potential flows or of creeping flows; it is also correct for steady initial conditions of general types of flows. For arbitrary initial conditions, the above functional may exist only if the admissible initial fields  $\mathbf{v}^c(s; \mathbf{x}, 0)$  are restricted by requiring each of them to be the result of a given configuration; thus, the probability density function (p.d.f.)  $f_N$  describing the entire system, will be allowed to include only  $Z_N$  in its arguments. Other special randomizing effects arise when Brownian motion is significant or when collisions have to be treated. They are also excluded from this study.

Thus, as in S.M., only the initial p.d.f.  $f_N(Z_N; 0)$ , given over  $\Gamma$ , will serve as the initial condition of the so-called Liouville equation whose solution is  $f_N(Z_N; t)$ ; this p.d.f. is symmetric with respect to the  $N$  inclusions (which are in fact assumed to be indistinguishable) and normalized to one. As a rule, such a p.d.f., which may provide any kind of averaged value, is too detailed a state variable. The reduced p.d.f.  $f_r$  of order  $r$  ( $r = 1, 2, \dots$ ), which gives the probability of observing the  $r$  first inclusions at the points  $\mathbf{z}_1, \dots, \mathbf{z}_r$  (an  $r$ -configuration in a reduced part of the  $\Gamma$ -space) and which are defined by integrating over the coordinates of the  $N - r$  remaining inclusions, are more suitable; they are solutions

of the B.B.G.K.Y. hierarchy defined over the various  $r$ -time Cartesian products of  $\mu_{z_i}$ .

Since no two inclusions of the same coordinates are distinguishable, it is more physically meaningful to see whether  $r$  prescribed values  $(\mathbf{z}, \mathbf{z}^\circ, \dots)$  are occupied by any set of  $r$  inclusions irrespective of their labels, instead of following an  $r$ -configuration. The  $9, 18, \dots$ - D spaces ("observation spaces"), where such *mechanical states* are observed, are denoted by  $\mu_z, \mu_z \otimes \mu_{z^\circ}, \dots$  and various dispersed phase state variables and their equations will be defined over them in Sec. 4. Due to the non-overlapping condition, some regions must in fact be cut out and the resulting spaces are then denoted  $\mu_{zz^\circ}, \mu_{zz^\circ z^{\circ\circ}} \dots$ . Such a notation change will be extended to "restricted" Cartesian products of ordinary physical spaces. Over  $\mu_{zz^\circ z^{\circ\circ}} \dots$  we introduce  $f^{(r)}(\mathbf{z}, \mathbf{z}^\circ, \mathbf{z}^{\circ\circ}, \dots)$ , the mean number of mechanical states per unit volume, which is equal to  $A_N^r f_r(\mathbf{z}, \mathbf{z}^\circ, \mathbf{z}^{\circ\circ}, \dots)$ , where  $A_N^r = N! / (N - r)!$  is the number of configurations leaving a mechanical state unchanged.

The basic tool of this approach is the "fine-grained density" defined by:

$$f_i(s; \mathbf{z}, t) = \delta\{\mathbf{z} - \mathbf{z}_i(s, t)\} = \delta\{\mathbf{x} - \mathbf{x}_i(s, t)\} \delta\{\mathbf{u} - \mathbf{u}_i(s, t)\} \delta\{\boldsymbol{\omega} - \boldsymbol{\omega}_i(s, t)\}.$$

Such an irregular field of events allows a fine-grained number density  $n(s; \mathbf{z}, t) = \sum f_i(s; \mathbf{z}, t)$  to be defined, including the contributions of all inclusions. Defining  $f^{(1)}$  as an average number density in the  $\mu_z$ -space, we can write  $f^{(1)} = E[n] = NE[f_1] = Nf_1$ . The mean values of the products  $n(\mathbf{z})[n(\mathbf{z}^\circ) - \delta(\mathbf{z} - \mathbf{z}^\circ)]$ ,  $n(\mathbf{z})[n(\mathbf{z}^{\circ\circ}) - \delta(\mathbf{z} - \mathbf{z}^{\circ\circ}) - \delta(\mathbf{z}^\circ - \mathbf{z}^{\circ\circ})] \dots$ , can be connected with higher-order distribution functions,  $f^{(2)}, f^{(3)} \dots$

### 3.2. General averaging formulas

Introducing the above fine-gradient density indeed offers one way of defining  $f_r$  or  $f^{(r)}$  without using  $f_N$ . As it is possible to derive equations for various fine-grained densities,  $f_1, f_1 f_2^\circ, f_1 f_2^\circ f_3^{\circ\circ} \dots$  (for brevity  $f_j^\circ = f_j(\mathbf{z}^\circ) \dots$ ) then, by averaging the latter, the first members of the B.B.G.K.Y. hierarchy can be obtained as far as desired in the observation spaces (KLIMONTOVICH [11]): the Liouville equation is by-passed. For the theory developed in this paper, the B.B.G.K.Y. hierarchy will be stopped at the second order equation. In the averaging process, dispersed phase fields  $\psi(\mathbf{z})$  appear as  $\left(\sum_i f_i\right) \psi(\mathbf{z}) =$

$$\sum_i f_i \psi(\mathbf{z}_i); \left(\sum_i \sum_{i \neq j} f_i f_j^\circ\right) \psi \zeta^\circ(\mathbf{z}, \mathbf{z}^\circ) = \sum_i \sum_{i \neq j} f_i f_j^\circ \psi(\mathbf{z}_i) \zeta(\mathbf{z}_j).$$

So, besides

$f_1(\mathbf{z}), f_2(\mathbf{z}, \mathbf{z}^\circ), \dots$  the first dispersed phase-averaged variables in the  $\mu_z$  and  $\mu_{zz^\circ}$  - spaces are:

$$(3.1) \quad \overline{\psi^1}(\mathbf{z}) = \sum_i E[f_i \psi(\mathbf{z}_i)] / \sum_i E[f_i] = E[f_1 \psi(\mathbf{z}_1)] / f_1,$$

$$(3.2) \quad \overline{\psi \zeta^{\circ 2}}(\mathbf{z}, \mathbf{z}^\circ) = \sum_i \sum_{i \neq j} E[f_i f_j^\circ \psi(\mathbf{z}_i) \zeta(\mathbf{z}_j)] / \sum_i \sum_{i \neq j} E[f_i f_j^\circ] \\ = E[f_1 f_2^\circ \psi(\mathbf{z}_1) \zeta(\mathbf{z}_2)] / f_2.$$

In most cases, averaged variables come out as straightforward explicit functions of  $\mathbf{z}$  and  $\mathbf{z}^\circ \dots$  and do not depend on time, i.e.  $\mathbf{u}_1 \omega_1$  and  $\mathbf{u}_1 \mathbf{u}_2$  become simply  $\mathbf{u} \omega$  and  $\mathbf{u} \mathbf{u}^\circ$ .

The next step in our approach will be to switch from observation spaces to ordinary 3-space domains or Cartesian products of them. Transforming each member of the B.B.G.K.Y. hierarchy into moment equations simplifies the dispersed phase dynamics formulation since the number of independent variables is considerably reduced. First moments equations are obtained by averaging the  $f_1$  equation over internal coordinates in  $\mu_z$ -space, higher order moments are introduced by averaging the  $f_2$  equation and so on. The most common moments are  $\phi^{(1)}$ , the (average) number density defined in  $\mathbf{V}_x^d$  and  $\phi^{(2)}$ , the pair number density defined in the pair physical space  $\mathbf{V}_{xx^\circ}^d$  (see above the definition of  $\mu_{zz^\circ}$  which is similar)

$$(3.3) \quad \phi^{(1)}(\mathbf{x}, t) = E \left[ \sum_i \varphi_i \right] = N E[\varphi_1] = N \phi_1(\mathbf{x}, t), \\ \phi^{(2)}(\mathbf{x}, \mathbf{x}^\circ, t) = E \left[ \sum_j \sum_{i \neq j} \varphi_i \varphi_j \right] = N(N-1) E[\varphi_1 \varphi_2] \\ = N(N-1) \phi_2(\mathbf{x}, \mathbf{x}^\circ, t),$$

where the fine-grained density  $\varphi_i(\mathbf{x}, t) = \delta\{\mathbf{x} - \mathbf{x}_i(s, t)\}$  has been introduced. Here and below, conditional averaged values of  $\psi$ , are denoted by an overscale; italic superscripts indicate the number of points occupied by inclusions in observation spaces while standard superscripts indicate the number of observation points in the physical spaces. Other moments of any order can be considered as inner products over various observation spaces. Let  $\langle f_i \rangle$  be such an  $i$ th order moment ( $i = 1, 2, \dots$ ) obtained by multiplying by  $f_i$  the quantity appearing to the left of

the semicolon. For instance, the first-order and second-order moments can be written:

$$(3.4) \quad \bar{\psi}^1(\mathbf{x}, t) = \sum_i E[\varphi_i \psi(\mathbf{x}_i)] / \sum_i E[\varphi_i] = \int \bar{\psi}^1 f_1 d\mathbf{u}d\boldsymbol{\omega} / \phi_1 = \langle \bar{\psi}^1; f_1 \rangle / \phi_1,$$

$$(3.5) \quad \begin{aligned} \overline{\psi \zeta^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ}, t) &= \sum_i \sum_{i \neq j} E[\varphi_i \varphi_j^{\circ} \psi(\mathbf{x}_i) \zeta(\mathbf{x}_j)] / \sum_i \sum_{i \neq j} E[\varphi_i \varphi_j^{\circ}] \\ &= \int \overline{\psi \zeta^{\circ 2}} f_2 d\mathbf{u}d\boldsymbol{\omega}d\mathbf{u}^{\circ}d\boldsymbol{\omega}^{\circ} / \phi_2 = \langle \overline{\psi \zeta^{\circ 2}}; f_2 \rangle / \phi_2, \end{aligned}$$

after having specified  $d\xi = d\mathbf{u}d\boldsymbol{\omega}$ . The most common averages are cross-correlations between inclusion velocities observed at two positions such as  $\overline{\mathbf{u}\mathbf{u}^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ})$ ,  $\overline{\boldsymbol{\omega}\mathbf{u}^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ})$  or  $\overline{\mathbf{u}\boldsymbol{\omega}^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ})$ . Given two observation locations  $\mathbf{x}$  and  $\mathbf{x}^{\circ}$ , each being occupied by an inclusion,  $\overline{\mathbf{u}^2}(\mathbf{x}, \mathbf{x}^{\circ})$  (respectively  $\overline{\mathbf{u}^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ})$ ) is the averaged velocity computed at  $\mathbf{x}$  (respectively at  $\mathbf{x}^{\circ}$ ). For the first case, where  $\zeta^{\circ} = 1$ ,  $\overline{\mathbf{u}^2}(\mathbf{x}, \mathbf{x}^{\circ})$  can be interpreted as the averaged velocity at  $\mathbf{x}$ , conditional upon the presence of an another inclusion at  $\mathbf{x}^{\circ}$ , this leads to the possible notation  $\overline{\mathbf{u}^2}(\mathbf{x}, \mathbf{x}^{\circ}) = \overline{\mathbf{u}^2}(\mathbf{x}|\mathbf{x}^{\circ})$ . Similarly we can write  $\overline{\mathbf{u}^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ}) = \overline{\mathbf{u}^{\circ 2}}(\mathbf{x}^{\circ}|\mathbf{x})$ . The correlations  $\overline{\mathbf{u}\mathbf{u}^2}(\mathbf{x}|\mathbf{x}^{\circ})$  and  $\overline{\boldsymbol{\omega}\mathbf{u}^2}(\mathbf{x}, \mathbf{x}^{\circ})$  are also averages of the same type. In contrast to the tensor  $\overline{\mathbf{u}\mathbf{u}^{\circ 2}}(\mathbf{x}, \mathbf{x}^{\circ})$ , the tensor  $\overline{\mathbf{u}\mathbf{u}^2}(\mathbf{x}|\mathbf{x}^{\circ})$  is not symmetrical with respect to  $\mathbf{x}$  and  $\mathbf{x}^{\circ}$ . In the following, when the arguments are not explicitly stated for the sake of simplicity, the superscript "o" (respectively no superscript) over a function, such as  $\overline{\mathbf{u}^{\circ 2}}$  (respectively  $\overline{\mathbf{u}^2}$ ) will indicate that its first spatial argument is  $\mathbf{x}^{\circ}$  (respectively  $\mathbf{x}$ ), the condition being specified at  $\mathbf{x}$  (respectively  $\mathbf{x}^{\circ}$ ).

On the other hand, continuous phase fine-grained variables, which appear as  $X^c \psi^c$  are only defined in the physical domain  $\mathbf{V}_x^c$  occupied by the mixture; they can also be averaged subject to various constraints. The event "there is the continuous phase at  $\mathbf{x}$ ", i.e.  $X^c = 1$ , has a finite probability equal to the average of  $X^c$ , the local continuous phase volume fraction or concentration  $\alpha^{c1}(\mathbf{x}) = E[X^c]$ . The simplest conditional average corresponding to this constraint is the standard conditional phase average of  $\psi^c$ , which has the notation  $\overline{\psi^c}^{c1}(\mathbf{x}) = E[X^c \psi^c] / \alpha^{c1}(\mathbf{x})$  so that conditional averaged values of  $\psi^c$  are also designated by an overscore. We will also need the conditioned concentration:

$$(3.6) \quad \begin{aligned} \alpha^{c2}(\mathbf{x}^{\circ}, t|\mathbf{x}) &= \sum_i E[X_i^c(\mathbf{x}^{\circ}, t)] / \phi^{(1)}(\mathbf{x}, t) \\ &= E[X_1^c(\mathbf{x}^{\circ}, t) \varphi_1(\mathbf{x}, t)] / \phi_1(\mathbf{x}, t) \end{aligned}$$

where  $X_i^c$  is a partial indicator function in which the  $i$ th inclusion is excluded

from  $X^c$  (see (2.6)); the second-order average is

$$(3.7) \quad \overline{\psi^{c2}}(\mathbf{x}^\circ|\mathbf{x}) = E[X_1^c \psi^c(\mathbf{x}^\circ) \phi_1(\mathbf{x})] / \alpha^{c2}(\mathbf{x}^\circ|\mathbf{x}) \phi_1(\mathbf{x}).$$

The overscore bears an index indicating the number of constraints, e.g.  $X^c = 1$ , plus the number of neighbouring points occupied by inclusions. All the above conditioned variables (3.6) and (3.7) are defined over a region denoted by  $\mathbf{V}_{x,x^\circ}^c$  which is  $\mathbf{V}_{x^\circ}^c \otimes \mathbf{V}_x^d$ , from which some inaccessible region due to the finite inclusions size (i.e.  $|\mathbf{x}^\circ - \mathbf{x}| < 2a$ ) in the carrying flow has been removed. Likewise, averages such as  $\overline{\psi^{c3}}(\mathbf{x}^{\circ\circ}|\mathbf{x}, \mathbf{x}^\circ)$  will be introduced at the third order, i.e. for two fixed inclusions. Defining them means introducing an additional partial indicator function  $X_{i,j}^c$  in which the  $i$ th and  $j$ th inclusions are excluded from  $X^c$ . The relevant space (denoted  $\mathbf{V}_{x,x^\circ,x^{\circ\circ}}^c$ ) is  $\mathbf{V}_{x^{\circ\circ}}^c \otimes \mathbf{V}_{x^\circ,x}^d$  from which some region is removed, and so on. Finally, composite averages such as  $\alpha^{c2}(\mathbf{x}^\circ|\mathbf{z})$  and  $\overline{\psi^{c2}}(\mathbf{x}^\circ|\mathbf{z})$  will be encountered.

It is worth recalling that a dispersed phase description can also be produced paralleling the description we have just given for continuous phase modelling. It would be suitable for standard two-phase flow models. In these models, dispersed phase variables as well as continuous phase ones are defined directly as moments in ordinary physical 3D-space from equations like those presented in Sec. 2.2 for the continuous phase. The averages associated with  $X^d$  (the dispersed phase volume fraction) and  $\mathbf{v}^d$  obtained from a standard averaging process for local instantaneous dispersed equations are simply:

$$(3.8) \quad E[X^d(\mathbf{x})] = \alpha^{d1}(\mathbf{x}) = 1 - \alpha^{c1}(\mathbf{x}) \quad \text{and} \quad E[X^d \mathbf{v}^d(\mathbf{x}) | X^d = 1] \\ = E[\mathbf{v}^d X^d] / \alpha^{d1} = \overline{\mathbf{v}^d}(\mathbf{x}).$$

$$(3.9) \quad E[X^d(\mathbf{x}^\circ)|\mathbf{x}_1 = \mathbf{x}] = \alpha^{d2}(\mathbf{x}^\circ|\mathbf{x}) = 1 - \alpha^{c2}(\mathbf{x}^\circ|\mathbf{x}) \quad \text{and} \\ E[X^d \mathbf{v}^d(\mathbf{x}^\circ)|\mathbf{x}_1 = \mathbf{x}, X^d = 1] = \overline{\mathbf{v}^d}(\mathbf{x}^\circ|\mathbf{x}).$$

In our approach, these standard dispersed phase variables will appear incidentally in interaction terms; they will be considered as provisional quantities which will be transformed into variables such as (3.4) and (3.5); thus the way of describing both phases will ultimately be unsymmetrical.

#### 4. Averaging process for the dispersed phase

The Klimontovich approach begins by a preparatory step which amounts to formulating local instant field equations in the sense of generalised functions (g.f.); it parallels the approach for the continuous phase in Sec. 2.2.

#### 4.1. The Klimontovich equations

To obtain the equation for  $f_1$  (CERCIGNANI [5]) in  $\mu_{\mathbf{z}}$ , we first determine the derivative with respect to time of this fine-grained function; inserting (2.1), (2.2) and (2.3), we obtain:

$$(4.1) \quad \frac{\partial f_1}{\partial t} + \mathbf{u}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}} + [m^{-1}\mathbf{F}_1 + \mathbf{g}] \cdot \frac{\partial f_1}{\partial \mathbf{u}} + I^{-1}\mathbf{K}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{\omega}} = 0.$$

It must be stressed that taking into account collisions would have introduced extra terms which would have meant generalising our analysis.  $\mathbf{F}_1$  and  $\mathbf{K}_1$  are composite quantities since they involve properties associated with both phases. On the one hand,  $\mathbf{F}_1$  and  $\mathbf{K}_1$  represent the resulting stress and torque exerted by the continuous phase upon the first inclusion with coordinates  $\mathbf{z}_1$ . On the other hand, their effects are described by two linear functionals of  $p^c[Z_N; \bar{\mathbf{x}}, t]$  and  $\mathbf{v}^c[Z_N; \bar{\mathbf{x}}, t]$ , where  $\bar{\mathbf{x}}$  is such that  $(\bar{\mathbf{x}} - \mathbf{x}_1)/a = \mathbf{n}$ . Let  $d\Omega$  be an element of solid angle on the sphere of unit radius, i.e.  $a^2 d\Omega = dS$ . The only points which contribute to the integral are thus  $\bar{\mathbf{x}} = \mathbf{x}_1 + a\mathbf{n}$  for all  $\mathbf{n}$  on this sphere:

$$(4.2) \quad \mathbf{F}_1(Z_N; t) = a^2 \int_{S(\mathbf{x}_1)} \{\mathbf{n} \cdot X^c \mathbb{T}^c\}(Z_N; \bar{\mathbf{x}}, t) d\Omega,$$

$$(4.3) \quad \mathbf{K}_1(Z_N; t) = a^3 \int_{S(\mathbf{x}_1)} \mathbf{n} \wedge \{\mathbf{n} \cdot X^c \mathbb{T}^c\}(Z_N; \bar{\mathbf{x}}, t) d\Omega,$$

$$= a^3 \mathcal{E} : \int_{S(\mathbf{x}_1)} \{\mathbf{nn} \cdot X^c \mathbb{T}^c\}(Z_N; \bar{\mathbf{x}}, t) d\Omega,$$

where  $\mathcal{E}$  is the antisymmetric alternating tensor. Equation (4.1) can be rearranged as:

$$(4.4) \quad \frac{\partial f_1}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (f_1 \mathbf{u}_1) + m^{-1} \frac{\partial}{\partial \mathbf{u}} \cdot (f_1 \mathbf{F}_1) + \frac{\partial}{\partial \mathbf{u}} \cdot (f_1 \mathbf{g})$$

$$+ I^{-1} \frac{\partial}{\partial \boldsymbol{\omega}} \cdot (f_1 \mathbf{K}_1) = 0.$$

This is the first equation of the Klimontovich hierarchy written in  $\mu_{\mathbf{z}}$ -space. Following the same line of calculations, we can obtain the equation for  $f_1 f_2$  in  $\mu_{\mathbf{z}\mathbf{z}^0}$ -space:

$$\begin{aligned}
 (4.5) \quad & d \frac{\partial f_1 f_2}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (f_1 f_2 \mathbf{u}_1) + \frac{\partial}{\partial \mathbf{x}^\circ} \cdot (f_1 f_2 \mathbf{u}_2) + m^1 \frac{\partial}{\partial \mathbf{u}} \cdot (f_1 f_2 \mathbf{F}_1) \\
 & + m^{-1} \frac{\partial}{\partial \mathbf{u}^\circ} \cdot (f_1 f_2 \mathbf{F}_2) + \frac{\partial}{\partial \mathbf{u}} \cdot (f_1 f_2 \mathbf{g}) + \frac{\partial}{\partial \mathbf{u}^\circ} \cdot (f_1 f_2 \mathbf{g}) \\
 & + I^{-1} \frac{\partial}{\partial \boldsymbol{\omega}} \cdot (f_1 f_2 \mathbf{K}_1) + I^{-1} \frac{\partial}{\partial \boldsymbol{\omega}^\circ} \cdot (f_1 f_2 \mathbf{K}_2) = 0.
 \end{aligned}$$

$\mathbf{F}_2$  and  $\mathbf{K}_2$  have the same structure as  $\mathbf{F}_1$  and  $\mathbf{K}_1$ ,  $S(\mathbf{x}_1)$  being changed into  $S(\mathbf{x}_2)$ . The procedure could have been continued to give the next product  $f_1 f_2 f_3$  and so on. Finally, the equations for the fine-grained number densities  $n(\mathbf{z})$  and  $n(\mathbf{z})[n(\mathbf{z}^\circ) - \delta(\mathbf{z} - \mathbf{z}^\circ)]$  can be obtained by combining equations for  $f_i$  and  $f_i f_j$  which are similar to (4.4) and (4.5).

#### 4.2. The revisited B.B.G.K.Y. hierarchy

The B.B.G.K.Y. hierarchy is obtained by averaging the entire sequence of Klimontovich equations. As far as g.f. are concerned, it is well known that  $E[O(g)] = O(E[g])$ ,  $O$  being any differential operator whatever. The equation for  $f_1$  is derived first.

$$(4.6) \quad \frac{\partial}{\partial t} f_1 + \frac{\partial}{\partial \mathbf{x}} \cdot f_1 \mathbf{u} + m^{-1} \frac{\partial}{\partial \mathbf{u}} \cdot f_1 \bar{\mathbf{F}}^I + \frac{\partial}{\partial \mathbf{u}} \cdot f_1 \mathbf{g} + I^{-1} \frac{\partial}{\partial \boldsymbol{\omega}} \cdot f_1 \bar{\mathbf{K}}^I = 0,$$

where the averaged force and the averaged torque exerted upon an inclusion which is known to be centred at  $\mathbf{x}$  and to have internal coordinates equal to  $\boldsymbol{\xi}$  are defined by:

$$\begin{aligned}
 (4.7) \quad \bar{\mathbf{F}}^I(\mathbf{z}) &= E[f_1 \mathbf{F}_1] / f_1 = a^2 \int_{S(\mathbf{x})} \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c2}}(\mathbf{x} + a\mathbf{n}|\mathbf{z}) d\Omega, \\
 \bar{\mathbf{K}}^I(\mathbf{z}) &= E[f_1 \mathbf{K}_1] / f_1 = a^3 \mathcal{E} : \int_{S(\mathbf{x})} \mathbf{nn} \cdot \overline{X^c \mathbb{T}^{c2}}(\mathbf{x} + a\mathbf{n}|\mathbf{z}) d\Omega.
 \end{aligned}$$

The local averaged force around the test inclusion is obtained by selecting formula (2.13) to express the fine-grained stress field:

$$\begin{aligned}
 (4.8) \quad \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c2}}(\tilde{\mathbf{x}}|\mathbf{z}) &= E[\{f_1 \mathbf{n} \cdot X_1^c \mathbb{T}^c\}] / f_1(\mathbf{z}) = -\alpha^{c2} \bar{p}^{c2} \mathbf{n} \\
 &+ 2\mu^c \mathbf{n} \cdot \mathbb{D}[\bar{\mathbf{v}}^{c2} + \alpha^{d2}(\bar{\mathbf{v}}^{d2} - \bar{\mathbf{v}}^{c2})].
 \end{aligned}$$

The same formula will be used throughout the article in any kind of averaging stress field around test inclusions. The positions of the inclusion centres which contribute to this integral are such that  $\tilde{\mathbf{x}} = \mathbf{x} + a\mathbf{n}$ .

The above averaging procedure can be extended to derive the equation for  $f_2$ :

$$(4.9) \quad \frac{\partial}{\partial t} f_2 + \frac{\partial}{\partial \mathbf{x}} \cdot f_2 \mathbf{u} + \frac{\partial}{\partial \mathbf{x}^\circ} \cdot f_2 \mathbf{u}^\circ + m^{-1} \frac{\partial}{\partial \mathbf{u}} \cdot f_2 \overline{\mathbf{F}}^2 + m^{-1} \frac{\partial}{\partial \mathbf{u}^\circ} \cdot f_2 \overline{\mathbf{F}}^{\circ 2} \\ + \frac{\partial}{\partial \mathbf{u}} \cdot f_2 \mathbf{g} + \frac{\partial}{\partial \mathbf{u}^\circ} \cdot f_2 \mathbf{g} + I^{-1} \frac{\partial}{\partial \boldsymbol{\omega}} \cdot f_2 \overline{\mathbf{K}}^2 + I^{-1} \frac{\partial}{\partial \boldsymbol{\omega}^\circ} \cdot f_2 \overline{\mathbf{K}}^{\circ 2} = 0,$$

where the averaged force exerted at time  $t$  upon an inclusion which is known to be located at  $\mathbf{z}$  and to have a companion inclusion at  $\mathbf{z}^\circ$  is:

$$(4.10) \quad \overline{\mathbf{F}}^2(\mathbf{z}|\mathbf{z}^\circ) = E[f_1 f_2 \mathbf{F}_1] / f_2 = a^2 \int_{S(\mathbf{x})} \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c3}}(\mathbf{x} + a\mathbf{n}|\mathbf{z}, \mathbf{z}^\circ) d\Omega$$

where the local averaged stress at the surface of the test inclusion is:

$$(4.11) \quad \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c3}}(\tilde{\mathbf{x}}|\mathbf{z}, \mathbf{z}^\circ) = E[f_1 f_2 \mathbf{n} \cdot X_{1,2}^c \mathbb{T}^c] / f_2(\mathbf{z}, \mathbf{z}^\circ) = -\alpha^{c3} \overline{p}^{c3} \mathbf{n} \\ + 2\mu^c \mathbf{n} \cdot \mathbb{D}[\overline{\mathbf{v}}^{c3} + \alpha^{d3}(\overline{\mathbf{v}}^{d3} - \overline{\mathbf{v}}^{c3})].$$

The position of the inclusion centres which contribute to this integral are such that  $\tilde{\mathbf{x}} = \mathbf{x} + a\mathbf{n}$ .

The point here is to distinguish between averaging  $f_1 f_2 \mathbf{F}_1$  which leads to (5.15) and averaging  $f_1 f_2 \mathbf{F}_2$  which leads to  $\overline{\mathbf{F}}^{\circ 2} = E[f_1 f_2 \mathbf{F}_2] / f_2 = \overline{\mathbf{F}}^2(\mathbf{z}^\circ|\mathbf{z})$ . Both quantities differ because  $\overline{\mathbf{F}}^2(\cdot|\cdot)$  is not symmetrical with respect to its two spatial arguments.

All the results which have just been obtained for the averaged force can be extended directly to the averaged torque. Furthermore the equations for  $f^{(1)}$  and  $f^{(2)}$  can be obtained readily by averaging the equations for the afore-mentioned fine-grained number densities.

### 4.3. Moment equations

The first two members of the B.B.G.K.Y. hierarchy above are written in  $\mu_z$  and  $\mu_{zz}$ -spaces either in terms of  $f_1$  and  $f_2$  or in terms of  $f^{(1)}$  and  $f^{(2)}$ . The latter form will be adopted. Transforming them into moment equations over the physical spaces  $\mathbf{V}_x^d$  and  $\mathbf{V}_{xx^\circ}^d$  instead of the  $\mu_z$  and  $\mu_{zz^\circ}$ -spaces is a standard process. The first-order moment equations of interest here are the averaged simple number density  $\phi^{(1)}$ , the averaged linear velocity  $\overline{\mathbf{u}}^1$  and the averaged angular velocity  $\overline{\boldsymbol{\omega}}^1$ :

$$(4.12) \quad \frac{\partial}{\partial t} \phi^{(1)} + \frac{\partial}{\partial \mathbf{x}} \cdot (\phi^{(1)} \overline{\mathbf{u}}^1) = 0,$$

$$(4.13) \quad \frac{\partial}{\partial t} \left( \phi^{(1)} \bar{\mathbf{u}}^1 \right) + \frac{\partial}{\partial \mathbf{x}} \cdot \sum_{i=1}^N E[\varphi_i \mathbf{u}_i \mathbf{u}_i] = \phi^{(1)} \mathbf{g} + m^{-1} \phi^{(1)} \bar{\mathbf{F}}^1,$$

$$(4.14) \quad \frac{\partial}{\partial t} \left( \phi^{(1)} \bar{\boldsymbol{\omega}}^1 \right) + \frac{\partial}{\partial \mathbf{x}} \cdot \sum_{i=1}^N E[\varphi_i \boldsymbol{\omega}_i \mathbf{u}_i] = I^{-1} \phi^{(1)} \bar{\mathbf{K}}^1,$$

where the resultant force  $\bar{\mathbf{F}}^1$  and torque  $\bar{\mathbf{K}}^1$  experienced by the first inclusion located at  $\mathbf{x}$ , whatever its translational or rotational velocities, were defined using (3.7) as:

$$(4.15) \quad \bar{\mathbf{F}}^1(\mathbf{x}) = \langle \bar{\mathbf{F}}^1; f_1 \rangle / \phi_1 = a^2 \int_{S(\mathbf{x})} \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c2}}(\mathbf{x} + a\mathbf{n}|\mathbf{x}) d\Omega,$$

$$(4.16) \quad \bar{\mathbf{K}}^1(\mathbf{x}) = \langle \bar{\mathbf{K}}^1; f_1 \rangle / \phi_1 = a^3 \mathcal{E} : \int_{S(\mathbf{x})} \mathbf{nn} \cdot \overline{X^c \mathbb{T}^{c2}} - (\mathbf{x} + a\mathbf{n}|\mathbf{x}) d\Omega,$$

where the local averaged stress at the test inclusion surface is defined by:

$$(4.17) \quad \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c2}}(\bar{\mathbf{x}}, t|\mathbf{x}) = \mathbf{n} \cdot E[\varphi_1 X_1^c \mathbb{T}^c] / \phi_1(\mathbf{x}) = -\alpha^{c2} \bar{p}^{c2} \mathbf{n} \\ + 2\mu^c \mathbf{n} \cdot \{ \mathbb{D}[\bar{\mathbf{v}}^{c2} + \alpha^{d2}(\bar{\mathbf{v}}^{d2} - \bar{\mathbf{v}}^{c2})] \}.$$

The expression of this stress will be simplified in Sec. 5.3.

We also need the equation governing the second-order moments. Starting from (4.9), we determine the second-order moment equations for the three variables, the averaged pair number density  $\phi^{(2)}(\mathbf{x}, \mathbf{x}^\circ, t)$ , the linear velocity  $\bar{\mathbf{u}}^{\circ 2}$  and angular velocity  $\bar{\boldsymbol{\omega}}^{\circ 2}$ , which are conditionally averaged upon the presence of an inclusion at  $\mathbf{x}$

$$(4.18) \quad \frac{\partial}{\partial t} \phi^{(2)} + \frac{\partial}{\partial \mathbf{x}} \cdot \left( \phi^{(2)} \bar{\mathbf{u}}^2 \right) + \frac{\partial}{\partial \mathbf{x}^\circ} \cdot \left( \phi^{(2)} \bar{\mathbf{u}}^{\circ 2} \right) = 0,$$

$$(4.19) \quad \frac{\partial}{\partial t} \left( \phi^{(2)} \bar{\mathbf{u}}^{\circ 2} \right) + \frac{\partial}{\partial \mathbf{x}^\circ} \cdot \sum_{i=1} \sum_{j \neq i} E[\varphi_i \varphi_j \mathbf{u}_j \mathbf{u}_j] \\ + \frac{\partial}{\partial \mathbf{x}} \cdot \sum_{i=1} \sum_{j \neq i} E[\varphi_i \varphi_j \mathbf{u}_i \mathbf{u}_j] = \phi^{(2)} \mathbf{g} + m^{-1} \phi^{(2)} \bar{\mathbf{F}}^{\circ 2},$$

$$(4.20) \quad \frac{\partial}{\partial t} \left( \phi^{(2)} \bar{\boldsymbol{\omega}}^{\circ 2} \right) + \frac{\partial}{\partial \mathbf{x}^\circ} \cdot \sum_{i=1} \sum_{j \neq i} E[\varphi_i \varphi_j \mathbf{u}_j \boldsymbol{\omega}_j] \\ + \frac{\partial}{\partial \mathbf{x}} \cdot \sum_{i=1} \sum_{j \neq i} E[\varphi_i \varphi_j \mathbf{u}_i \boldsymbol{\omega}_j] = I^{-1} \phi^{(2)} \bar{\mathbf{K}}^{\circ 2},$$

where the resultant force  $\overline{\mathbf{F}^{\circ 2}}$  and torque  $\overline{\mathbf{K}^{\circ 2}}$  experienced by the test inclusion located at  $\mathbf{x}^\circ$  and conditionally averaged upon the presence of another inclusion at  $\mathbf{x}$ , are:

$$(4.21) \quad \overline{\mathbf{F}^{\circ 2}} = \overline{\mathbf{F}^2}(\mathbf{x}^\circ|\mathbf{x}) = \langle \overline{\mathbf{F}^{\circ 2}}; f_2 \rangle / \phi_2 = a^2 \int_{S(\mathbf{x})} \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c\beta}}(\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x}, \mathbf{x}^\circ) d\Omega,$$

$$(4.22) \quad \overline{\mathbf{K}^{\circ 2}} = \overline{\mathbf{K}^2}(\mathbf{x}^\circ|\mathbf{x}) = \langle \overline{\mathbf{K}^{\circ 2}}; f_2 \rangle / \phi_2 \\ = a^3 \mathcal{E} : \int_{S(\mathbf{x})} \mathbf{nn} \cdot \overline{X^c \mathbb{T}^{c\beta}}(\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x}, \mathbf{x}^\circ) d\Omega,$$

where the local averaged stress at the test inclusion surface is defined by

$$(4.23) \quad \mathbf{n} \cdot \overline{X^c \mathbb{T}^{c\beta}}(\tilde{\mathbf{x}}|\mathbf{x}, \mathbf{x}^\circ) = \mathbf{n} \cdot E[\varphi_1 \varphi_2 X_{1,2}^c \mathbb{T}^c] / \phi_2(\mathbf{x}, \mathbf{x}^\circ) \\ = -\alpha^{c3} \overline{p}^{c3} \mathbf{n} + 2\mu^c \mathbf{n} \cdot \mathbb{D}[\overline{\mathbf{v}^{c\beta}} + \alpha^{d3}(\overline{\mathbf{v}^{d3}} - \overline{\mathbf{v}^{c3}})].$$

#### 4.4. The dispersed phase pseudo-turbulent tensors

When considering first order Eqs. (4.13) and (4.14), two types of correlation functions appear, namely  $E[\varphi_i \mathbf{u}_i \mathbf{u}_i]$  and  $E[\varphi_i \mathbf{u}_i \boldsymbol{\omega}_i]$ . As usual, these averages should be transformed into products of their averages plus a component due to pulsation effects around these averages. To this end, a fluctuation field is defined via  $\varphi_1(\mathbf{x}) \mathbf{u}_1 = \varphi_1(\mathbf{x}) \overline{\mathbf{u}}^1 + \varphi_1(\mathbf{x}) \mathbf{u}'_1$  for the first correlation and, using (3.4), it may be noted that

$$(4.24) \quad \sum_{i=1}^N E[\varphi_i \mathbf{u}'_i \mathbf{u}'_i] = N E[\varphi_1 \mathbf{u}'_1 \mathbf{u}'_1] = N E[\varphi_1 (\mathbf{u}_1 - \overline{\mathbf{u}}^1)(\mathbf{u}_1 - \overline{\mathbf{u}}^1)] = \phi^{(1)} \overline{\mathbf{u}}^1 \overline{\mathbf{u}}^1 \\ - 2\overline{\mathbf{u}}^1 N E(\varphi_1 \mathbf{u}_1) + N E[\varphi_1 \mathbf{u}_1 \mathbf{u}_1] = \sum_{i=1}^N E[\varphi_i \mathbf{u}_i \mathbf{u}_i] - \phi^{(1)} \overline{\mathbf{u}}^1 \overline{\mathbf{u}}^1.$$

So the overall correlation function which is the first term in the r.h.s. may be considered as the sum of two terms: the mean flow convection term  $\phi^{(1)} \overline{\mathbf{u}}^1 \overline{\mathbf{u}}^1$  of the dispersed phase and a second one which measures the linear velocity pulsations experienced by any inclusion passing through  $\mathbf{x}$ . It is the first-order agitation (or pseudo-turbulent) tensor of the dispersed phase:

$$(4.25) \quad \mathbb{A}_{uu}^1(\mathbf{x}) = \sum_{i=1}^N E[\varphi_i \mathbf{u}'_i \mathbf{u}'_i]$$

where the superscript indicates the order of the fluctuating field and where the subscript  $uu$  indicates which phase velocities are considered to be correlated: here the dispersed phase velocity is correlated with itself. Later (Sec. 5.3) similar functions will be introduced for the continuous phase velocity ( $vv$ ). Finally, a second fluctuation field  $\boldsymbol{\omega}'_1$  can be introduced and a second first-order pseudo-turbulent tensor  $\mathbb{A}_{uu}^1 = \sum_{i=1}^N E[\varphi_i \boldsymbol{\omega}'_i \mathbf{u}'_i]$  can be defined.

Other velocity variance tensors are expected to occur. When considering second-order Eqs. (4.19) and (4.20), four types of correlation function appear, namely  $E[\varphi_i \varphi_j \mathbf{u}_i \mathbf{u}_j]$ ,  $E[\varphi_i \varphi_j \mathbf{u}_i \boldsymbol{\omega}_j]$  on one side, and  $E[\varphi_i \varphi_j \mathbf{u}_i \mathbf{u}_j]$ ,  $E[\varphi_i \varphi_j \mathbf{u}_j \boldsymbol{\omega}_j]$  on the other side. Extra fluctuation fields have to be defined in order to obtain breakdowns such as (4.24). To begin with, they are  $\varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}^\circ) \mathbf{u}_i = \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}^\circ) \bar{\mathbf{u}}^2 + \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}^\circ) \mathbf{u}_i''$  and  $\varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}^\circ) \mathbf{u}_j = \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}^\circ) \bar{\mathbf{u}}^{\circ 2} + \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}^\circ) \mathbf{u}_j''$ . Inserting the latter field into the second order correlation function  $E[\varphi_i \varphi_j \mathbf{u}_j \mathbf{u}_j]$  gives:

$$(4.26) \quad \sum_{i=1} \sum_{j \neq i} E[\varphi_i \varphi_j \mathbf{u}_j \mathbf{u}_j] = N(N-1) E[\varphi_1 \varphi_2 \bar{\mathbf{u}}^{\circ 2} \bar{\mathbf{u}}^{\circ 2}] \\ + N(N-1) E[\varphi_1 \varphi_2 \mathbf{u}_2'' \mathbf{u}_2''] = \phi^{(2)} \bar{\mathbf{u}}^{\circ 2} \bar{\mathbf{u}}^{\circ 2} + \mathbb{A}_{uu^\circ}^2(\mathbf{x}^\circ, \mathbf{x}),$$

where a second-order unsymmetrical tensor  $\mathbb{A}_{uu^\circ}^2(\mathbf{x}^\circ, \mathbf{x})$  which measures the rotationless agitation experienced by any inclusion fixed at  $\mathbf{x}^\circ$  provided that there is a second inclusion at  $\mathbf{x}$ .

Inserting both fluctuation fields into  $E[\varphi_i \varphi_j \mathbf{u}_i \mathbf{u}_j]$  leads to a similar breakdown:

$$(4.27) \quad \sum_{i=1} \sum_{j \neq i} E[\varphi_i \varphi_j \mathbf{u}_i \mathbf{u}_j] = N(N-1) E[\varphi_1 \varphi_2 \bar{\mathbf{u}}^2 \bar{\mathbf{u}}^{\circ 2}] \\ + N(N-1) E[\varphi_1 \varphi_2 \mathbf{u}_1'' \mathbf{u}_2''] = \phi^{(2)} \bar{\mathbf{u}}^2 \bar{\mathbf{u}}^{\circ 2} + \mathbb{A}_{uu^\circ}^2(\mathbf{x}^\circ, \mathbf{x}),$$

where a second-order symmetrical tensor  $\mathbb{A}_{uu^\circ}^2(\mathbf{x}^\circ, \mathbf{x})$  which measures the linear velocity correlation between any pair of inclusions located at  $\mathbf{x}$  and  $\mathbf{x}^\circ$ .

Finally, extra fluctuation fields  $\boldsymbol{\omega}_i''$  and  $\boldsymbol{\omega}_j''$ , can be introduced to describe the rotational contribution to inclusion agitation and to display (unsymmetrical) second order pseudo-turbulent tensors  $\mathbb{A}_{\omega^\circ u}^2$  and  $\mathbb{A}_{\omega^\circ u^\circ}^2$ .

It is obvious that the physical origin of these tensors is the same. It has nothing to do with usual turbulence in a single-phase flow. Interactions between inclusions via a medium evolving with its own dynamics set up a highly nonlinear process which reveals itself in an apparently random pulsating motion of both phases super-imposed on their mean motion (unconditioned and conditioned by different test inclusions). Moreover, as far as the continuous phase is concerned,

fluctuations result from local distortions in fluid flow streamlines, including possibly the wakes caused by the submerged inclusions. Note that the latter origin has already been pointed out by BUYEVICH and SHCHELCHKOVA [4].

#### 4.5. The simplified form of the momentum equations

Breaking down the pseudo-turbulent correlation functions allows the momentum equations of any order to be simplified by means of some straightforward transformations. Indeed, the simplified forms follow from both dispersed phase continuity equations of the same order. For the first order:

$$(4.28) \quad \frac{\partial}{\partial t} \bar{\mathbf{u}}^1 + \bar{\mathbf{u}}^1 \cdot \frac{\partial}{\partial \mathbf{x}} \bar{\mathbf{u}}^1 = -(\phi^{(1)})^{-1} \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}_{uu}^1 + m^{-1} \bar{\mathbf{F}}^1 + \mathbf{g},$$

$$(4.29) \quad \frac{\partial}{\partial t} \bar{\omega}^1 + \bar{\mathbf{u}}^1 \cdot \frac{\partial}{\partial \mathbf{x}} \bar{\omega}^1 = -(\phi^{(1)})^{-1} \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}_{\omega u}^1 + I^{-1} \bar{\mathbf{K}}^1,$$

and for the second order:

$$(4.30) \quad \frac{\partial}{\partial t} \overline{\mathbf{u}^{\circ 2}} + \overline{\mathbf{u}^{\circ 2}} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}} \overline{\mathbf{u}^{\circ 2}} + \overline{\mathbf{u}}^2 \cdot \frac{\partial}{\partial \mathbf{x}} \overline{\mathbf{u}^{\circ 2}} = -(\phi^{(2)})^{-1} \frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbb{A}_{u^{\circ} u^{\circ}}^2 \\ - (\phi^{(2)})^{-1} \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}_{uu^{\circ}}^2 + m^{-1} \overline{\mathbf{F}}^{\circ 2} + \mathbf{g},$$

$$(4.31) \quad \frac{\partial}{\partial t} \overline{\omega^{\circ 2}} + \overline{\mathbf{u}^{\circ 2}} \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}} \overline{\omega^{\circ 2}} + \overline{\mathbf{u}}^2 \cdot \frac{\partial}{\partial \mathbf{x}} \overline{\omega^{\circ 2}} = -(\phi^{(2)})^{-1} \frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot \mathbb{A}_{\omega^{\circ} u^{\circ}}^2 \\ - (\phi^{(2)})^{-1} \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}_{\omega^{\circ} u}^2 + I^{-1} \overline{\mathbf{K}}^{\circ 2}.$$

Many differences can be pointed out with analogous kinetic equations (in their original or moment form) which appear in classical S.M. The most striking one lies in the expression (i.e. 4.15, 4.16, 4.21 and 4.22) of the resultant forces and torques experienced by the test inclusion located at  $\mathbf{x}$  or else at  $\mathbf{x}^{\circ}$ , and conditionally averaged depending on whether or not another inclusion is present at  $\mathbf{x}$ . They do not rely on inclusion interactions but on the adjacent carrying phase effects (conditionally averaged).

These momentum equations must be supplemented by the continuity Eqs. (4.12) and (4.18) which have just been used. The latter is an equation for the averaged pair density and it should be replaced by:

$$(4.32) \quad \frac{\partial \chi_2^{\circ}}{\partial t} + \frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot [\chi_2^{\circ} \overline{\mathbf{u}^{\circ 2}}] = -\overline{\mathbf{u}}^2 \cdot \frac{\partial \chi_2^{\circ}}{\partial \mathbf{x}} - (\chi_2^{\circ} / \phi_1) \frac{\partial}{\partial \mathbf{x}} \cdot [(\overline{\mathbf{u}}^2 - \overline{\mathbf{u}}^1) \phi_1]$$

where the conditional density is defined as

$$(4.33) \quad \chi_2^{\circ} = \chi_2(\mathbf{x}^{\circ} | \mathbf{x}) = \phi_2(\mathbf{x}, \mathbf{x}^{\circ}) / \phi_1(\mathbf{x})$$

## 4.6. Boundary conditions

Two types of condition on the boundaries of  $\mathcal{V}_x^d$  and on the external boundaries of  $\mathcal{V}_{x,x^\circ}^d$  have to be distinguished depending on whether these boundaries are close to an impermeable wall (i.e.  $\partial\mathcal{V}_w^c$ ) or are permeable fluid limits (i.e.  $\partial\mathcal{V}_f^c$ ). Near a wall, the soft collision assumption introduced in Sec. 2 entails velocity continuity conditions:

$$(4.34) \quad \bar{\mathbf{u}}^1(\mathbf{x} + a\mathbf{n}^b) - a\bar{\omega}^1(\mathbf{x} + a\mathbf{n}^b) \wedge \mathbf{n}^b = \mathbf{v}^b(\mathbf{x}), \quad \mathbf{x} \text{ on } \partial\mathcal{V}_w^c,$$

$$\bar{\mathbf{u}}^2(\mathbf{x}^\circ + a\mathbf{n}^b|\mathbf{x}) - a\bar{\omega}^2(\mathbf{x}^\circ + a\mathbf{n}^b|\mathbf{x}) \wedge \mathbf{n}^b = \mathbf{v}^b(\mathbf{x}^\circ), \quad \mathbf{x}^\circ \text{ on } \partial\mathcal{V}_w^c.$$

On the internal boundaries of  $\mathcal{V}_{x,x^\circ}^d$  a similar velocity continuity condition holds:

$$(4.35) \quad \bar{\mathbf{u}}^2(\mathbf{x} + 2a\mathbf{n}|\mathbf{x}) - a\bar{\omega}^2(\mathbf{x} + 2a\mathbf{n}|\mathbf{x}) \wedge \mathbf{n} = \bar{\mathbf{u}}^2(\mathbf{x}|\mathbf{x} + 2a\mathbf{n}) \\ + a\bar{\omega}^2(\mathbf{x}|\mathbf{x} + 2a\mathbf{n}) \wedge \mathbf{n},$$

where  $\mathbf{n} = \mathbf{x}^\circ - \mathbf{x}/|\mathbf{x}^\circ - \mathbf{x}|$  is the unit normal along the centre line. No condition can be prescribed for the density fields such as  $\phi_1(\mathbf{x})$  and  $\chi_2(\mathbf{x}^\circ|\mathbf{x})$  on all these types of boundaries.

When they pass the fluid boundary of  $\partial\mathcal{V}^d$ , which coincides with  $\partial\mathcal{V}_f^c$  the linear  $\mathbf{u}^f$  and angular  $\boldsymbol{\omega}^f$  velocities of the inclusions must be given:

$$(4.36) \quad \bar{\mathbf{u}}^1(\mathbf{x}) = \mathbf{u}^f(\mathbf{x}), \quad \mathbf{x} \text{ on } \partial\mathcal{V}_f^c \quad \text{and} \quad \bar{\mathbf{u}}^2(\mathbf{x}^\circ|\mathbf{x}) = \mathbf{u}^f(\mathbf{x}^\circ), \quad \mathbf{x}^\circ \text{ on } \partial\mathcal{V}_f^c$$

$$(4.37) \quad \bar{\boldsymbol{\omega}}^1(\mathbf{x}) = \boldsymbol{\omega}^f(\mathbf{x}), \quad \mathbf{x} \text{ on } \partial\mathcal{V}_f^c \quad \text{and} \quad \bar{\boldsymbol{\omega}}^2(\mathbf{x}^\circ|\mathbf{x}) = \boldsymbol{\omega}^f(\mathbf{x}^\circ), \quad \mathbf{x}^\circ \text{ on } \partial\mathcal{V}_f^c$$

Densities have also to be specified on this fluid part:

$$(4.38) \quad \phi_1(\mathbf{x}) = \varphi^f(\mathbf{x}), \quad \mathbf{x} \text{ on } \partial\mathcal{V}_f^c \quad \text{and} \quad \chi_{\circ 2}(\mathbf{x}^\circ|\mathbf{x}) = \varphi^f(\mathbf{x}^\circ), \quad \mathbf{x}^\circ \text{ on } \partial\mathcal{V}_f^c$$

so that the rate at which they are injected into the system under study or picked off can be known by using (4.36). The overall flux imposed on  $\partial\mathcal{V}_f^c$  is assumed

to satisfy  $\int_{\partial\mathcal{V}_f^c} \phi_1 \bar{\mathbf{u}}^1 \cdot \mathbf{n}^f dS = 0$  at any time (where  $\mathbf{n}^f$  denotes the unit normal

interior to  $\partial\mathcal{V}_f^c$ ). Hence the total number of inclusions is conserved as expressed by the normalization conditions for  $\phi_1(\mathbf{x}^\circ)$  and  $\chi_2(\mathbf{x}^\circ|\mathbf{x})$  over  $\mathcal{V}_{x^\circ}^d$  and  $\mathcal{V}_{x,x^\circ}^d$  respectively:

$$(4.39) \quad \int_{\mathcal{V}_x^d} \phi_1(\mathbf{x}) d\mathbf{x} = 1, \quad \int_{\mathcal{V}_{x,x^\circ}^d} \chi_2(\mathbf{x}^\circ|\mathbf{x}) d\mathbf{x}^\circ = 1.$$

## 5. Averaging process for the continuous phase

### 5.1. The extra fine-grained densities equations

The parallel between the fine-grained equations for both phases is achieved by deriving, for positions in the continuous phase, new equations in the g.f. sense relative to one neighbouring fixed inclusion centre, two fixed inclusion centres, and so on...; these govern mixed fine-grained variables such as  $\varphi_i(\mathbf{x}, t)X_i^c(\mathbf{x}^\circ, t)$ ,  $\varphi_i(\mathbf{x}, t)X_i^c\psi^c(\mathbf{x}^\circ, t)$  and  $\varphi_i(\mathbf{x}^\circ, t)\varphi_j(\mathbf{x}, t)X_{i,j}^c(\mathbf{x}^\circ, t)$ ,  $\varphi_i(\mathbf{x}^\circ, t)\varphi_j(\mathbf{x}, t)X_{i,j}^c\psi^c(\mathbf{x}^\circ, t)$  respectively, and they complete the fine-grained equations of Sec. 2.2. By averaging them, a whole sequence of conditionally averaged equations can be produced. In our theory this sequence will be presented up to the third order. LUNDGREN [15] proposed a similar hierarchy of averaged equations directly, without obtaining fine-grained equations beforehand. Furthermore, as the dispersed phase in his case was the immobile matrix of a porous medium, he did not observe some extra terms induced by the motion of test inclusions.

To obtain first the equations to the second order, i.e. with one fixed inclusion, say the first one, consider the following extended equations:

$$(5.1) \quad \left\{ \varphi_1 X_1^c \frac{\partial}{\partial \mathbf{x}^\circ} \cdot (\mathbf{v}^c) \right\} = 0,$$

$$(5.2) \quad \left\{ \varphi_1 X_1^c \frac{\partial \mathbf{v}^c}{\partial t} \right\} + \left\{ \varphi_1 X_1^c \frac{\partial}{\partial \mathbf{x}^\circ} \cdot (\mathbf{v}^c \mathbf{v}^c) \right\} - \left\{ \varphi_1 X_1^c \frac{\partial}{\partial \mathbf{x}^\circ} \cdot \mathbb{T}^c \right\} / \rho^c \\ - \left\{ \varphi_1 X_1^c \mathbf{g} \right\} = 0.$$

As in Sec. 2.2, each g.f. associated with derivatives in the usual sense of functions (i.e.  $\{\varphi_1 X_1^c \partial \mathbf{v}^c / \partial t\}$ ) can be transformed into derivatives in the sense of g.f. of the same functions (i.e.  $\partial \varphi_1 X_1^c \mathbf{v}^c / \partial t$ ). First, the continuity equation conditioned by the presence of an inclusion centre at point  $\mathbf{x}$  presents an unexpected extra volume source term in its r.h.s., due to the motion of the test inclusion:

$$(5.3) \quad \frac{\partial \varphi_1 X_1^c}{\partial t} + \frac{\partial}{\partial \mathbf{x}^\circ} \cdot (\varphi_1 X_1^c \mathbf{v}^c) = - \frac{\partial}{\partial \mathbf{x}} \cdot (\varphi_1 X_1^c \mathbf{u}_1).$$

The momentum equation conditioned by the presence of an inclusion centre at point  $\mathbf{x}$  is obtained by treating (5.2) in the same way and by introducing the partial surface Dirac g.f.  $\delta_{\Sigma}^1$  from which the first inclusion surface is excluded:

$$\begin{aligned}
(5.4) \quad & \frac{\partial \varphi_1 X_1^c \mathbf{v}^c}{\partial t} + \frac{\partial}{\partial \mathbf{x}^o} \cdot (\varphi_1 X_1^c \mathbf{v}^c \mathbf{v}^c) = -\frac{1}{\rho^c} \frac{\partial}{\partial \mathbf{x}^o} (\varphi_1 X_1^c p^c) \\
& + \nu^c \Delta^o (\varphi_1 X_1^c \mathbf{v}^c) + \nu^c \frac{\partial}{\partial \mathbf{x}^o} \left[ \frac{\partial}{\partial \mathbf{x}^o} \cdot (\varphi_1 X_1^c \mathbf{v}^c) \right] + 2\nu^c \frac{\partial}{\partial \mathbf{x}^o} \cdot (\mathbb{F}^c \varphi_1 \delta_\Sigma^1) \\
& + \frac{1}{\rho^c} \mathbf{n}^c \cdot \mathbb{T}^c \varphi_1 \delta_\Sigma^1 - \frac{\partial}{\partial \mathbf{x}} \cdot [\varphi_1 X_1^c \mathbf{u}_1 \mathbf{v}^c] + \varphi_1 X_1^c \mathbf{g},
\end{aligned}$$

where  $\mathbb{F}^c \varphi_1 \delta_\Sigma^1$  is the fine-grained extra deformation tensor and  $\mathbf{n}^c \cdot \mathbb{T}^c \varphi_1 \delta_\Sigma^1$  is the fine-grained interfacial stress exerted upon the inclusions, when the first inclusion centre is at point  $\mathbf{x}$ . Again, our method allows an extra momentum source term to appear due to the motion of the test inclusion.

To obtain the third-order continuous phase equations, i.e. with two fixed inclusions, the following extended equations are considered:

$$(5.5) \quad \left\{ \varphi_1 \varphi_2 X_{1,2}^c \frac{\partial}{\partial \mathbf{x}^{oo}} \cdot (\mathbf{v}^c) \right\} = 0,$$

$$\begin{aligned}
(5.6) \quad & \left\{ \varphi_1 \varphi_2 X_{1,2}^c \frac{\partial \mathbf{v}^c}{\partial t} \right\} + \left\{ \varphi_1 \varphi_2 X_{1,2}^c \frac{\partial}{\partial \mathbf{x}^{oo}} \cdot (\mathbf{v}^c \mathbf{v}^c) \right\} \\
& - \left\{ \varphi_1 \varphi_2 X_{1,2}^c \frac{\partial}{\partial \mathbf{x}^{oo}} \cdot \mathbb{T}^c \right\} / \rho^c - \{ \varphi_1 \varphi_2 X_{1,2}^c \mathbf{g} \} = 0.
\end{aligned}$$

Using transformations like those used above for the second-order equations, one obtains fine-grained continuity and momentum equations similar to (5.3) and (5.4) respectively.

## 5.2. The revisited Lundgren hierarchy

The Lundgren hierarchy is obtained by averaging the entire sequence of fine-grained equations written for the continuous phase. The property  $E[O(f)] = O(E[f])$  will be used again repeatedly. Averaging Eqs. (2.9) and (2.14), and using the various definitions of hydrodynamic variables given in Sec. 3, the first-order equations, valid over  $\mathcal{V}_x^c$ , are:

$$(5.7) \quad \frac{\partial \alpha^{c1}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\alpha^{c1} \bar{\mathbf{v}}^{c1}) = 0$$

$$\begin{aligned}
(5.8) \quad & \frac{\partial \alpha^{c1} \bar{\mathbf{v}}^{c1}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot E[X^c \mathbf{v}^c \mathbf{v}^c] = -\frac{1}{\rho^c} \frac{\partial}{\partial \mathbf{x}} (\alpha^{c1} \bar{p}^{c1}) + \nu^c \Delta (\alpha^{c1} \bar{\mathbf{v}}^{c1}) \\
& + \nu^c \frac{\partial}{\partial \mathbf{x}} \left[ \frac{\partial}{\partial \mathbf{x}} \cdot (\alpha^{c1} \bar{\mathbf{v}}^{c1}) \right] + 2\nu^c \frac{\partial}{\partial \mathbf{x}} \cdot E[\mathbb{F}^c \delta_\Sigma] + \frac{1}{\rho^c} E[\mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma] + \alpha^{c1} \mathbf{g}.
\end{aligned}$$

Next, we can derive the second-order equations, valid over  $\mathcal{V}_{x,x^\circ}^c$ , by summing the continuity Eq. (5.3) and the momentum Eq. (5.4) over all fixed inclusions and averaging them. Repeating the process, the third-order equations, valid over  $\mathcal{V}_{x,x^\circ,x^{\circ\circ}}^c$ , are obtained by summing and averaging the corresponding continuity momentum equations.

Now we turn our attention to the important problem of the non-closedness of the Lundgren hierarchy. This characteristic is apparent when examining the interfacial terms in the momentum equations at each order. Considering for example Eq. (5.8), these terms, i.e.  $2\nu^c \partial \bar{\partial} x \cdot (E[\mathbb{F}^c \delta_\Sigma])$  and  $E[\mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma] / \rho^c$ , are respectively the averaged extra-deformation tensor and the averaged interfacial force density. The presence of the Dirac function inside the average operator indicates some conditional averaging; as we shall see more precisely in the next paper, this means that one inclusion has to be fixed. At the second order, clearly, these corresponding interaction terms involve one more fixed inclusion and so on.

### 5.3. Boundary conditions

Continuity equations and non-slip conditions hold on boundaries  $\partial \mathcal{V}^c$  of  $\mathcal{V}_x^c$ , and on the external boundaries of  $\mathcal{V}_{x,x^\circ}^c$ ,  $\mathcal{V}_{x,x^\circ,x^{\circ\circ}}^c$ . Consequently various conditional averagings of (2.15) give:

$$(5.9) \quad \bar{v}^{c1}(\mathbf{x}) = \mathbf{v}^b(\mathbf{x}), \quad \mathbf{x} \text{ on } \partial \mathcal{V}^c; \quad \bar{v}^{c2}(\mathbf{x}^\circ | \mathbf{x}) = \mathbf{v}^b(\mathbf{x}^\circ), \quad \mathbf{x}^\circ \text{ on } \partial \mathcal{V}^c$$

$$\text{and } \bar{v}^{c3}(\mathbf{x}^{\circ\circ} | \mathbf{x}^\circ, \mathbf{x}) = \mathbf{v}^b(\mathbf{x}^{\circ\circ}), \quad \mathbf{x}^{\circ\circ} \text{ on } \partial \mathcal{V}^c.$$

Conditions of this type have also to be specified on the internal boundaries of  $\mathcal{V}_{x,x^\circ}^c$ ,  $\mathcal{V}_{x,x^\circ,x^{\circ\circ}}^c$ . They are based on fine grained mass balance equations and non-slip conditions valid on the test inclusions. The corresponding averaged forms at  $\mathbf{x}^\circ = \mathbf{x} + a\mathbf{n}$  and at  $\mathbf{x}^{\circ\circ} = \mathbf{x}^\circ + a\mathbf{n}$ , are (see 2.4):

$$(5.10) \quad \bar{v}^{c2}(\mathbf{x} + a\mathbf{n} | \mathbf{x}) = \bar{\mathbf{u}}^1(\mathbf{x}) + a \bar{\boldsymbol{\omega}}^1(\mathbf{x}) \wedge \mathbf{n},$$

$$\bar{v}^{c3}(\mathbf{x}^\circ + a\mathbf{n} | \mathbf{x}^\circ, \mathbf{x}) = \bar{\mathbf{u}}^2(\mathbf{x}^\circ | \mathbf{x}) + a \bar{\boldsymbol{\omega}}^2(\mathbf{x}^\circ | \mathbf{x}) \wedge \mathbf{n}.$$

An extra average boundary condition is obtained by interchanging  $\mathbf{x}^\circ$  and  $\mathbf{x}$  in the last equation. In this way all the equations of the Lundgren hierarchy are connected to the moment equations of the dispersed phase.

Strictly speaking, there are no particular boundary conditions for  $\alpha^{c1}$ ,  $\alpha^{c2}$ , and  $\alpha^{c3}$  which might be associated with field equations such as (5.7) and (5.8), while there are conditions for the densities such as (4.38). Nevertheless, to simplify interfacial momentum conditions  $\alpha^{cj} = 1$ , ( $j = 1, 2, 3, \dots$ ) on the walls  $\partial \mathcal{V}_w^c$  will be used. This condition is valid if the inclusions keep their spherical shape

when they collide instantaneously. Similar conditions are valid on the surface of the test inclusion. All these conditions are dynamic in nature and correspond to an idealised mechanical response of the inclusion material. For instance, they allow the local averaged stress tensor at the surface  $\mathbf{x}^\circ = \mathbf{x} + a\mathbf{n}$  of the test inclusion centred at  $\mathbf{x}$  to be simplified, see (4.17), to give:

$$(5.11) \quad \overline{X^c \mathbb{T}^{c2}}(\mathbf{x} + a\mathbf{n}|\mathbf{x}) = E[\varphi_1 X^c \mathbb{T}^c] / \phi_1(\mathbf{x}) = -\bar{p}^{c2} \mathbb{I} + 2\mu^c \left\{ \mathbb{D}(\bar{\mathbf{v}}^{c2}) + \left[ (\bar{\mathbf{v}}^{d2} - \bar{\mathbf{v}}^{c2}) \frac{\partial}{\partial \mathbf{x}} \alpha^{d2} \right]^s \right\}.$$

Here we have distinguished an extra viscous term with a non-zero gradient of  $\alpha^{d2}$ . It is represented by a symmetrical tensor involving the relative velocity. The velocity difference depends on momentum exchanges during collisions. In our case, it should be recalled that collisions are assumed to be soft enough not to give rise to significant pressure impulses and the conditional averaged continuous phase velocity merges with the inclusion velocity (5.10) at the point of impact; as a consequence, this extra term will be neglected. Of course, all the results which have just been obtained for the resultant averaged stress can be extended directly to the resultant averaged torque.

Likewise, the local averaged stress tensor (4.23) at the surface  $\mathbf{x}^{\circ\circ} = \mathbf{x}^\circ + a\mathbf{n}$  of the test inclusion centred at  $\mathbf{x}^\circ$ , when another inclusion is centred at  $\mathbf{x}$ , becomes:

$$(5.12) \quad \overline{X^c \mathbb{T}^{c3}}(\mathbf{x}^\circ + a\mathbf{n}|\mathbf{x}, \mathbf{x}^\circ) = E[\varphi_1 \varphi_2 X^c \mathbb{T}^c] / \phi_2(\mathbf{x}, \mathbf{x}^\circ) \\ = -\bar{p}^{c3} \mathbb{I} + 2\mu^c \left\{ \mathbb{D}(\bar{\mathbf{v}}^{c3}) + \left[ (\bar{\mathbf{v}}^{d3} - \bar{\mathbf{v}}^{c3}) \frac{\partial}{\partial \mathbf{x}} \alpha^{d3} \right]^s \right\},$$

where the last viscous term is neglected as above. An extra averaged boundary condition is obtained by interchanging  $\mathbf{x}^\circ$  and  $\mathbf{x}$ .

#### 5.4. The pseudo-turbulent tensors in the continuous phase

In the three first-order momentum equations, we are faced with correlations between continuous phase velocities such as  $E[X^c \mathbf{v}^c \mathbf{v}^c(\mathbf{x})]$ ,  $E[\varphi_i X_i^c \mathbf{v}^c \mathbf{v}^c(\mathbf{x}^\circ)]$  and  $E[\varphi_i \varphi_j X_{i,j}^c \mathbf{v}^c \mathbf{v}^c(\mathbf{x}^{\circ\circ})]$ . These functions can be broken down like the corresponding dispersed phase correlations. The first-order fluctuation field is given via  $X^c \mathbf{v}^c(\mathbf{x}) = X^c \bar{\mathbf{v}}^{c1}(\mathbf{x}) + X^c \mathbf{v}^{c'}(\mathbf{x})$  and yields

$$(5.13) \quad E[X^c \mathbf{v}^c \mathbf{v}^c] = E[X^c \bar{\mathbf{v}}^{c1} \bar{\mathbf{v}}^{c1}] + E[X^c \mathbf{v}^{c'} \mathbf{v}^{c'}] = \alpha^{c1} \bar{\mathbf{v}}^{c1} \bar{\mathbf{v}}^{c1}(\mathbf{x}) \\ + E[X^c \mathbf{v}^{c'} \mathbf{v}^{c'}],$$

since the conditional average  $\mathbf{v}'^{c1}$  is zero and  $(X^c)^2 = X^c$ . The first term in the r.h.s. of (5.13) gives rise to the mean flow convection term of the continuous phase in the first order momentum equation. The second one, which measures the agitation in the continuous phase, is the first-order agitation or pseudo-turbulent tensor relative to this phase:

$$(5.14) \quad \mathbb{A}_{vv}^1(\mathbf{x}) = E[X^c \mathbf{v}^c \mathbf{v}^c].$$

Now, let us consider the second-order fluctuation field defined by

$$(5.15) \quad \varphi_i(\mathbf{x}) X_i^c \mathbf{v}^c(\mathbf{x}^\circ) = \varphi_i(\mathbf{x}) X_i^c \bar{\mathbf{v}}^{c2}(\mathbf{x}^\circ | \mathbf{x}) + \varphi_i(\mathbf{x}) X_i^c \mathbf{v}^{c''}(\mathbf{x}^\circ).$$

Inserting this equation into the overall second-order correlation function gives

$$(5.16) \quad \sum_{i=1}^N E[\varphi_i X_i^c \mathbf{v}^c \mathbf{v}^c] = N E[\varphi_1 X_1^c \bar{\mathbf{v}}^{c2} \bar{\mathbf{v}}^{c2}] + \sum_{i=1}^N E[\varphi_i X_i^c \mathbf{v}^{c''} \mathbf{v}^{c''}] \\ = \phi^{(1)}(\mathbf{x}) \alpha^{c2}(\mathbf{x}^\circ | \mathbf{x}) \bar{\mathbf{v}}^{c2} \bar{\mathbf{v}}^{c2}(\mathbf{x}^\circ | \mathbf{x}) + \mathbb{A}_{v^\circ v^\circ}^2(\mathbf{x}^\circ | \mathbf{x}),$$

where the second-order agitation tensor for the continuous phase  $\mathbb{A}_{v^\circ v^\circ}^2(\mathbf{x}^\circ | \mathbf{x})$  has been introduced.

Likewise at the third-order, we arrive at

$$(5.17) \quad \sum_i^N \sum_{i \neq j}^{N-1} E[\varphi_i \varphi_j X_{i,j}^c \mathbf{v}^c \mathbf{v}^c] = N(N-1) E[\varphi_1 X_{1,2}^c \bar{\mathbf{v}}^{c2} \bar{\mathbf{v}}^{c2}] \\ + \sum_{i=1}^N \sum_{i \neq j}^{N-1} E[\varphi_i \varphi_j X_{i,j}^c \mathbf{v}^{c''} \mathbf{v}^{c''}] = \phi^{(2)}(\mathbf{x}^\circ, \mathbf{x}) \alpha^{c3}(\mathbf{x}^{\circ\circ} | \mathbf{x}^\circ, \mathbf{x}) \bar{\mathbf{v}}^{c3} \bar{\mathbf{v}}^{c3}(\mathbf{x}^{\circ\circ} | \mathbf{x}^\circ, \mathbf{x}) \\ + \mathbb{A}_{v^{\circ\circ} v^{\circ\circ}}^3(\mathbf{x}^{\circ\circ} | \mathbf{x}^\circ, \mathbf{x}).$$

Furthermore, test inclusion motions generate at any order new correlation terms which are revealed by our approach. In mass equations, there are  $E[X_{i,j}^c \varphi_i \mathbf{u}_i]$ ,  $E[X_{i,j}^c \varphi_i \varphi_j \mathbf{u}_i]$  and  $E[X_{i,j}^c \varphi_i \varphi_j \mathbf{u}_j]$ . Some simple transformations are helpful to interpret these terms and these will be proposed in a future paper (part III). These same work will be carried out for  $E[X_i^c \varphi_i \mathbf{u}_i \mathbf{v}^c]$ ,  $E[X_{i,j}^c \varphi_i \varphi_j \mathbf{u}_i \mathbf{v}^c]$  and  $E[X_{i,j}^c \varphi_i \varphi_j \mathbf{u}_j \mathbf{v}^c]$ , which appear in the momentum equations. Cross-correlation agitation tensors between both phases ( $vu$ ) have to be defined in this way.

### 5.5. The simplified form of the continuity and momentum equations

With all these definitions at hand, the momentum equations for the three first orders can now be simplified by means of certain transformations that are similar

to those used in Sec. 4.5 for the dispersed phase equations and which are standard in two-phase flow modelling. They are based on using continuity equations. Moreover, all the pressure and viscous terms in the momentum equations will be broken down in such a way that all the terms which look like single-phase flow terms will be placed in the l.h.s. and extra terms which are specific to a two-phase flow case will be placed in the r.h.s.

Consider the first order. By substituting (5.7), (5.13) and (5.14) in (5.8) it is easily found that the first-order momentum equation at  $\mathbf{x}$  becomes:

$$(5.18) \quad \alpha^{c1} \left( \frac{\partial \bar{\mathbf{v}}^{c1}}{\partial t} + \bar{\mathbf{v}}^{c1} \cdot \frac{\partial}{\partial \mathbf{x}} \bar{\mathbf{v}}^{c1} + \left( \frac{\partial}{\partial \mathbf{x}} \bar{p}^{c1} \right) / \rho^c - \nu^c \Delta(\bar{\mathbf{v}}^{c1}) \right. \\ \left. - \nu^c \frac{\partial}{\partial \mathbf{x}} \left[ \frac{\partial}{\partial \mathbf{x}} \cdot \bar{\mathbf{v}}^{c1} \right] - \mathbf{g} \right) = (\bar{p}^{c1} \mathbb{I} / \rho^c - 2\nu^c \mathbb{D}(\bar{\mathbf{v}}^{c1})) \cdot \frac{\partial}{\partial \mathbf{x}} \alpha^{d1} - 2\nu^c \frac{\partial}{\partial \mathbf{x}} \\ \cdot \left[ \bar{\mathbf{v}}^{c1} \frac{\partial}{\partial \mathbf{x}} \alpha^{d1} \right]^s + 2\nu^c \frac{\partial}{\partial \mathbf{x}} \cdot E[\mathbb{F}^c \delta_\Sigma] + E[\mathbf{n}^c \cdot \mathbb{T}^c \delta_\Sigma] / \rho^c - \frac{\partial}{\partial \mathbf{x}} \cdot \mathbb{A}_{vv}^1.$$

The second and third-order momentum equations

$$(5.19) \quad \alpha^{c2} \left\{ \frac{\partial \bar{\mathbf{v}}^{c2}}{\partial t} + \bar{\mathbf{v}}^{c2} \cdot \frac{\partial}{\partial \mathbf{x}^o} \bar{\mathbf{v}}^{c2} + \left( \frac{\partial}{\partial \mathbf{x}^o} \bar{p}^{c2} \right) / \rho^c - \nu^c \Delta^o(\bar{\mathbf{v}}^{c2}) \right. \\ \left. - \nu^c \frac{\partial}{\partial \mathbf{x}^o} \left[ \frac{\partial}{\partial \mathbf{x}^o} \cdot \bar{\mathbf{v}}^{c2} \right] - \mathbf{g} \right\} = (\bar{p}^{c2} \mathbb{I} / \rho^c - 2\nu^c \mathbb{D}^o(\bar{\mathbf{v}}^{c2})) \cdot \frac{\partial}{\partial \mathbf{x}^o} \alpha^{d2} \\ - 2\nu^c \frac{\partial}{\partial \mathbf{x}^o} \cdot \left[ \bar{\mathbf{v}}^{c2} \frac{\partial}{\partial \mathbf{x}^o} \alpha^{d2} \right]^s + 2\nu^c (\phi_1)^{-1} \frac{\partial}{\partial \mathbf{x}^o} \cdot E[\mathbb{F}^c \varphi_1 \delta_\Sigma^1] \\ + (\rho^c \phi_1)^{-1} E[\mathbf{n}^c \cdot \mathbb{T}^c \varphi_1 \delta_\Sigma^1] - (\phi^{(1)})^{-1} \frac{\partial}{\partial \mathbf{x}^o} \cdot \mathbb{A}_{vv^o}^2 \\ - (\phi_1)^{-1} \frac{\partial}{\partial \mathbf{x}} \cdot E[\varphi_1 X_1^c \mathbf{u}_1 \mathbf{v}^c] + (\phi_1)^{-1} \bar{\mathbf{v}}^{c2} \cdot \frac{\partial}{\partial \mathbf{x}} E[\varphi_1 X_1^c \mathbf{u}_1] \\ (5.20) \quad \alpha^{c3} \left\{ \frac{\partial \bar{\mathbf{v}}^{c3}}{\partial t} + \bar{\mathbf{v}}^{c3} \cdot \frac{\partial}{\partial \mathbf{x}^{oo}} \bar{\mathbf{v}}^{c3} + \left( \frac{\partial}{\partial \mathbf{x}^{oo}} \bar{p}^{c3} \right) / \rho^c - \nu^c \Delta^{oo}(\bar{\mathbf{v}}^{c3}) \right. \\ \left. - \nu^c \frac{\partial}{\partial \mathbf{x}^{oo}} \left[ \frac{\partial}{\partial \mathbf{x}^{oo}} \cdot \bar{\mathbf{v}}^{c3} \right] - \mathbf{g} \right\} = (\bar{p}^{c3} \mathbb{I} / \rho^c - 2\nu^c \mathbb{D}^{oo}(\bar{\mathbf{v}}^{c3})) \cdot \frac{\partial}{\partial \mathbf{x}^{oo}} \alpha^{d3}$$

$$\begin{aligned}
 (5.20) \quad & -2\nu^c \frac{\partial}{\partial \mathbf{x}^{\circ\circ}} \cdot \left[ \bar{\mathbf{v}}^c \frac{\partial}{\partial \mathbf{x}^{\circ\circ}} \alpha^{d3} \right]^s + 2\nu^c (\phi_2)^{-1} \frac{\partial}{\partial \mathbf{x}^{\circ\circ}} \cdot E[\mathbb{F}^c \varphi_1 \varphi_2 \delta_{\Sigma}^{1,2}] \\
 [\text{cont.}] \quad & + (\rho^c \phi_2)^{-1} E[\mathbf{n}^c \cdot \mathbb{T}^c \varphi_1 \varphi_2 \delta_{\Sigma}^{1,2}] - (\phi_2^{(2)})^{-1} \frac{\partial}{\partial \mathbf{x}^{\circ\circ}} \cdot \mathbb{A}_{v^{\circ\circ} v^{\circ\circ}}^3 \\
 & - (\phi_2)^{-1} \frac{\partial}{\partial \mathbf{x}} \cdot E[X_{1,2}^c \varphi_1 \varphi_2 \mathbf{u}_1 \mathbf{v}^c] - (\phi_2)^{-1} \frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot E[X_{1,2}^c \varphi_1 \varphi_2 \mathbf{u}_2 \mathbf{v}^c] \\
 & + (\phi_2)^{-1} \bar{\mathbf{v}}^c \cdot \frac{\partial}{\partial \mathbf{x}} E[X_{1,2}^c \varphi_1 \varphi_2 \mathbf{u}_1] + (\phi_2)^{-1} \bar{\mathbf{v}}^c \cdot \frac{\partial}{\partial \mathbf{x}^{\circ}} E[X_{1,2}^c \varphi_1 \varphi_2 \mathbf{u}_2]
 \end{aligned}$$

may also be simplified in a similar way thanks to the averaged continuous phase continuity equations at the corresponding order, derived from Sec. 5.1, i.e.

$$(5.21) \quad \frac{\partial \alpha^{c2} \phi_1}{\partial t} + \frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot (\alpha^{c2} \phi_1 \bar{\mathbf{v}}^{c2}) = -\frac{\partial}{\partial \mathbf{x}} \cdot E[\varphi_1 X_{1,2}^c \mathbf{u}_1],$$

$$\begin{aligned}
 (5.22) \quad & \frac{\partial \alpha^{c3} \phi_2}{\partial t} + \frac{\partial}{\partial \mathbf{x}^{\circ\circ}} \cdot (\alpha^{c3} \phi_2 \bar{\mathbf{v}}^{c3}) = -\frac{\partial}{\partial \mathbf{x}} \cdot E[\varphi_1 \varphi_2 X_{1,2}^c \mathbf{u}_1] \\
 & - \frac{\partial}{\partial \mathbf{x}^{\circ}} \cdot E[\varphi_1 \varphi_2 X_{1,2}^c \mathbf{u}_2].
 \end{aligned}$$

Moreover, the continuous phase continuity equations themselves can also be simplified from the dispersed phase continuity Eqs. (4.12) and (4.18).

## 6. Conclusions

This part I presents a general statistical method for deriving averaged equations for non-turbulent dispersed flows. It belongs to a developing class of methods which mixes the kinetic theory of gases and classical continuum mechanics approaches. Probability in our case refers merely to repeated trials which are themselves a practical consequence of apprehending systems with a large number of degrees of freedom. The only randomising effect comes from the initial conditions of the dispersed phase. Brownian motion and collisions between inclusions are not taken into account.

In standard two-fluid models, the dispersed phase is poorly defined by two single moments, the averaged local volume fraction and the averaged velocity. Here, it is possible to use a natural way of introducing hidden characteristics of the dispersed phase. For instance, the averaged angular velocity field has been introduced. Provided inclusions can be defined by a finite number of parameters with their own equation of evolution, other types of inclusion may be envisaged,

e.g. oscillating bubbles in an acoustic field, non-spherical particles orientating themselves in a shear flow, particles having a dipole interacting with an external electromagnetic field, etc. As a rule, an infinite number of coordinates are strictly required in more complex cases, such as spherical fluid inclusions with some viscosity, immersed in another fluid and inclusions with some deformability (thus requiring conditions to be prescribed for the component of stress normal to the inclusion surface); in fact, only a finite number of them prove to be necessary. It must be admitted that in many real cases, particles often have irregular shapes which may be either permanent (solid particles) or continuously developing (distorted bubbles). Thus, it may be too complicated to enlarge the set of geometrical characteristics of a particle to many extra parameters. Because of these mathematical and physical difficulties, our method is obviously not able to treat all practical flow problems even if they are dilute and laminar. However, in many cases, it continues to serve one important purpose. It can be used as a reliable guide for proposing new closure laws and to assess their validity, either by indicating their structure or by providing the order of magnitude of certain coefficients.

The obtained hierarchies now have to be truncated simultaneously using the same perturbation method involving a small parameter. In many statistical theories devoted to two-phase flows, the assumption of diluteness has been introduced. Our approach presented in a future paper is also based on the same parameter, i.e. the spatially averaged dispersed phase volume fraction. Note that the precision of our approach is somewhat restricted from the outset since only the first order equations of these hierarchies (second- and third-order equations for the dispersed and continuous phases respectively) can be derived in practice, at least if nonlinear terms are kept. On the other hand, there will be no *a priori* assumption about homogeneity of the dispersed phase volume fraction since all evolution equations controlling this phase are available. However, the equations defined at the end of this paper are not yet ready to be treated by any asymptotic method. In subsequent papers they will be simplified beforehand and all the terms appearing in them will receive a straightforward interpretation.

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