Deflection relationships between the homogeneous Kirchhoff plate theory and different functionally graded plate theories

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WE DERIVE FIELD EQUATIONS for a functionally graded plate whose deformations are governed by either the first-order shear deformation theory or the third-order shear-deformation theory. These equations are further simplified for a simply supported polygonal plate. An exact relationship is established between the deflection of the functionally graded plate and that of an equivalent homogeneous Kirchhoff plate. This relationship is used to explicitly express the displacements of a plate particle according to the first-order shear deformation theory in terms of the deflection of a homogeneous Kirchhoff plate. These relationships can readily be used to obtain similar correspondences between the deflections of a transversely isotropic laminated plate and a homogeneous Kirchhoff plate.

Key Words: deflection relationship, functionally graded plate, laminated plate.

1. Introduction

LAMINATED COMPOSITE MATERIALS are commonly used in engineering structures. A sudden change in material properties at the interfaces can result in locally large deformations which may trigger the initiation and propagation of a microcrack in a lamina. One way to overcome this is to use functionally graded materials in which material properties vary continuously. This is achieved by gradually changing the volume fraction of the constituent materials usually only in one (the thickness) direction to obtain a smooth variation of material properties and an optimum response to external thermomechanical loads (REDDY and CHIN [12], PRAVEEN and REDDY [9]).

An interesting issue for plates made of functionally gradient materials is the determination of relationships between their deflections predicted by various shear deformation plate theories and that given by the classical Kirchhoff plate theory. Such relationships have been found for sandwich plates (Hu [2], Liu and Cheng [7], Wang [13]), single-layer homogeneous plates (Wang and Alwis [15], Reddy and Wang [13] and laminated plates materially and geometrically symmetric about the midplane (Cheng and Kitipornchai [1]). For plates symmetric about the midsurface, the stretching and bending deformation modes are uncoupled and hence can be separately analyzed. This, however, is not the case for functionally graded plates whose material properties are generally not symmetric about the midsurface. Here we study deformations of a thin plate made of a functionally graded material and seek relationships between its deflections predicted by two shear deformation theories and that given by the classical Kirchhoff plate theory.

2. Field equations

Consider an undeformed plate of uniform thickness h. We use a rectangular Cartesian coordinate system $\{x_i\}(i = 1, 2, 3)$, with the plane $x_3 = 0$ coincident with the mid-surface of the plate. Hereafter, a comma followed by a subscript i denotes the partial derivative with respect to x_i, and a repeated index implies summation over the range of the index with Latin indices ranging from 1 to 3 and Greek indices from 1 to 2.

The displacement field in the third-order plate theory (HSDT) proposed by Reddy [10], the first-order shear deformation plate theory (FSDT) and the classical laminated plate theory (CLT) can be written as

$$(2.1) \qquad v_\alpha(x_i) = u_\alpha - x_3 u_{3,\alpha} + g\varphi_\alpha, \quad v_3(x_i) = u_3,$$

where u_α, u_3 and φ_α are independent of x_3 and

$$(2.2) \qquad g(x_3) = x_3 \left(1 - \frac{4x_3^2}{3h^2}\right).$$

Note that the hypothesis (2.1) is a special case of that proposed by Kączkowski [4]; the reader is also referred to the survey article by Jemielita [3]. Substitution from (2.2) into (2.1) gives the displacement field for the HSDT and the choices $g(x_3) = x_3$ and $g(x_3) = 0$ give, respectively, the displacement fields for the FSDT and the CLT.

For the linear bending problem of a functionally graded plate subjected to an arbitrary distributed normal load $q(x_\alpha)$ on its surface, the field equations are

$$(2.3) \qquad N_{\alpha\beta,\beta} = 0, \quad M_{\alpha\beta,\alpha\beta} + q = 0, \quad P_{\alpha\beta,\beta} - R_\alpha = 0,$$

where

$$(2.4) \qquad [N_{\alpha\beta}, \ M_{\alpha\beta}, \ P_{\alpha\beta}] = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}[1, \ x_3, \ g]dx_3, \quad R_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3}g_{,3}dx_3,$$

$$(2.5) \qquad \sigma_{\alpha\beta} = H_{\alpha\beta\omega\rho}e_{\omega\rho}, \quad \sigma_{\alpha 3} = 2E_{\alpha 3\omega 3}e_{\omega 3},$$

$$(2.6) \qquad e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

Here σ_{33} is assumed to be negligible. For an isotropic material (Librescu [5])

$$(2.7) \qquad H_{\alpha\beta\omega\rho} = \frac{\nu E}{1-\nu^2}\delta_{\alpha\beta}\delta_{\omega\rho} + \frac{E}{2(1+\nu)}(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}), \quad E_{\alpha 3\omega 3} = \bar{\mu}\delta_{\alpha\omega},$$

where δ_{ij} is the Kronecker delta, E, ν and $\bar{\mu}$ denote respectively, Young's modulus, Poisson's ratio, and the shear modulus. Here we have purposely not set $\bar{\mu} = E/2(1+\nu)$ so that the results may be applicable to a transversely isotropic plate. For the functionally graded plate, the material properties are assumed to vary in the thickness direction only,

$$(2.8) \qquad E = E(x_3), \quad \nu = \nu(x_3), \quad \bar{\mu} = \bar{\mu}(x_3).$$

For a plate made of different isotropic laminae, the material moduli are piecewise constant functions of x_3. Using Eqs. (2.1), (2.5) and (2.6), Eq. (2.4) may alternatively be written as

$$(2.9) \qquad \begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{bmatrix} = (\mathbf{a} - \mathbf{b}) \begin{bmatrix} u_{\omega,\omega} \\ -u_{3,\omega\omega} \\ \varphi_{\omega,\omega} \end{bmatrix} \delta_{\alpha\beta} + \mathbf{b} \begin{bmatrix} \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \\ -u_{3,\alpha\beta} \\ \frac{1}{2}(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}) \end{bmatrix},$$

$$R_\alpha = c\varphi_\alpha,$$

where

$$\mathbf{a} = \int_{-h/2}^{h/2} \mathbf{F}\frac{E}{1-\nu^2}dx_3, \quad \mathbf{b} = \int_{-h/2}^{h/2} \mathbf{F}\frac{E}{1+\nu}dx_3, \quad c = \int_{-h/2}^{h/2} (g_{,3})^2\bar{\mu}dx_3,$$

$$(2.10) \qquad \mathbf{a} = \begin{bmatrix} a_0 & a_4 & a_5 \\ a_4 & a_1 & a_2 \\ a_5 & a_2 & a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 & b_4 & b_5 \\ b_4 & b_1 & b_2 \\ b_5 & b_2 & b_3 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & x_3 & g \\ x_3 & x_3^2 & x_3 g \\ g & x_3 g & g^2 \end{bmatrix}.$$

Note that a_4, a_5, b_4 and b_5 vanish for a plate materially and geometrically symmetric about its midsurface. Substitution from (2.9) into (2.3) results in the following field equations in terms of the five displacement functions u_α, u_3 and φ_α:

$$(2.11) \quad \frac{1}{2}b_0 u_{\alpha,\beta\beta} + (a_0 - \frac{1}{2}b_0)u_{\beta,\beta\alpha} - a_4 u_{3,\alpha\beta\beta} + \frac{1}{2}b_5 \varphi_{\alpha,\beta\beta}$$
$$+ \left(a_5 - \frac{1}{2}b_5\right)\varphi_{\beta,\beta\alpha} = 0,$$

$$(2.12) \quad a_4 u_{\alpha,\alpha\beta\beta} - a_1 u_{3,\alpha\alpha\beta\beta} + a_2 \varphi_{\alpha,\alpha\beta\beta} + q = 0,$$

$$(2.13) \quad \frac{1}{2}b_5 u_{\alpha,\beta\beta} + (a_5 - \frac{1}{2}b_5)u_{\beta,\beta\alpha} - a_2 u_{3,\alpha\beta\beta} + \frac{1}{2}b_3 \varphi_{\alpha,\beta\beta}$$
$$+ (a_3 - \frac{1}{2}b_3)\varphi_{\beta,\beta\alpha} - c\varphi_\alpha = 0.$$

Furthermore, substitution for $u_{\alpha,\beta\beta}$ from (2.11) into (2.12) and (2.13) yields

$$(2.14) \quad (a_4^2 - a_0 a_1)u_{3,\alpha\alpha\beta\beta} + (a_0 a_2 - a_4 a_5)\varphi_{\alpha,\alpha\beta\beta} + a_0 q = 0,$$

$$(2.15) \quad (a_5 b_0 - a_0 b_5)u_{\beta,\beta\alpha} + (a_4 b_5 - a_2 b_0)u_{3,\alpha\beta\beta} + \frac{1}{2}(b_0 b_3 - b_5^2)\varphi_{\alpha,\beta\beta}$$
$$+ (a_3 b_0 - a_5 b_5 + \frac{1}{2}b_5^2 - \frac{1}{2}b_0 b_3)\varphi_{\beta,\beta\alpha} - b_0 c\varphi_\alpha = 0.$$

To simplify the field equations, two new potential functions, w and f, are introduced such that

$$(2.16) \quad \varphi_\alpha = (u_3 + w)_{,\alpha} + \epsilon_{\alpha\omega} f_{,\omega},$$

where $\epsilon_{\alpha\omega}$ is the two-dimensional permutation tensor. Even though Eq. (2.16) uniquely defines φ_α, however, w and f are not uniquely determined from it. This is because the Cauchy-Riemann equation

$$(2.17) \quad (u_3^* + w^*)_{,\alpha} + \epsilon_{\alpha\omega} f_{,\omega}^* = 0$$

always has a solution $[f^* + i(u_3^* + w^*)]$ which is an analytic function of the complex variable $(x_1 + ix_2)$. The expression (2.16) for φ_α remains unchanged when $(u_3 + w)$ and f are simultaneously incremented by $(u_3^* + w^*)$ and f^* respectively.

Substituting for φ_α from Eq. (2.16) into Eqs. (2.11), (2.14) and (2.15) we obtain

$$(2.18) \quad \frac{1}{2}b_0 u_{\alpha,\beta\beta} + (a_0 - \frac{1}{2}b_0)u_{\beta,\beta\alpha} - a_4 u_{3,\alpha\beta\beta} + a_5(u_3 + w)_{,\alpha\beta\beta}$$
$$+ \frac{1}{2}b_5 \epsilon_{\alpha\omega} f_{,\omega\beta\beta} = 0,$$

(2.19) $[(a_4^2 - a_0a_1 + a_0a_2 - a_4a_5)u_3 + (a_0a_2 - a_4a_5)w],_{\alpha\alpha\beta\beta} + a_0q = 0,$

(2.20) $[(a_5b_0 - a_0b_5)u_{\beta,\beta} + (a_4b_5 - a_2b_0)u_{3,\beta\beta} + (a_3b_0 - a_5b_5)(u_3 + w),_{\beta\beta}$

$$-b_0c(u_3 + w)],_\alpha + \epsilon_{\alpha\omega}[\frac{1}{2}(b_0b_3 - b_5^2)f,_{\beta\beta} - b_0cf],_\omega = 0.$$

Equation (2.20) is the Cauchy-Riemann equation, which can equivalently be written as

(2.21) $\frac{1}{2}(b_0b_3 - b_5^2)f,_{\beta\beta} - b_0cf + i[(a_5b_0 - a_0b_5)u_{\beta,\beta}$

$$+(a_4b_5 - a_2b_0 + a_3b_0 - a_5b_5)u_{3,\beta\beta} - b_0cu_3$$

$$+(a_3b_0 - a_5b_5)w,_{\beta\beta} - b_0cw] = H(x_1 + ix_2),$$

where $H(x_1 + ix_2)$ is an analytic function. Furthermore, viewing Eq. (2.21) as a nonhomogeneous partial differential equation for unknowns (u_α, u_3, w, f), its solution is the sum of a homogeneous general solution and a particular solution. Since both the real and imaginary parts of $H(x_1 + ix_2)$ are harmonic functions, the particular solution $(u_\alpha^*, u_3^*, w^*, f^*)$ can be taken as

(2.22) $u_\alpha^* = 0, \quad u_3^* = 0, \quad -b_0c(f^* + iw^*) = H(x_1 + ix_2),$

which satisfies the Cauchy-Riemann condition (2.17). We note that the particular solution makes trivial contributions to u_α, u_3 and φ_α, and hence to displacements and stresses of the functionally graded plate. Consequently, it is discarded. The homogeneous part of Eq. (2.21) gives

(2.23) $(a_5b_0 - a_0b_5)u_{\beta,\beta} + (a_4b_5 - a_2b_0 + a_3b_0 - a_5b_5)u_{3,\beta\beta}$

$$-b_0cu_3 + (a_3b_0 - a_5b_5)w,_{\beta\beta} - b_0cw = 0,$$

(2.24) $$\frac{1}{2}(b_0b_3 - b_5^2)f,_{\beta\beta} - b_0cf = 0.$$

Let

(2.25) $$\hat{u}_\alpha = u_\alpha + \frac{1}{a_0}[(a_5 - a_4)u_3 + a_5w],_\alpha + \frac{b_5}{b_0}\epsilon_{\alpha\omega}f,_\omega,$$

and using it to eliminate u_α from Eqs. (2.18) and (2.23), we rewrite Eqs. (2.18), (2.19), (2.23) and (2.24) as

(2.26) $$\frac{1}{2}b_0\hat{u}_{\alpha,\beta\beta} + (a_0 - \frac{1}{2}b_0)\hat{u}_{\beta,\beta\alpha} = 0,$$

(2.27) $[(\bar{a}_2 - \bar{a}_1)u_3 + \bar{a}_2 w]_{,\alpha\alpha\beta\beta} + q = 0,$

(2.28) $(a_5 - \dfrac{a_0 b_5}{b_0})\hat{u}_{\beta,\beta} + (\bar{a}_3 - \bar{a}_2)u_{3,\beta\beta} - cu_3 + \bar{a}_3 w_{,\beta\beta} - cw = 0,$

(2.29) $\dfrac{1}{2}\bar{b}_3 f_{,\beta\beta} - cf = 0,$

where

(2.30) $\bar{a}_1 = a_1 - \dfrac{a_4^2}{a_0}, \quad \bar{a}_2 = a_2 - \dfrac{a_4 a_5}{a_0}, \quad \bar{a}_3 = a_3 - \dfrac{a_5^2}{a_0}, \quad \bar{b}_3 = b_3 - \dfrac{b_5^2}{b_0}.$

It follows from Eqs. (2.10) and the usual assumptions, $E > 0$, $-1 < \nu < \dfrac{1}{2}$ that $a_0 > 0$, $b_0 > 0$. This form of field equations is convenient for seeking fundamental solutions. Note that the unknown function f has been uncoupled from the other four unknowns $\hat{u}_\alpha$, u_3 and w in the field Eqs. (2.26) – (2.29), but is still coupled with them in Eq. (2.25) and hence in most of boundary conditions. However, as shown below, for simply supported functionally graded polygonal plates, the unknown f can be totally decoupled and hence separately determined.

3. Simply supported rectilinear edges

We now consider a simply supported polygonal plate, and express boundary conditions as

(3.1) $N_{NN} = 0, \quad M_{NN} = 0, \quad P_{NN} = 0,$

(3.2) $u_3 = 0, \quad u_T = 0, \quad \varphi_T = 0,$

where the upper case subscripts N and T denote, respectively, the normal and tangential directions on the boundary. No implicit summation applies to the repeated upper case subscripts. Also, note that $u_3 = 0$ implies $u_{3,T} = 0$, and

(3.3) $\begin{bmatrix} N_{NN} \\ M_{NN} \\ P_{NN} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{a} \begin{bmatrix} u_{N,N} \\ -u_{3,NN} \\ \varphi_{N,N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

We recall Gram's inequality (Mitrinovic and Vasic [8])

(3.4) $\det(\mathbf{G}) \geq 0,$

where $\mathbf{G} = (G_{ij})$ is a $n \times n$ matrix with elements defined by

(3.5) $G_{ij} = \displaystyle\int_a^b f_i f_j dx_3,$

and the equality in (3.4) holds if and only if the real and integrable functions $f_i(x_3)$ ($x_3 \in [a,b]$; $i = 1,\ldots,n$) are linearly dependent. The Gram theorem implies that

$$(3.6) \qquad \qquad \det(\mathbf{a}) > 0,$$

for the HSDT. Therefore, Eq. (3.3) gives

$$(3.7) \qquad \qquad u_{N,N} = 0, \quad u_{3,NN} = 0, \quad \varphi_{N,N} = 0.$$

Using Eqs. (2.16) and (2.23), Eqs. (3.1) and (3.7) can be written as

$$(3.8) \qquad u_3 = 0, \quad u_{3,NN} = 0, \quad w = 0, \quad w_{,NN} = 0, \quad f_{,N} = 0, \quad u_T = 0,$$

$$u_{N,N} = 0,$$

where only three of the first four and the last three of Eqs. (3.8) are necessary for finding a solution of the bending problem. Note that the unknowns u_α, u_3, w and f are uncoupled in the boundary condition for a simply supported polygonal plate.

4. Deflection relations between different theories

The solution of Eq. (2.29) under the boundary condition $(3.8)_5$ is

$$(4.1) \qquad \qquad f = 0.$$

Note that Eq. (2.29) has the null solution (3.9) only when the polygonal plate is simply supported. Thus for the bending problem of a simply supported functionally graded polygonal plate, only four functions u_α, u_3 and w need to be determined.

Recalling Eqs. (2.5) and (3.8), the boundary conditions associated with the field Eq. (2.6) for a simply supported polygonal plate are

$$(4.2) \qquad \qquad \hat{u}_T = 0, \quad \hat{u}_{N,N} = 0,$$

and the solution of this boundary value problem is $\hat{u}_\alpha = 0$, i.e.,

$$(4.3) \qquad \qquad u_\alpha = -[(a_5 - a_4)u_3 + a_5 w]_{,\alpha}/a_0.$$

The field equations for u_3 and w are the biharmonic Eq. (2.27) and the second-order Eq. (2.28). Equation (2.28) upon using $\hat{u}_\alpha = 0$ yields

$$(4.4) \qquad \qquad (\bar{a}_3 - \bar{a}_2)u_{3,\beta\beta} - cu_3 + \bar{a}_3 w_{,\beta\beta} - cw = 0;$$

and the associated boundary conditions are three of the Eqs. $(3.8)_{1-4}$.

Note that the field Eqs. (2.27) and (4.4) have the same forms as those for a plate (Cheng and Kitipornchai [1]) symmetric about its mid-plane and thus can be regarded as equations for such an equivalent plate with parameters given by Eq. (2.30). The over-barred quantities $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$ defined by (2.30) are the constants of a functionally graded plate equivalent to those of a plate symmetric about the midsurface because $a_4 = a_5 = 0$ for such a plate. This implies that a solution of the bending problem for a simply supported and polygonal functionally graded plate can be equivalently obtained from the solution of the corresponding problem for an identical plate symmetric about the midsurface. The in-plane displacements are then obtained from Eq. (4.3).

We now consider the classical Kirchhoff theory for functionally graded plates. Setting $g(x_3) = 0$ in Eq. (2.10) yields

$$(4.5) \qquad a_2 = a_3 = a_5 = c = 0,$$

or

$$(4.6) \qquad \bar{a}_2 = \bar{a}_3 = c = 0,$$

and Eq. (4.4) is trivially satisfied. We conclude from Eq. (4.3) that the in-plane displacements are given by

$$(4.7) \qquad u_\alpha^K = \frac{a_4}{a_0} u_{3,\alpha}^K,$$

and Eq. (2.27) reduces to

$$(4.8) \qquad -\bar{a}_1 u_{3,\alpha\alpha\beta\beta}^K + q = 0.$$

This is the Kirchhoff field equation for the bending deformation of the functionally graded plate with simply supported rectilinear edges. The superscript K on a variable signifies its value for the Kirchhoff plate theory. The boundary conditions on simply supported rectilinear edges are

$$(4.9) \qquad u_3^K = 0, \quad u_{3,NN}^K = 0.$$

Based on the uniqueness of the solution of the boundary-value problem defined by Eqs. (4.8) and (4.9), and the analogy between the field Eqs. (2.27) and (4.8) and between the boundary conditions $(3.8)_{1-4}$ and (4.9), it can be concluded that

$$(4.10) \qquad (\bar{a}_2 - \bar{a}_1)u_3 + \bar{a}_2 w = -\bar{a}_1 u_3^K.$$

Eliminating the function w from Eqs. (4.4) and (4.10), we obtain

$$(4.11) \qquad (\bar{a}_1 \bar{a}_3 - \bar{a}_2^2)u_{3,\alpha\alpha} - c\bar{a}_1 u_3 = \bar{a}_1 \bar{a}_3 u_{3,\alpha\alpha}^K - c\bar{a}_1 u_3^K.$$

This is an exact relationship between the deflections of the HSDT and the Kirchhoff theories for simply supported polygonal plates made of functionally graded

materials. If the deflection u_3^K for the Kirchhoff theory is known, the deflection u_3 of the HSDT can be computed from the second-order Eq. (4.11) and the boundary condition $(3.8)_1$. Other unknown functions w and u_α are then simply obtained from Eqs. (4.10) and (4.3).

Furthermore, it is seen that the form of Eq. (4.8) is precisely the same as that of the equation governing the bending deformations of a homogeneous Kirchhoff thin plate with bending rigidity $\bar{a}_1$ and subjected to the normal pressure q. Thus the deflection of the functionally graded plate using the HSDT has been connected with the deflection of a homogeneous Kirchhoff thin plate. As there are solutions available for a classical homogeneous thin plate, the calculation of the deflection of the functionally graded plate using the relatively more sophisticated HSDT reduces to solving the second-order differential Eq. (4.11), which is a much easier task than solving the original problem.

The aforestated calculation is even further simplified if one uses the FSDT for the functionally graded plate. In this case, taking $g(x_3) = x_3$ in Eq. (2.10) we get

$$(4.12) \qquad a_1 = a_2 = a_3, \quad a_4 = a_5,$$

or

$$(4.13) \qquad \bar{a}_1 = \bar{a}_2 = \bar{a}_3.$$

Explicit expressions for u_3^F, w^F and u_α^F in terms of the Kirchhoff deflection u_3^K obtained from Eqs. (4.11), (4.10) and (4.3) are

$$(4.14) \qquad u_3^F = u_3^K - \frac{\bar{a}_1}{c^F} u_{3,\alpha\alpha}^K, \quad w^F = -u_3^K, \quad u_\alpha^F = \frac{a_4}{a_0} u_{3,\alpha}^K,$$

where

$$(4.15) \qquad c^F = \kappa \int\limits_{-h/2}^{h/2} \bar{\mu}\, dx_3,$$

and κ is the shear correction factor. Therefore, once the deflection of the homogeneous Kirchhoff plate of rigidity $\bar{a}_1$ is known, the solution of the FSDT is readily obtained through simple algebraic and differential manipulations of the deflection of the Kirchhoff plate. It should be noted that unless $q = 0$, Eq. $(3.8)_2$ is not an essential boundary condition for simply supported edges in the FSDT for which $\det(\mathbf{a}) = 0$. The proof of this statement is omitted.

Our results also apply to a plate made of a transversely isotropic material because we have not required that $\bar{\mu} = E/2(1 + \nu)$. For a transversely isotropic plate with its plane of isotropy parallel to the mid-plane, E and ν are designated, respectively, as Young's modulus and Poisson's ratio in the plane of isotropy and $\bar{\mu}$ as the transverse shear modulus. A typical example is a laminated composite

plate with transversely isotropic laminae. Such composite laminates are widely used in missiles and re-entry vehicles because their special thermomecianical properties provide thermal protection and high flexibility in transverse shear (Librescu and Stein [6]).

For laminated plates made of transversely isotropic materials and symmetric about their midsurfaces, Cheng and Kitipornchai [1] have established relationships between the deflections of a plate according to the HSDT, FSIT and the classical plate theory. Equations (4.11), (4.10), (4.3) and (4.14) represent generalizations of such relationships to a plate that is not symmetric about its mid-plane.

5. An example

Consider a functionally graded rectangular plate simply supported at edges $x_1 = 0, a$ and $x_2 = 0, b$. Under the action of the normal pressure

$$(5.1) \qquad q = Q \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

the deflection due to the bending deformations of the functionally gradec plate according to the HSDT and the Kirchhoff plate theory is assumed to be given by

$$(5.2) \qquad [u_3 \ u_3^K] = [U_3 \ U_3^K] \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

where U_3 and U_3^K are respectively, the central deflections of the plate in the HSDT and classical theory. In view of the relation (4.11) between the deflections of the two theories, we have

$$(5.3) \qquad u_3 = (1 + \beta)U_3^K \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

where
$$(5.4) \qquad \beta = \frac{\bar{a}_2^2(a^2 + b^2)\pi^2}{(\bar{a}_1 \bar{a}_3 - \bar{a}_2^2)(a^2 + b^2)\pi^2 + c\bar{a}_1 a^2 b^2}$$

characterizes the difference in the two deflections. This parameter depends only on the geometry and the material properties of the functionally graded plate. The through-thickness in-plane displacements of the HSDT are given by

$$(5.5) \qquad v_1 = \frac{\gamma \pi}{a} U_3^K \cos \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b}, \quad v_2 = \frac{\gamma \pi}{b} U_3^K \sin \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b},$$

where
$$(5.6) \qquad \gamma(x_3) = \left(\frac{a_4}{a_0} - x_3\right)(1 + \beta) + \left(g - \frac{a_5}{a_0}\right)\frac{\bar{a}_1}{\bar{a}_2}\beta$$

is a function of x_3 and characterizes the through-thickness variation of the in-plane displacements. For the FSDT parameters β and γ are given by

$$(5.7) \qquad \beta^F = \frac{\bar{a}_1 \pi^2}{c^F} \left(\frac{1}{a^2} + \frac{1}{b^2} \right), \quad \gamma^F(x_3) = \frac{a_4}{a_0} - x_3.$$

It can be shown that γ for the Kirchhoff plate theory is the same as that for the FSDT. Since there is no interlayer between two different materials in a functionally graded plate, computation of the out-of-plane shear stresses $\sigma_{\alpha 3}$ and the normal stress σ_{33} is not important which, compared with the longitudinal stresses, are of small orders of magnitude. The longitudinal stresses are given by

$$\sigma_{11} = -\frac{\gamma E \pi^2}{1 - \nu^2} \left(\frac{1}{a^2} + \frac{\nu}{b^2} \right) U_3^K \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

$$(5.8) \qquad \sigma_{22} = -\frac{\gamma E \pi^2}{1 - \nu^2} \left(\frac{\nu}{a^2} + \frac{1}{b^2} \right) U_3^K \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{b},$$

$$\sigma_{12} = \frac{\gamma E \pi^2}{(1 + \nu) a b} U_3^K \cos \frac{\pi x_1}{a} \cos \frac{\pi x_2}{b}.$$

The functionally graded materials are usually made by mixing two distinct material phases, such as a metal and a ceramic. The effective material properties can be obtained from the "rule of mixture"

$$(5.9) \qquad P_{\text{eff}} V = P_m V_m + P_c V_c, \quad V_m + V_c = 1,$$

where P stands for the material property, V for the volume fraction, and subscripts m, c and eff stand, respectively, for the metal, ceramic and the effective. A more accurate determination of the macroscopic material properties requires a better understanding of the microstructure and deformation of the constituents. The relation $(5.9)_1$ is exact for the mass density.

The volume fraction of the ceramic phase is assumed to be given by

$$(5.10) \qquad V_c = \left(\frac{h + 2x_3}{2h} \right)^n.$$

Figure 1 shows the through-thickness variation of the volume fraction of the ceramic for $n = 0.2, 0.5, 1, 2, 5$. Note that the bottom surface of the plate is metal-rich and the top surface ceramic-rich.

The dimensionless through-thickness in-plane displacement and the longitudinal stress are defined by

$$(5.11) \qquad \bar{v}_1 = \frac{v_1(0, b/2, x_3)}{U_3^K}, \qquad \bar{\sigma}_{11} = \frac{a \sigma_{11}(a/2, b/2, x_3)}{E^* U_3^K},$$
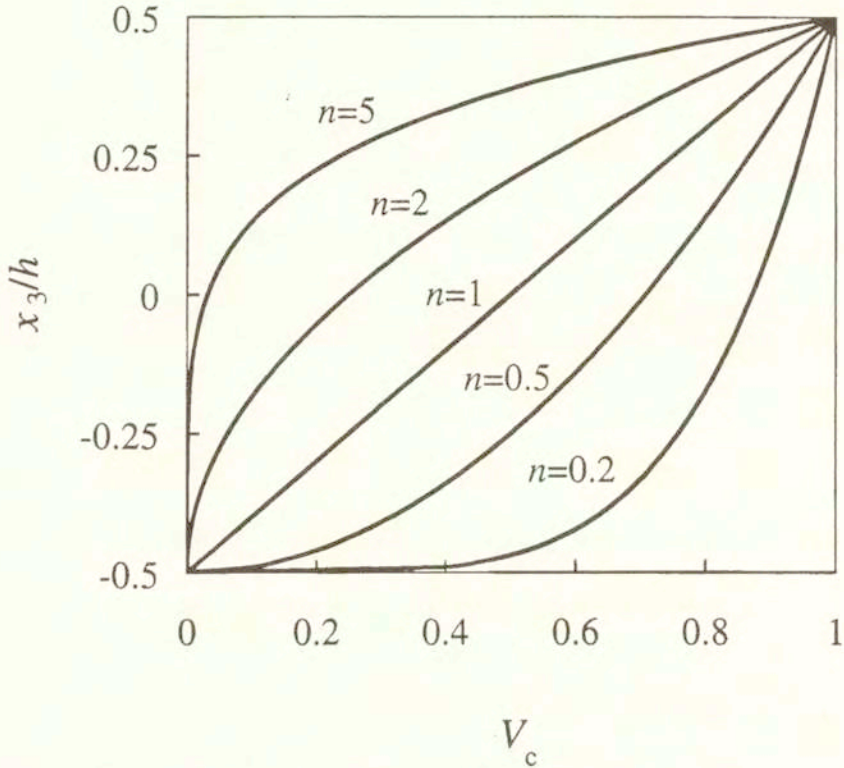
FIG. 1. Through-the-thickness distribution of the volume fraction of the ceramic phase in the functionally gradient plate.

where E^* is set equal to 1 GPa. We take the shear correction factor $\kappa = 5/6$ in the FSDT. Note that the value $5/6$ of the shear correction factor was proposed for a homogeneous and isotropic plate; its use in a functionally graded plate may not be very realistic. The functionally graded material is a mixture of aluminum and zirconia (PRAVEEN and REDDY [9]), and we take

$$(5.12) \quad E_m = 70\text{GPa}, \quad E_c = 151\text{GPa}, \quad \nu_m = \nu_c = 0.3, \quad a = b = 10h.$$

For simplicity, Poisson's ratio for both aluminum and zirconia is assigned the same value; it is equivalent to the assumption that the effective value of the shear modulus is also derived from Eq. (5.9).

Table 1 lists values of U_3/U_3^K and U_3^F/U_3^K for $n = 0, 0.2, 0.5, 1, 2$ and 5. Here U_3^F equals the central deflection according to the FSDT. Figures 2 and 3 show the through-thickness distributions of the non-dimensional in-plane displacement $\bar{v}_1$ and the longitudinal stress $\bar{\sigma}_{11}$ obtained by using (a) the HSDT and (b) the FSDT. These variables are nondimensionalized (e.g. see Eq. (5.11)) so once the central deflection of the effective homogeneous Kirchhoff plate is known, the displacements and stresses of the functionally graded plate can be determined.
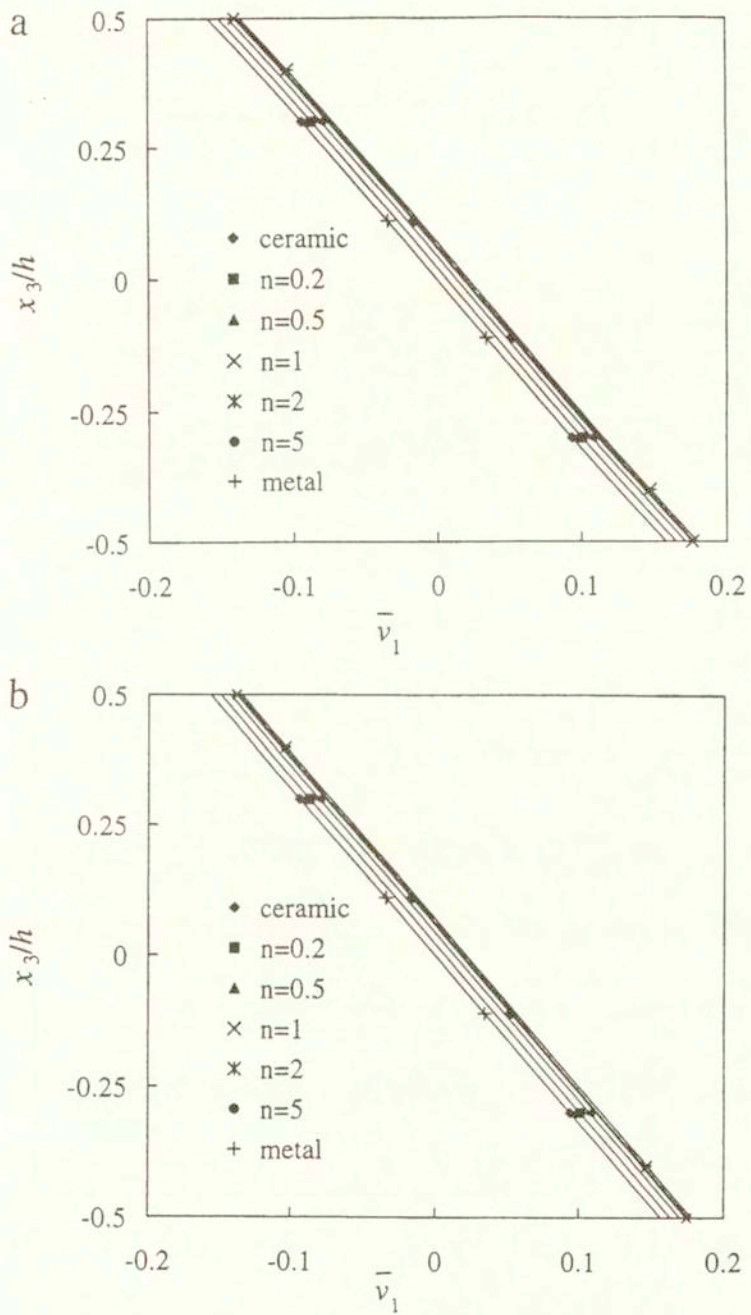
FIG. 2. Through-the-thickness distribution of the dimensionless in-plane displacement of the functionally graded square plate ($a = 10h$) using (a) the third-order plate theory and (b) the first-order plate theory.
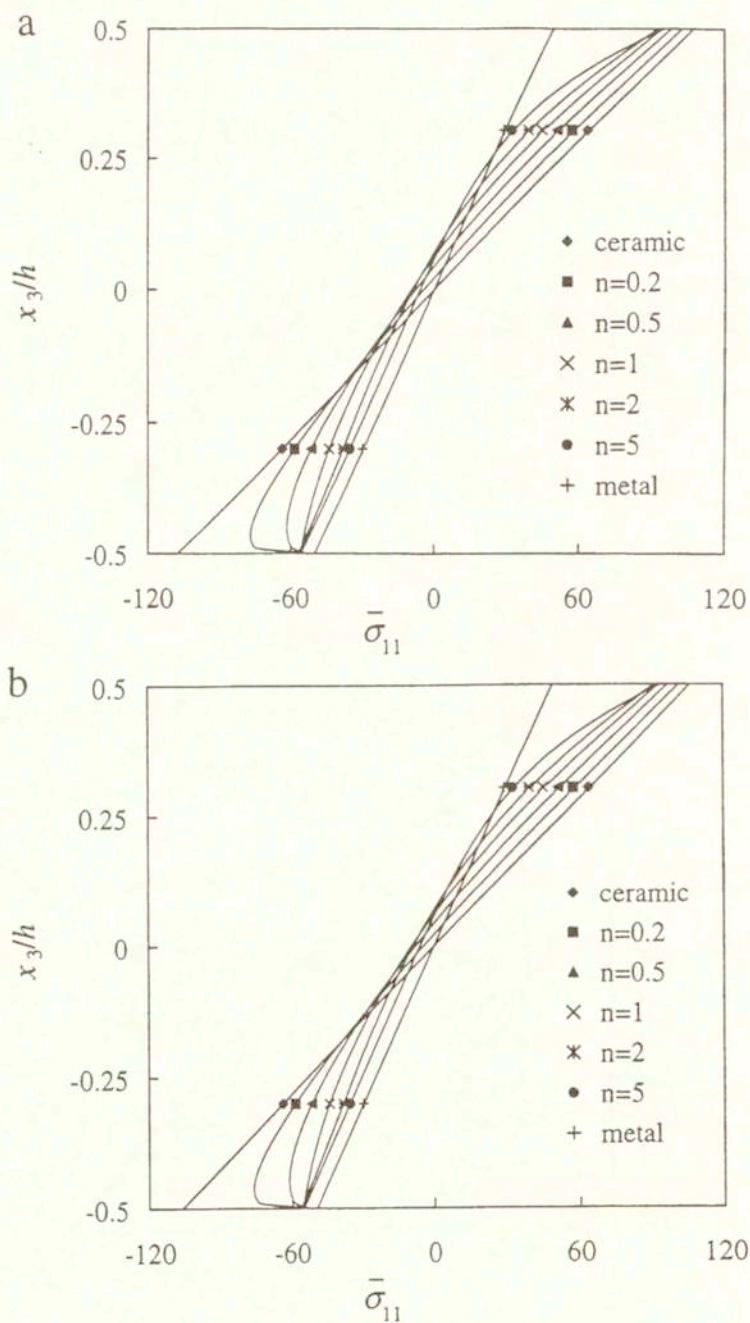
[155]

FIG. 3. Through-the-thickness distribution of the dimensionless longitudinal stress of the functionally graded square plate ($a = 10h$) using (a) the third-order plate theory and (b) the first-order plate theory.

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Table 1. Central deflection of a functionally gradient square plate according to the three theories.

n	0	0.2	0.5	1	2	5
U_3/U_3^K	1.056360	1.054326	1.053157	1.053836	1.057850	1.064430
U_3^F/U_3^K	1.056398	1.054802	1.053711	1.053872	1.056260	1.060650

It is clear from the values listed in Table 1 that for each value of n, the classical Kirchhoff plate theory underestimates the central deflection of the plate by about 5.5% as compared to that given by either one of the other two plate theories studied herein. Results plotted in Figs. 2 and 3 reveal that the in-plane displacement $\bar{v}_1$ and the stress $\bar{\sigma}_{11}$ calculated from the two theories essentially coincide with each other. Thus for the problem studied herein results predicted by the FSDT are accurate enough for all practical purposes. This is because the through-thickness distribution of the in-plane displacement for the HSDT is nearly affine and agrees with that assumed in the FSDT. The FSDT obviates the need to solve the second-order differential Eq. (4.11).

The curves depicting the through-thickness distributions of the in-plane displacement $\bar{v}_1$ are parallel to each other for all values of the volume fraction V_c of the ceramic. The largest deviation between the values of $\bar{v}_1$ for an equivalent homogeneous plate and a functionally gradient plate occurs for $n = 2$. The maximum value of $|\bar{\sigma}_{11}|$ depends upon V_c. For $n = 0.2$ and 0.5, the magnitude of the compressive $\bar{\sigma}_{11}$ is maximum at a point a little above the lower surface of the plate. However, for other values of n, the magnitude of $\bar{\sigma}_{11}$ is maximum at a point on the top and bottom surfaces of the plate, as is the case for a homogeneous plate.

6. Conclusions

Two potential functions have been used to derive a set of equations that govern the deformations of a functionally graded plate. The deflections of a simply supported functionally graded polygonal plate given by the first-order shear deformation theory (FSDT) and the higher-order shear deformation theory (HSDT) have been related to that of an equivalent homogeneous Kirchhoff plate. These relationships are valid for a laminated plate that is not necessarily symmetric about its midsurface, and have been used to compute results for a simply supported square metal-ceramic plate.

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