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Nonlinearity and memory effects in low temperature heat propagation

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IN ORDER TO ACCOUNT for low temperature heat propagation phenomena in crystals of sodium fluoride and bismuth, we employ a thermodynamic model for rigid materials involving a vector-field internal state variable. The model is either wavelike or diffusive, depending on the temperature regime considered.

1. Introduction

IN THIS PAPER we continue an investigation ([12, 21]) of the effects of nonlinearity and memory on the propagation of heat waves through crystalline materials at low temperatures. The work is intended to extend the interpretation of the experimental results of [5, 6, 15] and [18] in the context of thermodynamics with internal state parameters. These experimental results gave evidence, as had been seen previously in ^{3}He ([1]), that there existed second sound in crystals of sodium fluoride and bismuth at temperatures below where the materials reached their peak thermal conductivity (approximately 18.5 K and 4.5 K respectively). There was an absence of any such wavelike thermal phenomena at higher temperatures, where only diffusive heat propagation was observed. The speed, U_E, of small-amplitude waves tended to zero as the temperature approached the temperature of peak conductivity from below, while appearing, at least in the case of bismuth, to tend to a finite limit as the temperature fell towards 0 K (no useful measurements could be made below 10 K in sodium fluoride because of interference from transverse and longitudinal mechanical waves). This material-dependent temperature of peak thermal conductivity, or *critical* temperature, is denoted by ϑ_λ.

Our present approach is different from our earlier results, which were built on the ideas in [11, 13]. The internal variable[1] responsible for the effective memory in heat propagation will here be a vector field, $\mathbf{p}$, rather than the *semi-empirical* temperature, β, a scalar field. The advantage of this setting is that mathematically, the system of equations derived for a rigid conductor becomes a first order hyperbolic system, a balance law in $\mathbf{p}$ and ϑ, which reduces to a nonlinearly damped two by two (rather than three by three) system in the one-dimensional case. Physically, it also becomes more straightforward to apply experimental data (the observed speed of second sound waves and heat conductivity as functions of temperature) to constitutive equations, avoiding the necessity of integration and making invertibility assumptions, which further resolves a difficulty at zero kelvin where the use of simple algebraic constitutive functions could lead to an unphysical singularity, and lets us extend our analysis to bismuth.

The method of describing thermal wave phenomena in inelastic bodies using vector-valued internal state variables was introduced by KOSIŃSKI [9], and one of the authors, [22]. Related ideas were also introduced in [4] and [16] (see [14]). Here we employ this approach in order to propose physically motivated constitutive equations for bismuth and sodium fluoride which depend on ϑ_λ, and investigate the relation between ϑ_λ and a further temperature ϑ_m where the system loses genuine nonlinearity. Corresponding results have been obtained in [12, 21] for sodium fluoride, using the scalar-valued internal state variable, β. Loss of genuine nonlinearity has also been found in the setting of extended thermodynamics ([19, 20]). In that context, it was speculated that the temperature at which this occurs plays a role in solids analogous to the lambda point in liquid helium at which the speed of second sound vanishes. The analysis based on our constitutive equations shows that these temperatures are however different.

We now attempt to motivate the ideas contained in the derivation, which will be presented fully in Sec. 2. Consider the classical constitutive equation for heat flux, $\mathbf{q} = -k_0 \nabla \vartheta$, where $\mathbf{q}$ denotes heat flux, k_0 the conductivity and ϑ the absolute temperature. One approach for allowing $\mathbf{q}$ to become dependent the on effect of memory is to consider if as a functional of the history of the temperature gradient,

$$\mathbf{q} = -\frac{k_0}{\tau} \int_{-\infty}^{t} e^{-\frac{1}{\tau}(t-s)} \nabla \vartheta(\mathbf{x}, s) ds, \tag{1.1}$$

where τ denotes a relaxation time. An extensive discussion of this and more modern ideas can be found in [7] and [8]. It follows easily from (1.1) that one

[1]) Internal parameters were introduced to constitutive models for thermodynamics of solids by COLEMAN and GURTIN, [2].

obtains the well-known Maxwell-Cattaneo equation

$$\tau \mathbf{q}_t + \mathbf{q} = -k_0 \nabla \vartheta. \tag{1.2}$$

In order to extend the idea behind (1.1), define

$$\mathbf{p} = \frac{1}{\tau} \int_{-\infty}^{t} e^{-\frac{1}{\tau}(t-s)} \nabla \vartheta(\mathbf{x}, s) ds, \tag{1.3}$$

so that (1.1) is equivalent to

$$q = -k_0 \mathbf{p}, \tag{1.4}$$

$$\mathbf{p}_t = -\frac{1}{\tau}\mathbf{p} + \frac{1}{\tau}\nabla \vartheta \tag{1.5}$$

which in the steady-state case, $\mathbf{p}_t = \mathbf{0}$, implies the classical relation

$$\mathbf{q} = -k_0 \nabla \vartheta. \tag{1.6}$$

Next consider the more general relations

$$\begin{aligned} \mathbf{q} &= -\alpha(\vartheta) \int_{-\infty}^{t} e^{-b(t-s)} \nabla f_1(\vartheta)(\mathbf{x}, s) ds, \\ \mathbf{p} &= \int_{-\infty}^{t} e^{-b(t-s)} \nabla f_1(\vartheta)(\mathbf{x}, s) ds, \end{aligned} \tag{1.7}$$

with $b > 0$, which are equivalent to

$$\mathbf{q} = -\alpha(\vartheta)\mathbf{p}, \tag{1.8}$$

$$\mathbf{p}_t = -b\mathbf{p} + g_1(\vartheta)\nabla \vartheta, \tag{1.9}$$

where $f_1'(\vartheta) = g_1(\vartheta)$ and, for $\mathbf{p}_t = \mathbf{0}$,

$$\mathbf{q} = -\frac{\alpha(\vartheta) g_1(\vartheta)}{b} \nabla \vartheta \equiv -\mathcal{K}(\vartheta) \nabla \vartheta. \tag{1.10}$$

It will be seen in the next section that there is a thermodynamic relationship, (2.18), showing that

$$\alpha(\vartheta) = \psi_{20} \vartheta^2 g_1(\vartheta), \tag{1.11}$$

where ψ_{20} is a constant coming from the Helmholtz free energy function ψ,

$$\psi = \psi_1(\vartheta) + \frac{1}{2}\psi_{20}\vartheta|\mathbf{p}|^2. \tag{1.12}$$

Using (1.10) and (1.11), the steady-state thermal conductivity coefficient $\mathcal{K}(\vartheta)$ is given by

$$\mathcal{K}(\vartheta) = \frac{\psi_{20}}{b}\,(\vartheta g_1(\vartheta))^2, \tag{1.13}$$

and, as will be seen in the next section, the speed of second sound satisfies

$$U_E^2 = \psi_{20}\,\frac{(g_1(\vartheta))^2}{c_0\vartheta}, \tag{1.14}$$

where c_0 is a constant, cf. (2.26), (2.27). Recalling the critical temperature ϑ_λ,

$$U_E^2 \to 0 \text{ as } \vartheta \to \vartheta_\lambda -, \tag{1.15}$$

(1.14) implies that $g_1(\vartheta) \to 0$ as $\vartheta \to \vartheta_\lambda -$, from which (1.13) gives

$$\mathcal{K}(\vartheta) \to 0 \text{ as } \vartheta \to \vartheta_\lambda - . \tag{1.16}$$

Unfortunately, (1.16) is incompatible with experimental evidence ([5]) which shows a large peak in thermal conductivity at ϑ_λ, and a further generalization of either (1.7), or of (1.8) and (1.9) is needed in order to account for this fact. We replace (1.8) and (1.9) with

$$\mathbf{q} = -\alpha(\vartheta)\mathbf{p}, \tag{1.17}$$

$$\mathbf{p}_t = g_1(\vartheta)\nabla\vartheta + g_2(\vartheta)\mathbf{p}. \tag{1.18}$$

If, in addition, we wish to take into account the "effective", Fourier conductivity $k(\vartheta)$, [7], which is considered as part of the heat flux law in Jeffrey's type materials, where

$$\mathbf{q} = -k(\vartheta)\nabla\vartheta - \alpha_0 \int_{-\infty}^{t} e^{-b(t-s)}\nabla\vartheta(\mathbf{x}, s)ds, \quad \alpha_0 = \text{constant}, \tag{1.19}$$

we can replace the second term on the right in (1.19) by the right side of the expression for $\mathbf{q}$ in (1.7), and the constitutive equation for $\mathbf{q}$ becomes

$$\mathbf{q} = -k(\vartheta)\nabla\vartheta - \alpha(\vartheta)\mathbf{p}. \tag{1.20}$$

Equations (1.18) and (1.20) make it possible to account for small diffusive effects such as the broadening which can be seen in travelling pulses of second sound as the temperature increases towards ϑ_λ.

The approach introduced here, based on (1.20) and (1.18), is equivalent to that in [12] and [21] only if $\mathbf{p}$ is a gradient field, $\mathbf{p} = \nabla\beta$ for the scalar field β representing a *semi-empirical* temperature. This is however possible only if the evolution equation for $\mathbf{p}$ is of the form (1.9), when (1.20) and (1.18) reduce to

$$\mathbf{q} = -k(\vartheta)\nabla\vartheta - \alpha(\vartheta)\nabla\beta, \tag{1.21}$$

$$\beta_t = f_1(\vartheta) - b\beta. \tag{1.22}$$

In our case however, and in general, the models are distinct.

We present the general framework of the model in the next section and provide examples of constitutive functions which we relate to experimental data in Sec. 3. In the final Section, we will use these functions in examining three-dimensional weakly discontinuous plane waves propagating through crystals.

2. The governing system of equations

In the following section there are parallels to the derivation of [21] concerning the thermodynamics of materials which allow the propagation of low temperature heat pulses. For clarity, we will give a derivation of the present model and note differences where they occur, and refer the reader to that paper for more detailed information.

As in [21], we let ϑ_λ be the critical temperature below which second sound is observed. The vector field, $\mathbf{p}$, as defined in the Introduction, is related to the absolute temperature, ϑ, and its gradient through the initial value problem (1.18) together with appropriate initial data. That is (see [22]),

$$\mathbf{p}_t = g_1(\vartheta)\nabla\vartheta + g_2(\vartheta)\mathbf{p}, \tag{2.1}$$

$$\mathbf{p}(\mathbf{x}, 0) = \mathbf{p}_0(\mathbf{x}). \tag{2.2}$$

The free energy per unit volume, ψ, entropy density, η, and heat flux, $\mathbf{q}$, are related by

$$\psi(\vartheta, \mathbf{p}) = \psi_1(\vartheta) + \frac{1}{2}\psi_2(\vartheta)|\mathbf{p}|^2, \tag{2.3}$$

$$\mathbf{q} = -k(\vartheta)\nabla\vartheta - \alpha(\vartheta)\mathbf{p}, \tag{2.4}$$

and

$$\eta = -\partial_\vartheta\psi(\vartheta, \mathbf{p}). \tag{2.5}$$

The free energy is connected to the internal energy per unit volume, ε, by

$$\psi = \varepsilon - \eta\vartheta. \tag{2.6}$$

Balance of energy and the second law of thermodynamics imply

$$\varepsilon_t + \nabla \cdot \mathbf{q} = r, \tag{2.7}$$

and

$$\eta_t + \nabla \cdot \left(\frac{\mathbf{q}}{\vartheta}\right) \geq \frac{r}{\vartheta}, \tag{2.8}$$

where r is the body heat supply per unit volume.

Using (2.1), (2.3) – (2.6) and (2.8) as in [21], we find

$$\psi_2(\vartheta)g_2(\vartheta)|\mathbf{p}|^2 + \left(\psi_2(\vartheta)g_1(\vartheta) - \frac{\alpha(\vartheta)}{\vartheta}\right)\nabla\vartheta \cdot \mathbf{p} - \frac{1}{\vartheta}k(\vartheta)|\nabla\vartheta|^2 \leq 0. \tag{2.9}$$

This inequality is satisfied for arbitrary choices of $\nabla\vartheta$ and $\mathbf{p}$ if and only if

$$\psi_2(\vartheta)g_2(\vartheta) \leq 0, \quad k(\vartheta) \geq 0 \tag{2.10}$$

and

$$\left(\psi_2(\vartheta)g_1(\vartheta) - \frac{\alpha(\vartheta)}{\vartheta}\right)^2 \leq -4\frac{k(\vartheta)}{\vartheta}\psi_2(\vartheta)g_2(\vartheta). \tag{2.11}$$

By setting

$$\alpha(\vartheta) = \vartheta\psi_2(\vartheta)g_1(\vartheta), \tag{2.12}$$

(2.11) is satisfied for arbitrary choices of admissible $k(\vartheta)$ including $k(\vartheta) = 0$

As in [21], we adopt a particular form, (1.12), for (2.3),

$$\psi_2(\vartheta) = \psi_{20}\vartheta, \quad \psi_{20} > 0. \tag{2.13}$$

For this choice of ψ, the internal energy ε is related to ψ_1 by

$$\varepsilon(\vartheta) = \psi_1(\vartheta) - \psi_1'(\vartheta)\vartheta. \tag{2.14}$$

and the specific heat $c_v > 0$ in terms of ϑ by

$$c_v(\vartheta) = \varepsilon'(\vartheta). \tag{2.15}$$

Finally, in the absence of a body heat supply, Eqs. (2.7), (2.4), (2.15), and (2.1) give

$$c_v(\vartheta)\vartheta_t - \nabla \cdot (k(\vartheta)\nabla\vartheta + \alpha(\vartheta)\mathbf{p}) = 0, \tag{2.16}$$

$$\mathbf{p}_t = g_1(\vartheta)\nabla\vartheta + g_2(\vartheta)\mathbf{p}, \tag{2.17}$$

where by (2.10), (2.11) and (2.13),

$$\alpha(\vartheta) = \psi_{20}\vartheta^2 g_1(\vartheta), \tag{2.18}$$

$$k(\vartheta) \geq 0 \quad \text{and} \quad g_2(\vartheta) \leq 0. \tag{2.19}$$

By letting $\mathbf{p}_t = 0$ in (2.17),

$$g_1(\vartheta)\nabla\vartheta = -g_2(\vartheta)\mathbf{p}, \tag{2.20}$$

and taking $\mathbf{p}$ and $\nabla\vartheta$ to point in the same direction in this case, implies

$$g_1(\vartheta) \geq 0 \tag{2.21}$$

from (2.19). The steady-state conductivity coefficient, $\mathcal{K}(\vartheta)$ (in contrast with (1.10), (1.13)) for (2.4) becomes

$$\mathbf{q}(\vartheta) = -\left(k(\vartheta) - \frac{\psi_{20}\vartheta^2 g_1^2(\vartheta)}{g_2(\vartheta)}\right)\nabla\vartheta = -\mathcal{K}(\vartheta)\nabla\vartheta. \tag{2.22}$$

In order for the equations derived above to hold over the critical temperature ϑ_λ, as well as under, it is sufficient to make the following assumption concerning the constitutive functions, $g_1, g_2 \in C(\mathbb{R}_+)$,

$$\lim_{\vartheta\to\vartheta_\lambda -} \frac{g_1^2(\vartheta)}{g_2(\vartheta)} > 0 \quad \text{and} \quad g_i(\vartheta) = 0, \ \ i = 1, 2, \quad \vartheta \geq \vartheta_\lambda. \tag{2.23}$$

This ensures both the required Fourier conductivity above ϑ_λ (see (2.4), with $\alpha(\vartheta) = 0$ by (2.8)), and from (2.22), a conductivity peak as $\vartheta \to \vartheta_\lambda -$.[2] For $\vartheta \geq \vartheta_\lambda$, Eqs. (2.16) and (2.17) reduce to

$$c_v(\vartheta)\vartheta_t - \nabla\cdot(k(\vartheta)\nabla\vartheta) = 0, \tag{2.24}$$

and

$$\mathbf{p}_t = 0. \tag{2.25}$$

Thus the vector field $\mathbf{p}$ no longer possesses any time dependence and decouples from the field equation for temperature. In the absence of viscosity, $k(\vartheta)$, and the transition temperature, ϑ_λ, the system (2.16), (2.17) in ϑ and $\mathbf{p}$ is equivalent to that obtained in terms of ϑ and $\mathbf{q}$ by MORRO and RUGGERI ([17]).

For $\vartheta < \vartheta_\lambda$, neglecting the influence of $k(\vartheta)$ in (2.4) (the inviscid limit), the system (2.16), (2.17) is a hyperbolic balance law. The expression for the second

[2]) It may be useful in certain situations to modify (2.23) with a transition layer $(\vartheta_\lambda, \vartheta_\lambda + \varepsilon)$ over which $\frac{g_1^2(\vartheta)}{g_2(\vartheta)}$ decays to zero smoothly.

sound speed U_E is the nonzero characteristic speed λ in (4.20), which provides the speed of weakly discontinuous waves propagating into a state for which $\mathbf{q} = 0$,

$$U_E^2 = \frac{\alpha(\vartheta) g_1(\vartheta)}{c_v(\vartheta)} = \frac{\psi_{20} \vartheta^2 g_1(\vartheta)^2}{c_v(\vartheta)}, \tag{2.26}$$

where we have used relation (2.18).

Specific constitutive functions may be derived from experimental data ([5, 6, 15] and [18]). Data for U_E give the function $g_1(\vartheta)$, while data for the steady-state conductivity $\mathcal{K}(\vartheta)$ give $g_2(\vartheta)$, (cf. (2.22) with $k(\vartheta) = 0$). For the heat capacity we will use Debye's law,

$$c_v(\vartheta) = c_0 \vartheta^3, \quad c_0 > 0. \tag{2.27}$$

We note, in particular, that (2.22) and (2.26) provide very simple connections between the constitutive functions g_1, g_2, and the steady-state conductivity and second sound speed (cf. [12, 21]).

3. Constitutive functions

In order to derive a specific example for $g_1(\vartheta)$, we will use an empirical relation employed to interpolate experimental data for NaF and Bi ([3]),

$$U_E(\vartheta)^{-2} = A + B\vartheta^n, \tag{3.1}$$

where, for ϑ measured in degrees kelvin, and U_E in cm/sec,

$$n = 3.10, \quad A = 9.09 \times 10^{-12}, \quad B = 2.22 \times 10^{-15}, \quad 10 \text{ K} \leq \vartheta \leq 18 \text{ K}, \tag{3.2}$$

for NaF, and

$$n = 3.75, \quad A = 9.07 \times 10^{-11}, \quad B = 7.58 \times 10^{-13}, \quad 1 \text{ K} \leq \vartheta \leq 4 \text{ K}, \tag{3.3}$$

for Bi.

Equation (3.1) fits the available experimental data ([5, 6, 15] and [18]) for the range of temperatures where second sound is detected, that is, for a set of temperatures below ϑ_λ. For high purity crystals of Bi and NaF, we take the values of ϑ_λ to be 4.5K and 18.5K, respectively. We will take, for $\vartheta < \vartheta_\lambda$,

$$g_1(\vartheta) = g_{10}(\vartheta)(\vartheta_\lambda - \vartheta)_+^{r_1}, \quad g_{10}(\vartheta) > 0, \quad g_{10}(\vartheta_\lambda) \neq 0, \tag{3.4}$$

and

$$g_2(\vartheta) = g_{20}(\vartheta)(\vartheta_\lambda - \vartheta)_+^{r_2}, \quad g_{20}(\vartheta) < 0, \quad g_{20}(\vartheta_\lambda) \neq 0, \tag{3.5}$$

subject to (2.23), where $z_+^r \equiv z^r H(z)$ and $H(z)$ denotes the Heaviside step function.

In this case we obtain the following formulae for U_E (see (2.26)) and the conductivity $\mathcal{K}(\vartheta)$ with $k(\vartheta) = 0$ (see (2.22)) using (2.27),

$$U_E(\vartheta)^2 = \psi_{20} \frac{g_{10}(\vartheta)^2}{c_0 \vartheta} (\vartheta_\lambda - \vartheta)_+^{2r_1}, \tag{3.6}$$

and

$$\mathcal{K}(\vartheta) = -\psi_{20} \frac{\vartheta^2 g_{10}(\vartheta)^2}{g_{20}(\vartheta)} (\vartheta_\lambda - \vartheta)_+^{2r_1 - r_2}. \tag{3.7}$$

Here the case $2r_1 = r_2$ admits a finite peak in conductivity at $\vartheta = \vartheta_\lambda$, while for $2r_1 < r_2$ the peak is infinite. Observation of second sound in bismuth as $\vartheta \to 0$ ([18]) indicates that U_E reaches a finite, nonzero limit there. We reflect this by choosing

$$g_{10}(\vartheta) = a\vartheta^{\frac{1}{2}}, \quad a > 0, \tag{3.8}$$

for both Bi and NaF, even though the data appear unavailable at temperatures below about 10K for NaF because of interference by mechanical waves. Then (3.6) reduces to

$$U_E(\vartheta)^2 = \frac{\psi_{20}\, a^2}{c_0} (\vartheta_\lambda - \vartheta)_+^{2r_1}. \tag{3.9}$$

We can now use the experimental data contained in (3.1) to obtain values for the parameters in (3.9). For NaF, we choose $r_1 = 1/5$ and

$$U_E = 0.186((18.5 - \vartheta)_+^{1/5} \tag{3.10}$$

where here U_E is measured in cm/μsec and $\vartheta_\lambda = 18.5$K. A comparison of (3.10) with (3.1) and (3.2) is given in Fig. 1.

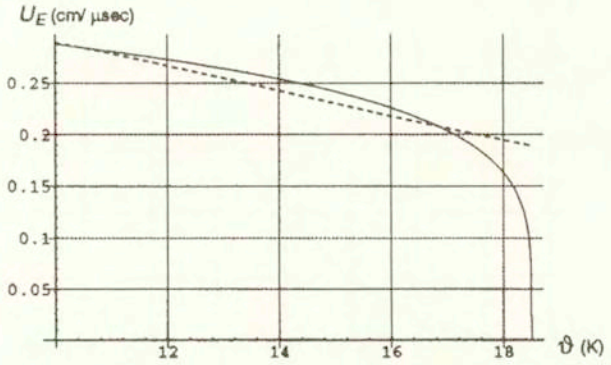

FIG. 1. NaF, second sound speed (solid curve), $U_E = 0.186(18.5 - \vartheta)_+^{1/5}$, together with data (dotted curve), $U_E = (9.09 + 0.00222\vartheta^{3.1})^{-1/2}$, (COLEMAN and NEWMAN, [3]).

For Bi, we take $r_1 = 1/4, \vartheta_\lambda = 4.5$ and

$$U_E = 0.078(4.5 - \vartheta)_+^{1/4}. \tag{3.11}$$

Equation (3.11) with (3.1) and (3.3) illustrates Fig. 2.

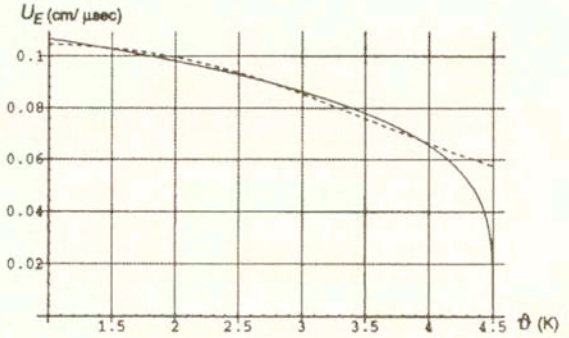

FIG. 2. Bi, second sound speed (solid curve), $U_E = 0.078(4.5 - \vartheta)_+^{1/4}$, together with data (dotted curve), $U_E = (90.7 + 0.758\vartheta^{3.75})^{-1/2}$, (COLEMAN and NEWMAN, [3]).

These simple power law constitutive functions lead to reasonable approximations to the data for second sound. The drops close to the critical temperatures reflect the vanishing of the speed of second sound at these temperatures, while the empirical data-interpolation functions become invalid above ϑ_λ.

Next we obtain $g_{20}(\vartheta)$ in the case of NaF. Having $g_{10}(\vartheta)$ by (3.8), and $r_1 = 1/5$ for NaF, one can obtain $g_{20}(\vartheta)$ with $r_2 = 2/5$ by using experimental data for heat conductivity together with (3.7). Since crystalline materials are known to have cubic temperature dependence close to zero kelvin (see [6] for NaF), we take

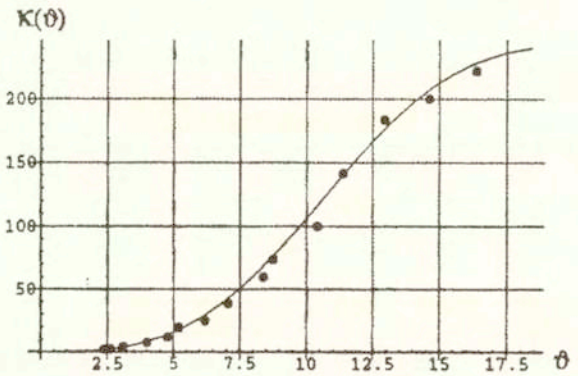

FIG. 3. NaF, conductivity $\mathcal{K}(\vartheta) = 0.127\,\vartheta^3(1 + 0.00002\vartheta^4)^{-1}\,\mathrm{watt\,K^{-1}cm^{-1}}$, and experimental data after [6].

$$g_{20}(\vartheta) = -b(1 + \epsilon\vartheta^4), \quad b > 0, \quad |\epsilon| << 1, \tag{3.12}$$

and (3.7) gives

$$\mathcal{K}(\vartheta) = \frac{\psi_{20}\, a^2 \vartheta^3}{b(1 + \epsilon\vartheta^4)}\,. \tag{3.13}$$

Figure 3 illustrates Eq. (3.13) together with data of thermal conductivity against temperature for NaF, for $\psi_{20}\, a^2/b = 0.127$ and $\epsilon = 2.10^{-5}$.

4. Weakly discontinuous plane waves

In this section, we will consider the system of four Eqs. (2.16), (2.17), with $k(\vartheta) = 0$ and $\vartheta < \vartheta_\lambda$,

$$\vartheta_t - \frac{\alpha'(\vartheta)}{c_v(\vartheta)}\, \mathbf{p} \cdot \nabla\vartheta - \frac{\alpha(\vartheta)}{c_v(\vartheta)}\, \nabla \cdot \mathbf{p} = 0, \tag{4.1}$$

$$\mathbf{p}_t - g_1(\vartheta)\nabla\vartheta = g_2(\vartheta)\mathbf{p}, \tag{4.2}$$

where $\mathbf{p} = (p^1, p^2, p^3)$, and ϑ are continuous in $(\mathbf{x}, t)$.

We represent a three-dimensional characteristic surface $\mathcal{S}(t)$ in implicit form by

$$\mathcal{S}(t) = \{\mathbf{x} \in \mathbb{R}^3 : \mathcal{G}(\mathbf{x}, t) = 0\}, \tag{4.3}$$

where the unit normal $\mathbf{n}$ and normal speed λ are given by

$$\mathbf{n} = \frac{\nabla\mathcal{G}}{|\nabla\mathcal{G}|}, \quad \lambda = -\frac{\mathcal{G}_t}{|\nabla\mathcal{G}|}. \tag{4.4}$$

Across $\mathcal{S}(t)$, $\mathbf{p}$ and ϑ are continuous,

$$[\![\mathbf{p}]\!] = \mathbf{0}, \quad [\![\vartheta]\!] = 0, \tag{4.5}$$

but their derivatives experience a jump discontinuity. Evaluating (4.1) and (4.2) across $\mathcal{S}(t)$ gives

$$[\![\vartheta_t]\!] - \frac{\alpha'(\vartheta^+)}{c_v(\vartheta^+)}\, \mathbf{p}^+ \cdot [\![\nabla\vartheta]\!] - \frac{\alpha(\vartheta^+)}{c_v(\vartheta^+)}\, [\![\nabla \cdot \mathbf{p}]\!] = 0, \tag{4.6}$$

$$[\![\mathbf{p}_t]\!] - g_1(\vartheta^+)[\![\nabla\vartheta]\!] = 0, \tag{4.7}$$

where ϑ^+ and $\mathbf{p}^+$ are the values of ϑ and $\mathbf{p}$ ahead of the wave.

Next we define the directional derivative of a vector field $\mathbf{U}$ by

$$\frac{d}{dt}[\![\mathbf{U}]\!] = [\![\mathbf{U}_t]\!] + \lambda n^k [\![\mathbf{U}_{,k}]\!], \tag{4.8}$$

where $\mathbf{U}_{,k} = \dfrac{\partial}{\partial x^k}$, $k = 1, 2, 3$, with $\mathbf{n}$ and λ from (4.4). For $\mathbf{U}$ continuous across $\mathcal{S}(t)$, we therefore have

$$[\![\mathbf{U}_t]\!] = -\lambda n^k [\![\mathbf{U}_{,k}]\!]. \tag{4.9}$$

Let $\mathbf{a}$ denote the jump of derivatives $\mathbf{U}_{,k}$ across $\mathcal{S}(t)$ via

$$[\![\mathbf{U}_{,k}]\!] = \mathbf{a}\, n_k. \tag{4.10}$$

Taking $\mathbf{U} = (\vartheta, p^1, p^2, p^3)$ and $\mathbf{a} = (r_0, r^1, r^2, r^3)$, (4.10) can be expressed as

$$[\![\nabla\vartheta]\!] = r_0\mathbf{n}, \tag{4.11}$$

$$[\![\nabla\mathbf{p}]\!] = \mathbf{r} \otimes \mathbf{n}, \quad \mathbf{r} = (r^1, r^2, r^3). \tag{4.12}$$

By combining the compatibility condition (4.9) with (4.7) for ϑ and $\mathbf{p}$, we can relate r_0 to $\mathbf{r}$,

$$\mathbf{r} = -r_0 \frac{g_1(\vartheta^+)}{\lambda}\, \mathbf{n}. \tag{4.13}$$

This allows us to represent the jumps of the first derivatives of ϑ and $\mathbf{p}$ in terms of r_0 ,

$$[\![\vartheta_t]\!] = -\lambda r_0\,, \tag{4.14}$$

$$[\![\nabla\vartheta]\!] = r_0\mathbf{n}\,, \tag{4.15}$$

$$[\![\mathbf{p}_t]\!] = r_0 g_1^+ \mathbf{n}\,, \tag{4.16}$$

$$[\![\nabla\mathbf{p}]\!] = -r_0 \frac{g_1^+}{\lambda}\mathbf{n} \otimes \mathbf{n}\,, \tag{4.17}$$

where we abbreviate $g_1(\vartheta^+)$ with g_1^+, and will continue similarly for any functions of ϑ evaluated across $\mathcal{S}(t)$. Substituting (4.14) – (4.17) into (4.6) and (4.7) gives the following equation for λ when $r_0 \neq 0$,

$$\lambda^2 + \lambda \frac{\alpha'^+}{c_v^+}\, \mathbf{p}^+ \cdot \mathbf{n} - \frac{\alpha^+ g_1^+}{c_v^+} = 0. \tag{4.18}$$

From (2.18), (2.19) and (2.21) we therefore have real characteristic speeds, depending on the state $(\vartheta^+, \mathbf{p}^+)$ ahead of the wave, and the normal $\mathbf{n}$ to the wavefront. If $\vartheta^+ < \vartheta_\lambda$ is constant, the evolution Eq. (4.2) implies that $\mathbf{p}^+$ satisfies

$$\mathbf{p}^+ = \mathbf{p}_0 e^{g_2^+ t} \tag{4.19}$$

with $g_2^+ < 0$. We take $\mathbf{p}_0 = \mathbf{0}$, which leads to $\lambda = \lambda(\vartheta^+)$,

$$\lambda^2 = \frac{\alpha^+ g_1^+}{c_v^+}. \tag{4.20}$$

For the remainder of this section we consider plane waves propagating into an equilibrium state,

$$\vartheta^+ = \text{constant}, \quad \mathbf{p}^+ = \mathbf{0}, \quad \mathbf{n} = \text{constant}. \tag{4.21}$$

To derive an equation for the amplitude, r_0, we differentiate (4.1), (4.2) with respect to t and then compute the jump across $\mathcal{S}(t)$. Using the relation

$$[\![\vartheta_{tt}]\!] = -\lambda \frac{dr_0}{dt} - \lambda \mathbf{n} \cdot [\![\nabla \vartheta_t]\!] \tag{4.22}$$

and

$$[\![\mathbf{p}_{tt}]\!] = g_1^+ \mathbf{n} \frac{dr_0}{dt} - \lambda \mathbf{n} \cdot [\![\nabla \mathbf{p}_t]\!] \tag{4.23}$$

which come from (4.8) and (4.14) – (4.17), we obtain

$$-\lambda c_v \frac{dr_0}{dt} + r_0^2 (c_v' \lambda^2 - 2\alpha' g_1) - \lambda c_v \mathbf{n} \cdot [\![\nabla \vartheta_t]\!] - \alpha [\![\nabla \cdot \mathbf{p}_t]\!] = 0, \tag{4.24}$$

$$g_1 \mathbf{n} \frac{dr_0}{dt} - \lambda \mathbf{n} \cdot [\![\nabla \mathbf{p}_t]\!] + \lambda g_1' r_0^2 \mathbf{n} - g_1 [\![\nabla \vartheta_t]\!] = g_1 g_2 r_0 \mathbf{n}. \tag{4.25}$$

For simplicity of notation, we have omitted the symbol "+" in the coefficients above. Taking the scalar product of (4.25) with $\frac{1}{g_1}\mathbf{n}$, we calculate

$$\mathbf{n} \cdot [\![\nabla \vartheta_t]\!] = \frac{dr_0}{dt} - \frac{\lambda}{g_1} \mathbf{n} \otimes \mathbf{n} \cdot [\![\nabla \mathbf{p}_t]\!] + \frac{\lambda g_1'}{g_1} r_0^2 - g_2 r_0. \tag{4.26}$$

Using (4.26) in (4.24) gives

$$-2\lambda c_v \frac{dr_0}{dt} + r_0^2 \left(c_v' \lambda^2 - 2\alpha' g_1 - \lambda^2 c_v \frac{g_1'}{g_1} \right) + \lambda c_v g_2 r_0 + \lambda^2 c_v \frac{1}{g_1} \mathbf{n} \otimes \mathbf{n} \cdot [\![\nabla \mathbf{p}_t]\!] - \alpha [\![\nabla \cdot \mathbf{p}_t]\!] = 0. \tag{4.27}$$

The final two terms in (4.27) reduce to

$$\alpha \mathbf{n} \otimes \mathbf{n} \cdot [\![\nabla \mathbf{p}_t]\!] - \alpha [\![\nabla \cdot \mathbf{p}_t]\!], \tag{4.28}$$

after applying relation (4.20) for λ^2. Arguing as in [10] (Chapter 2.6), one finds that

$$\mathbf{n} \otimes \mathbf{n} \cdot [\![\nabla \mathbf{p}_t]\!] = \mathbf{n} \cdot \left(\frac{d}{dt} \left[\!\left[\frac{\partial \mathbf{p}}{\partial n} \right]\!\right] - \left[\!\left[\lambda \frac{\partial^2 \mathbf{p}}{\partial n^2} \right]\!\right] \right) = [\![\nabla \cdot \mathbf{p}_t]\!]. \tag{4.29}$$

As a result, (4.27) can be written in terms of $[\![\vartheta_t]\!]$, using (4.14) and (4.20),

$$\frac{d}{dt}[\![\vartheta_t]\!] + \mathcal{A}[\![\vartheta_t]\!]^2 + \mathcal{B}[\![\vartheta_t]\!] = 0 \tag{4.30}$$

where the coefficients $\mathcal{A}$ and $\mathcal{B}$ are given (see (2.18), (2.19) and (2.21)) by

$$\mathcal{A} = \frac{1}{2}\left(\frac{c_v'}{c_v} - \frac{4}{\vartheta^+} - \frac{3g_1'}{g_1}\right), \tag{4.31}$$

$$\mathcal{B} = -\frac{1}{2}g_2 > 0. \tag{4.32}$$

Finally we examine (4.31) for the special case (2.27), (3.4) with $g_{10} = a\vartheta^{1/2}$. This gives

$$\mathcal{A} = \frac{1}{2}\left(\frac{3r_1}{\vartheta_\lambda - \vartheta^+} - \frac{5}{2\vartheta^+}\right). \tag{4.33}$$

As in [21], we obtain a temperature $\vartheta^+ = \vartheta_m$ at which $\mathcal{A} = 0$, here

$$\vartheta_m = \frac{5\vartheta_\lambda}{6r_1 + 5}. \tag{4.34}$$

Since $\mathcal{B} > 0$, the solutions $[\![\vartheta_t]\!]$ of (4.30) tend to infinity in finite time whenever $\mathcal{A}[\![\vartheta_t]\!]_0 + \mathcal{B} < 0$, with initial condition $[\![\vartheta_t]\!]_0 = \vartheta_t^-(0)$. We now have an analogous result on the finite time blowup of three-dimensional plane temperature-rate waves to the one-dimensional case in [21], with a *hot* temperature-rate wave having $\vartheta_t^-(0) > 0$, and a *cold* temperature-rate wave having $\vartheta_t^-(0) < 0$.

THEOREM *Let $\vartheta^+ < \vartheta_\lambda$ be the temperature in front of the temperature-rate wave, and let $[\![\vartheta_t]\!]_0 = \vartheta_t^-(0)$. Then,*

1. *For $\vartheta_m < \vartheta^+ < \vartheta_\lambda$ (i.e. $\mathcal{A} > 0$), the amplitude $[\![\vartheta_t]\!]$ blows up in finite time if $\vartheta_t^-(0) < -\frac{B}{A} < 0$ (cold temperature-rate wave);*

2. *For $0 < \vartheta^+ < \vartheta_m$ (i.e. $\mathcal{A} < 0$), the amplitude $[\![\vartheta_t]\!]$ blows up in finite time if $\vartheta_t^-(0) > -\frac{B}{A} > 0$ (hot temperature-rate wave);*

3. *For $\vartheta^+ = \vartheta_m$ (i.e. $\mathcal{A} = 0$), the amplitude $[\![\vartheta_t]\!]$ is a decreasing function of time, and blow-up does not occur.*

We remark that at ϑ_m, the system (4.1), (4.2) loses genuine nonlinearity. Such a temperature has also been found by RUGGERI *et al.* ([19, 20]), whose derivation is based on (3.1) for U_E and so does not reflect the fact that second sound is not observed above certain temperatures. As a result, they do not obtain a relationship of the type (4.34) since there is no analogue of the temperature ϑ_λ playing the part, in solids, of the lambda point for liquid helium.

As in [21], one may also show that no one-dimensional shocks can propagate into equilibrium states for which $\vartheta^+ = \vartheta_m$. From (3.10) and (3.11) respectively, ϑ_m is found to be 14.9 K for sodium fluoride and 3.46 K for bismuth.

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