Arch. Mech., **51**, 5, pp. 633–641, Warszawa 1999

BRIEF NOTES

Electrification capillary stability of a hollow jet

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THE PRESENT WORK extends Chandrasekhar's theory CHANDRASEKHAR [2] of axisymmetric capillary instability of a hollow jet. Here the instability of this model is investigated for all (non)-axisymmetric perturbation modes under the combined effect of the capillary and electrification forces. The electrification dispersion relation has been derived, studied analytically and numerically, and the (un-) stable domains are identified. Some reported results are recovered as limiting cases. The principle of the exchange of instability is valid. The capillary instability of a hollow jet becomes worse in the presence of the electrification forces.

1. Introduction

THE IDEA OF A HOLLOW JET model which is a gas cylinder (of zero inertia) submerged in an infinite liquid, is due to RAYLEIGH [1]. CHANDRASEKHAR [2] reported the capillary dispersion relation of such model for axisymmetric perturbation mode $m = 0$ (m is the azimuthal wavenumber) only, see also DRAZIN and REID [3]. CHENG [4] analyzed the capillary instability of this model, taking into account (or not) the gas inertia force. Concerning more detailed studies of pure hydrodynamic stability for $m = 0$, we may refer to the complete analysis of LIN and LAIN [5] and LEE and WANG [6]. The hydromagnetic stability of a hollow jet has been developed by RADWAN [7]. The latter author in [8] has examined the rotating forces effects on the capillary instability of a hollow jet. The model of a hollow jet will describe the phenomena observed in nature, such as e.g. gas escaping from below an oil layer or a jet formed up when gas is pumped into a fluid.

The purpose of the present paper is to examine the effect of the electrification forces on the capillary instability of a hollow cylinder. This will be carried out for general cylindrical wave propagation upon using the energy conservation principle in the form different from that used in our previous papers.

The most interesting issue in this work is that both the potential energy of surface tension and that of electrification have the surface area of the gas-liquid interface as an extensive variable, but the sign of intensive variables of each potential energy are opposite. Hence, it is expected that the competition between the capillary and the electrification instabilities may show a variety of stability characteristics.

To our best knowledge, the present electrification problem has not been treated or even approached, up to now, in the literature.

2. Basic state

Consider a circular gas cylinder (of radius a) dispersed in an infinite liquid. Following CHANDRASEKHAR [2] we assume that the liquid inertia force predominat over that of the gas cylinder, i.e. the gas motion could be ignored relative to that of the liquid in the perturbed state. But at the same time we have to be sure that the constant gas pressure in the unperturbed state is of considerable value, otherwise the model will collapse and this is not our case, see Eq. (2.12) below. The liquid could be water, water solutions containing salt or even glycerin while the gas could be air, helium or freon 12, see KENDALL [9]. The liquid is assumed to be non-viscous and incompressible. An electric potential V_0 is applied along the dielectric gas-liquid interface. We shall use the cylindrical polar coordinates (r, ϕ, z) with the z-axis coinciding with the axis of the gas cylinder. The present model of a hollow jet is acting upon the capillary, pressure gradient and electrification forces. The gravitation force effects are not considered here.

The basic equations required for investigating the stability of the present problem (using the SI unit system) inside of the liquid are:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\nabla^2 W = D; \tag{2.3}$$

along the gas-liquid interface

$$p_s = T\left(\mathbf{r}_1^{-1} + \mathbf{r}_2^{-1}\right), \tag{2.4}$$

$$\left(\mathbf{r}_1^{-1} + \mathbf{r}_2^{-1}\right) = \nabla\cdot\mathbf{n}. \tag{2.5}$$

Here $\rho, \mathbf{u}$ and p are the liquid mass density, velocity vector and kinetic pressure, p_s is the curvature pressure due to the capillary force, r_1 and r_2 are the principal radii of curvature, W is the electrification potential, D is the electric charge density term due to the electrification at the gas-liquid interface and will be zero here, T is the surface tension coefficient and $\mathbf{n}$ is a unit outward vector normal to the gas-liquid interface, given by

$$\mathbf{n} = \frac{\nabla f}{\sqrt{\nabla f \cdot \nabla f}}, \tag{2.6}$$

where

$$-f(r, \phi, z; t) = 0 \tag{2.7}$$

is the equation of motion of the gas-liquid boundary surface.

The unperturbed state, characterized by $\mathbf{u} = 0$, $\partial/\partial\phi = 0$ and $\partial/\partial z = 0$, has been studied. The unperturbed basic quantities are given by

$$p_0 = \text{const}, \tag{2.8}$$

$$p_{0s} = -T/a, \tag{2.9}$$

$$W_0 = V_0(\ln r/\ln a). \tag{2.10}$$

Upon applying the balance of the total pressures across the gas-liquid interface at $r = a$, we obtain

$$p_0 = p_g + \varepsilon_0 \left(V_0/(a \ln a)\right)^2 - T/a, \tag{2.11}$$

where p_g is the gas constant pressure in the unperturbed state.
For $p_0 > 0$, it must be

$$p_g + \varepsilon_0 \left(V_0/(a \ln a)\right)^2 > T/a \tag{2.12}$$

otherwise the model under consideration will collapse towards a hollow jet of a radius smaller than a.

3. Eigenvalue problem

For small departure from the initial unperturbed state, the perturbation equations describing the oscillation of the hollow jet model are obtained by solving Eqs. (2.1) – (2.5). The constants of integration are determined upon applying appropriate boundary conditions across the perturbed interface. The kinetic and potentials energies of the fluids are computed. Moreover, upon using the energy conservation principle, the following dispersion relation is derived:

$$\omega^2 = \left(\frac{T}{\rho a^3}(m^2 + x^2 - 1) + \frac{\varepsilon_0 V_0^2}{\rho a^4 (\ln a)}\left(1 + \frac{x K_m'(x)}{K_m(x)}\right)\right)\left(\frac{x K_m'(x)}{K_m(x)}\right). \tag{3.1}$$

Equation (3.1) is the desired stability criterion of the present model. It relates the growth rate ω with the longitudinal and azimuthal wavenumbers x and m, the second kind of the modified Bessel function $K_m(x)$ and its derivative of order m, the permittivity ε_0 of the fluid medium and other parameters T, a, V_0 and ρ of the problem.

In absence of the electrification effects ($V_0 = 0$) and simultaneously assuming that the fluid disturbance is longitudinally axisymmetric $m = 0$, dispersion relation (3.1) reduces to that indicated by RAYLIEGH [1] and just given by CHANDRASEKHAR [2].

If we assume that the perturbation of the gas-liquid interface could be axisymmetric and non-axisymmetric $m \geqslant 0$, and at the same time $V_0 = 0$, Eq. (3.1) degenerates to RADWAN'S result [8] if the magnetic field effects were neglected in reference [8].

It is recommended that all quantities can be expressed in dimensionless form using the radius a of the jet, the surface tension coefficient T, the mass density ρ of the liquid and the electrification potential V_0 as scalar values. Taking into account that the quantity $(T/\rho a^3)$ as well as $(\varepsilon_0 V_0^2/(\rho a^4 \ln a))$ has a dimension of $(\text{time})^{-2}$, we introduce

$$N^2 = \frac{\omega^2}{T/\rho a^3}, \tag{3.2}$$

$$\Gamma = \frac{\varepsilon_0 V_0^2}{aT(\ln a)} \tag{3.3}$$

so that the eigenvalue relation (3.1) can be written in the dimensionless form

$$N^2 = (m^2 + x^2 - 1 \left(\frac{xK'_m(x)}{K_m(x)}\right) + \Gamma \left(1 + \frac{xK'_m(x)}{K_m(x)}\right)\left(\frac{xK'_m(x)}{K_m(x)}\right). \tag{3.4}$$

In the axisymmetric perturbation mode $m = 0$, the relation (3.4) reduces to

$$N^2 = (1 - x^2)\left(\frac{xK_1(x)}{K_0(x)}\right) + \Gamma\left(1 - \frac{xK_1(x)}{K_0(x)}\right)\left(\frac{xK_1(x)}{K_0(x)}\right). \tag{3.5}$$

4. Stability discussion

4.1. Hydrodynamic stability

This is the case in which the hollow fluid jet is uncharged. The stability criterion which describes such a case is given in its general form, from Eq. (3.1) at $V_0 = 0$, by

$$L^2 = (m^2 + x^2 - 1)\left(\frac{xK'_m(x)}{K_m(x)}\right), \tag{4.1}$$

with

$$L^2 = \frac{\omega^2}{T/\rho a^3}, \qquad \text{as} \quad V_0 = 0. \tag{4.1$'$}$$

In order to discuss the stability here and in other sections, it is found more convenient to write down certain propertiess of I_m and K_m and their derivatives.

For each non-zero real value of x and $m \geqslant 0$, cf. ABRAMOWITZ and STEGUN [11], we have

$$I_m(x) \succ 0, \tag{4.2}$$

$$K_m(x) \succ 0, \tag{4.3}$$

where $I_m(x)$ is always positive and monotonically increasing while $K_m(x)$ is monotonic decreasing but never negative. The recurrence relations of the modified Beesel functions are

$$2I'_m(x) = I_{m-1}(x) + I_{m+1}(x), \tag{4.4}$$

$$2K'_m(x) = -K_{m-1}(x) - K_{m+1}(x). \tag{4.5}$$

Using Eqs. (4.4), (4.5) and (4.2), (4.3) we obtain

$$I'_m(x) \succ 0, \tag{4.6}$$

$$K'_m(x) \prec 0, \tag{4.7}$$

for each $x \neq 0$ and $m \geqslant 0$. In view of the inequalities (4.3) and (4.7), for $x \neq 0$ we obtain

$$x\left(K'_m(x)/K_m(x)\right) < 0. \tag{4.8}$$

Now, let us return to the dispersion relation (3.5). By taking into account the inequality (4.8), Eq. (3.5) yield that

$$L^2 < 0 \qquad \text{for} \qquad m \geqslant 1 \qquad \text{as} \quad x \neq 0, \tag{4.9}$$

while for $m = 0$, we have

$$L^2 > 0 \qquad \text{as} \qquad -1 \prec x \prec 1, \tag{4.10}$$

$$L^2 \leqslant 0 \qquad \text{as} \qquad x \geqslant 1 \quad \text{or} \quad x \leqslant -1. \tag{4.11}$$

This means that a hollow cylindrical jet is stable for all non-axisymmetric modes m of all short and long wavelengths, and also for sausage mode $m = 0$ whose wavelength $\lambda(= 2\pi/k)$ is shorter than the circumference $2\pi a$ of the gas core jet. The hollow jet is capillary unstable only for axisymmetric mode $m = 0$ whose wavelength λ is longer than $2\pi a$ where the case when $\lambda = 2\pi a$ is that of marginal stability.

In order to verify the foregoing analytical results, the capillary dispersion relation (3.5) is calculated numerically for the most unstable mode $m = 0$.

The values of the quadratic dimensionless temporal amplification L are tabulated and presented graphically as a function of x, see Fig. 1. The data reveal the following conclusions. The capillary unstable domain of the hollow jet is the only interval $0 < x < 1$. The maximum mode of instability occurs at $x = 0.484$. The numerical values of L^2 increase rapidly for very small values of x and reach the maximum at $x = 0.50$; then they are fastly decreasing and change the sign at $x = 1$. The values of L^2 are negative for all values of $x \geqslant 1$. The point $x = 1$ represents a transition from stable to unstable domains.

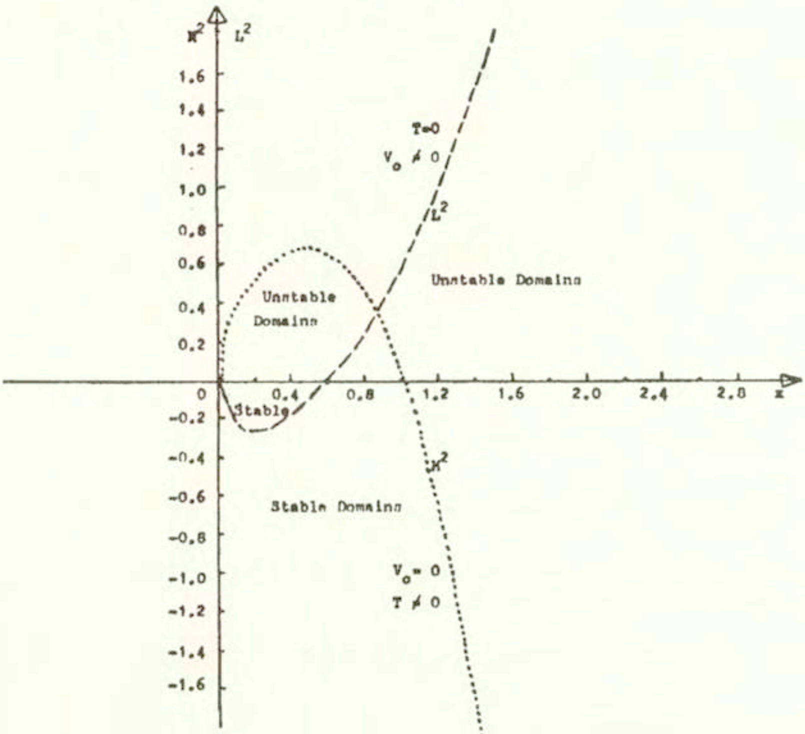

FIG. 1.

4.2. Electrification stability

In this case we assume that the hollow jet is acting upon the electrification force only and we neglect the capillary force influence. The dispersion relation follows from the general characteristic Eq. (3.1) with $T = 0$ in the form

$$M^2 = \left(1 + \frac{xK_m'(x)}{K_m(x)}\right)\left(\frac{xK_m'(x)}{K_m(x)}\right), \tag{4.12}$$

with

$$M^2 = \varepsilon_0 V_0^2/(\rho a^4 \ln a). \tag{4.12)'$$

Applying the inequality (4.8) to the relation (4.12) we find that M^2 is negative, hence the electrification hollow jet is stable iff the inequality

$$\left(1 + \frac{xK'_m(x)}{K_m(x)}\right) \geqslant 0 \tag{4.13}$$

is satisfied, and vice versa where the equality in (4.13) corresponds to the neutral stability states. From the viewpoint of the inequalities (4.3) and (4.7), the condition (4.13) may be rewritten as

$$K_m(x) \geqslant x\left|K'_m(x)\right|. \tag{4.14}$$

It is difficult to identify analytically whether the condition (4.13) is satisfied or not. However, in order to avoid these difficulties, it is recommended that we should analyse (4.12) numerically.

In the axisymmetric sausage mode $m = 0$, it is found that the inequality (4.13) is satisfied in the ranges $0 < x < 0.595088$ and so as $0.59509 < x < \infty$. Therefore the electrification force is stabilizing in the domain $0 < x \leqslant 0.595088$ and destabilizing in the neughboring domains $0.59509 < x < \infty$. The point at which $x = 0.595088$ is that of a transition from a stability state to one of instability, see Fig. 1. In the non-axisymmetric perturbation mode $m = 1$, it is found that the inequality (4.13) is not satisfied for all short and long wavelengths. This means that the electrification force has a strong destabilizing influence on the charged hollow jet.

4.3. Hydro-electrification stability discussions

In this general case the model of a hollow jet is acting upon the combined effects of the electrification and capillary forces. The stability criterion required for investigation of such a case is given by the characteristic Eq. (3.4) in its general form. The latter could be discussed with the aid of the Subsec. (4.1) as $(T \neq 0, V_0 = 0)$ and (4.2) as $(T = 0, V_0 \neq 0)$. The first case of hydrodynamic investigations reveal that the capillary stable domains are

$$1 \leqslant x < \infty, \qquad \text{as} \qquad m = 0, \tag{4.15}$$

$$0 < x < \infty \qquad \text{as} \qquad m \geqslant 1, \tag{4.16}$$

while the only unstable domain is

$$0 < x < 1, \qquad \text{as} \qquad m = 0. \tag{4.17}$$

The second case of electrification analysis indicates that the model is stable in the domain

$$0 < x \leqslant 0.59452, \qquad \text{as} \qquad m = 0, \tag{4.18}$$

while it is unstable in the domains

$$0.5945 < x < \infty, \qquad \text{as} \qquad m = 0, \tag{4.19}$$

$$0 < x < \infty, \qquad \text{as} \qquad m \geqslant 1. \tag{4.20}$$

In the case of consideration of the combined effects of both the capillary and electrification forces, it is found that the unstable results are not as it could be expected.

The dispersion relation (3.4) as $m = 0$ has been investigated numerically for different values of the dimensionless electrification factor

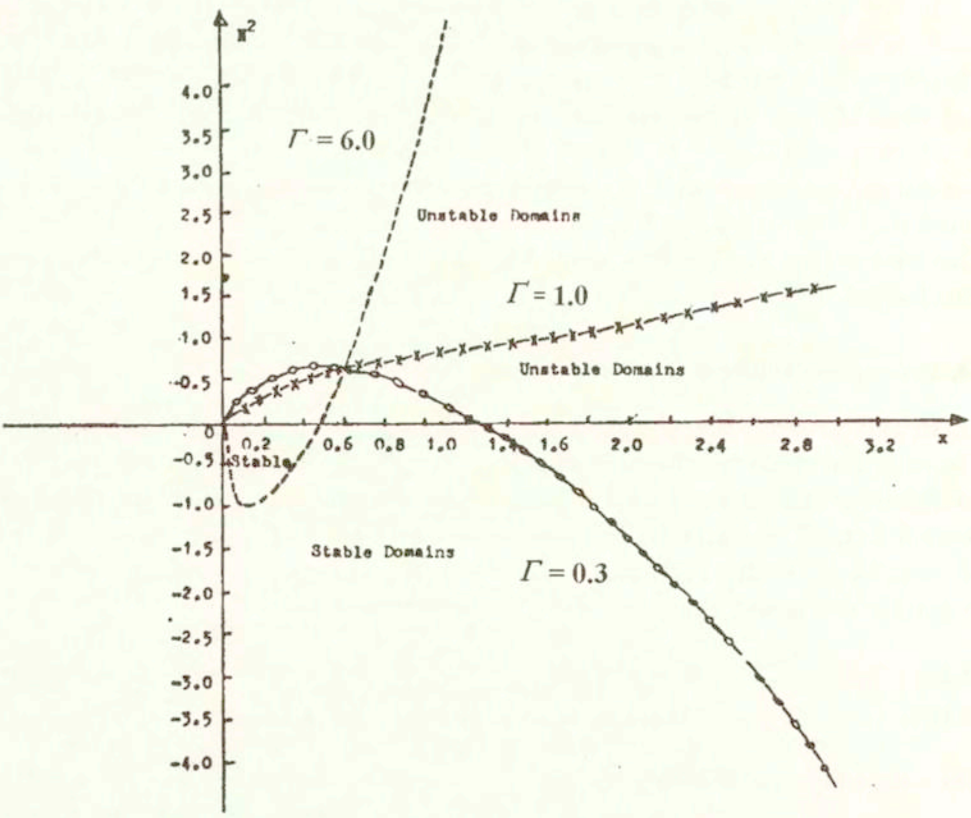

FIG. 2.

(4.21)
$$\Gamma = \left(\frac{\varepsilon_0 V_0^2}{\rho a^4 (\ln a)}\right) \Big/ \left(\frac{T}{\rho a^3}\right) = \left(\frac{\varepsilon_0 V_0^2}{Ta(\ln a)}\right).$$

The numerical data are illustrated in the range $0 \leqslant x \leqslant 3.0$ of short and long wavelengths, see Fig. 2. It is found that the model is completely unstable for $\Gamma = 1$ for all short and long wavelengths. As $\Gamma = 0.3$, the model is unstable for small values of x, i.e. for very long wavelengths as $0 < x \leqslant 1.1$, while it is stable in the neighboring domain $1.2 \leqslant x \leqslant 3.0$. The latter domain increases with increasing x values. For $\Gamma = 6.0$ the situation has been reversed: the model is stable for very long wavelengths as $0 < x \leqslant 0.4$ while it is unstable for short wavelengths $0.5 \leqslant x < 3.0$. This change is due to the influence of the electrification force. This indicates that the principle of exchange of instability is valid in the present case.

Acknowledgement

The authors would like to thank the Editor and the Referes for improving the paper according their crucial comments.

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Received July 13, 1998; revised version July 30, 1999.