

# Comparison of two entropy principles and their applications in granular flows with/without fluid

*Dedicated to Prof. Y.-H. Pao  
on the occasion of his seventieth birthday*

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TWO ENTROPY PRINCIPLES that are commonly used are: (i) the Clausius-Duhem inequality with the procedure of exploitation due to Coleman-Noll (CD-CN), and (ii) the entropy principle of Müller-Liu (ML). CD-CN makes *a priori* postulates about the entropy flux and entropy supply and assumes external source terms in (most) balance laws. ML postulate the entropy flux to be a general constitutive variable and treat all field equations as constraints for the exploitation of the entropy principle. These and further differences are explained, and results are presented with the use of both principles for (i) a granular solid with a scalar structure equation, and (ii) for a saturated mixture of granular/fluid constituents with scalar structure equations for each constituent. It is shown that the two entropy principles yield different results. It is further indicated which theories are likely to be problematic when the CD-CN approach is used. These theories are then applied to analyses of steady fully-developed gravity flows down an inclined plane.

**Key words:** entropy principle, granular flow, solid-fluid mixture, constitutive equations, gravitational flow.

## 1. Introduction

TO SOME EXTENT, modern continuum thermodynamics amounts to a collection of “thermodynamical theories” sharing common premises and common methodology. There are the theories of elastic materials, of viscous materials, of materials with memory, of mixtures, and so on. It is generally the case that, in the context of each theory, one considers all processes (compatible with classical conservation laws) that bodies composed of the prescribed material might admit. Moreover, there exist for the theory some universal physical principles that have been abstracted from experience. Therefore one can reduce the generality of the constitutive rela-

tions of dependent material variables by relying upon these principles. The most important of these principles is the second law of thermodynamics.

Mathematicians interested in continuum thermodynamics are generally not aware of the differences in the various postulations of the second law of thermodynamics. Virtually the same is true for many continuum mechanicians; in particular it is surprising how shallowly and mechanically many continuum mechanicians handle the second law. It appears that they have superficially learned how Coleman-Noll apply the Clausius-Duhem inequality and use it as a machine to generate inferences with little contemplation whether the deduced results make physically sense. In this paper we will make an attempt to explain how the basic postulates of two forms of the entropy principle differ from one another and then demonstrate that they yield different results. It is these results which allow us to favour one set of basic postulates over the other. The two entropy principles are the

- (i) generalized Clausius-Duhem – Coleman-Noll approach (CD-CN),
- (ii) Müller-Liu entropy principle (ML).

We will make clear below what we mean by “generalized Clausius-Duhem approach”. Our demonstration of the essential steps in these two principles will include only the most important mathematical steps and omit significant details that would detract from the main ideas. The reader can fill in these details himself by reading the pertinent literature.

In Section 2, we present the two approaches of the entropy principles according to a set of generalized field equations and constitutive relations and compare their differences. Section 3 is devoted to the representation of the constitutive equations of both (i) a dry granular material and (ii) a multiphase mixture from thermodynamic considerations of the Müller-Liu approach. Furthermore, these are compared with those of CD-CN. In order to assess the implications of the theories, we consider in Sec. 4 a specific boundary-value problem, namely gravity-flow down an inclined plane for the dry granular material and the solid-fluid mixture. In Section 5, this paper is summarized.

## 2. On entropy principles

In this section we will explain how the two exploitations of the entropy principle are made and what postulates are underlying them. We then demonstrate how they differ from one another. The constitutive class for which this comparison is implemented is a restricted one in which constitutive relations express a dependent variable as a function of its independent variables (and not a functional), or for which a constitutive relation may be expressed as a differential



equation among some variables. The constitutive class fathomed by this assumption is still very large and covers most solid, fluid and mixture theories including many dealing with hereditary effects.

### 2.1. Basic equations with source terms

Consider a field theory for a number of field variables  $\mathbf{u} = (\mathbf{u}_i, \mathbf{u}_d)$  defined over the body. Let  $u_i$  be the independent fields, i.e., those field variables for which the theory provides field equations. Let, moreover,  $\mathbf{u}_d$  be the dependent field variables which are functionally expressed in terms of the independent fields. Let  $\mathbf{s}$ ,  $s^\varepsilon$  be source terms, arbitrary known functions defined over the body and over time.

Any continuum mechanical field theory consists of the following statements:

- Balance laws

$$(2.1) \quad \mathcal{F}(\mathbf{u}_i, \mathbf{u}_d) - \mathbf{s} = \mathbf{0}, \quad f^\varepsilon(u_i, u_d) - s^\varepsilon = 0.$$

These are for instance the balance laws of momentum, angular momentum and energy, but in electromagneto-mechanical applications they can also include some of the Maxwell equations. In (2.1) we have singled out one scalar-type equation – the one with the superscript  $\varepsilon$  – from the others; this is the energy equation.  $\mathcal{F}$  and  $f$  denote functional differential operators involving differentiations of space and time.

- Constraint relations and source-free balance laws

It is often so that the field variables are subjected to constraint conditions which are either of kinematic or thermomechanical nature. These constraint conditions are also expressible as functional relations between the field variables  $(\mathbf{u}_i, \mathbf{u}_d)$ ,

$$(2.2) \quad \mathcal{C}(\mathbf{u}_i, \mathbf{u}_d) = 0.$$

For example, an incompressible material is kinematically constrained by the equation  $\det \mathbf{F} = 1$  where  $\mathbf{F}$  is the deformation gradient or  $\det \mathbf{C} = 1$ , where  $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ . The corresponding Eq. (2.2) is

$$(2.3) \quad \text{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = 0 \quad \text{or} \quad \dot{\gamma} = 0,$$

where  $\gamma$  is the material mass density. Another example of a constraint condition is the *saturation condition* in a porous mixture of a solid and a fluid. It states that the water fills the entire pore space. If  $\nu_f$  and  $\nu_s$  are the fluid and solid volume fractions, then the constraint condition requires  $\nu_f + \nu_s = 1$  or

$$(2.4) \quad \frac{d}{dt}(\nu_f + \nu_s) = 0.$$

Source-free balance relations are also of the form expressed by Eq. (2.2). The conservation equation of mass is of such a form, or the constituent balance laws of mass in a mixture of a finite number of constituents. In these latter laws production terms can enter due to phase changes or chemical reactions. These are no source terms as their origin is within the body and not external.

It is often so that authors introduce external source terms in evolution equations of the type (2.2) to make them of type (2.1). In most situations the reason is mathematical, but there is no justification on physical grounds to do so. For instance, to add an arbitrary source term to a balance law of mass is physically not justifiable. Neither can balance relations for hidden variables have such external source terms simply because they express something about the microstructure of the body which is entirely internal to the body. Equations like this are the equilibrated force balances in granular theories, the balance laws for configurational forces used in connection with phase changes, the spin balances in polar theories such as micropolar, micromorphic and liquid crystal theories etc.

• Constitutive relations

The constitutive relations are functional relations between the dependent fields  $\mathbf{u}_d$  and the independent fields  $\mathbf{u}_i$ . When  $\mathbf{u}_d$  are expressed as functional relations of  $\mathbf{u}_i$ , they read

$$(2.5) \quad \mathbf{u}_d = \mathcal{M}(\mathbf{u}_i).$$

It is for these relations that we have divided the field variables into  $\mathbf{u}_i$  and  $\mathbf{u}_d$ . Examples are an equation of the stress tensor in terms of the strain tensor in an elastic constitutive relation, or the heat flux expressed as being affine to the temperature gradient.

In most continuum thermodynamic theories it is stipulated that the balance laws and constitutive relations together define a well-posed problem; in other words, with appropriate initial and boundary conditions these equations are supposed to yield unique functions of space and time for the field variables, at least for some finite non-zero interval of time. When constraint conditions are added, additional variables enter the theory, which represent the constraint stresses or forces that must be applied to guarantee the maintainance of the constraint conditions. These additional fields are not contained in  $\mathbf{u}_i$  and  $\mathbf{u}_d$  of (2.5).

Combining (2.1), (2.2) and (2.5) yields

$$(2.6) \quad \left. \begin{aligned} \mathcal{F}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) - \mathbf{s} &= \mathbf{0}, \\ f^\varepsilon(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) - s^\varepsilon &= 0, \end{aligned} \right\} \mathbf{IF}(\mathbf{u}_i) - \mathbf{s} = \mathbf{0},$$

$$\mathbf{C}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) = \mathbf{0}.$$

These equations are called *field equations*. Any set of  $\mathbf{u}_i$  that satisfies the Eq. (2.6) is called a *thermodynamic process*. In reality the constitutive functions (2.5) are



not arbitrary, they should obey universal physical principles, i.e., one can reduce the generality of these functions by relying upon these physical principles. The most important of these principles is the second law of thermodynamics, which we now introduce in the form of the *entropy principle*.

There exists an entropy density  $\eta$ , entropy flux  $\Phi$ , entropy production density  $\pi^\eta$  and entropy supply density  $s^\eta$ , which obey a balance law. The second law of thermodynamics requires that the following inequality is satisfied,

$$(2.7) \quad \pi^\eta := \mathcal{H}(\eta, \Phi) - s^\eta \geq 0.$$

Now, any process which satisfies (2.7) (via the constitutive relations) represents a so-called physically admissible process. The entropy inequality, however, must not hold for arbitrary fields  $\mathbf{u}_i$ , but only for thermodynamic processes, i.e. solutions of the field equations. The *working principle* is therefore that all thermodynamic processes must satisfy (2.7) or all fields which satisfy (2.6) must in addition satisfy (2.7). We must point out that as long as  $\eta$ ,  $\Phi$ ,  $s^\eta$  are not related to any of the quantities in (2.6), the second law is an empty statement. *Various second laws differ by the method how this link is made.* Here, we will shortly present two evaluation methods of the second law.

## 2.2. Generalized Coleman-Noll evaluation of the Clausius-Duhem inequality

It is assumed that

- there exists an absolute temperature  $\theta$ ,
- there exist *a priori postulates* by which the entropy supply rate density  $s^\eta$  and the entropy flux  $\Phi$  are connected to some field variables of Eqs. (2.1) and (2.2). For instance, in the classical Clausius-Duhem inequality one postulates the relations

$$(2.8) \quad s^\eta = \frac{s^\varepsilon}{\theta}, \quad \Phi = \Phi^{\mathcal{M}}(\mathbf{u}_i) = \frac{\mathbf{q}}{\theta}(\mathbf{u}_i),$$

where  $s^\varepsilon$ ,  $\mathbf{q}$  represent the energy supply density and energy flux density vector. Most authors use (2.8) if they apply the CD-CN procedure for the exploitation of the entropy inequality. In mixture theories, however, supporters of the CD-approach generally recognize that entropy flux and heat flux need not necessarily be collinear. In those instances entropy and energy balance statements are formally written down for each constituent, and (2.8) is replaced by

$$(2.9) \quad s_a^\eta = \frac{s_a^\varepsilon}{\theta_a}, \quad \Phi_a = \Phi_a^{\mathcal{M}}(\mathbf{u}_i) = \frac{\mathbf{q}_a(\mathbf{u}_i)}{\theta_a}, \quad a \in (1, \dots, N),$$

where  $a$  is a counting index for the number of constituents. The second law of thermodynamics is then here expressed as a statement concerning the entropy balance as a whole; it requires its entropy production to be non-negative. We

will encounter this example below. Finally, notice that in (2.9) we have assumed each constituent to possess its own temperature. Of course we may also specialize these relations to constituents having the same temperatures

- $\eta$  is a constitutive quantity with constitutive relation

$$(2.10) \quad \eta = \eta^{\mathcal{M}}(\mathbf{u}_i).$$

Combination of the energy Eq. (2.1)<sub>2</sub> and the entropy inequality (2.7) by use of the postulate (2.8) yields

$$(2.11) \quad \underbrace{\mathcal{H}(\eta^{\mathcal{M}}(\mathbf{u}_i), \phi^{\mathcal{M}}(\mathbf{u}_i)) - \frac{1}{\theta} f_{\varepsilon}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i))}_{\mathbb{H}(\mathbf{u}_i)} \geq 0 \quad \text{or} \quad \mathbb{H}(\mathbf{u}_i) \geq 0,$$

which should be satisfied for all thermodynamic processes. This form of the entropy inequality no longer contains any source terms.

A clear formulation of the fundamental approach to the exploitation of the Second Law is due to COLEMAN and NOLL [1]. It is as follows: The “universe” is such that there can always be found a neighbourhood of a material point such that the sources  $\mathbf{s}$  may have any value. In the CN-CD approach one assumes all balance Eqs. (2.1) to contain free source terms, therefore only relations (2.2) constrain the independent  $\mathbf{u}_i$  in the exploitation of the inequality (2.11). It has been shown by LIU [7] that for constitutive relations of the class restricted above, satisfaction of (2.11) for all fields constrained by (2.2) is equivalent to satisfying the inequality

$$(2.12) \quad \mathbb{H}(\mathbf{u}_i) - \boldsymbol{\lambda}_{\mathcal{C}} \cdot \mathcal{C}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) \geq 0, \quad \forall \mathbf{u}_i$$

for unconstrained fields  $u_i$ , where  $\boldsymbol{\lambda}_{\mathcal{C}}$  represent the corresponding Lagrange multipliers. When the constitutive Eqs. (2.5) are introduced into inequality (2.12) and all the indicated differentiations are performed, this inequality can be written in the form

$$(2.13) \quad \mathbf{a}(\mathbf{u}_i) \cdot (D\mathbf{u}_i) + b(\mathbf{u}_i) \geq 0,$$

where  $D\mathbf{u}_i$  represent new emerging temporal and spatial derivatives of the independent variables  $\mathbf{u}_i$ <sup>1</sup>, which are not included in the constitutive relations (2.5), hence inequality (2.13) is linear in  $D\mathbf{u}_i$ . Since the inequality must hold for all fields  $\mathbf{u}_i$  and the variables  $D\mathbf{u}_i$  can hence take any values, the inequality could be violated unless

$$(2.14) \quad \mathbf{a}(\mathbf{u}_i) = 0, \quad b(\mathbf{u}_i) \geq 0.$$

<sup>1</sup> The variables  $D\mathbf{u}_i$  may and generally do arise in the balance laws (2.1), but since source terms are present, these equations can always be fulfilled by selecting the external sources accordingly. Thus these equations do not influence inequality (2.13).



We recall that the main purpose of the entropy principle is to derive restrictions upon the constitutive relation (2.5). With relations (2.14) the following results can be obtained:

- reduced dependences of constitutive relations,
- thermostatic equilibrium relations for constitutive quantities,
- thermodynamic potential relations,
- Gibbs relation.

In particular, entropy, internal energy and free energies depend in general only on a reduced number of variables, always those of thermostatic equilibrium. Thus, these variables have dependences in non-equilibrium as if the non-equilibrium states would correspond to an equilibrium. This is a disadvantage and perhaps also questionable, because statistical mechanics shows that the non-equilibrium entropy should depend on non-equilibrium variables, such as strain rate and temperature gradient, if the Enskog procedure is pushed to second iterates. Incidentally, non-collinearity of the entropy flux to the heat flux is also shown by the same Enskog procedure.

Some important points relating to the CD-CN approach should be made:

- When there are no constraints, only the energy equation has an influence on the result (2.14).
- To preserve the property that all balance equations contain free source terms, authors often invent source terms without physical motivation, e.g., for mass balances, structure balance laws, etc. In such cases the results obtained from the Coleman-Noll exploitation of the entropy inequality are dubious.
- When besides  $\theta$  also  $\dot{\theta}$  is an independent constitutive variable in the constitutive relations (2.5), this approach is *a priori* in doubt because the existence of absolute temperature is questionable under those circumstances except in equilibrium.
- When mixtures with distinct constituent temperatures are considered, the method is equally in doubt.

### 2.3. Müller-Liu's entropy principle

In the CD-CN approach, the flux and the supply of entropy are related *a priori* to the flux and supply of heat. And, free sources are assumed for all balance equations except perhaps the balance of mass. In order to relax these assumptions, MÜLLER [9] proposed an entropy principle in which the entropy and its flux are both *a priori* unrestricted constitutive quantities. LIU [7] introduced Lagrange multipliers to consider the influences of all balance laws on the entropy inequality, by which the exploitation of the general entropy inequality is much facilitated.

It is assumed that

- $\theta$  is an empirical temperature,

• the entropy density  $\eta$  and the entropy flux  $\Phi$  are general constitutive relations and no *a priori* postulates are introduced,

$$(2.15) \quad \eta = \eta^{\mathcal{M}}(\mathbf{u}_i), \quad \Phi = \Phi^{\mathcal{M}}(\mathbf{u}_i),$$

• source terms do not affect the material behaviour.

To satisfy the entropy inequality (2.7) for all thermodynamic processes, all field Eqs. (2.6) serve as constraints for the inequality (2.7). It follows that

$$(2.16) \quad \mathcal{H}(\eta^{\mathcal{M}}(\mathbf{u}_i), \Phi^{\mathcal{M}}(\mathbf{u}_i)) - \mathbf{\Lambda} \cdot \mathbf{IF}(\mathbf{u}_i) - \lambda_{\mathcal{C}} \cdot \mathcal{C}(\mathbf{u}_i, \mathcal{M}(\mathbf{u}_i)) + (\mathbf{\Lambda} \cdot \mathbf{s} - s^\eta) \geq 0,$$

where  $\mathbf{\Lambda} = (\lambda, \lambda^\varepsilon)$ ,  $\lambda_{\mathcal{C}}$  represent Lagrange multipliers. The above third assumption requires

$$(2.17) \quad s^\eta = \mathbf{\Lambda} \cdot \mathbf{s},$$

so that the entropy supply  $s^\eta$  is known as soon as  $\mathbf{\Lambda}$  is determined. By evaluation of the entropy inequality (2.16) for a given constitutive class, the following variables or relations can be obtained:

- Lagrange multipliers  $\mathbf{\Lambda}$ ,  $\lambda_{\mathcal{C}}$ ,
- reduced dependences of constitutive relations,
- thermostatic equilibrium relations for constitutive quantities,
- Gibbs relation,
- thermodynamic potential relations.

It is important to emphasize that these results differ from those of the classical evaluation of the entropy inequality of Coleman-Noll in the following respects:

- This second law holds for open and for closed systems.
- Results are in many cases the same as for the CD-CN approach, but not when the theories are complex. **As a rule:** Differences are likely to occur when structural variables enter the formulation such as for the Cosserat continua, liquid crystals, gradient theories, porous media.
- Experience shows that when results between the two entropy principles differ, those obtained by the Müller-Liu principle are generally physically better founded.

In particular we note that entropy, internal and free energies may depend on non-equilibrium variables yielding a different Gibbs relation than that obtained with the CD-CN approach. As a rule, the differences occur primarily in thermodynamic non-equilibrium, but not exclusively. For instance, in the theory of liquid crystals the orientation field of the rodlike molecules in equilibrium is determined by the entropy flux contribution that is not collinear with the heat flux vector. If the Clausius-Duhem inequality were true, the orientation field in thermodynamic equilibrium would be arbitrary, and hence chaotic. We would never be able to read on our laptop screen what we write if the screen is a liquid crystal display.



In granular media of elongated particles (rice) the situation must be very much the same.

### 3. Consequences of the entropy principles

In this section, results are presented that are obtained with the use of the entropy principle of Müller-Liu and compared with those of the classical evaluation of the entropy inequality of Coleman-Noll for a granular solid with a scalar structure equation, and a saturated mixture of granular/fluid constituents with scalar structure equations for each constituent.

The necessary thermal and mechanical field variables are introduced as primitive quantities. Specifically, there exists a kinematic variable, the volume fraction or volume distribution function  $\nu$  (see e.g. GOODMAN and COWIN [4], WANG and HUTTER [17] for a granular material and PASSMAN *et al.* [13], WANG and HUTTER [18] for a solid-fluid mixture). It is complemented by the distributed mass density (true mass density)  $\gamma$ , the stress tensor  $\mathbf{T}$ , body force  $\mathbf{b}$ , specific internal energy  $\varepsilon$ , heat flux vector  $\mathbf{q}$  and heat supply  $r$ . In addition, to account for the energy flux and energy supply associated with the time rate of change of the volume distribution, a higher order stress and body force were introduced by GOODMAN and COWIN [4]. An equilibrated inertia  $k$ , equilibrated stress vector  $\mathbf{h}$  and intrinsic equilibrated body force  $f$  are introduced. For a solid-fluid mixture, the above listed variables should be denoted with an added subscript  $a$  for each constituent  $a$ , with  $a = s$  for the solid and  $a = f$  for the fluid, respectively.

#### 3.1. Granular material

For a granular material, the distributed solid body must satisfy the basic laws of motion of continuum mechanics. Accordingly, the following balance equations must be satisfied:

$$\begin{aligned}
 \mathcal{R} &:= \overline{\dot{\gamma}\nu} + \gamma\nu \operatorname{div} \mathbf{v} = 0, \\
 \mathbf{M} &:= \gamma\nu\dot{\nu} - \operatorname{div} \mathbf{T} - \gamma\nu\mathbf{b} = \mathbf{0}, \\
 \mathcal{N} &:= \gamma\nu k\ddot{\nu} - \operatorname{div} \mathbf{h} - \gamma\nu f = 0, \\
 \mathcal{E} &:= \gamma\nu\dot{\varepsilon} - \mathbf{T} \cdot \mathbf{D} - \mathbf{h} \cdot \operatorname{grad} \dot{\nu} + \gamma\nu f\dot{\nu} + \operatorname{div} \mathbf{q} - \gamma\nu r = 0,
 \end{aligned}
 \tag{3.1}$$

where  $\overline{(\bullet)}$  indicates the material time derivative. The balance Eqs. (3.1)<sub>1,2</sub> are analogous to the classical balance equations of mass and linear momentum. The third equation is a scalar structure equation, which describes the balance of equilibrated force (see GOODMAN and COWIN, [4]). The conservation of energy (3.1)<sub>4</sub> differs from the traditional statements by considering the works of equilibrated

force. (3.1)<sub>1,3</sub> are source-free equations and thus belong to the class (2.2); (3.1)<sub>2,4</sub> do have source terms,  $\mathbf{b}$  and  $r$ . We also point out that GOODMAN and COWIN [4] and PASSMAN *et al.* [13] also introduce a source term in (3.1)<sub>3</sub>, thus making this equation to have no influence in the CD-CN entropy principle.

For the granular material, the following independent constitutive variables are postulated:

$$(3.2) \quad \mathcal{C} = \hat{\mathcal{C}}(\nu, \text{grad } \nu, \dot{\nu}, \gamma, \theta, \text{grad } \theta, \mathbf{D})$$

for the dependent constitutive variables  $\mathcal{C} = \{\psi, \eta, \mathbf{T}, \mathbf{h}, f, \mathbf{q}, \Phi\}$ . The forms of these constitutive relations are reduced by the entropy inequality (2.7), which here can be written as

$$(3.3) \quad \Pi = \gamma \nu \dot{\eta} + \text{div } \Phi - \gamma \nu s \geq 0.$$

According to Müller-Liu's entropy principle, the following inequality must be satisfied for all physical processes

$$(3.4) \quad \Pi = \rho \dot{\eta} + \text{div } \Phi - \rho s - 1/\theta \left\{ \lambda^\nu \mathcal{R} + \boldsymbol{\lambda}^\nu \cdot \mathbf{M} + \lambda^k \mathcal{N} \right\} - \lambda^\varepsilon \mathcal{E} \geq 0,$$

in which the balance relations (3.1) appear as constraints on the entropy inequality, where  $\lambda^\nu$ ,  $\boldsymbol{\lambda}^\nu$ ,  $\lambda^k$  and  $\lambda^\varepsilon$  represent the Lagrange multipliers. For convenience, a factor  $1/\theta$  has been extracted above from  $\lambda^\nu$ ,  $\boldsymbol{\lambda}^\nu$  and  $\lambda^k$ .

Substituting (3.2) into (3.4) and assuming material isotropy, the corresponding restrictions on forms such as (3.2) have been obtained elsewhere (WANG and HUTTER, [17]). By assumptions that the inner free energy, which is defined by  $\psi = \varepsilon - \theta \eta$ , does not depend on  $\dot{\nu}$ ,  $\psi \neq \hat{\psi}(\bullet, \dot{\nu})$ , and supposing the Lagrange multiplier for the energy equation to be  $\lambda^\varepsilon = 1/\theta^2$ , we can obtain the expressions for the Lagrange multipliers

$$(3.5) \quad \boldsymbol{\lambda}^\nu = \mathbf{0}, \quad \lambda^k = 0, \quad \lambda^\nu = -\gamma \frac{\partial \psi}{\partial \gamma}$$

and the reduced constitutive relations

$$(3.6) \quad \begin{aligned} \psi &= \hat{\psi}(\nu, \text{grad } \nu \cdot \text{grad } \nu, \gamma, \theta), & \Phi &= \mathbf{q}/\theta, \\ \mathbf{h} &= \gamma \nu \frac{\partial \psi}{\partial \text{grad } \nu} = \mathcal{A} \text{grad } \nu & \text{with } \mathcal{A} &= 2\gamma \nu \frac{\partial \psi}{\partial (\text{grad } \nu \cdot \text{grad } \nu)}. \end{aligned}$$

In thermodynamic equilibrium, which is defined by  $(\dot{\nu}, \text{grad } \theta, \mathbf{D}) = \mathbf{0}$  and denoted by the superscript  $E$ , the stress  $\mathbf{T}$ , the heat flux  $\mathbf{q}$  and the intrinsic equilibrated body force  $f$  can be expressed as

$$(3.7) \quad \mathbf{T}^E = -\nu p \mathbf{I} - \mathcal{A} \text{grad } \nu \otimes \text{grad } \nu, \quad \mathbf{q}^E = \mathbf{0}, \quad f^E = \frac{p - \beta}{\gamma \nu},$$

<sup>2</sup> This assumption is not reasonable in cases when  $\dot{\theta}$  should also be an independent constitutive variable. Since we will not include such a dependence, the a priori assignment  $\lambda^\varepsilon = 1/\theta$  is justifiable on the basis that Müller and Liu have proved it in LIU and MÜLLER [8].



where  $p$  represents the thermodynamic pressure and  $\beta$  the configuration pressure,

$$(3.8) \quad p = \gamma^2 \frac{\partial \psi}{\partial \gamma}, \quad \beta = \gamma \nu \frac{\partial \psi}{\partial \nu}.$$

For incompressible granular grains  $\gamma = \text{const}$ , so  $p$  is an independent field variable and can no longer be determined by the free energy  $\psi$  as expressed in (3.8)<sub>1</sub>. One can prove this point by means of two different methods. *One is:* We return to the constitutive assumptions (3.2) and note that, in view of the restriction  $\gamma = \text{const}$ , the list of variables appearing in (3.2) is no longer independent. We delete  $\gamma$  from the constitutive equations and repeat the above analysis. *The other method* is based on the method of Lagrange multipliers. We begin with the same constitutive postulates (3.2), but consider  $\dot{\gamma} = 0$  as a new constraint, which can be combined to the entropy inequality (3.4) with a new Lagrange multiplier, and then repeat the above evaluation of the entropy inequality. The two approaches yield the same results.

Some results are summarized in the following points:

- The entropy flux  $\Phi$  is in general not collinear with the heat flux with  $(1/\theta)$  as a factor. Only when the Helmholtz free energy is assumed to be not a function of  $\dot{\nu}$  [ $\psi \neq \psi(\bullet, \dot{\nu})$  – note the rule of equipresence may be violated in this case], the classical result  $\Phi = \mathbf{q}/\theta$  does hold.
- If a free source term in the equilibrated force balance would have been permitted, one would have proved  $\Phi = \mathbf{q}/\theta$  under all circumstances.
- $\mathcal{A} \neq 0$  gives rise to Mohr-Coulomb yield stresses in thermodynamic equilibrium provided the volume fraction is *non-uniform*.
- CD-CN and M-L yield different results *under dynamic, but not static conditions*.

### 3.2. Granular-fluid mixture theory

Similar to the process in Subsec. 3.1 for a granular material, we can obtain reduced constitutive relations for a saturated solid-fluid mixture. Details can be found in WANG and HUTTER [18]. The corresponding balance equations are

$$(3.9) \quad \begin{aligned} \mathcal{R}_a &:= \dot{\rho}_a + \rho_a \text{div} \mathbf{v}_a = 0, \\ \mathbf{M}_a &:= \rho_a \dot{\mathbf{v}}_a - \text{div} \mathbf{T}_a - \rho_a \mathbf{b}_a - \mathbf{m}_a^+ = 0, \\ \mathcal{N}_a &:= \rho_a k_a \dot{\nu}_a - \text{div} \mathbf{h}_a - \rho_a f_a = 0, \\ \mathcal{E} &:= \rho \dot{\epsilon} + \text{div} \mathbf{q} - \mathbf{T} \cdot \mathbf{D} - \sum \mathbf{h}_a \cdot \text{grad} \dot{\nu}_a + \sum \rho_a f_a \dot{\nu}_a - \rho r = 0 \end{aligned}$$

for the solid phase ( $a = 1 = s$ ) and the fluid phase ( $a = 2 = f$ ), respectively. Here,  $\dot{f}_a = \partial f / \partial t + (\text{grad} f_a) \cdot \mathbf{v}_a = \dot{f}_a + (\text{grad} f_a) \cdot \mathbf{u}_a$  ( $\mathbf{u}_a = \mathbf{v}_a \cdot \mathbf{v}$ ) is the material

time derivative with respect to  $\mathbf{v}_a$ , while  $\dot{f}_a = \partial f / \partial t + (\text{grad } f_a) \cdot \mathbf{v}$  is the material time derivative with respect to the mixture velocity,  $\mathbf{v} \cdot \mathbf{m}_a^+$  is internal growth of linear momentum of the constituent  $a$  with the condition  $\mathbf{m}_s^+ + \mathbf{m}_f^+ = 0$ . It is assumed that the energy exchange between the fluid and the solid constituents is so efficient that the mixture can be characterized by a single temperature  $\theta$ , and we need then work only with the energy balance of the mixture (3.9)<sub>4</sub> and the entropy inequality of the mixture

$$(3.10) \quad \Pi = \rho \dot{\eta} + \text{div } \boldsymbol{\Phi} - \rho s \geq 0.$$

The constituent and mixture fields and fluxes are connected by the sum relations

$$(3.11) \quad \begin{aligned} \rho &= \sum \rho_a, \quad \mathbf{v} = \sum \xi_a \mathbf{v}_a, \quad \varepsilon = \varepsilon_I + \frac{1}{2} \sum \xi_a \mathbf{u}_a \cdot \mathbf{u}_a, \quad r = \sum \xi_a r_a, \\ \eta &= \sum \xi_a \eta_a, \quad s = \sum \xi_a s_a, \quad \Pi = \sum \eta_a^+, \quad \mathbf{T} = \sum (\mathbf{T}_a - \rho_a \mathbf{u}_a \otimes \mathbf{u}_a), \\ \boldsymbol{\Phi} &= \sum (\boldsymbol{\Phi}_a + \rho_a \eta_a \mathbf{u}_a), \quad \mathbf{q} = \sum \left\{ \mathbf{q}_a - \left[ \mathbf{T}_a - \rho_a (\varepsilon_a + \frac{1}{2} \mathbf{u}_a \cdot \mathbf{u}_a) \mathbf{I} \right] \mathbf{u}_a \right\} \end{aligned}$$

with  $\sum = \sum_{a=1}^2$  and  $\xi_a = \rho_a / \rho$ .

A physical process must simultaneously satisfy (3.9) and (3.10) as well as other possible additional constraint relations, such as that of saturation

$$(3.12) \quad \nu_s + \nu_f = 1 \quad \longrightarrow \quad \mathcal{S} := \sum (\dot{\nu}_a - \mathbf{u}_a \cdot \text{grad } \nu_a) = 0.$$

According to the Müller-Liu approach of the entropy inequality, one can account for all these requirements by requesting

$$(3.13) \quad \begin{aligned} \Pi = \rho \dot{\eta} + \text{div } \boldsymbol{\Phi} - \rho s - 1/\theta \left\{ \sum \lambda_a^\nu \mathcal{R}_a + \sum \boldsymbol{\lambda}_a^\nu \cdot \mathbf{M}_a + \sum \lambda_a^k \mathcal{N}_a + \pi \mathcal{S} \right\} \\ - \lambda^\varepsilon \mathcal{E} \geq 0. \end{aligned}$$

This entropy inequality (3.13) applies to the general class of two-phase media. Each class is characterized by particular constitutive postulates. For the fluid-saturated granular material, the following independent constitutive variables are postulated

$$(3.14) \quad \mathcal{C}_s = \hat{\mathcal{C}}_s(\mathcal{S}_s), \quad \mathcal{C}_f = \hat{\mathcal{C}}_f(\mathcal{S}_f), \quad \mathbf{m}_s^+ = \hat{\mathbf{m}}_s^+(\mathcal{F}_s, \mathcal{F}_f) = -\mathbf{m}_f^+$$

for  $\mathcal{C}_a \in \{\varepsilon_a, \eta_a, \mathbf{T}_a, \mathbf{h}_a, \mathbf{q}_a, \boldsymbol{\Phi}_a\}$ , as well as the mechanical interactions  $\mathbf{m}_s^+$ , with

$$(3.15) \quad \begin{aligned} \mathcal{S}_a &= (\nu_a, \text{grad } \nu_a, \dot{\nu}_a, \gamma_a, \text{grad } \gamma_a, \theta, \text{grad } \theta, \mathbf{D}_a), \\ \mathcal{F}_a &= (\nu_a, \text{grad } \nu_a, \dot{\nu}_a, \gamma_a, \text{grad } \gamma_a, \theta, \text{grad } \theta, \mathbf{u}_a, \mathbf{D}_a, \mathbf{W}_a), \end{aligned}$$



where  $\mathbf{W}_a$  is the skew-symmetric part of  $\text{grad } \mathbf{v}_a$ ,  $\mathcal{W}_a$  the difference  $\mathcal{W}_a = \mathbf{W}_a - \mathbf{W}$ , with  $\mathbf{W} = \text{skw}(\text{grad } \mathbf{v})$ , respectively. Here, the principles of phase separation and material objectivity have been assumed.

Substituting (3.14) into (3.13) and assuming material isotropy, the corresponding restrictions on the constitutive relations (3.14) have been obtained elsewhere (WANG and HUTTER, [18]). They are expressions for the constituent entropy flux  $\boldsymbol{\phi}_a$  and the equilibrated stress vector  $\mathbf{h}_a$  as well as the dependence of the constituent inner specific free energy  $\psi_a$ , which is defined by  $\psi_a = \varepsilon_a - \theta\eta$ , viz.,

$$(3.16) \quad \begin{aligned} \boldsymbol{\phi}_a &= \mathbf{q}_a/\theta, & \psi_a &= \hat{\psi}_a(\nu_a, \text{grad } \nu_a \cdot \text{grad } \nu_a, \gamma_a, \theta), \\ \mathbf{h}_a &= \rho_a \frac{\partial \psi_a}{\partial \text{grad } \nu_a} = \mathcal{A}_a \text{grad } \nu_a, & \text{with } \mathcal{A}_a &= 2\rho_a \frac{\partial \psi_a}{\partial (\text{grad } \nu_a \cdot \text{grad } \nu_a)} \end{aligned}$$

and the expressions for the heat flux  $\mathbf{q}$ , the intrinsic equilibrated body force  $f_a$ , the stress  $\mathbf{T}_a$  and internal growths of linear momentum  $\mathbf{m}_a^+$  in thermodynamic equilibrium (denoted by the superscript  $E$ , defined by  $(\dot{\nu}_s, \dot{\nu}_f, \text{grad } \theta, \mathbf{v}_s, \mathbf{v}_f, \mathbf{D}_s, \mathbf{D}_N) = 0$ )

$$(3.17) \quad \begin{aligned} \mathbf{q}^E &= \mathbf{0}, & f_a^E &= \frac{p_a - \beta_a}{\gamma_a \nu_a} - \frac{\pi}{\gamma_a \nu_a}, \\ \mathbf{T}_a^E &= -\nu_a (p_a + \gamma_a (\psi_I - \psi_a)) \mathbf{I} - \mathcal{A}_a \text{grad } \nu_a \otimes \text{grad } \nu_a, \\ \mathbf{m}_s^{+E} &= \pi \text{grad } \nu_s + (\psi_I - \psi_s)(1 - \xi_s) \text{grad } (\nu_s \gamma_s) \\ &&& - (\psi_I - \psi_f) \xi_f \text{grad } (\nu_f \gamma_f) = -\mathbf{m}_f^{+E}, \end{aligned}$$

where  $\psi_I = \xi_s \psi_s + \xi_f \psi_f$  is the mixture inner free energy. The variables  $\pi$ ,  $\beta_a$  and  $p_a$  all have the meaning of pressure. As the Lagrange multiplier associated with the saturation constraint  $\pi$  is called the *saturation pressure*, which is a new independent variable for a saturation mixture.  $\beta_a$  is the *configuration pressure* and  $p_a$  the *thermodynamic pressure*, respectively, which can be expressed as

$$(3.18) \quad \beta_a := \rho_a \frac{\partial \psi_a}{\partial \nu_a}, \quad p_a := \gamma_a^2 \frac{\partial \psi_a}{\partial \gamma_a},$$

where the expression for  $p_a$  is only suitable for compressible constituents; otherwise, for incompressible constituents, i.e. constituents whose true mass density does not change,  $p_a$  is an independent field variable and can no longer be determined by the free energy  $\psi_a$  as expressed in (3.18)<sub>2</sub>.

It is important to emphasize that these results differ from those obtained by evaluation of the entropy inequality using the Coleman-Noll approach for a solid-fluid mixture (PASSMAN *et al.*, [13]).

• The partial entropy flux  $\boldsymbol{\phi}_a$  is in general not collinear with the heat flux with  $(1/\theta)$  as a prefactor. Only in the case the Helmholtz free energy is assumed *not* to be a function of  $\dot{\nu}_a, \forall a$ , the result  $\boldsymbol{\phi}_a = \mathbf{q}_a/\theta$  is obtained. But even then  $\boldsymbol{\phi} \neq \mathbf{q}/\theta$ , as

$$\boldsymbol{\phi} = \frac{\mathbf{q}}{\theta} - \frac{1}{\theta} \sum_a \left\{ \rho_a \psi_a \mathbf{u}_a - \left( \mathbf{T}_a - \frac{\rho_a}{2} (\mathbf{u}_a \cdot \mathbf{u}_a) \mathbf{1} \right) \mathbf{u}_a \right\}.$$

• If the CD-CN approach is used *with* a source term in the equilibrated force balances,  $\boldsymbol{\phi}_a = \mathbf{q}_a/\theta$  would have been obtained even with  $\dot{\nu}_a$  as independent constitutive variables. This has been done so by PASSMANN *et al.* [13].

• If the same theory is developed using the CD-CN approach, i.e. by *a priori* setting  $\boldsymbol{\phi}_a = \mathbf{q}_a/\theta$  and introducing a source term in the balance of equilibrated forces, then the equilibrium results are

$$(3.19) \quad \begin{aligned} \mathbf{q}^E &= 0, & f_a^E &= \frac{1}{\gamma_a \nu_a} (p_a - \beta_a - \pi), \\ \mathbf{T}_a^E &= -\nu_a p_a \mathbf{1} - \mathcal{A}_a \text{grad } \nu_a \otimes \text{grad } \nu_a, & \mathbf{m}_s^{+E} &= \pi \text{grad } \nu_s = -\mathbf{m}_f^{+E}, \end{aligned}$$

and they are different from (3.17).

• When  $\mathcal{A}_a = 0, \forall a \in (1, \dots, N)$ , no shear stresses can be supported in equilibrium. Therefore, volume fraction gradients as independent constitutive variables are important.

• Formally, for  $\mathcal{A}_a = 0, \forall a \in (1, \dots, N)$ , the above formulas for the stresses and the interaction forces do not agree with the corresponding formulas obtained by SVENDSEN and HUTTER [16], even though with  $\mathbf{h}_a = \mathbf{0}$  and  $\dot{\nu}_a = 0, \forall a \in (1, \dots, N)$ , the two formulations are the same.

This result is no surprise: It says that “the limit of a theory need not be the theory of the limit”.

Finally it should be stated that the reduced entropy production rate is independent of the saturation pressure. So, the constraint variable does not produce entropy, as it should for physical reasons, for details see WANG and HUTTER [18].

#### 4. Application in inclined gravity-flow problem

In Sec. 3 we derived the equilibrium expressions of the stresses, heat fluxes and intrinsic equilibrated body forces by the restrictions of the entropy principle on the constitutive equations. We assumed that these quantities may be decomposed into the thermodynamic equilibrium parts (denoted by the superscript  $E$ ) and the dynamic parts (denoted by the superscript  $D$ ). Furthermore, for the dynamic parts a linear theory is considered, in which the dynamic parts of these quantities are linear in the dynamic variables. By substituting these expressions into the



field equations we can obtain a system of the field equations to analyse the cases of steady isothermal flows of a layer of uniform thickness  $L$  of a granular material, as well as a granular-fluid mixture down a rough plate inclined at an angle  $\theta$  to the horizontal (in our computations we take  $\theta = 40^\circ$ ), as shown in Fig. 1.

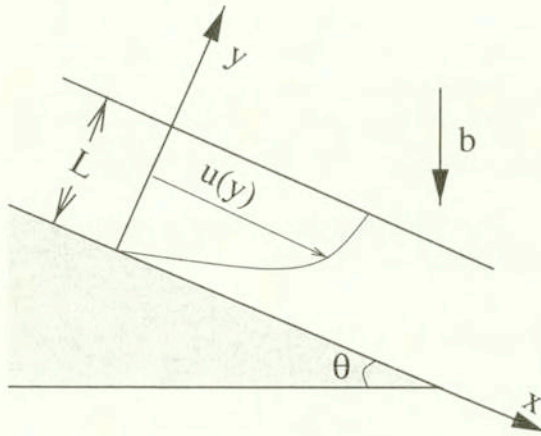


FIG. 1. Inclined gravity-flow and coordinate system.

4.1. Granular materials

Under the assumptions

$$(4.1) \quad \mathbf{T} = \mathbf{T}^E + \mathbf{T}^D, \quad \mathbf{q} = \mathbf{q}^E + \mathbf{q}^D, \quad f = f^E + f^D$$

and a linear theory for the dynamic parts

$$(4.2) \quad \mathbf{q}^D = -\kappa \text{grad } \theta, \quad \mathbf{T}^D = \xi \dot{\nu} \mathbf{I} + \lambda (\text{tr} \mathbf{D}) \mathbf{I} + 2\mu \mathbf{D}, \quad f^D = -\zeta \dot{\nu} - \delta \text{tr} \mathbf{D},$$

we obtain with the use of (3.7), (4.2) in (4.1) the constitutive equations

$$(4.3) \quad \begin{aligned} \mathbf{T} &= [-p + \lambda \text{tr} \mathbf{D} + \xi \dot{\nu}] \mathbf{I} - \mathcal{A} \text{grad } \nu \otimes \text{grad } \nu + 2\mu \mathbf{D}, \\ \mathbf{q} &= -\kappa \text{grad } \theta, \quad f = \frac{p}{\gamma \nu} - \frac{\beta}{\gamma \nu} - \delta \text{tr} \mathbf{D} - \zeta \dot{\nu}. \end{aligned}$$

Substituting (4.3) into the field Eq. (3.1), using the expressions for the inner free energy  $\psi$  (PASSMAN *et al.*, [12]) and the viscosity  $\mu$  (PASSMAN *et al.*, [12, 14]; SAVAGE, [15]),

$$(4.4) \quad \begin{aligned} \gamma \nu \psi &= a_0 (\nu - \nu_c)^2 + \alpha_0 \left( \frac{\nu_c}{\nu_\infty - \nu} \right)^2 \text{grad } \nu \cdot \text{grad } \nu, \\ \mu &= \mu_0 \left( \frac{\nu_c}{\nu_\infty - \nu} \right)^4, \end{aligned}$$

where  $\nu_\infty$  is the volume fraction corresponding to the densest possible packing of the material, and  $\nu_c$  is the critical volume fraction (GOODMAN and COWIN, [3]), assuming  $\mathbf{v} = [u(y), 0, 0]$  and introducing the following dimensionless variables

$$(4.5) \quad \bar{y} = \frac{y}{\lambda}, \quad \bar{p} = \frac{p}{a_0}, \quad \bar{u} = u / \left( \frac{\gamma_0 b \lambda^2}{\mu_0} \right),$$

where  $\lambda$  is an internal length scale

$$(4.6) \quad \lambda = \sqrt{\frac{\alpha_0}{a_0}},$$

we can conveniently formulate the granular flow problem down an inclined plane in terms of the dimensionless equations

$$(4.7) \quad \frac{d}{d\bar{y}} \left[ \nu \bar{p} + 2 \left( \frac{\nu_c}{\nu_\infty - \nu} \right)^2 \left( \frac{d\nu}{d\bar{y}} \right)^2 \right] + S \cos \theta \nu = 0,$$

$$(4.8) \quad \frac{d}{d\bar{y}} \left[ \left( \frac{\nu_c}{\nu_\infty - \nu} \right)^4 \frac{d\bar{u}}{d\bar{y}} \right] + \nu \sin \theta = 0,$$

$$(4.9) \quad \frac{d}{d\bar{y}} \left[ 2 \left( \frac{\nu_c}{\nu_\infty - \nu} \right)^2 \frac{d\nu}{d\bar{y}} \right] + \bar{p} - \frac{1}{\nu} \left[ (\nu - \nu_c)^2 - \left( \frac{\nu_c}{\nu_\infty - \nu} \right)^2 \left( \frac{3\nu - \nu_\infty}{\nu - \nu_\infty} \right) \left( \frac{d\nu}{d\bar{y}} \right)^2 \right] = 0$$

with the non-dimensional boundary conditions

$$(4.10) \quad \nu(0) = \nu_0, \quad \bar{u}(0) = 0, \quad \frac{d\nu}{d\bar{y}}(\bar{L}) = 0; \quad \frac{\partial \bar{u}}{\partial \bar{y}}(\bar{L}) = 0; \quad \bar{p}(\bar{L}) = 0,$$

where the dimensionless parameter  $S$  represents the grain size, and  $\bar{L}$  the dimensionless flow thickness, defined as

$$(4.11) \quad S = \frac{\gamma b \lambda}{a_0}, \quad \bar{L} = \frac{L}{\lambda},$$

respectively. Evidently, the problem is characterized by five dimensionless parameters  $\nu_0$ ,  $\nu_\infty$ ,  $\nu_c$ ,  $S$  and  $\bar{L}$ . The variation of the first three is rather narrow and will not be studied here, but the remaining two typify the grain size and the flow depth in terms of the internal length scale. We choose

$$(4.12) \quad \nu_\infty = 0.644, \quad \nu_c = 0.555, \quad \nu_0 = 0.51,$$



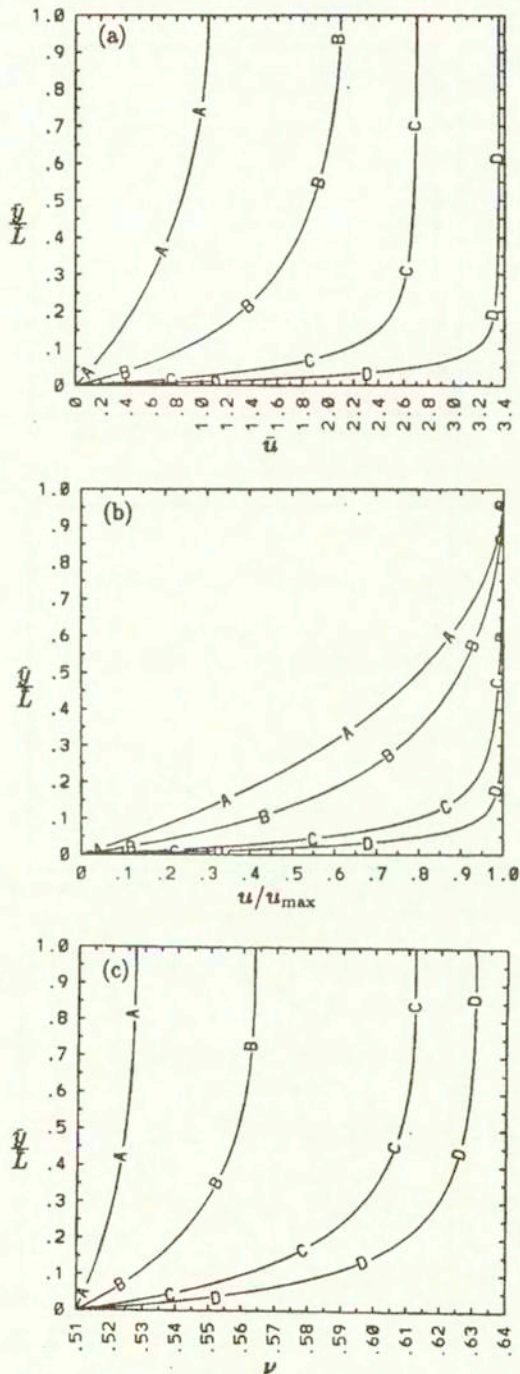


FIG. 2. Non-dimensional velocity profiles (a), normalized velocity profiles (b) and volume fraction profiles (c) for a fixed value of the parameter  $S = 0.1$  and various values of  $\bar{L}$ :  $\bar{L} = 5$  (A); 10 (B); 20 (C) and 30 (D).

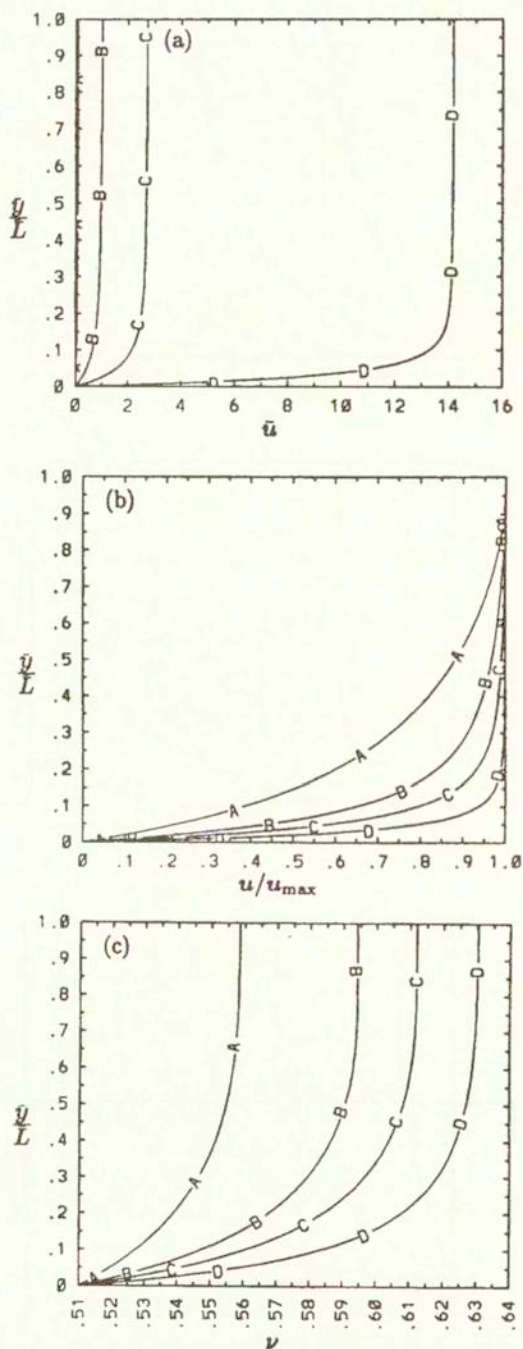


FIG. 3. Non-dimensional velocity profiles (a), normalized velocity profiles (b) and volume fraction profiles (c) for a fixed value of  $\bar{L} = 20$  and various values of  $S$ :  $S = 0.01$  (A);  $0.05$  (B);  $0.1$  (C) and  $0.3$  (D).



values as given by SAVAGE [15], appropriate to natural angular beach sand with diameters of particles from 0.318 mm to 0.414 mm. Some typical results calculated for the volume distribution, dimensionless velocity for this gravity-flow problem, are shown in Figs. 2, 3 for various values of the dimensionless granular flow thickness  $\bar{L}$  and the parameter  $S$ , indicating the grain size, respectively.

From Figs. 2, 3 the following results can be deduced:

- The granular velocity and the volume fraction depend strongly on the flow layer thickness and the grain size.
- For a fixed grain size (a fixed  $S$ ), if the layer thickness is small, the shear can extend from the bottom to the free surface, which behaves much like an incompressible fluid, and the volume fraction has only a small change across the depth, whereas for thicker grain flow the flow structure is far from an incompressible fluid, in which in a large region near the free surface of the grain flow is similar to that of a plug flow, with a nearly constant velocity and less changed volume fraction, the shear layers close to the bottom, where dilatation has occurred, may be very thin.
- For a fixed  $\bar{L}$ <sup>3</sup>, indicating the ratio of the flow thickness to the grain size, fine grains (small values of  $S$ ) show a velocity profile similar to that of a fluid. Increasing the grain size increases the manifestation of the granular character of the material, a plug flow near the free surface and a large shear with large dilatation near the bottom.
- The granular velocity increases with increasing flow thickness.

#### 4.2. Granular-fluid mixture

Similarly, we assume that  $\mathbf{T}_a$ ,  $f_a$  and  $\mathbf{m}_a^+$  may be decomposed according to

$$(4.13) \quad \mathbf{T}_a = \mathbf{T}_a^E + \mathbf{T}_a^D, \quad f_a = f_a^E + f_a^D, \quad \mathbf{m}_a^+ = \mathbf{m}_a^{+E} + \mathbf{m}_a^{+D},$$

where  $\mathbf{T}_a^E$ ,  $f_a^E$  and  $\mathbf{m}_a^{+E}$  represent the thermodynamic equilibrium parts, as displayed in (3.17), while  $\mathbf{T}_a^D$ ,  $f_a^D$  and  $\mathbf{m}_a^{+D}$  are their dynamic contributions. For the dynamic parts, scalar-, vector-, and tensor-valued quantities are assumed to depend explicitly and linearly on scalar-, vector-, and tensor-valued independent dynamic variables, respectively, by the forms

$$(4.14) \quad \mathbf{T}_a^D = 2\mu_a \mathbf{D}_a, \quad f_a^D = \lambda_a \dot{\nu}_a, \quad \mathbf{m}_s^{+D} = -m_D(\mathbf{v}_s - \mathbf{v}_f) = -\mathbf{m}_f^{+D},$$

where  $\mu_a$ ,  $\lambda_a$ ,  $m_D$  are the functions (PASSMAN *et al.*, [14]; WANG and HUTTER, [18])

$$(4.15) \quad \mu_s = \bar{\mu}_s \left( \frac{\nu_c}{\nu_\infty - \nu_s} \right)^4, \quad \mu_f = \nu_f^2 \bar{\mu}_f, \quad m_D = \nu_s(1 - \nu_s)D$$

<sup>3</sup>) Here, a fixed  $\bar{L}$  means, if the grain size varies, that the flow thickness should be changed accordingly.

with  $\bar{\mu}_s$ ,  $\bar{\mu}_f$  and  $D$  being positive constants. To obtain the explicit expressions of  $\mathbf{T}_a$ ,  $\mathbf{m}_a^+$  and  $f_a$ , a representation for the specific free energy  $\psi_a$  for each constituent  $a$  is needed. Following PASSMAN *et al.* [14], we choose the expressions

$$(4.16) \quad \begin{aligned} \nu_s \gamma_s \psi_s &= a_s [\nu_s - \nu_c]^2 + \bar{\alpha}_s \left( \frac{\nu_c}{\nu_\infty - \nu_s} \right)^2 (\text{grad } \nu_s \cdot \text{grad } \nu_s), \\ \nu_f \gamma_f \psi_f &= a_f [\nu_f - (1 - \nu_c)]^2. \end{aligned}$$

For the steady motion down the inclined plate, by assuming

$$(4.17) \quad \begin{aligned} \mathbf{v}_s &= [u_s(y), 0, 0], & \mathbf{v}_f &= [u_f(y), 0, 0], & \nu_s &= \nu_s(y), & \nu_f &= \nu_f(y), \\ p_s &= p_s(y), & p_f &= p_f(y), & \pi &= \pi(y), \\ \mathbf{b}_s &= \mathbf{b}_f = [b \sin \theta, -b \cos \theta, 0] \end{aligned}$$

and introducing the following dimensionless variables:

$$(4.18) \quad \begin{aligned} \bar{y} &= \frac{y}{\lambda_s}, & \bar{p}_s &= \frac{p_s}{a_s}, & \bar{p}_f &= \frac{p_f}{a_s}, & \bar{\beta}_s &= \frac{\beta_s}{a_s}, & \bar{\beta}_f &= \frac{\beta_f}{a_s}, \\ \bar{\pi} &= \frac{\pi}{a_s}, & \bar{u}_s &= u_s / \left( \frac{\gamma_s g \lambda_s^2}{\bar{\mu}_s} \right), & \bar{u}_f &= u_f / \left( \frac{\gamma_s g \lambda_s^2}{\bar{\mu}_s} \right), \end{aligned}$$

where  $\lambda_s$  is an internal length scale of grains

$$(4.19) \quad \lambda_s = \sqrt{\frac{\bar{\alpha}_s}{a_s}},$$

we can obtain the following dimensionless governing differential equations:

$$(4.20) \quad \nu_s + \nu_f = 1,$$

$$(4.21) \quad \begin{aligned} \frac{d}{d\bar{y}} \left\{ \nu_s \bar{p}_s + \xi_f \left[ \left( \frac{\nu_s \zeta_a}{\nu_f \zeta_\gamma} - 1 \right) (\nu_s - \nu_c)^2 - \frac{1}{(\nu_\infty - \nu_s)^2} \left( \frac{d^2 \nu_s}{d\bar{y}^2} \right) \right] \right. \\ \left. + 2 \left( \frac{\nu_c}{\nu_\infty - \nu_s} \right)^2 \left( \frac{d\nu_s}{d\bar{y}} \right)^2 \right\} - \left\{ \pi + \frac{(\xi_f^2 - \zeta_\gamma \xi_s^2) \xi_f}{\nu_s} \right. \\ \left. \left[ \left( \frac{\nu_s \zeta_a}{\nu_f \zeta_\gamma} - 1 \right) (\nu_s - \nu_c)^2 - \frac{1}{(\nu_\infty - \nu_s)^2} \left( \frac{d^2 \nu_s}{d\bar{y}^2} \right) \right] \right\} \frac{d\nu_s}{d\bar{y}} + S \nu_s \cos \theta = 0, \end{aligned}$$



$$(4.22) \quad \frac{d}{d\bar{y}} \left\{ \nu_f \bar{p}_f + \xi_s \frac{\nu_f}{\nu_s} \zeta_\gamma \left[ \left( \frac{\nu_s \zeta_a}{\nu_f \zeta_\gamma} - 1 \right) (\nu_s - \nu_c)^2 - \frac{1}{(\nu_\infty - \nu_s)^2} \left( \frac{d^2 \nu_s}{d\bar{y}^2} \right) \right] \right. \\ \left. - \left\{ \pi + \frac{(\xi_f^2 - \zeta_\gamma \xi_s^2) \xi_f}{\nu_s} \left[ \left( \frac{\nu_s \zeta_a}{\nu_f \zeta_\gamma} - 1 \right) (\nu_s - \nu_c)^2 - \frac{1}{(\nu_\infty - \nu_s)^2} \left( \frac{d^2 \nu_s}{d\bar{y}^2} \right) \right] \right\} \frac{d\nu_f}{d\bar{y}} + \zeta_\gamma S \nu_f \cos \theta = 0, \right.$$

$$(4.23) \quad \frac{d}{d\bar{y}} \left[ \left( \frac{\nu_c}{\nu_\infty - \nu_s} \right)^4 \frac{d\bar{u}_s}{d\bar{y}} \right] - \bar{D} \nu_s (1 - \nu_s) (\bar{u}_s - \bar{u}_f) + \nu_s \sin \theta = 0,$$

$$(4.24) \quad \frac{d}{d\bar{y}} \left[ \zeta_\mu \nu_f^2 \frac{d\bar{u}_f}{d\bar{y}} \right] + \bar{D} \nu_s (1 - \nu_s) (\bar{u}_s - \bar{u}_f) + \zeta_\gamma \nu_f \sin \theta = 0,$$

$$(4.25) \quad \bar{\pi} = \bar{p}_f - \bar{\beta}_f,$$

$$(4.26) \quad \bar{\beta}_s - \bar{\beta}_f = \bar{p}_s - \bar{p}_f + \frac{d}{d\bar{y}} \left[ 2 \left( \frac{\nu_c}{\nu_\infty - \nu_s} \right)^2 \frac{d\nu_s}{d\bar{y}} \right]$$

with the expressions for the dimensionless configuration pressures  $\bar{\beta}_s$  and  $\bar{\beta}_f$  as follows:

$$\nu_s \bar{\beta}_s = (\nu_s^2 - \nu_c^2) + \frac{3\nu_s - \nu_c}{(\nu_\infty - \nu_s)^3} \left( \frac{d\nu_s}{d\bar{y}} \right)^2, \quad \nu_f \bar{\beta}_f = \zeta_a (\nu_s^2 - \nu_c^2),$$

where the dimensionless parameters  $S$ ,  $\bar{L}$ ,  $\bar{D}$ ,  $\zeta_\gamma$ ,  $\zeta_\mu$  and  $\zeta_a$  are defined as

$$(4.27) \quad S = \frac{\gamma b \lambda_s}{a_s}, \quad \bar{L} = \frac{L}{\lambda_s}, \quad \bar{D} = D / \left( \frac{\bar{\mu}_s}{\lambda_s^2} \right), \quad \zeta_\gamma = \frac{\gamma_f}{\gamma_s}, \\ \zeta_\mu = \frac{\mu_f}{\bar{\mu}_s}, \quad \zeta_a = \frac{a_f}{a_s},$$

respectively. Together with  $\nu_0$ ,  $\nu_c$ ,  $\nu_\infty$  these are nine dimensionless quantities. Here  $S$  represents a dimensionless scale of the grain size,  $\bar{L}$  the dimensionless flow thickness,  $\bar{D}$  the dimensionless drag coefficient,  $\zeta_\gamma$  the ratio of the fluid material density to that of the solid grain,  $\zeta_\mu$  the ratio of the viscosities and  $\zeta_a$  the ratio of the "energy storage capacities" of two constituents. (4.20) – (4.27) is a system of seven equations with seven unknowns  $\nu_s$ ,  $\nu_f$ ,  $\bar{\pi}$ ,  $\bar{p}_s$ ,  $\bar{p}_f$ ,  $\bar{u}_s$ ,  $\bar{u}_f$ , which will be solved subjected to the boundary conditions

$$(4.28) \quad \nu_s(0) = \nu_0, \quad \nu_f(0) = 1 - \nu_0, \quad \bar{u}_s(0) = 0, \quad \bar{u}_f(0) = 0, \\ \frac{d\nu_s}{d\bar{y}}(\bar{L}) = 0, \quad \frac{d\nu_f}{d\bar{y}}(\bar{L}) = 0, \quad \frac{\partial \bar{u}_s}{\partial \bar{y}}(\bar{L}) = 0, \quad \frac{\partial \bar{u}_f}{\partial \bar{y}}(\bar{L}) = 0, \\ \bar{p}_s(\bar{L}) = 0, \quad \bar{p}_f(\bar{L}) = 0.$$

We choose to investigate the case with estimated parameters corresponding to a mixture of water with natural beach sand with the given non-dimensional parameters listed in Table 1 (see SAVAGE, [15]; PASSMAN *et al.*, [14]).

**Table 1. Values of non-dimensional parameters arising in the field equations**

$\zeta_\gamma$	$\zeta_a$	$\zeta_\mu$	$\bar{D}$	$\nu_\infty$	$\nu_c$	$\nu_0$
0.45	1.0	0.01	20	0.644	0.555	0.51

Profiles of the solid volume fraction  $\nu_s$  and the solid, fluid velocities are illustrated in Figs. 4, 5 for various values of the two remaining dimensionless quantities: the non-dimensional flow thickness  $\bar{L}$  and the parameter  $S$ . These figures illustrate the following behaviour of the solid-fluid mixture flow:

- The solid, fluid velocities and the solid volume fraction are strongly dependent on the flow layer thickness and the grain size.

- A relatively thin layer thickness shows a marked variation of the solid volume fraction  $\nu_s$  across the whole depth, while thicker flow shows an increasing tendency of  $\nu_s$  to increase quickly to an asymptotic value toward the free surface. In a relatively narrow zone near the bottom, where the shear is the largest, dilatation occurs. But the absolute thickness of this dilatation layer seems to be less dependent on the flow thickness.

- For small flow thickness and small grain size the shear can extend to the whole flow region. As  $\bar{L}$  increases or  $S$  increases (at a fixed  $\bar{L}$ , this means that both the grain size and the flow layer thickness increase), there is an increasing tendency that the shear is limited to a narrow layer near the bottom with high dilatancy; above the layer the velocity can be regarded as constant. In this case for such a granular solid-fluid mixture flow down an inclined plane, one can often assume that only a portion of the flow which is close to the base is fluidized, while the upper portion is passively moving with the speed of the particles at the upper edge of this fluidized layer.

- In Table 2 the dimensionless solid and fluid velocities in the mixture and the solid velocities in the dry granular material at the free surface (e.g. maximum velocities) are displayed. Evidently, in the solid-fluid mixture, if the thickness is relatively small, the surface velocities increase by increasing the flow thickness, while for sufficiently large stream thickness, they depend only slightly on the stream thickness. On the contrary, in the dry granular solid, the solid velocity increases monotonously by increasing the flow thickness.

- Comparison of Figs. 2 – 5 shows that in the solid-fluid mixture the fluid velocity is always much larger than that of the solid; here the fluid is pulling the solid via the drag force interacting between them. Furthermore, this solid velocity in the mixture is always larger than that in a dry granular flow. These properties can be more easily inferred from Table 2.



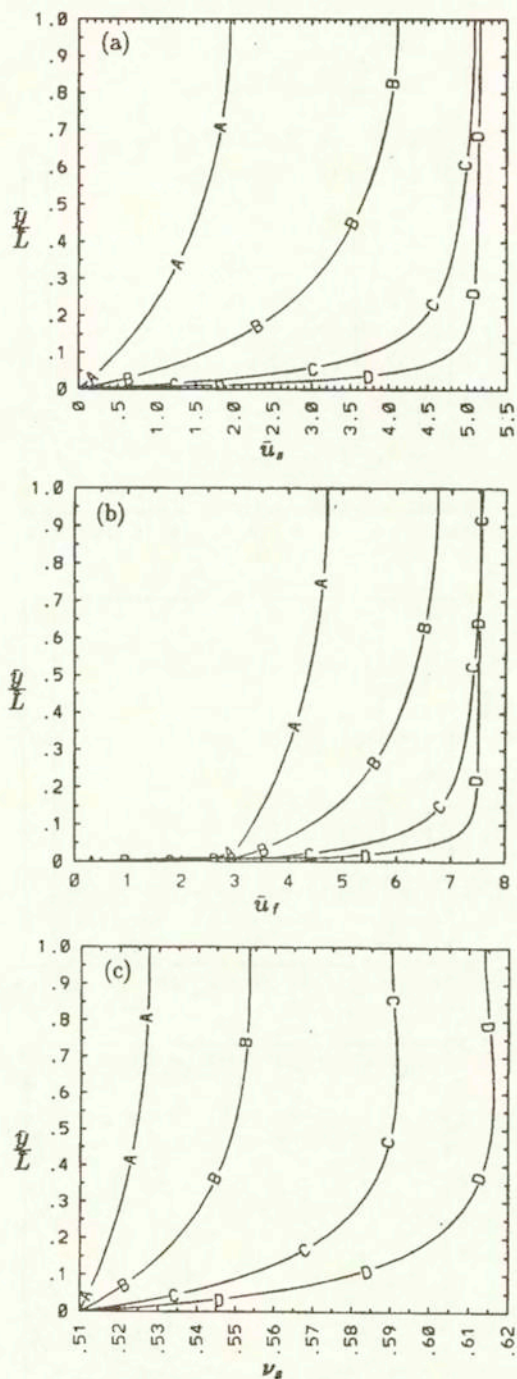
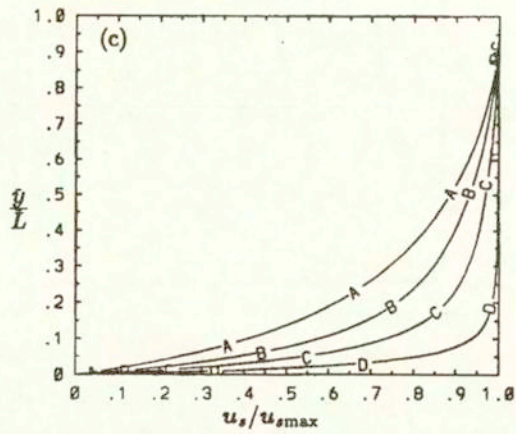
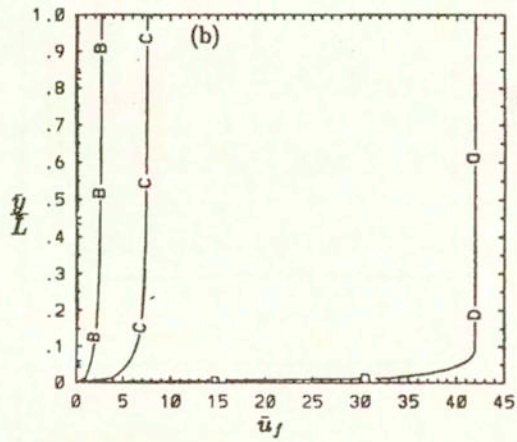
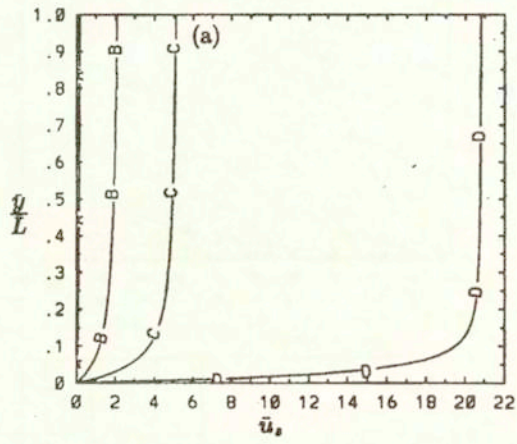


FIG. 4. Non-dimensional solid velocity profiles (a), fluid velocity profiles (b) and volume fraction profiles (c) for a fixed value of the parameter  $S = 0.1$  and various values of  $\bar{L}$ :  $\bar{L} = 5$  (A); 10 (B); 20 (C) and 30 (D).



[FIG. 5]



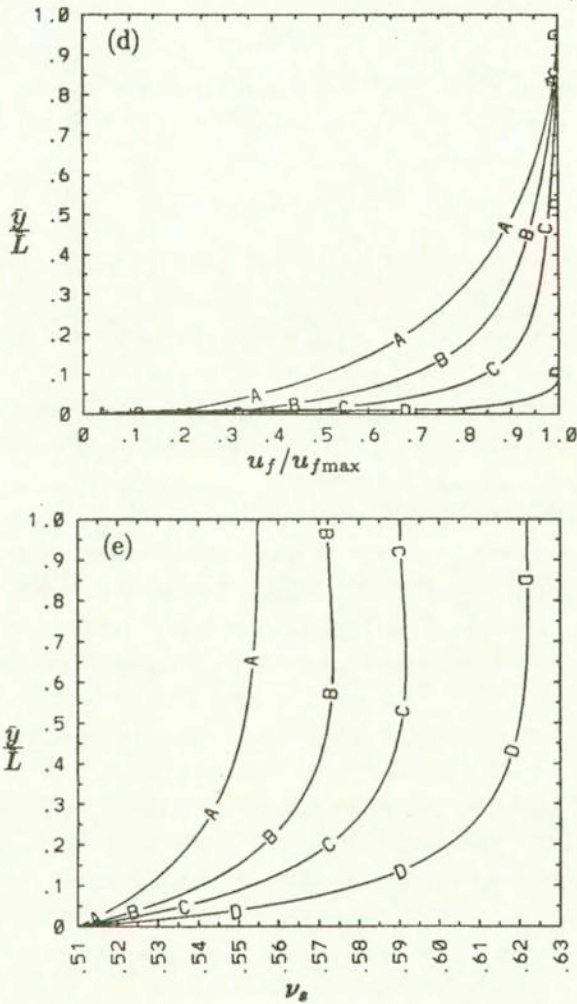


FIG. 5. Non-dimensional solid, fluid velocity profiles (a) (b), normalized solid, fluid velocity profiles (c) (d) and volume fraction profiles (e) for a fixed value of  $\bar{L} = 20$  and various values of  $S$ :  $S = 0.01$  (A);  $0.05$  (B);  $0.1$  (C) and  $0.3$  (D).

Table 2. Non-dimensional granular velocities in a dry granular flow, granular and fluid velocities in a granular-fluid mixture at the free surface for various values of the non-dimensional flow thickness  $\bar{L}$

	$\bar{L}$	5	10	20	30
In a granular flow	Granular velocity $\bar{u}$	1.05	2.10	2.70	3.36
In a granular-fluid	Granular velocity $\bar{u}_s$	1.95	4.11	5.09	5.16
mixture	Fluid velocity $\bar{u}_f$	4.72	6.75	7.57	7.55

- It might be mentioned that for the drag coefficient  $D = 0$  (the corresponding results are not displayed in this paper) the fluid phase behaves very similar to a viscous fluid; the shear of the fluid can extend to the whole flow region. On the other hand, the dominant shear of the solid flow may still be mainly restricted to the regions near the bottom.

## 5. Concluding remarks

In this paper we attempted to explain how the basic postulates of two forms of the entropy principle: (i) the generalized Clausius-Duhem – Coleman-Noll approach and (ii) the Müller-Liu entropy principle differ from one another. CD-CN makes *a priori* postulates about the entropy flux and entropy supply and assumes external source terms in (most) balance laws. ML postulate the entropy flux to be a general constitutive variable and treat all field equations as constraints for the exploitation of the entropy principle. It was demonstrated that they yielded different constitutive relations for the granular material with/without fluid. Results were presented with the use of both principles for a granular solid with a scalar structure equation and a saturated mixture of granular/fluid constituents with scalar structure equations for each constituent. These results allow us to favour one set of basic postulates over the other. These theories were then applied to the analysis of steady fully-developed gravity flows down an inclined plane. A series of non-dimensional field equations were derived. Numerical results show that for a large thickness of the flow and large grain sizes, dilatant shearing layers exist only near the bottom. In the zones far away from the bottom the shearing nearly vanishes, where each constituent moves as an entire body in a plug-flow manner, while for small thickness of the granular flow and fine grains, the behaviour of the granular flows is similar to that of a viscous fluid, the shear can extend from the bottom to the free surface.

Finally, some points should be emphasized:

- HUTTER *et al.* [5] demonstrated that the emerging solutions of a constitutive mixture theory, if it is obtained by the CD-CN exploitation of the entropy principle (EHLERS, [2]), are extremely restricted. There exists no solution for a simple gravity-driven shearing flow of viscous constituents. However, in this case, if the mixture theory is derived from the Müller-Liu approach of the entropy principle (SVENDSEN and HUTTER, [16]), this nonexistence of the solution can be avoided. This is tempting to favour the Müller-Liu approach of the entropy principle over the CD-CN on another account, not just according to basic postulates.

- Different authors do not unanimously agree upon the form of the scalar structure equations to describe the constituent volume fractions  $\nu_a$ . SVENDSEN and HUTTER [16], HUTTER *et al.* [16] treated the solid volume fraction as an



internal variable and write an evolution equation balancing its time rate of change with its production  $\pi_a$  in the form

$$\dot{\nu}_a = \pi_a.$$

A disadvantage of this form is that if the constituents are incompressible, this equation is no longer independent. In this case, it is the same as the mass balance. WILMAŃSKI [19], on the other hand, using statistical arguments on the microscale demonstrated that the Svendsen-Hutter equation needed to be complemented by a flux term, thus arriving at a complete balance law

$$\dot{\nu}_a = \operatorname{div} \mathbf{h}_a + \pi_a.$$

The two different entropy principles applied to the different granular mixture models give different results. Differences between these models should to be studied.

- CD-CN should be abandoned in the following classes of models: *Polar continua* (solids, anisotropic fluids, liquid crystals) because the spin balance has no free source terms; *Structured continua*; *Mixtures* (as is long known); *Coupled field theories*, etc.

- Many of these theories suffer from the necessity of formulating the boundary conditions which are physically not transparent.

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