Arch. Mech., **51**, 5, pp. 559–603, Warszawa 1999

Irreducible representations for constitutive equations of anisotropic solids I: crystal and quasicrystal classes D_{2mh}, D_{2m} and C_{2mv}

H. XIAO, O.T. BRUHNS, and A. MEYERS

Institute of Mechanics I, Ruhr-University Bochum
D-44780 Bochum, Germany

A SIMPLE, UNIFIED PROCEDURE is applied to derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations involving any finite number of vector variables and second order tensor variables. In the paper consisting of three parts, we consider all kinds of material symmetry groups as subgroups of the cylindrical group $D_{\infty h}$. This paper, together with a previous work, covers all kinds of material symmetric groups of solids, except for the five cubic crystal classes and the two icosahedral quasicrystal classes. In this part, our concern is with all crystal classes and quasicrystal classes D_{2mh}, D_{2m} and C_{2mv} for all integers $m \geqslant 2$.

1. Introduction

ANISOTROPIC SOLIDS, such as crystals, quasicrystals, composite materials and textured materials etc., manifest their macroscopic mechanical and physical behaviours in complicated and varied manners, e.g., elasticity, elastoplasticity, viscoelasticity, viscoplasticity, creep, damage, yielding, etc., as well as heat conductivity, electric and magnetic permeativity, piezoelectricity, electro- and magnetostrictions, photoelasticity, electromagnetic elasticity, etc. In continuum physics, such complicated and varied material behaviours are mathematically modelled by various forms of scalar-, vector- , skewsymmetric and symmetric second order tensor-valued functions of scalar and vector and second order tensor variables, which are commonly known as material constitutive equations, such as yield functions, elastic stored-energy functions and Helmholz free energy functions; Ohm's law of electric conduction, Fourier's law of heat conduction; electric field-stress relation, stress-electric field relation; stress-deformation relation, stress-strain rate relation, stress rate-strain rate relation, and evolution equations of internal state variables, etc. (see, e.g., NYE [18] for classical linear cases and TRUESDELL and NOLL [38] and ERINGEN and MAUGIN [12] for general cases). The principle of material objectivity and material symmetry require that constitutive equations of a solid obey a combined invariance restriction under the material symmetry group of this solid. As a rational basis of consistent mathematical modelling of complicated and varied material behaviours, it is desirable to obtain general reduced forms or representations of material constitutive equations under the just-stated invariance restriction. In the past decades, this aspect was extensively studied. Earlier, attention was concentrated on *polynomial representations* mainly for scalar-valued functions and for vector-valued and tensor-valued functions in some cases (see, e.g., the monographs by TRUESDELL and Noll [38], SPENCER [33], KIRAL and ERINGEN [14], BETTEN [4] and SMITH [26] for many related results). *Nonpolynomial representations* in a general sense were considered later for isotropic functions by WANG [40], SMITH [24], BOEHLER [6] and PENNISI and TROVATO [19] *et al.* and for anisotropic functions by LOKHIN and SEDOV [17], BOEHLER and RACLIN [10], BOEHLER [7 – 9] and LIU [16], *et al.*, and developed recently by RYCHLEWSKI [22], ZHANG and RYCHLEWSKI [54], ZHENG and SPENCER [59], as well as one of these authors (see XIAO [43 – 44, 47, 52]). Some results on polynomial representations can be found in the foregoing monographs and in ADKINS [1 – 2], SMITH and RIVLIN [28 – 29], PIPKIN and RIVLIN [20], SPENCER and RIVLIN [34 – 37], SPENCER [31 – 32], SMITH, SMITH and RIVLIN [30], SMITH and KIRAL [27], KIRAL and SMITH [15], SMITH [25], *et al.* Some recent results on nonpolynomial representations are given in ZHENG [55 – 56], ZHENG and BOEHLER [58], BISCHOFF-BEIERMANN and BRUHNS [5], JEMIOŁO and TELEGA [13], XIAO [42, 45 – 46, 48, 49 – 51], BRUHNS, XIAO and MEYERS [11], XIAO, BRUHNS and MEYERS [53]. Here it is not our intent to reproduce the huge body of literature. For details, refer to the monographs mentioned before and the recent reviews by BETTEN [3], RYCHLEWSKI and ZHANG [23] and ZHENG [57], as well as the relevant references therein.

Representations for material constitutive equations should be made as compact as possible. As compared with polynomial representations, nonpolynomial representations are not only more general both in notion and in scope, but may furnish more compact forms of reduced constitutive equations, as noted by WANG [40] for isotropic cases and by BOEHLER [7 – 10] for anisotropic cases. Although now many results in many cases are available, general aspects of tensor function representations, especially nonpolynomial representations, are still under investigation, which are concerned with any finite number of vector variables and tensor variables and all kinds of material symmetry groups including the 32 crystal classes and all denumerably infinitely many quasicrystal classes. In fact, except the well-known results for isotropic functions, until recently general results on irreducible nonpolynomial representations have been available only for such simple material symmetry groups as transverse isotropy groups and triclinic, monoclinic and rhombic (orthotropic) groups etc. (see ZHENG [55] and ZHENG and BOEHLER [58], JEMIOŁO and TELEGA [13]). General results for all kinds of material symmetry groups as subgroups of the transverse isotropy group $C_{\infty h}$, as well as some other particular results have been derived very recently by one of the authors (see the related references given before).

In a series of works, we aim to provide irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric second order tensor-valued anisotropic constitutive equations of any finite number of vector variables and second order tensor variables relative to all crystal and quasicrystal classes pertaining to the cylindrical group $D_{\infty h}$. The results for all crystal and quasicrystal classes as subgroups of the transverse isotropy group $C_{\infty h}$ are available in XIAO [50 – 51], as mentioned before. In a succeeding work consisting of three parts, we are concerned with all other crystal and quasicrystal classes. In the present part, we consider the crystal and quasicrystal classes D_{2mh}, D_{2m} and C_{2mv} for all integers $m \geqslant 2$, which will be given at suitable places respectively. In the other two parts that will appear, we shall treat the crystal and quasicrystal classes D_{2m+1d}, D_{2m+1}, C_{2m+1v}, D_{2m+1h} and D_{2md} for all integers $m \geqslant 1$, respectively. This series of works cover all kinds of anisotropic solids except cubic crystals and icosahedral quasicrystals. Throughout, we use the Schoenflies symbol to represent crystal and quasicrystal classes. For a detailed account of them, refer to, e.g., SPENCER [33] for crystal classes and VAINSHTEIN [39] for both crystal and quasicrystal classes.

2. Notations and preliminieries

Throughout, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}$, etc.; $\mathbf{W}$, $\mathbf{H}$, $\mathbf{\Omega}$, etc.; and $\mathbf{A}$, $\mathbf{B}$,$\mathbf{C}$, etc., are used to designate vectors, skewsymmetric second order tensors and symmetric second order tensors over a 3-dimensinal inner product space, respectively. R, V, Skw and Sym are used to denote the reals and the sets of all vectors, all skewsymmetric second order tensors and all symmetric second order tensors, respectively. Moreover, Orth (Orth^+) is used to represent the full (proper) orthogonal groups consisting of all orthogonal (proper orthogonal) tensors. The scalar product of two second order tensors $\mathbf{F}$ and $\mathbf{G}$ is denoted by $\mathrm{tr}\mathbf{F}\mathbf{G}^{\mathrm{T}} = \mathbf{F} : \mathbf{G} = F_{ij}G_{ij}$. For any two vectors $\mathbf{u}$, $\mathbf{v} \in V$, the notations $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \otimes \mathbf{v}$ are used to denote the scalar product, the vector product and the tensor product of the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively; the mixed product of three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{r}$ is signified by $[\mathbf{u}, \mathbf{v}, \mathbf{r}]$, i.e.

$$[\mathbf{u}, \mathbf{v}, \mathbf{r}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{r} = (\mathbf{v} \times \mathbf{r}) \cdot \mathbf{u} = (\mathbf{r} \times \mathbf{u}) \cdot \mathbf{v}\,; \tag{2.1}$$

and, finally, the notations $\mathbf{u} \wedge \mathbf{v}$ and $\mathbf{u} \vee \mathbf{v}$ are used to stand for the skewsymmetric and symmetric tensors defined by

$$\begin{cases} \mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} = \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}\,, \\ \mathbf{u} \vee \mathbf{v} = \mathbf{v} \vee \mathbf{u} = \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}\,. \end{cases} \tag{2.2}$$

2.1. Functional bases, generating sets and their irreducibility

Let g be a material symmetry group of solid materials, i.e. a subgroup of Orth. Scalar-valued function $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$, vector-valued function $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and skewsymmetric or symmetric second order tensor-valued function $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ of the a vector variables $\mathbf{u}_i \in V$, the b skewsymmetric tensor variables $\mathbf{W}_\sigma \in \mathrm{Skw}$ and the c symmetric tensor variables $\mathbf{A}_L \in \mathrm{Sym}$ are said to be invariant (for f) or form-invariant (for $\mathbf{h}$ and $\mathbf{F}$) under the group $g \subseteq \mathrm{Orth}$ if they, respectively, fulfil the invariance requirements

$$
\begin{aligned}
f(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L),\\
\mathbf{h}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L),\\
\mathbf{F}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)\mathbf{Q}^{\mathrm{T}},
\end{aligned}
$$

for any orthogonal tensor $\mathbf{Q} \in g$ and for any $(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L) \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. In each scalar-valued function $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and each vector-valued or tensor-valued functiuon $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ or $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ fulfilling the above invariance requirement will be called an *invariant* of $(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ under the group g and a form-invariant vector-valued or tensor-valued function under the group g, separately. In particular, the commonly-known isotropic functions refer to the case when $g = \mathrm{Orth}$. Every other case results in anisotropic functions.

The foregoing invariance requirements place restrictions on the forms of the tensor functions $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$, $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$. Finding general reduced forms or representations of tensor functions under such restrictions constitutes the central topic in the theory of representations for tensor functions as applied to material constitutive modelling. One of the main general facts in this field is as follows (see PIPKIN and WINEMAN [21, 41]; see also SPENCER [33]): There is a finite set of invariants under the group $g \subseteq \mathrm{Orth}$, $\{I_1, \cdots, I_r\}$, such that every invariant under g is expressible as a single-valued function of this set of invariants. A set of invariants with the just-mentioned property is known as a *functional basis* for the invariants under the group g. On the other hand, there is a finite set of form-invariant vector-valued or second order tensor-valued functions under the group $g \subseteq Orth$, $\{\psi_1, \cdots, \psi_s\}$, such that every form-invariant vector-valued or second order tensor-valued function ψ is expressible as a linear combination of this set of form-invariant vector-valued or second order tensor-valued functions with scalar coefficients that are invariants under g. A set of form-invariant vector-valued or tensor-valued functions with the just-mentioned property is known as a *generating set* for the form-invariant vector-valued or tensor-valued functions under the group g, each element of which is accordingly called a vector generator or a tensor generator.

Thus, finding general reduced forms or representations of the invariant f and the form-invariant vector-valued and tensor-valued functions $\mathbf{h}$ and $\mathbf{F}$ under a given material symmetry group $g \subseteq \mathrm{Orth}$ is equivalent to determining a functional basis (for f) and a *generating set* (for $\mathbf{h}$ and $\mathbf{F}$ separately) under the group g. Moreover, both functional bases and generating sets to be employed are further required to be *irreducible* in order to arrive at compact representations. A functional basis (resp. a generating set) under the group $g \subseteq \mathrm{Orth}$ is said to be irreducible if none of its proper subsets is again a functional basis (resp. a generating set) under the group g.

Let X be a set of vectors and second order tensors and let M be one of the spaces V, Skw and Sym. We use the notation $\Gamma(X)$ to denote the intersection of the symmetry groups of all vectors and second order tensors in the set X, called the symmetry group of X. Moreover, we use the notation $M(g)$, where $g \subseteq \mathrm{Orth}$ is an orthogonal subgroup, to designate the set of all vectors or tensors in M, each of which is invariant under the action of the group g. The former is a subgroup of Orth, while the latter is a g-invariant subspace of M. A criterion for generating sets is as follows (see XIAO [44]).

CRITERION 1. The form-invariant vector-valued or tensor-valued functions under the group $g \subseteq \mathrm{Orth}$, $\mathbf{G}_1(X), \cdots, \mathbf{G}_s(X)$, where the variables X pertain to a g-invariant domain $D \subseteq V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, form a generating set for the form-invariant vector-valued or tensor-valued functions under g defined on D if and only if the inequality

$$\mathrm{rank}\{\mathbf{G}_1(X), \cdots, \mathbf{G}_r(X)\} \geqslant \dim M(g \cap \Gamma(X)) \tag{2.3}$$

is satisfied for each $X \in D$, where $M = V$, Skw, Sym, respectively, when the vector-valued, skewsymmetric tensor-valued and symmetric tensor-valued functions are involved, respectively.

A useful property for generating sets is: Let $G(X) = \{\mathbf{G}_1(X), \cdots, \mathbf{G}_r(X)\}$ be any given generating set for form-invariant vector-valued or tensor-valued functions under the group $g \subseteq \mathrm{Orth}$, where the variables X pertain to a g-invariant domain $D \subseteq V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. Then the set $G(X)$ generates the *admissible range subspace* $M(\Gamma(X) \cap g)$ at each point $X \in D$, i.e.

$$\mathrm{span}G(X) = M(\Gamma(X) \cap g) \text{ for each } X \in D\,, \tag{2.4}$$

where M is the range of the tensor function in question. The above formula can be derived from THEOREMS 2.2 – 2.3 in XIAO [44].

Moreover, the following criterion for functional bases is well-known [21, 41, 33].

CRITERION 2. A set $I(X)$ of invariants of the variables $X \in D$ under the group $g \subseteq \mathrm{Orth}$ is a functional basis for the invariants of the variables $X \in D$ under the group $g \subseteq \mathrm{Orth}$ iff the variables $X \in D$ is determined to within an orthogonal tensor pertaining to g by this set, i.e. the condition $I(X') = I(X)$ for $X, X' \in D$ implies that X and X' pertain to the same g-orbit: $X' = \mathbf{Q} \star X$, $\mathbf{Q} \in g$.

In the above, the symbols rank$\mathcal{S}$ and dim$\tilde{\mathcal{S}}$, where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are a set of vectors or tensors and a vector or a tensor subspace, are used to designate the number of linearly independent elements in the set $\mathcal{S}$ and the dimension of the subspace $\tilde{\mathcal{S}}$, respectively.

We shall apply the aforementioned criterion to verify that a given set of vector-valued or tensor-valued functions defined on a given domain is a generating set required and, moreover, to check the irreducibility of this generating set. Towards the latter goal, it suffices to show that if each vector or tensor generator $\mathbf{G}_0$ is removed from a generating set $\mathcal{S}$ in question, then the proper subset $\mathcal{S}\backslash\{\mathbf{G}_0\}$ fails to fulfil the presented criterion at a point X_0. We shall call a generator $\mathbf{G}_0 \in \mathcal{S}$ to be irreducible if it has the property just indicated. Evidently, a generating set $\mathcal{S}$ is irreducible iff every generator $\mathbf{G}_0 \in \mathcal{S}$ is irreducible.

2.2. Symmetry groups of vectors and second order tensors

To apply the criterion given before, we need to evaluate symmetry groups of various kinds of sets of vectors and/or second order tensors and the values of the dimension dim$M(g)$ for $M = V$, Skw, Sym and for all subgroups $g \subseteq D_{\infty h}$. In this subsection, we shall provide some basic facts and results for future use.

Henceforth, the notation $\mathbf{R}_{\mathbf{u}}^{\theta}$ will be used to represent the right-handed rotation through the angle θ about an axis in the direction of the vector $\mathbf{u} \neq \mathbf{0}$.

The symmetry group of vector $\mathbf{0} \neq \mathbf{u} \in V$ and tensors $\mathbf{O} \neq \mathbf{W} \in \mathrm{Skw}$ and $\mathbf{A} \in \mathrm{Sym}$ are as follows:

$$\Gamma(\mathbf{u}) = \{\mathbf{R}_{\mathbf{u}}^{\theta}, -\mathbf{R}_{\mathbf{a}}^{\pi} \mid \theta \in R;\ \mathbf{a} \neq \mathbf{0},\ \mathbf{a} \cdot \mathbf{u} = 0\} \equiv C_{\infty v}(\mathbf{u}). \tag{2.5}$$

$$\Gamma(\mathbf{W}) = \{\pm\mathbf{R}_{\mathbf{w}}^{\theta} \mid \theta \in R\} \equiv C_{\infty h}(\mathbf{w}), \quad \mathbf{w} = \mathbf{E} : \mathbf{W}. \tag{2.6}$$

$$\Gamma(\mathbf{A}) = \begin{cases} \mathrm{Orth} \ \text{ if } \mathbf{A} = x\mathbf{I}, \\ D_{\infty h}(\mathbf{a}) \text{ if } \exists \mathbf{0} \neq \mathbf{a} \in V:\ \mathbf{A} = x\mathbf{I} + y\mathbf{a} \otimes \mathbf{a},\ y \neq 0, \\ D_{2h}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \text{ otherwise}. \end{cases} \tag{2.7}$$

In the last expression, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are used to denote three orthonormal eigenvectors of $\mathbf{A}$. From (2.7) it follows:

(2.8) $$C_{2h}(\mathbf{a}) \equiv \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{a}}\} \subset \Gamma(\mathbf{A}) \Longleftrightarrow \mathbf{a}$$
is an eigenvector of the symmetric tensor **A**.

Throughout, **E** is used to denote Levi-Civita tensor, i.e. the third order permutation tensor, and $(\mathbf{E} : \mathbf{W})_i = E_{ijk}W_{jk}$ is the axial vector of **W**. Besides, **I** is used to denote the second order identity tensor, $D_{\infty h}(\mathbf{a})$ the cylindrical group with the preferred axis **a**, i.e.

(2.9) $$D_{\infty h}(\mathbf{a}) = \Gamma(\mathbf{a} \otimes \mathbf{a}) = \{\pm\mathbf{R}^{\theta}_{\mathbf{a}}, \pm\mathbf{R}^{\pi}_{\mathbf{l}} \mid \theta \in R;\ \mathbf{l} \neq \mathbf{0},\ \mathbf{l} \cdot \mathbf{a} = 0\},$$

and moreover

(2.10) $$D_{2h}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{a}_1}, \pm\mathbf{R}^{\pi}_{\mathbf{a}_2}, \pm\mathbf{R}^{\pi}_{\mathbf{a}_3}\}.$$

Besides the above groups, the following groups will be used:

(2.11) $$C_{\infty}(\mathbf{n}) = C_{\infty h}(\mathbf{n}) \cap \mathrm{Orth}^+ = \{\mathbf{R}^{\theta}_{\mathbf{n}} \mid \theta \in R\},$$

(2.12) $$D_2(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = D_{2h}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \cap \mathrm{Orth}^+ = \{\mathbf{I}, \mathbf{R}^{\pi}_{\mathbf{a}_1}, \mathbf{R}^{\pi}_{\mathbf{a}_2}, \mathbf{R}^{\pi}_{\mathbf{a}_3}\},$$

(2.13) $$C_{2v}(\mathbf{a}_3, \mathbf{a}_1, \mathbf{a}_2) = \{\mathbf{I}, -\mathbf{R}^{\pi}_{\mathbf{a}_1}, -\mathbf{R}^{\pi}_{\mathbf{a}_2}, \mathbf{R}^{\pi}_{\mathbf{a}_3}\}, \quad S_2 = \{\pm\mathbf{I}\}, \quad C_1 = \{\mathbf{I}\},$$

(2.14) $$C_{2h}(\mathbf{r}) = \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{r}}\}, \quad C_2(\mathbf{r}) = \{\mathbf{I}, \mathbf{R}^{\pi}_{\mathbf{r}}\}, \quad C_{1h}(\mathbf{r}) = \{\mathbf{I}, -\mathbf{R}^{\pi}_{\mathbf{r}}\}.$$

Henceforth, we shall cite these subgroups with their defining vector(s) dropped, if no confusion arises.

Next, for all subgroups $g \subseteq D_{\infty h}$, we provide the values of the dimension $\dim M(g)$ for $M = V$, Skw, Sym respectively, by the following tables.

Table 1. $M = V$ **and** $g_{\infty v} = \{$ **All subgroups of** $C_{\infty v}$ **except** C_1 **and** $C_{1h}\}$

g	C_1	C_{1h}	$g_{\infty v}$	others
$\dim M(g)$	3	2	1	0

Table 2. $M = Skw$ **and** $g_{\infty h} = \{$**All subgroups of** $C_{\infty h}$ **except** C_1 **and** $S_2\}$

g	C_1	S_2	$g_{\infty h}$	others
$\dim M(g)$	3	3	1	0

Table 3. $M =$ **Sym**

g	C_1	S_2	C_2	C_{1h}	C_{2h}	D_2	C_{2v}	D_{2h}	others
$\dim M(g)$	6	6	4	4	4	3	3	3	2

2.3. $D_{\infty h}$-invariant decompositions of a vector and a second order tensor

Let **n** and **e** be two given orthonormal vectors. For vector $\mathbf{u} \in V$ and skew-symmetric and symmetric tensors $\mathbf{W} \in$ Skw and $\mathbf{A} \in$ Sym, the following decomposition formulas hold:

$$\begin{aligned} \mathbf{u} &= (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \overset{\circ}{\mathbf{u}} \\ \overset{\circ}{\mathbf{u}} &= \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\,; \end{aligned} \tag{2.15}$$

$$\begin{aligned} \mathbf{W} &= \frac{1}{2}(\mathrm{tr}\mathbf{WN})\mathbf{N} - \mathbf{n}\wedge\mathbf{Wn}\,, \\ \mathbf{N} &= \mathbf{En}\,; \end{aligned} \tag{2.16}$$

$$\begin{aligned} \mathbf{A} &= \overset{\circ}{\mathbf{A}} + \mathbf{A}_0\,, \\ \overset{\circ}{\mathbf{A}} &= \mathbf{A} - \mathbf{A}_0\,, \\ \mathbf{A}_0 &= \frac{1}{2}(3\mathbf{n}\cdot\mathbf{An} - \mathrm{tr}\mathbf{A})\mathbf{n}\otimes\mathbf{n} + \frac{1}{2}(\mathrm{tr}\mathbf{A} - \mathbf{n}\cdot\mathbf{An})\mathbf{I}\,. \end{aligned} \tag{2.17}$$

Let $\mathbf{D}_1, \cdots, \mathbf{D}_4$ be the four traceless tensors given by

$$\mathbf{D}_1 = \mathbf{e}\otimes\mathbf{e} - \mathbf{e}'\otimes\mathbf{e}', \quad \mathbf{D}_2 = \mathbf{e}\vee\mathbf{e}'; \quad \mathbf{D}_3 = \mathbf{n}\vee\mathbf{e}, \quad \mathbf{D}_4 = \mathbf{n}\vee\mathbf{e}', \tag{2.18}$$

where

$$\mathbf{e}' = \mathbf{n}\times\mathbf{e}\,, \tag{2.19}$$

i.e., the triplet $(\mathbf{e}, \mathbf{e}', \mathbf{n})$ constitutes a right-handed orthonormal system. Then each symmetric tensor $\mathbf{A} \in \mathrm{Sym}$ has the further decomposition

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_\mathbf{n} + \mathbf{A}_\mathbf{e}\,, \tag{2.20}$$

where the three tensors $\mathbf{A}_0$, $\mathbf{A}_\mathbf{n}$ and $\mathbf{A}_\mathbf{e}$ are mutually orthogonal and take forms

$$\mathbf{A}_\mathbf{n} = \mathbf{n}\vee\overset{\circ}{\mathbf{A}}\mathbf{n} = |\overset{\circ}{\mathbf{A}}\mathbf{n}|(\mathbf{D}_3\cos\phi(\mathbf{A}) + \mathbf{D}_4\sin\phi(\mathbf{A}))\,, \tag{2.21}$$

$$\mathbf{A}_\mathbf{e} = |\mathbf{q}(\mathbf{A})|(\mathbf{D}_1\cos\psi(\mathbf{A}) + \mathbf{D}_2\sin\psi(\mathbf{A}))\,. \tag{2.22}$$

Here and henceforth, $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$ is used to denote the norm of the vector $\mathbf{u}$, and $\mathbf{q}(\mathbf{A})$ to represent a vector depending on $\mathbf{A}$, defined by

$$\mathbf{q}(\mathbf{A}) = \frac{1}{2}((\mathrm{tr}\mathbf{AD}_1)\mathbf{e} + (\mathrm{tr}\mathbf{AD}_2)\mathbf{e}')\,. \tag{2.23}$$

Moreover, $\phi(\mathbf{A})$ and $\psi(\mathbf{A})$ are used to designate the *two angles formed by the two vectors* $\overset{\circ}{\mathbf{A}}\mathbf{n}$ *and* $\mathbf{e}$ *and by the two vectors* $\mathbf{q}(\mathbf{A})$ *and* $\mathbf{e}$ respectively, i.e.,

$$\phi(\mathbf{A}) = \langle\overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{e}\rangle, \quad \psi(\mathbf{A}) = \langle\mathbf{q}(\mathbf{A}), \mathbf{e}\rangle\,. \tag{2.24}$$

The decomposition formula (2.20) with $(2.17)_2$ and (2.21) - (2.22), which is invariant under the cylindrical group $D_{\infty h}(\mathbf{n})$, is a slight variant of that introduced by BISCHOFF-BEIERMANN and BRUHNS [5] (see also BRUHNS, XIAO and MEYERS [11] and XIAO [48 - 49]).

Finally, for each integer r and each vector $\mathbf{z}$ in the $\mathbf{n}$-plane we define two scalar functions $\alpha_r(\mathbf{z})$ and $\beta_r(\mathbf{z})$, a vector-valued function $\boldsymbol{\eta}_r(\mathbf{z})$ and a symmetric tensor-valued function $\boldsymbol{\Phi}_r(\mathbf{z})$ by

$$\alpha_r(\mathbf{z}) = |\mathbf{z}|^r \cos r\langle \mathbf{z}, \mathbf{e}\rangle, \qquad \beta_r(\mathbf{z}) = |\mathbf{z}|^r \sin r\langle \mathbf{z}, \mathbf{e}\rangle\,, \tag{2.25}$$

$$\boldsymbol{\eta}_r(\mathbf{z}) = \alpha_r(\mathbf{z})\mathbf{e} - \beta_r(\mathbf{z})\mathbf{e}'\,, \tag{2.26}$$

$$\boldsymbol{\Phi}_r(\mathbf{z}) = \alpha_r(\mathbf{z})\mathbf{D}_1 - \beta_r(\mathbf{z})\mathbf{D}_2\,. \tag{2.27}$$

The following formulas are useful:

$$\begin{aligned} \boldsymbol{\eta}_r(\mathbf{z}) &= \boldsymbol{\Phi}_{r-1}(\mathbf{z})\mathbf{z} = |\mathbf{z}|^{-2}(\alpha_{r+1}(\mathbf{z})\mathbf{z} - \beta_{r+1}(\mathbf{z})(\mathbf{n}\times\mathbf{z}))\,, \\ \alpha_r(\mathbf{z}) &= \boldsymbol{\eta}_{r-1}(\mathbf{z})\cdot\mathbf{z}\,, \\ \beta_r(\mathbf{z}) &= -[\mathbf{n}, \mathbf{z}, \boldsymbol{\eta}_{r-1}(\mathbf{z})]\,. \end{aligned} \tag{2.28}$$

For $\mathbf{A} \in \mathrm{Sym}$, $J(\mathbf{A})$ is used to denote a $C_{\infty h}(\mathbf{n})$-invariant given by

$$J(\mathbf{A}) = [\mathbf{n}, \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}] = |\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2|\mathbf{q}(\mathbf{A})|\sin(2\phi(\mathbf{A}) - \psi(\mathbf{A}))\,. \tag{2.29}$$

A useful fact concerning $J(\mathbf{A})$ is as follows:

$$J(\mathbf{A}) = 0 \iff \mathbf{n}\times\,\overset{\circ}{\mathbf{A}}\,\mathbf{n} \text{ is an eigenvector of } \mathbf{A}\,. \tag{2.30}$$

It should be noted that the two orthonormal vectors $\mathbf{n}$ and $\mathbf{e}$ will always be arranged to be in the directions of the principal symmetry axis and a two-fold symmetry axis of the material symmetry group $g \subseteq D_{\infty h}(\mathbf{n})$ concerned.

3. A unified procedure for constructing both functional bases and generating sets

Usually, it is not easy to derive representations for anisotropic functions of vectors and tensors, even for the case when only one vector or one tensor variable is involved, except for some simple anisotropy groups. This situation may be improved by recent results obtained by one of the authors. It has been proved (see XIAO [44, 52]) that irreducible generating sets for arbitrary-order tensor-valued anisotropic and isotropic functions of any finite number of vector variables and second order tensor variables, can be formed by union of generating sets for the same type of anisotropic or isotropic functions of certain subsets of not more than three variables. Therefore, the representation problem for the former may be reduced to that for the latter. Moreover, by developing the isotropic extention method for anisotropic functions by LOKHIN and SEDOV [17], BOEHLER *et al.* [7 – 10], LIU [16] and RYCHLEWSKI [22], *et al.*, it has been shown (see XIAO [43, 47]) that any type of rth-order tensor-valued anisotropic function of a set of vector variables and second order tensor variables for $r = 0,\ 1,\ 2$, may be presented as an rth-order tensor-valued isotropic function of an extended set of vector variables and second order tensor variables. Hence, complete representations for the former are obtainable from those for the latter by applying the well-known results for isotropic functions of vectors and second order tensors. Further, a unified procedure for constructing both functional bases and generating sets has been established recently by incorporating these facts and others (see XIAO [50 – 51]). For any given subgroup $g \subset D_{\infty h}$, this unified procedure is outlined as follows.

STEP 1: *Representations involving single variables* $\mathbf{x} = \mathbf{u},\ \mathbf{W},\ \mathbf{A}$. Determine irreducible representations (functional basis and generating sets) for invariants and form-invariant vector-valued and skewsymmetric and symmetric tensor-valued functions of each variable $\mathbf{x} \in \{\mathbf{u}, \mathbf{W}, \mathbf{A}\}$ under the subgroup g. Then, form the scalar products $\mathbf{r} \cdot \mathbf{h}_r(\mathbf{x})$ and $\mathbf{H} : \boldsymbol{\psi}_s(\mathbf{x}) = \mathrm{tr}\mathbf{H}\boldsymbol{\psi}_s(\mathbf{x})$ and $\mathbf{C} : \mathbf{F}_t(\mathbf{x}) = \mathrm{tr}\mathbf{C}\mathbf{F}_t(\mathbf{x})$ of a vector variable $\mathbf{r} \in V$ and each presented vector generator $\mathbf{h}_r(\mathbf{x})$, of a skewsymmetric tensor variable $\mathbf{H} \in \mathrm{Skw}$ and each presented skewsymmetric tensor generator $\boldsymbol{\psi}_s(\mathbf{x})$, and of a symmetric tensor variable $\mathbf{C} \in \mathrm{Sym}$ and each presented symmetric tensor generator $\mathbf{F}_t(\mathbf{x})$, respectively;

STEP 2: *Representations involving g-irreducible sets of two variables*, $(\mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{v}),\ (\mathbf{u}, \mathbf{W}),\ (\mathbf{u}, \mathbf{A}),\ (\mathbf{W}, \boldsymbol{\Omega}),\ (\mathbf{W}, \mathbf{A}),\ (\mathbf{A}, \mathbf{B})$. The process is the same as Step 1, except for the fact that the single variable $\mathbf{x}$ therein is replaced by the two variables $(\mathbf{x}, \mathbf{y})$ here. By a g-irreducible set of two variables $(\mathbf{x}, \mathbf{y})$ we mean a set $(\mathbf{x}, \mathbf{y})$ with the property

$$\Gamma(\mathbf{x}, \mathbf{y}) \cap g \neq \Gamma(\mathbf{x}') \cap g, \qquad \mathbf{x}' = \mathbf{x},\ \mathbf{y}\,; \tag{3.1}$$

The above condition and a similar condition below will be explained shortly.

STEP 3: *Representations involving g-irreducible sets of three variables*, $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{v}, \mathbf{W}),\ (\mathbf{u}, \mathbf{v}, \mathbf{A}),\ (\mathbf{u}, \mathbf{v}, \mathbf{r}),\ (\mathbf{u}, \mathbf{W}, \boldsymbol{\Omega}),\ (\mathbf{u}, \mathbf{W}, \mathbf{A})$ *and* $(\mathbf{u}, \mathbf{A}, \mathbf{B})$. The process is the same as STEP 1, except for the fact that the single variable $\mathbf{x}$ therein is replaced by the set $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ of three variables here. By a g-irreducible set of three variables $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ we mean a set $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ with the property

$$\Gamma(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cap g \neq \Gamma(\mathbf{x}', \mathbf{y}') \cap g \ \text{ for any } \ \mathbf{x}', \mathbf{y}' \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}\,. \tag{3.2}$$

STEP 4. Collect all invariants and generators obtained in STEP 1 – STEP 3 and let each variable involved run over the general set of variables, $(\mathbf{u}_1, \cdots, \mathbf{u}_a, \mathbf{W}_1, \cdots, \mathbf{W}_b, \mathbf{A}_1, \cdots, \mathbf{A}_c) \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. Then finally we obtain the desired general representations for invariants and form-invariant vector-valued and skewsymmetric and symmetric tensor-valued functions of any finite number of vector variables and second order tensor variables under the subgroup g.

The introduction of the g-irreducibility conditions (3.1) and (3.2) is mainly based on the fact that is given below. Consider any given symmetry group g and any given set of two variables of interest, $(\mathbf{x}, \mathbf{y})$. For the case when there is one of $\mathbf{x}$ and $\mathbf{y}$, i.e. $\mathbf{x}' \in \{\mathbf{x}, \mathbf{y}\}$, such that

$$g \cap \Gamma(\mathbf{x}') = g \cap \Gamma(\mathbf{x}, \mathbf{y}),$$

we have (see CRITERION 1 in Sec. 2)

$$\mathrm{rank} G(\mathbf{x}') \geqslant \dim M(g \cap \Gamma(\mathbf{x}')) = \dim M(g \cap \Gamma(\mathbf{x}, \mathbf{y}))$$

for a generating set $G(\mathbf{x}')$ for the g-form-invariant tensor functions of a single variable $\mathbf{x}'$ taking values in a g-invariant domain M. If a generating set $G(\mathbf{x}')$ for a single variable $\mathbf{x}'$, given at STEP 1, is incorporated into a generating set $G(\mathbf{x}, \mathbf{y})$ for two variables $(\mathbf{x}, \mathbf{y})$, i.e. $G(\mathbf{x}') \subset G(\mathbf{x}, \mathbf{y})$, then using the foregoing inequality we infer

$$\mathrm{rank} G(\mathbf{x}, \mathbf{y}) \geqslant \mathrm{rank} G(\mathbf{x}') \geqslant \dim M(g \cap \Gamma(\mathbf{x}, \mathbf{y})).$$

This shows that for the foregoing case concerning the two variables $(\mathbf{x}, \mathbf{y})$, a generating set for the single variable $\mathbf{x}$ or $\mathbf{y}$ is already sufficient in order to fulfil CRITERION 1, and no generators dependent on both $\mathbf{x}$ and $\mathbf{y}$ are needed. Thus, in constructing a generating set for two variables $(\mathbf{x}, \mathbf{y})$, the case indicated before is trivial and can be ignored, and hence it is enough to consider the case specified by (3.1). Similarly, in constructing a generating set for three variables $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, it is enough to consider the case specified by (3.2).

It will be seen that the *g-irreducibility conditions* (3.1) and (3.2), which specify particular forms of sets of two or three variables, may result in a considerable simplification of fulfilling the related steps, as has been shown in XIAO [50 – 51]. In contrast with this, it seems difficult to deal directly with the sets of two or three variables that are set free.

Moreover, when forming the scalar products of each variable $\mathbf{z} \in \{\mathbf{r}, \mathbf{H}, \mathbf{C}\}$ and the presented generators, some reduction can be made. Let X_0 be a g-irreducible set of two variables considered in the second step in the foregoing unified procedure, and let $\mathbf{z}_0 \in X_0$ and $\mathbf{z} \in \{\mathbf{r}, \mathbf{H}, \mathbf{C}\}$. Then $(\mathbf{z}_0, \mathbf{z})$ is also a set of two variables. If this set has been covered in fulfilling step 2, then $\mathbf{z}$ may be treated as being subjected to the condition:

$$(3.3) \qquad \Gamma(\mathbf{z}) \supset \Gamma(X_0) \cap g, \qquad \Gamma(\mathbf{z}_0, \mathbf{z}) \cap g \neq \Gamma(X_0, \mathbf{z}) \cap g .$$

The reason is as follows. The unified procedure is mainly based on the fact: The general domain $D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$ can be decomposed into union of certain *g-symmetry-reduced* subdomains: $D = D_1 \cup \cdots \cup D_s$ (see Theorem 3.1 in XIAO [44] and the related results in XIAO [52]). Each D_α has the property: for each set $X \in D_\alpha$ of vector and second order tensor variables, there is a subset $Z_0 \subset X$ with not more than three variables, such that $\Gamma(Z_0) \cap g = \Gamma(X) \cap g$. Thus, generating sets for $X \in D_\alpha$ are given by those for Z_0. At the same time, a functional basis for $X \in D_\alpha$ is given by

$$(3.4) \qquad I(X) = I(Z_0) \cup (\mathbf{r} \cdot V(Z_0)) \cup (\mathbf{H} : \mathrm{Skw}(Z_0)) \cup (\mathbf{C} : \mathrm{Sym}(Z_0)) ,$$

where $\mathbf{r}$, $\mathbf{H}$ and $\mathbf{C}$ are three generic variables running over all vectors, all skewsymmetric tensors and all symmetric tensors in the set $X \backslash Z_0$, respectively; $I(Z_0)$, $V(Z_0)$, $\mathrm{Skw}(Z_0)$ and $\mathrm{Sym}(Z_0)$ are, respectively, a functional basis and vector, skewsymmetric tensor and symmetric tensor generating sets for Z_0 under the group g; and $\mathbf{r} \cdot V(Z_0)$, $\mathbf{H} : Skw(Z_0)$ and $\mathbf{C} : Sym(Z_0)$ are three sets of the invariants formed by the inner products between the forgoing generic variables and the generators in the foregoing three generating sets, respectively. As step 4 indicates, the results for all subdomains D_α collectively supply the desired general results for the whole domain D. Now we explain the reduction indicated before. In the just-mentioned process, let $Z_0 = X_0 \subset X$ be a set of two variables. Then $(3.3)_1$ is evidently true for each $\mathbf{z} \in X$. Suppose that $(3.3)_2$ is not true. Then, by using $\Gamma(X_0) \cap g = \Gamma(X) \cap g$ we infer

$$\Gamma(\mathbf{z}_0, \mathbf{z}) \cap g = \Gamma(X) \cap g,$$

where $\mathbf{z}_0 \in X_0$ and $\mathbf{z} \in \{\mathbf{r}, \mathbf{H}, \mathbf{C}\} \subset X$. As a result, by replacing Z_0 with $(\mathbf{z}_0, \mathbf{z})$ in (3.4) we obtain a functional basis for $X \in D_\alpha$. Thus, if the set $(\mathbf{z}_0, \mathbf{z})$ has been covered, the just-mentioned basis makes the basis $I(X)$ given by (3.4) redundant. It is the just-shown fact that results in the reduction condition (3.3).

It should be pointed out that the reduction condition (3.3) is not necessary for the aforementioned unified procedure. However, for some of the sets of two variables, taking this condition into consideration will be helpful to remove some redundant invariants from the scalar products, as will be seen (e.g., see Sec. 4.2(**vi**)).

Applying the aforementioned unified procedure, in this part and the other two parts we shall derive irreducible nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vector variables and second order tensor variables under all crystal and quasicrystal classes as subgroups of the cylindrical group $D_{\infty h}$.

It should be pointed out that the generating sets thus obtained are irreducible, while the functional bases thus obtained need not be so. The irreducibility of the latter will be examined elsewhere.

Here, we confine ourselves to material symmetries of solids, which are characterized by finite and continuously infinite subgroups of the 3-dimensional full orthogonal group. Other kinds of material symmetries are possible, such as those of liquid crystals etc., which are characterized by subgroups of the 3-dimensional unimodular group $\mathcal{U}(3)$. The above procedure based upon the notion of isotropic extension may be extended to cover the latter kinds of material symmetry groups. The main basis in this more general aspect has been laid down by RYCHLEWSKI [22], in which the existence and reality of isotropic extension in the most general form has been proved with an arbitrary group acting on an arbitrary set, not necessarilly restricted to the subgroups of the 3-dimensional full orthogonal group.

4. Crystal and quasicrystal classes D_{2mh} for $m \geqslant 2$

The classes D_{2mh} are of the form

$$D_{2mh}(\mathbf{n}, \mathbf{e}) = \{\pm \mathbf{R}_{\mathbf{n}}^{2k\pi/2m},\ \pm \mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m}\mathbf{e},\ k = 0, 1, \cdots, 2m-1\}. \tag{4.1}$$

These classes include the crystal classes D_{4h} and D_{6h} as the particular cases when $m = 2,\ 3$. For the sake of simplicity, henceforth we shall use $\mathbf{l}$ to represent one of the *two-fold axis vectors* $\mathbf{l}_0, \mathbf{l}_1, \cdots, \mathbf{l}_{2m-1}$. The latter constitute an equipartition of a unit circle on the $\mathbf{n}$-plane.

4.1. Single variables

(i) A single vector $\mathbf{u}$

Each anisotropic function of the vector variable $\mathbf{u}$ under D_{2mh} may be extended as an isotropic function of the extended set of three variables, $(\mathbf{u}, \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$ (see THEOREM 2 in XIAO [47]). Applying this fact and the related results for isotropic functions and following the unified procedure in Sec. 3, we construct the following table.

V	$\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\ \mathring{\mathbf{u}},\ \boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}})\}\ (\equiv V_{2m}(\mathbf{u}))$.
Skw	$\{\beta_{2m}(\mathring{\mathbf{u}})\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}})\}\ (\equiv \mathrm{Skw}_{2m}(\mathbf{u}))$.
Sym	$\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}}, \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathring{\mathbf{u}})\}$ $(\equiv \mathrm{Sym}_{2m}(\mathbf{u}))$.

$$R \quad (\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n}), \overset{o}{\mathbf{r}}\cdot\overset{o}{\mathbf{u}}, \overset{o}{\mathbf{r}}\cdot\eta_{2m-1}(\overset{o}{\mathbf{u}});$$

$$(\mathrm{tr}\mathbf{HN})\beta_{2m}(\overset{o}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\overset{o}{\mathbf{u}}\cdot\mathbf{Hn}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{o}{\mathbf{u}})\cdot\mathbf{Hn};$$

$$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, \overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{C}}\overset{o}{\mathbf{u}}, \mathrm{tr}\overset{o}{\mathbf{C}}\,\mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\overset{o}{\mathbf{u}}\cdot\overset{o}{\mathbf{C}}\,\mathbf{n},$$

$$(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{o}{\mathbf{u}})\cdot\overset{o}{\mathbf{C}}\,\mathbf{n};$$

$$\{(\mathbf{u}\cdot\mathbf{n})^2, |\overset{o}{\mathbf{u}}|^2, \alpha_{2m}(\overset{o}{\mathbf{u}})\} \ (\equiv I_{2m}(\mathbf{u})).$$

Throughout Parts I-III, we replace the scalar product $\mathbf{r}\cdot\boldsymbol{\psi}$ and $\mathrm{tr}\mathbf{CG}$, where $\boldsymbol{\psi}$ and $\mathbf{G}$ are a vector generator and a symmetric tensor generator, with $\overset{o}{\mathbf{r}}\cdot\boldsymbol{\psi}$ and $\mathrm{tr}\overset{o}{\mathbf{C}}\,\mathbf{G}$ respectively, if the invariant $(\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\psi}$ and the two invariants $\mathbf{n}\cdot\mathbf{Gn}$ and $\mathrm{tr}\mathbf{G}$ are redundant. In fact, by using the decomposition formula (2.15) and (2.17) we have

$$\mathbf{r}\cdot\boldsymbol{\psi} = \overset{o}{\mathbf{r}}\cdot\boldsymbol{\psi} + (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\boldsymbol{\psi}, \qquad \mathrm{tr}\mathbf{CG} = \mathrm{tr}\overset{o}{\mathbf{C}}\,\mathbf{G} + p_1(\mathbf{C})\mathbf{n}\cdot\mathbf{Gn} + p_2(\mathbf{C})\mathrm{tr}\mathbf{G},$$

where $p_{1,2}(\mathbf{C}) = \dfrac{1}{2}(\mathrm{tr}\mathbf{C} \pm \mathbf{n}\cdot\mathbf{Cn})$ are two $D_{\infty h}(\mathbf{n})$-invariants of $\mathbf{C}$.

We need to show that the presented sets $V_{2m}(\mathbf{u})$, $\mathrm{Skw}_{2m}(\mathbf{u})$, $\mathrm{Sym}_{2m}(\mathbf{u})$ and $I_{2m}(\mathbf{u})$ furnish the desired generating sets and functional basis. In fact, by applying the related results for isotropic functions, we derive complete representations for vector-, 2nd order tensor- and scalar-valued isotropic functions of the extended set of variables, $(\mathbf{u}, \mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}}), \mathbf{n}\otimes\mathbf{n})$, as follows:

$$\mathbf{u}, (\mathbf{n}\otimes\mathbf{n})\mathbf{u}, \quad \mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})\mathbf{u};$$

$$\mathbf{u}\wedge(\mathbf{n}\otimes\mathbf{n})\mathbf{u}, \ \mathbf{u}\wedge(\mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})\mathbf{u}), \quad \mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})\mathbf{u}\wedge(\mathbf{n}\otimes\mathbf{n})\mathbf{u};$$

$$\mathbf{I}, \ \mathbf{n}\otimes\mathbf{n}, \ \mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}}), \ \mathbf{u}\vee(\mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})\mathbf{u}), \ \mathbf{u}\otimes\mathbf{u}, \ \mathbf{u}\vee(\mathbf{n}\otimes\mathbf{n})\mathbf{u};$$

$$|\mathbf{u}|^2, \ \mathbf{u}\cdot(\mathbf{n}\otimes\mathbf{n})\mathbf{u}, \quad \mathbf{u}\cdot\mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})\mathbf{u}.$$

In deriving the above results, many obviously redundant generators and invariants have been removed by using the facts

$$(\mathbf{n}\otimes\mathbf{n})^2 = \mathbf{n}\otimes\mathbf{n}, \ \mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})^2 = |\overset{o}{\mathbf{u}}|^{4m-4}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n}), \ \mathbf{\Phi}_{2m-2}(\overset{o}{\mathbf{u}})\mathbf{n} = \mathbf{0}.$$

By using the formulas (2.15) and (2.28), from the results given above one may easily derive the results listed in the aforementioned table.

It can readily be proved that the functional basis given is irreducible. Moreover, it is evident that the three presented generating sets are irreducible. In fact, each of them is minimal.

(ii) A single skewsymmetric tensor $\mathbf{W}$

Every vector-valued anisotropic function of the variable $\mathbf{W}\in\mathrm{Skw}$ under D_{2mh} vanishes. Each scalar-valued or second order tensor-valued anisotropic function of

$\mathbf{W}$ under D_{2mh} may be extended as a scalar-valued or second order tensor-valued isotropic function of the extended set of variables, $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$ (see Theorem 2 in XIAO [47]). Applying this fact and the related results for isotropic functions, we construct the following table.

$$\text{Skw}\quad \{\mathbf{W}, \mathbf{n}\wedge\boldsymbol{\eta}_{2m-1}(\mathbf{Wn}), \mathbf{\Phi}_{2m-2}(\mathbf{Wn})\mathbf{W}^2 - \mathbf{W}^2\mathbf{\Phi}_{2m-2}(\mathbf{Wn})\} \quad (\equiv \text{Skw}_{2m}(\mathbf{W})).$$

$$\text{Sym}\quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{n}\vee\mathbf{Wn}, \mathbf{W}^2, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{\Phi}_{2m-2}(\mathbf{Wn})\mathbf{W} - \mathbf{W}\mathbf{\Phi}_{2m-2}(\mathbf{Wn})\} \quad (\equiv Sym_{2m}(\mathbf{W})).$$

$$R\quad \mathrm{tr}\mathbf{HW}, \boldsymbol{\eta}_{2m-2}(\mathbf{Wn})\cdot\mathbf{Hn}, \mathrm{tr}\mathbf{HW}^2\mathbf{\Phi}_{2m-2}(\mathbf{Wn});$$

$$\mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{Cn}, (\overset{\circ}{\mathbf{C}}\,\mathbf{n})\cdot(\mathbf{Wn}), \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{W}^2, \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{\Phi}_{2m-2}(\mathbf{Wn}),$$

$$\mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{\Phi}_{2m-2}(\mathbf{Wn});$$

$$\{(\mathrm{tr}\mathbf{WN})^2,\ |\mathbf{Wn}|^2, \alpha_{2m}(\mathbf{Wn}),\ (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathbf{Wn})\}\ (\equiv I_{2m}(\mathbf{W})).$$

We show that the sets $I_{2m}(\mathbf{W})$, $\text{Skw}_{2m}(\mathbf{W})$ and $\text{Sym}_{2m}(\mathbf{W})$ provide the desired functional basis and generating sets. First, we consider the skewsymmetric tensor-valued function. Suppose $\mathbf{Wn} = \mathbf{0}$. Then the symmetry group $\Gamma(\mathbf{W})$ is given by $C_{\infty h}(\mathbf{n})$ if $\mathbf{W}\neq\mathbf{O}$ or by Orth if $\mathbf{W}=\mathbf{O}$. From Table 2 given in Sec. 2, it is evident that the presented set $\text{Skw}_{2m}(\mathbf{W})$ obeys the criterion (2.3). Suppose $\mathbf{Wn}\neq\mathbf{0}$. Then we have $\Gamma(\mathbf{\Phi}_{2m-2}(\mathbf{Wn}))\subseteq D_{\infty h}(\mathbf{n})$, and hence

$$\Gamma(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n}) = \Gamma(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn})).$$

From the latter and the criterion (2.3), we infer that a generating set for the two variables $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}))$ offers a generating set for the three variables $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$. By using the related result for isotropic functions we know that the former is just given by the set $\text{Skw}_{2m}(\mathbf{W})$.

Next, we consider the scalar-valued function. By using the related result for isotropic functions, we know that a functional basis for the three variables $(\mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathbf{n}\otimes\mathbf{n})$ is given by $|\mathbf{Wn}|^2$, $\mathrm{tr}\mathbf{W}^2$, $\mathrm{tr}\mathbf{W}^2\mathbf{\Phi}$, $\mathrm{tr}\mathbf{W}^2\mathbf{\Phi}^2$, $\mathrm{tr}\mathbf{\Phi}^2\mathbf{W}^2\mathbf{\Phi}\mathbf{W}$ with $\mathbf{\Phi} = \mathbf{\Phi}_{2m-2}(\mathbf{Wn})$. In deriving the above results, many obviously redundant invariants have been removed by using

$$(\mathbf{n}\otimes\mathbf{n})^2 = \mathbf{n}\otimes\mathbf{n},\ \mathbf{\Phi}_{2m-2}(\mathbf{Wn})\mathbf{n} = \mathbf{0},\ \mathbf{\Phi}_{2m-2}(\mathbf{Wn})^2 = |\mathbf{Wn}|^{4m-4}(\mathbf{I}-\mathbf{n}\otimes\mathbf{n}).$$

Again by using the above facts and the formula (2.28) and the identity (see (2.16))

$$\mathbf{W}^2 = \frac{1}{2}(\mathrm{tr}\mathbf{WN})\mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn}) - \mathbf{Wn}\otimes\mathbf{Wn} - |\mathbf{Wn}|^2\mathbf{n}\otimes\mathbf{n} - \frac{1}{4}(\mathrm{tr}\mathbf{WN})^2(\mathbf{I}-\mathbf{n}\otimes\mathbf{n}), \tag{4.2}$$

we further deduce that, of the five invariants given before, the first three yield the three invariant $|\mathbf{Wn}|^2$, $(\mathrm{tr}\mathbf{WN})^2$ and $\alpha_{2m}(\mathbf{Wn})$, the fourth is redundant, and the last yields the invariant $(\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathbf{Wn})$.

Finally, we prove that the set $\mathrm{Sym}_{2m}(\mathbf{W})$ is a desired generating set by showing that this set obeys the criterion (2.3). It can easily be proved that the set $\mathrm{Sym}_{2m}(\mathbf{W})$ obeys (2.3) when $\mathbf{Wn} = \mathbf{0}$. In what follows we suppose $\mathbf{Wn} \neq \mathbf{0}$. Then the three vectors

$$\mathbf{n},\ \mathbf{e}_1 = \mathbf{Wn},\ \mathbf{e}_2 = \mathbf{n} \times \mathbf{Wn}\,, \tag{4.3}$$

form an orthogonalized basis of the space V, and hence the six symmetric tensors $\mathbf{n}\otimes\mathbf{n}$, $\mathbf{I}-\mathbf{n}\otimes\mathbf{n}$, $\mathbf{n}\vee\mathbf{e}_1$, $\mathbf{C}_1 = \mathbf{n}\vee\mathbf{e}_2$, $\mathbf{C}_2 = \mathbf{e}_1\vee\mathbf{e}_2$ and $\mathbf{C}_3 = \mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2$ form an orthogonalized basis of the space Sym. Of the six generators in the set $\mathrm{Sym}_{2m}(\mathbf{W})$, the first three yield the first three tensors in the just-mentioned basis. Now consider the latter three generators in the set $\mathrm{Sym}_{2m}(\mathbf{W})$, denoted by $\mathbf{G}_1 = \mathbf{\Phi}_{2m-2}(\mathbf{W})$, $\mathbf{G}_2 = \mathbf{W}^2$, $\mathbf{G}_2 = \mathbf{W}\mathbf{\Phi}_{2m-2}(\mathbf{W}) - \mathbf{\Phi}_{2m-2}(\mathbf{W})\mathbf{W}$. We have

$$\Delta = \begin{vmatrix} \mathrm{tr}\mathbf{G}_1\mathbf{C}_1 & \mathrm{tr}\mathbf{G}_1\mathbf{C}_2 & \mathrm{tr}\mathbf{G}_1\mathbf{C}_3 \\ \mathrm{tr}\mathbf{G}_2\mathbf{C}_1 & \mathrm{tr}\mathbf{G}_2\mathbf{C}_2 & \mathrm{tr}\mathbf{G}_2\mathbf{C}_3 \\ \mathrm{tr}\mathbf{G}_3\mathbf{C}_1 & \mathrm{tr}\mathbf{G}_3\mathbf{C}_2 & \mathrm{tr}\mathbf{G}_3\mathbf{C}_3 \end{vmatrix} = \begin{vmatrix} 0 & -2\beta_{2m}(\mathbf{Wn}) & 2\alpha_{2m}(\mathbf{Wn}) \\ xy^2 & 0 & -y^4 \\ 2\beta_{2m} & 2x\alpha_{2m}(\mathbf{Wn}) & 2x\beta_{2m}(\mathbf{Wn}) \end{vmatrix},$$

i.e., $\Delta = 4y^4(x^2y^{4m-2} + (\beta_{2m}(\mathbf{Wn}))^2)$ with $x = \mathrm{tr}\mathbf{WN}$ and $y = |\mathbf{Wn}|$. Hence, we infer

$$\mathrm{rankSym}_{2m}(\mathbf{W}) = \begin{cases} 4 \text{ if } \Delta = 0\,, \\ 6 \text{ if } \Delta \neq 0\,, \end{cases}$$

for $\mathbf{Wn} \neq \mathbf{0}$. From the latter and

$$\Delta = 0 \Longrightarrow \mathbf{W} = c\mathbf{El}_k \Longrightarrow \Gamma(\mathbf{W}) \cap D_{2mh}(\mathbf{n}, \mathbf{e}) = C_{2h}(\mathbf{l}_k),$$

as well as from Table 3 given in Sec. 2, we conclude that the set $\mathrm{Sym}_{2m}(\mathbf{W})$ obeys the criterion (2.3) when $\mathbf{Wn} \neq \mathbf{0}$.

Both the generating sets $\mathrm{Skw}_{2m}(\mathbf{W})$ and $\mathrm{Sym}_{2m}(\mathbf{W})$ are minimal and hence irreducible.

(iii) A single symmetric tensor **A**

$$\text{Skw} \quad \{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{N}, \mathbf{n}\wedge \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\wedge \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) + |\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m-4}J(\mathbf{A})\mathbf{N},$$

$$|\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m-2}\mathbf{n}\wedge \overset{\circ}{\mathbf{A}}{}^2\mathbf{n} + \beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\mathbf{N}\} \ (\equiv \mathrm{Skw}_{2m}(\mathbf{A})).$$

$$\text{Sym} \quad \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \overset{\circ}{\mathbf{A}}\,\mathbf{n}\otimes \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$

$$\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A})), \mathbf{n}\vee\mathbf{A}_{\mathbf{e}}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})\} \ (\equiv \mathrm{Sym}_{2m}(\mathbf{A})).$$

$$R \quad (\mathrm{tr}\mathbf{HN})\beta_m(\mathbf{q}(\mathbf{A})), (\mathbf{Hn})\cdot \overset{\circ}{\mathbf{A}}\,\mathbf{n}, \eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}) \cdot \mathbf{Hn}$$
$$-|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m-4}(\mathrm{tr}\mathbf{HN})J(\mathbf{A}),$$
$$|\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^{2m-2}(\mathbf{Hn})\cdot \overset{\circ}{\mathbf{A}}{}^2\mathbf{n} - (\mathrm{tr}\mathbf{HN})\beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n});$$
$$\{\mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\mathbf{A}, |\overset{\circ}{\mathbf{A}}\,\mathbf{n}|^2, |\mathbf{q}(\mathbf{A})|^2, \alpha_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}), \alpha_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\cdot \overset{\circ}{\mathbf{A}}{}^3\mathbf{n},$$
$$\mathrm{tr}\mathbf{A}_{\mathbf{n}}^2\Phi_{m-1}(\mathbf{q}(\mathbf{A}))\} \; (\equiv I_{2m}(\mathbf{A}))\,.$$

In the above table and in the tables given in (iv)-(xiv), we do not supply the invariants depending on two or three symmetric tensors that are derived from the scalar products of the symmetric tensor variable $\mathbf{C} \in \mathrm{Sym}$ and the presented symmetric tensor generators. In the final general result that will be given by Theorem 1 we shall directly quote the result established by Theorem 1 in XIAO, BRUHNS and MEYERS [53], which is simpler and more compact than the foregoing invariants from the scalar products.

It is known (see XIAO [48 – 49]) that the sets $\mathrm{Sym}_{2m}(\mathbf{A})$ and $I_{2m}(\mathbf{A})$ given above are an irreducible generating set and an irreducible functional basis for symmetric tensor-valued and scalar-valued anisotropic functions of a symmetric tensor $\mathbf{A}$ under D_{2mh} for each $m \geqslant 2$. Hence, in what follows we only need to prove that the set $\mathrm{Skw}_{2m}(\mathbf{A})$ given is an irreducible generating set for skewsymmetric tensor-valued functions of $\mathbf{A}$ under D_{2mh}, i.e. it obeys the criterion (2.3). In fact, by using (2.28) – (2.29) we derive the equalities

$$\mathbf{G}_1 = -|\mathbf{z}|^{-2}\beta_{2m}(\mathbf{z})\mathbf{n}\wedge(\mathbf{n}\times\mathbf{z}) + |\mathbf{z}|^{2m-4}J(\mathbf{A})\mathbf{N} + |\mathbf{z}|^{-2}\alpha_{2m}(\mathbf{z})\mathbf{n}\wedge\mathbf{z},$$

$$\mathbf{G}_2 = |\mathbf{z}|^{2m-4}J(\mathbf{A})\mathbf{n}\wedge(\mathbf{n}\times\mathbf{z}) + \beta_{2m}\mathbf{N} + |\mathbf{z}|^{2m-4}(\mathbf{n}\cdot \overset{\circ}{\mathbf{A}}{}^3\mathbf{n})\mathbf{n}\wedge\mathbf{z}, \quad \mathbf{z} = \overset{\circ}{\mathbf{A}}\,\mathbf{n}.$$

Here and below, $\mathbf{G}_1$ and $\mathbf{G}_2$ are used to denote the last two generators in the set $\mathrm{Skw}_{2m}(\mathbf{A})$, respectively. Observing that the coefficient determinant of the two generators with respect to the two tensors $\mathbf{N}$ and $\mathbf{n}\wedge(\mathbf{n}\times\mathbf{z})$ is given by $\Delta = J(\mathbf{A})^2|\mathbf{z}|^{4m-8} + |\mathbf{z}|^{-2}(\beta_{2m}(\mathbf{z}))^2$, we deduce

$$\text{(4.4)} \qquad \mathrm{rank}\;\;\mathrm{Skw}_{2m}(\mathbf{A}) \geqslant \begin{cases} \mathrm{rank}\{\mathbf{n}\wedge\mathbf{z}, \mathbf{G}_1, \mathbf{G}_2\} \\ = \mathrm{rank}\{\mathbf{N}, \mathbf{n}\wedge\mathbf{z}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{z})\} = 3 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \Delta \neq 0, \\ \mathrm{rank}\{\mathbf{n}\wedge\mathbf{z}\} = 1 \qquad\qquad \text{if } \Delta = 0,\ \mathbf{z}\neq\mathbf{0}, \\ \mathrm{rank}\{\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{N}\} = 1 \;\; \text{if } \mathbf{z}=\mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A}))\neq 0, \\ 0 \qquad\qquad\qquad\qquad\quad\;\; \text{if } |\mathbf{z}| = \beta_m(\mathbf{q}(\mathbf{A})) = \mathbf{0}, \end{cases}$$

$$\Gamma(\mathbf{A}) \cap D_{2mh}(\mathbf{n},\mathbf{e}) \supset \begin{cases} C_{2h}(\mathbf{l}_k) & \text{if } \Delta = 0,\ \mathbf{z} \neq \mathbf{0}, \\ & \text{i.e. } \beta_{2m}(\mathbf{z}) = J(\mathbf{A}) = 0,\ \mathbf{z} \neq \mathbf{0}, \\ C_{2h}(\mathbf{n}) & \text{if } \mathbf{z} = \mathbf{0},\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0, \\ D_{2h}(\mathbf{n}, \mathbf{l}_k, \mathbf{n} \times \mathbf{l}_k) & \text{if } |\mathbf{z}| = \beta_m(\mathbf{q}(\mathbf{A})) = 0. \end{cases} \tag{4.5}$$

In deriving $(4.5)_1$, (2.30) is used. Thus, from (4.4) – (4.5) and Table 2 in Sec. 2 we infer that the set $\mathrm{Skw}_{2m}(\mathbf{A})$ obeys (2.3). Further, by considering $\mathbf{A}_1 = \mathbf{n} \vee (\mathbf{e}+\mathbf{l}_1)$ and $\mathbf{A}_2 = \mathbf{e} \vee \mathbf{l}_1$, we deduce that the last three generators (for $\mathbf{A}_1$) and the first generator (for $\mathbf{A}_2$) in the set $\mathrm{Skw}_{2m}(\mathbf{A})$ are irreducible.

4.2. D_{2mh}-irreducible sets of two variables

(iv) The D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v})$ of two vectors

$$V \quad V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})\,.$$

$$\mathrm{Skw} \quad \mathrm{Skw}_{2m}(\mathbf{u}) \cup Skw_{2m}(\mathbf{v}) \cup \{\mathbf{u} \wedge \mathbf{v}, |\mathbf{u}|^{2m-2}\mathbf{u} \wedge \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})$$
$$+|\mathbf{v}|^{2m-2}\mathbf{v} \wedge \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\}\ (\equiv \mathrm{Skw}_{2m}(\mathbf{u}, \mathbf{v})).$$

$$\mathrm{Sym} \quad \mathrm{Sym}_{2m}(\mathbf{u}) \cup \mathrm{Sym}_{2m}(\mathbf{v}) \cup \{\mathbf{u} \vee \mathbf{v}, |\mathbf{u}|^{2m-2}\mathbf{u} \vee \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})$$
$$+|\mathbf{v}|^{2m-2}\mathbf{v} \vee \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\}\quad (\equiv \mathrm{Sym}_{2m}(\mathbf{u},\mathbf{v}))\,.$$

$$R \quad \mathbf{r} \cdot V_{2m}(\mathbf{z}), \mathbf{H} : \mathrm{Skw}_{2m}(\mathbf{z}), \mathbf{C} : \mathrm{Sym}_{2m}(\mathbf{z}), \mathbf{z} = \mathbf{u},\ \mathbf{v};$$
$$\mathbf{u} \cdot \mathbf{H}\mathbf{v}, |\mathbf{u}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}}) \cdot \mathbf{H}\mathbf{u} + |\mathbf{v}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\ \mathbf{H}\mathbf{v};$$
$$\mathbf{u} \cdot \overset{\circ}{\mathbf{C}}\,\mathbf{v}, |\mathbf{u}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}}) \cdot \overset{\circ}{\mathbf{C}}\,\mathbf{u} + |\mathbf{v}|^{2m-2}\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}}) \cdot \overset{\circ}{\mathbf{C}}\,\mathbf{v};$$
$$I_{2m}(\mathbf{u}) \cup I_{2m}(\mathbf{v}) \cup \{(\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}), \overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{v}}, \overset{\circ}{\mathbf{u}} \cdot \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}}), \overset{\circ}{\mathbf{v}} \cdot \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}})\}.$$

To prove the above results, we first work out the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v})$, which is specified by (see (3.1)) $\Gamma(\mathbf{u}, \mathbf{v}) \cap D_{2mh} \neq \Gamma(\mathbf{z}) \cap D_{2mh}$, $\mathbf{z} = \mathbf{u}$, $\mathbf{v}$. Evidently, $\Gamma(\mathbf{z}) \cap D_{2mh}(\mathbf{n}, \mathbf{e}) \neq C_1$, i.e. $\mathrm{rank} V_{2m}(\mathbf{z}) \neq 3$ for $\mathbf{z} = \mathbf{u}$, $\mathbf{v}$. Hence, we have $(\mathbf{z} \cdot \mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{z}}) = 0$. The latter produces the following three disjoint cases for $\mathbf{z}$:

$$a\mathbf{n},\ a \neq 0; \quad a\mathbf{e} + b\mathbf{e}',\ a^2 + b^2 \neq 0; \quad a\mathbf{n} + b\mathbf{l},\ ab \neq 0\,. \tag{4.6}$$

Considering the combinations of the above forms and excluding the cases

$$\mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{n};\ \mathbf{u} = c\mathbf{n},\ \mathbf{v} = a\mathbf{n} + b\mathbf{l}; \mathbf{u} = c\mathbf{l}, \mathbf{v} = a\mathbf{n} + b\mathbf{l};$$

$$\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \mathbf{v} = c\mathbf{e} + d\mathbf{e}',\ \beta_{2m}(\mathbf{z}) \neq 0,\ \mathbf{z} = \mathbf{u} \text{ or } \mathbf{z} = \mathbf{v};$$

which violate the D_{2mh}-irreducibility condition for $(\mathbf{u},\mathbf{v})$, we derive the following four cases for the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v})$:

(c1) $\mathbf{u} = a\mathbf{n},\ \ \mathbf{v} = c\mathbf{e} + d\mathbf{e}',\ a(c^2+d^2) \neq 0;$

(c2) $\mathbf{u} = a\mathbf{e},\ \mathbf{v} = c\mathbf{l},\ \ \mathbf{l} \neq \mathbf{e},\ ac \neq 0;$

(c3) $\mathbf{u} = a\mathbf{e} + b\mathbf{e}',\ \ \mathbf{v} = c\mathbf{n} + d\mathbf{e},\ bcd \neq 0;$

(c4) $\mathbf{u} = a\mathbf{n} + b\mathbf{e},\ \mathbf{v} = c\mathbf{n} + d\mathbf{l},\ \mathbf{l} \neq \mathbf{e},\ abcd \neq 0.$

For most cases for g-irreducible sets considered here and later, there will be one or two unit vectors that can be chosen among the two-fold axis vectors of the group g. For the sake of simplicity and without losing generality we can fix one of them as we wish. For instance, in cases (c2)–(c4) above, one of the two-fold axis vectors involved is fixed as $\mathbf{e}\ (= \mathbf{l}_0)$.

For case (c1), we have

$$\Gamma(\mathbf{u},\mathbf{v}) \cap D_{2mh} = C_{1h}(\mathbf{l}) \text{ if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) = 0,$$

$$\operatorname{rank}(V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})) \geqslant \begin{cases} \operatorname{rank}\{\mathbf{u}, \overset{\circ}{\mathbf{v}}, \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 3 & \text{if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) \neq 0\,, \\ \operatorname{rank}\{\mathbf{u}, \overset{\circ}{\mathbf{v}}\} = 2 & \text{if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) = 0\,, \end{cases}$$

$$\operatorname{rank}\ \operatorname{Skw}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \begin{cases} \operatorname{rank}\{\mathbf{N}, \mathbf{u}\wedge\mathbf{v}, \mathbf{u}\wedge\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 3 & \\ & \text{if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) \neq 0\,, \\ \operatorname{rank}\{\mathbf{u}\wedge\mathbf{v}\} = 1 & \text{if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) = 0\,, \end{cases}$$

$$\operatorname{rank}\ \operatorname{Sym}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \begin{cases} \operatorname{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \boldsymbol{\Phi}_{2m-2}(\overset{\circ}{\mathbf{v}}), & \\ \qquad \mathbf{u}\vee\mathbf{v}, \mathbf{u}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 6 & \text{if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) \neq 0\,, \\ \operatorname{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, \mathbf{u}\vee\mathbf{v}\} = 4 & \\ & \text{if } \beta_{2m}(\overset{\circ}{\mathbf{v}}) = 0\,. \end{cases}$$

For case (c2) we have $\Gamma(\mathbf{u},\mathbf{v}) \cap D_{2mh} = C_{1h}(\mathbf{n})$ and

$$\operatorname{rank}(V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})) \geqslant \operatorname{rank}\{\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}\} = 2,$$

$$\operatorname{rank}\ \operatorname{Skw}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \operatorname{rank}\{\mathbf{u}\wedge\mathbf{v}\} = 1,$$

$$\operatorname{rank}\ \operatorname{Sym}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \operatorname{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{u}\vee\mathbf{v}\} = 4.$$

For case (c3)–(c4) we have

$$\operatorname{rank}(V_{2m}(\mathbf{u}) \cup V_{2m}(\mathbf{v})) \geqslant \operatorname{rank}\{\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\} = 3,$$

$$\operatorname{rank}\ \mathrm{Skw}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \operatorname{rank}\{(\mathbf{v}\cdot\mathbf{n})\mathbf{n}\wedge \overset{\circ}{\mathbf{v}}, \mathbf{u}\wedge\mathbf{v}, \mathbf{u}\wedge \boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 3,$$

$$\operatorname{rank}\ \mathrm{Sym}_{2m}(\mathbf{u},\mathbf{v}) \geqslant \operatorname{rank}\{\mathbf{I}, \mathbf{n}\cdot\mathbf{n}, \overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\vee\overset{\circ}{\mathbf{v}}, \mathbf{u}\vee\mathbf{v}, \mathbf{u}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{v}})\} = 6.$$

From the above results and Tables 1 – 3 in Sec. 2, we infer that the three presented sets of generators obey the criterion (2.3), and therefore they provide the desired vector, skewsymmetric tensor and symmetric tensor generating sets. Moreover, by means of case (c1)–(c4) it can easily be proved that the presented set $I_{2m}(\mathbf{u},\mathbf{v})$ is a functional basis for the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v})$.

Finally, by considering the pair $\mathbf{u}_0 = \mathbf{n}$ and $\mathbf{v}_0 = \mathbf{e} + \mathbf{l}_1$ we deduce that the respective last two generators in the two sets $\mathrm{Skw}_{2m}(\mathbf{u},\mathbf{v})$ and $\mathrm{Sym}_{2m}(\mathbf{u},\mathbf{v})$ are irreducible.

(v) The D_{2mh}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$ of two skewsymmetric tensors

$$\text{Skw}\quad \{\mathbf{W}, \boldsymbol{\Omega}, \mathbf{W}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{W}\}\ (\equiv \mathrm{Skw}_{2m}(\mathbf{W},\boldsymbol{\Omega}))\,.$$

$$\begin{aligned}\text{Sym}\quad & \mathrm{Sym}_{2m}(\mathbf{W}) \cup \mathrm{Sym}_{2m}(\boldsymbol{\Omega}) \cup \{\mathbf{W}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{W},\\ & |\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n} + |\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})\boldsymbol{\Omega}\mathbf{n}\vee\mathbf{N}\boldsymbol{\Omega}\mathbf{n}\}\\ & (\equiv \mathrm{Sym}_{2m}(\mathbf{W},\boldsymbol{\Omega})).\end{aligned}$$

$$\begin{aligned} R\quad & \mathrm{tr}\mathbf{H}\mathbf{W}, \mathrm{tr}\mathbf{H}\boldsymbol{\Omega}, \mathrm{tr}\mathbf{H}\mathbf{W}\boldsymbol{\Omega}; \mathbf{C} : Sym_{2m}(\mathbf{W}), \mathbf{C} : Sym_{2m}(\boldsymbol{\Omega}), \mathrm{tr}\overset{\circ}{\mathbf{C}}\,\mathbf{W}\boldsymbol{\Omega},\\ & |\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})[\mathbf{n}, \mathbf{W}\mathbf{n}, \overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{n}] + |\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})[\mathbf{n}, \boldsymbol{\Omega}\mathbf{n}, \overset{\circ}{\mathbf{C}}\,\boldsymbol{\Omega}\mathbf{n}];\\ & I_{2m}(\mathbf{W}) \cup I_{2m}(\boldsymbol{\Omega}) \cup \{\mathrm{tr}\mathbf{W}\boldsymbol{\Omega}\}\ (\equiv I_{2m}(\mathbf{W},\boldsymbol{\Omega})).\end{aligned}$$

To prove the above results, we first work out the D_{2mh}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$, which is specified by (see (3.1)) $\Gamma(\mathbf{W},\boldsymbol{\Omega}) \cap D_{2mh} \neq \Gamma(\mathbf{z}) \cap D_{2mh}$, $\mathbf{z} = \mathbf{W}$, $\boldsymbol{\Omega}$. It is evident that $\Gamma(\mathbf{z}) \cap D_{2mh} \neq S_2$ for $\mathbf{z} = \mathbf{W}$, $\boldsymbol{\Omega}$. The latter implies that either $\mathbf{R}_{\mathbf{n}}^{\pi}$ or $\mathbf{R}_{\mathbf{l}}^{\pi}$ pertains to the symmetry group $\Gamma(\mathbf{z})$. Hence, the skewsymmetric tensor $\mathbf{z} \in \{\mathbf{W},\boldsymbol{\Omega}\}$ takes one of the forms:

$$a\mathbf{E}\mathbf{n},\ a \neq 0,; \quad a\mathbf{E}\mathbf{l},\ a \neq 0\,. \tag{4.7}$$

Considering the combinations of the above forms, we derive the following two cases for the D_{2mh}-irreducible set $(\mathbf{W},\boldsymbol{\Omega})$:

(c1) $\mathbf{W} = a\mathbf{E}\mathbf{n}$, $\boldsymbol{\Omega} = b\mathbf{E}\mathbf{e}$, $ab \neq 0$;

(c2) $\mathbf{W} = a\mathbf{E}\mathbf{e}$, $\boldsymbol{\Omega} = b\mathbf{E}\mathbf{l}$, $\mathbf{l} \neq \mathbf{e}$, $ab \neq 0$.

For the above two cases, it may easily be understood that the variables $(\mathbf{W}, \boldsymbol{\Omega})$ can be determined to within an orthogonal tensor $\mathbf{Q} \in D_{2mh}$ by the presented set $I_{2m}(\mathbf{W}, \boldsymbol{\Omega})$, and hence the latter is a desired functional basis. Moreover, for the sets $(\mathbf{W}, \boldsymbol{\Omega})$ at issue, we have $\Gamma(\mathbf{W}, \boldsymbol{\Omega}) \cap D_{2mh} = \Gamma(\mathbf{W}, \boldsymbol{\Omega}) = S_2$. Hence, a generating set for skewsymmetric tensor-valued isotropic functions of $(\mathbf{W}, \boldsymbol{\Omega})$ offers a desired generating set. The former is just the presented set $\mathrm{Skw}_{2m}(\mathbf{W}, \boldsymbol{\Omega})$. This set is obviously irreducible.

We show that the presented set $\mathrm{Sym}_{2m}(\mathbf{W}, \boldsymbol{\Omega})$ supplies a desired symmetric tensor generating set. In fact, we have

$$\begin{aligned}\text{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{W}, \boldsymbol{\Omega}) \geqslant \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \boldsymbol{\Omega}^2, \mathbf{n}\vee\boldsymbol{\Omega}\mathbf{n}, \mathbf{W}\boldsymbol{\Omega}+\boldsymbol{\Omega}\mathbf{W},\\ \boldsymbol{\Omega}\mathbf{n}\vee\mathbf{N}\boldsymbol{\Omega}\mathbf{n}\} = 6,\end{aligned}$$

$$\begin{aligned}\text{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{W}, \boldsymbol{\Omega}) \geqslant \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{W}^2, \mathbf{n}\vee\mathbf{W}\mathbf{n}, \mathbf{n}\vee\boldsymbol{\Omega}\mathbf{n},\\ \mathbf{W}\boldsymbol{\Omega}+\boldsymbol{\Omega}\mathbf{W}\} = 6,\end{aligned}$$

for cases (c1) and (c2) separately. Since dim Sym $= 6$, these indicate that the set $\mathrm{Sym}_{2m}(\mathbf{W}, \boldsymbol{\Omega})$ obeys the criterion (2.3). Further, by considering case (c1), we deduce that the last two generators in the set $\mathrm{Sym}_{2m}(\mathbf{W}, \boldsymbol{\Omega})$ are irreducible.

(vi) The D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$ of a skewsymmetric tensor and a symmetric tensor

Skw $\{\mathbf{W}, \mathring{\mathbf{A}}\,\mathbf{W}+\mathbf{W}\,\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}^2-\mathbf{W}^2\,\mathring{\mathbf{A}}\}$ $(\equiv \mathrm{Skw}_{2m}(\mathbf{W}, \mathbf{A}))$.

Sym $\mathrm{Sym}_{2m}(\mathbf{W}) \cup \mathrm{Sym}_{2m}(\mathbf{A}) \cup \{\mathring{\mathbf{A}}\,\mathbf{W}-\mathbf{W}\,\mathring{\mathbf{A}}, (\mathrm{tr}\mathbf{W}\mathbf{N})\,\mathring{\mathbf{A}}\,\mathbf{n}\vee\mathbf{N}\,\mathring{\mathbf{A}}\,\mathbf{n}\}$ $(\equiv \mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{A}))$.

R $\mathrm{tr}\mathbf{H}\mathbf{W}; \mathbf{C} : \mathrm{Sym}_{2m}(\mathbf{W}), \mathrm{tr}\mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{W}, (\mathrm{tr}\mathbf{W}\mathbf{N})[\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{n}];$

$$I_{2m}(\mathbf{W}) \cup I_{2m}(\mathbf{A}) \cup \{(\mathbf{W}\mathbf{n})\cdot(\mathring{\mathbf{A}}\,\mathbf{n}), \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}^2,$$

$$|\,\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-2}(\mathbf{W}\mathbf{n})\cdot\mathring{\mathbf{A}}^{\,2}\mathbf{n} - (\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n})\} \ (\equiv I_{2m}(\mathbf{W}, \mathbf{A}))\ .$$

In the above table, the skewsymmetric tensor variable $\mathbf{H}$ is regarded as being subjected to the condition: $\mathbf{H} = c\mathbf{W}$. In fact, from cases (c1)–(c3) derived later and the condition (see (3.3))

$$\Gamma(\mathbf{W}, \mathbf{H}) \cap D_{2mh} \neq \Gamma(\mathbf{W}, \mathbf{A}, \mathbf{H}) \cap D_{2mh}\ (= S_2),$$

we derive $\mathbf{H} = c\mathbf{W}$. The other case for $\mathbf{H}$ has been covered by (v).

The proof for the presented results is as follows. First, we work out the D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$, which is specified by (see (3.1)) $\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2mh} \neq \Gamma(\mathbf{z}) \cap$

D_{2mh}, $\mathbf{z} = \mathbf{W}$, $\mathbf{A}$. It is evident that $\Gamma(\mathbf{z}) \cap D_{2mh} \neq S_2$, $\mathbf{z} = \mathbf{W}$, $\mathbf{A}$. The latter implies that either $\mathbf{R}_{\mathbf{n}}^{\pi}$ or $\mathbf{R}_{\mathbf{l}}^{\pi}$ pertains to the symmetry group $\Gamma(\mathbf{z})$ for each $\mathbf{z} \in \{\mathbf{W}, \mathbf{A}\}$. Hence, the skewsymmetric tensor $\mathbf{W}$ takes one of the forms given by (4.7), and $\overset{\circ}{\mathbf{A}}$ takes one of the forms:

$$a\mathbf{D}_1 + b\mathbf{D}_2,\ a^2 + b^2 \neq 0; \quad a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}') + b\mathbf{n} \vee \mathbf{l}',\ b \neq 0. \tag{4.8}$$

Here and henceforth, for each vector $\mathbf{u}$, the notation $\mathbf{u}'$ is used to represent a vector given by

$$\mathbf{u}' = \mathbf{n} \times \mathbf{u}.$$

Considering the combinations of the forms given by (4.7) and (4.8) and excluding the cases

$$\mathbf{W} = c\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ \beta_m(\mathbf{q}(\mathbf{A})) \neq 0;$$

$$\mathbf{W} = c\mathbf{E}\mathbf{l},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}');$$

$$\mathbf{W} = c\mathbf{E}\mathbf{l},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}') + b\mathbf{n} \vee \mathbf{l}';$$

which violate the D_{2mh}-irreducibility condition for $(\mathbf{W}, \mathbf{A})$, we derive the following four cases for the D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$:

$$\text{(c1)}\ \mathbf{W} = f\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1,\ fa \neq 0;$$

$$\text{(c2)}\ \mathbf{W} = f\mathbf{E}\mathbf{n},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_4,\ fb \neq 0;$$

$$\text{(c3)}\ \mathbf{W} = f\mathbf{E}\mathbf{e},\ \overset{\circ}{\mathbf{A}} = a\mathbf{D}_1 + b\mathbf{D}_2,\ fb \neq 0;$$

$$\text{(c4)}\ \mathbf{W} = f\mathbf{E}\mathbf{e},\ \overset{\circ}{\mathbf{A}} = a(\mathbf{l} \otimes \mathbf{l} - \mathbf{l}' \otimes \mathbf{l}') + b\mathbf{n} \vee \mathbf{l}',\ \mathbf{l} \neq \mathbf{e},\ fb \neq 0.$$

For cases (c1)–(c4), we have $\Gamma(\mathbf{W}, \mathbf{A}) \cap D_{2mh} = \Gamma(\mathbf{W}, \overset{\circ}{\mathbf{A}})$. From this fact and the criterion (2.3) it follows that generating sets for tensor-valued anisotropic functions of the D_{2mh}-irreducible set $(\mathbf{W}, \mathbf{A})$ are obtainable from those for tensor-valued isotropic functions of $(\mathbf{W}, \overset{\circ}{\mathbf{A}})$. As a result, by applying the related result for isotropic functions we know that the presented set $\mathrm{Skw}_{2m}(\mathbf{W}, \mathbf{A})$ supplies a desired irreducible skewsymmetric tensor generating set.

Now we show that the presented set $\mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{A})$ obeys the criterion (2.3). Case (c1) can be treated easily. For case (c2) we have

$$\text{rank}\ \ \mathrm{Sym}_{2m}(\mathbf{W}, \mathbf{A}) \geqslant \text{rank}\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{A}}, \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{A}}\,\mathbf{n}),$$
$$\overset{\circ}{\mathbf{A}}\,\mathbf{W} - \mathbf{W}\,\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\,\mathbf{n} \vee \mathbf{N}\,\overset{\circ}{\mathbf{A}}\,\mathbf{n}\} = 6,$$

and for cases (c3)–(c4) we have

$$\text{rank } \operatorname{Sym}_{2m}(\mathbf{W},\mathbf{A}) \geqslant \text{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{W}^2, \mathbf{n}\vee\mathbf{Wn}, \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}}\mathbf{W}-\mathbf{W}\overset{\circ}{\mathbf{A}}\} = 6.$$

Thus, we infer that the presented set $\operatorname{Sym}_{2m}(\mathbf{W},\mathbf{A})$ obeys the criterion (2.3), and hence it provides a desired symmetric tensor generating set. Moreover, by considering case (c2) we deduce that the last two generators in the set $\operatorname{Sym}_{2m}(\mathbf{W},\mathbf{A})$ are irreducible.

Next, we are concerned with the presented set $I_{2m}(\mathbf{W},\mathbf{A})$ of invariants. Let

$$I'(\mathbf{W},\mathbf{A}) = \{(\mathrm{tr}\mathbf{WN})^2, |\mathbf{Wn}|^2, \mathrm{tr}\mathbf{A}, \mathbf{n}\cdot\mathbf{An}, |\overset{\circ}{\mathbf{A}}\mathbf{n}|^2, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}{}^3\mathbf{n}, |\mathbf{q}(\mathbf{A})|^2, (\mathbf{Wn})\cdot(\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathrm{tr}\overset{\circ}{\mathbf{A}}\mathbf{W}^2, (\mathbf{Wn})\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{n}\}.$$

We shall prove that the latter offers a functional basis of the D_{2mh}-irreducible set $(\mathbf{W},\mathbf{A})$ under the group $D_{\infty h}(\mathbf{n})$ and this basis is determined by the presented set $I_{2m}(\mathbf{W},\mathbf{A})$. In fact, the just-mentioned basis is obtainable from an isotropic functional basis of $(\mathbf{W},\overset{\circ}{\mathbf{A}},\mathbf{n}\otimes\mathbf{n})$ (see BOEHLER [8]), plus the two invariants $\mathrm{tr}\mathbf{A}$ and $\mathbf{n}\cdot\mathbf{An}$. Applying the related result for isotropic functions we know that the just-mentioned isotropic functional basis is given by

$$\mathrm{tr}\overset{\circ}{\mathbf{A}}\mathbf{W}^2, (\mathbf{Wn})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n}, \ (\mathbf{Wn})\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{n}, \ \mathrm{tr}\overset{\circ}{\mathbf{A}}{}^2\mathbf{W}^2, \mathrm{tr}\mathbf{W}^2\overset{\circ}{\mathbf{A}}{}^2\mathbf{W}\overset{\circ}{\mathbf{A}}, \mathrm{tr}\mathbf{W}^2\overset{\circ}{\mathbf{A}}\mathbf{W}(\mathbf{n}\otimes\mathbf{n}),$$

as well as by the invariants of a single variable $\mathbf{W}$ or $\mathbf{A}$ in $I'(\mathbf{W},\mathbf{A})$. Each of the latter has been covered or can be determined by the bases $I_{2m}(\mathbf{W})$ or $I_{2m}(\mathbf{A})$. The first three invariants above yield the last three invariants in the set $I'(\mathbf{W},\mathbf{A})$. The last three invariants above are redundant. In fact, the just-mentioned fact can be proved easily for cases (c1)–(c3). For case (c4) we have

$$I = (\mathbf{Wn})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n} = fb(\mathbf{l}\cdot\mathbf{e}), \ \mathrm{tr}\overset{\circ}{\mathbf{A}}{}^2\mathbf{W}^2 = -I^2 - f^2(a^2+b^2),$$

$$\mathrm{tr}\mathbf{W}^2\overset{\circ}{\mathbf{A}}\mathbf{W}(\mathbf{n}\otimes\mathbf{n}) = -f^2 I,$$

$$\mathrm{tr}\mathbf{W}^2\overset{\circ}{\mathbf{A}}{}^2\mathbf{W}\overset{\circ}{\mathbf{A}} = b^{-2}(b^2-2a^2)(f^2b^2-I^2)I,$$

with $f^2 = |\mathbf{Wn}|^2$, $a^2 = |\mathbf{q}(\mathbf{A})|^2$, $b^2 = |\overset{\circ}{\mathbf{A}}\mathbf{n}|^2$ and $fb \neq 0$. Then, we deduce that the foregoing fact is also true for case (c4).

Hence, we conclude that the set $I'(\mathbf{W},\mathbf{A})$ given before is a functional basis of the D_{2mh}-irreducible set $(\mathbf{W},\mathbf{A})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$. Moreover, for the D_{2mh}-irreducible set $(\mathbf{W},\mathbf{A})$, the last invariant in the basis $I'(\mathbf{W},\mathbf{A})$ is given by the last invariant in the set $I_{2m}(\mathbf{W},\mathbf{A})$ (note here that $(\mathrm{tr}\mathbf{WN})\beta_{2m}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})=0$).

Then, applying the just-proved fact, for two D_{2mh}-irreducible sets $(\mathbf{W},\mathbf{A})$ and $(\mathbf{W}',\mathbf{A}')$ we deduce

$$I_{2m}(\mathbf{W}',\mathbf{A}')=I_{2m}(\mathbf{W},\mathbf{A})\Longrightarrow I'(\mathbf{W}',\mathbf{A}')=I'(\mathbf{W},\mathbf{A})$$

$$\Longrightarrow \exists\mathbf{Q}\in D_{\infty h}(\mathbf{n}):\ \mathbf{W}'=\mathbf{QWQ}^{\mathrm{T}},\ \mathbf{A}'=\mathbf{QAQ}^{\mathrm{T}}.$$

Moreover, we have

$$I_{2m}(\mathbf{W}',\mathbf{A}')=I_{2m}(\mathbf{W},\mathbf{A})\Longrightarrow I_{2m}(\mathbf{W}')=I_{2m}(\mathbf{W}),\ I_{2m}(\mathbf{A}')=I_{2m}(\mathbf{A}),$$

$$\Longrightarrow \exists\mathbf{R}_1,\mathbf{R}_2\in D_{2mh}:\ \mathbf{W}'=\mathbf{R}_1\mathbf{W}\mathbf{R}_1^{\mathrm{T}},\ \mathbf{A}'=\mathbf{R}_2\mathbf{A}\mathbf{R}_2^{\mathrm{T}}.$$

From these facts we derive

$$\mathbf{R}_1^{\mathrm{T}}\mathbf{Q}\in\Gamma(\mathbf{W})\cap D_{\infty h}(\mathbf{n}),\ \mathbf{R}_2^{\mathrm{T}}\mathbf{Q}\in\Gamma(\mathbf{A})\cap D_{\infty h}(\mathbf{n}).$$

From the latter and the facts: $\Gamma(\mathbf{A})\cap D_{\infty h}(\mathbf{n})=D_{2h}(\mathbf{n},\mathbf{e},\mathbf{e}')$ and

$$\Gamma(\mathbf{A})\cap D_{\infty h}(\mathbf{n})=C_{2h}(\mathbf{e}),\qquad \Gamma(\mathbf{W})\cap D_{\infty h}(\mathbf{n})=C_{2h}(\mathbf{e}),$$

$$\Gamma(\mathbf{W})\cap D_{\infty h}(\mathbf{n})=C_{2h}(\mathbf{l}),$$

for cases (c1)–(c4) respectively, we infer that $\mathbf{Q}\in D_{2mh}$ for cases (c1)–(c4).

Thus, we conclude that $I_{2m}(\mathbf{W},\mathbf{A})$ is a functional basis of the D_{2mh}-irreducible $(\mathbf{W},\mathbf{A})$ under the group D_{2mh}.

REMARK. The above procedure can be used to deal with functional bases for other kinds of g-irreducible sets $(\mathbf{x},\mathbf{y})$ of two variables in future. Accordingly, henceforth for each similar case we need only to show that a presented set of invariants for a g-irreducible set $(\mathbf{x},\mathbf{y})$ determines a functional basis of $(\mathbf{x},\mathbf{y})$ under the transverse isotropy group $C_{\infty v}(\mathbf{n})$ (for $g\subset C_{\infty v}(\mathbf{n})$) or $D_{\infty h}(\mathbf{n})$ (for other g).

(vii) The D_{2mh}-irreducible set $(\mathbf{A},\mathbf{B})$ of two symmetric tensors

$$\begin{aligned}\mathrm{Skw}\quad & \mathrm{Skw}_{2m}(\mathbf{A})\cup\mathrm{Skw}_{2m}(\mathbf{B})\cup\{\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}-\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}},\\ & \overset{\circ}{\mathbf{B}}\,\mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}\,\mathbf{n},\overset{\circ}{\mathbf{A}}\,\mathbf{n}\wedge\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}\,\mathbf{n}\}\ (\equiv\mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B})).\end{aligned}$$

$$\text{Sym}\quad \text{Sym}_{2m}(\mathbf{A})\cup\text{Sym}_{2m}(\mathbf{B})\cup\{\mathring{\mathbf{A}}\mathring{\mathbf{B}}+\mathring{\mathbf{B}}\mathring{\mathbf{A}}\}\ (\equiv \text{Sym}_{2m}(\mathbf{A},\mathbf{B})).$$

$$R\quad I_{2m}(\mathbf{A})\cup I_{2m}(\mathbf{B})\cup\{\text{tr}\mathbf{A}_\mathbf{n}\mathbf{B}_\mathbf{n},\text{tr}\mathbf{A}_\mathbf{e}\mathbf{B}_\mathbf{e},\text{tr}\mathring{\mathbf{A}}^{2}\mathring{\mathbf{B}},\text{tr}\mathring{\mathbf{B}}^{2}\mathring{\mathbf{A}}\}\ (\equiv I_{2m}(\mathbf{A},\mathbf{B}))\,.$$

In the above table, the scalar product between each presented skewsymmetric tensor generator and the skewsymmetric tensor variable $\mathbf{H}$ has been omitted. In fact, for any $\mathbf{O}\neq\mathbf{H}\in\text{Skw}$ and any $\mathbf{A},\ \mathbf{B}\in\text{Sym}$ we have $\Gamma(\mathbf{z}_0,\mathbf{H})=\Gamma(\mathbf{A},\mathbf{B},\mathbf{H})$, $\mathbf{z}_0\in\{\mathbf{A},\mathbf{B}\}$, which violates the condition (3.3) with $X_0=(\mathbf{A},\mathbf{B})$ and $\mathbf{z}=\mathbf{H}$ and $g=D_{2mh}$. The case when $\mathbf{H}\neq\mathbf{O}$ has been covered by (**vi**).

To prove the presented results, we first work out the D_{2mh}-irreducible set $(\mathbf{A},\mathbf{B})$, specified by (see (3.1)) $\Gamma(\mathbf{A},\mathbf{B})\cap D_{2mh}\neq\Gamma(\mathbf{z})\cap D_{2mh},\ \mathbf{z}=\mathbf{A},\ \mathbf{B}$. It is evident that $\Gamma(\mathbf{z})\cap D_{2mh}\neq S_2$. The latter implies that each symmetric tensor $\mathbf{z}\in\{\mathbf{A},\mathbf{B}\}$ take one of the forms given by (4.8). Thus, considering the combinations of the forms given by (4.8) and excluding the cases

$$\mathring{\mathbf{A}}=a\mathbf{D}_1+b\mathbf{D}_2,\ \mathring{\mathbf{B}}=c\mathbf{D}_1+d\mathbf{D}_2,\ \beta_m(\mathbf{q}(\mathbf{z}))\neq0,\ \mathbf{z}=\mathbf{A}\text{ or }\mathbf{z}=\mathbf{B};$$

$$\mathring{\mathbf{A}}=a(\mathbf{l}\otimes\mathbf{l}-\mathbf{l}'\otimes\mathbf{l}'),\ \mathring{\mathbf{B}}=c(\mathbf{l}\otimes\mathbf{l}-\mathbf{l}'\otimes\mathbf{l}')+d\mathbf{n}\vee\mathbf{l}';$$

which violate the D_{2mh}-irreducibility condition for $(\mathbf{A},\mathbf{B})$, we derive the following three disjoint cases for the D_{2mh}-irreducible set $(\mathbf{A},\mathbf{B})$:

$$\text{(c1)}\quad \mathring{\mathbf{A}}=a\mathbf{D}_1,\ \mathring{\mathbf{B}}=b(\mathbf{l}\otimes\mathbf{l}-\mathbf{l}'\otimes\mathbf{l}'),\ \mathbf{l}\neq\mathbf{e},\ \mathbf{e}',\ ab\neq0;$$

$$\text{(c2)}\quad \mathring{\mathbf{A}}=a\mathbf{D}_1+b\mathbf{D}_2,\ \mathring{\mathbf{B}}=c\mathbf{D}_1+d\mathbf{D}_4,\ bd\neq0;$$

$$\text{(c3)}\quad \mathring{\mathbf{A}}=a\mathbf{D}_1+b\mathbf{D}_4,\ \mathring{\mathbf{B}}=c(\mathbf{l}\otimes\mathbf{l}-\mathbf{l}'\otimes\mathbf{l}')+d\mathbf{n}\vee\mathbf{l}',\ \mathbf{l}\neq\mathbf{e},\ bd\neq0.$$

Then, for case (c1) we have $\Gamma(\mathbf{A},\mathbf{B})\cap D_{2mh}=C_{2h}(\mathbf{n})$ and

$$\text{rank}\ \ \text{Skw}_{2m}(\mathbf{A},\mathbf{B})\geqslant\text{rank}\{\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}}\}=1,$$

$$\text{rank}\ \ \text{Sym}_{2m}(\mathbf{A},\mathbf{B})\geqslant\text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}},\mathring{\mathbf{B}}\}=4.$$

For case (c2) we have

$$\text{rank}\ \ \text{Skw}_{2m}(\mathbf{A},\mathbf{B})\geqslant\text{rank}\{\mathbf{n}\wedge\mathring{\mathbf{B}}\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}},\mathbf{B}\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\mathbf{n}\}=3,$$

$$\text{rank}\ \ \text{Sym}_{2m}(\mathbf{A},\mathbf{B})\geqslant\text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}},\mathring{\mathbf{B}},\mathring{\mathbf{A}}\mathbf{n}\otimes\mathring{\mathbf{A}}\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{B}}+\mathring{\mathbf{B}}\mathring{\mathbf{A}}\}=6.$$

For case (c3), we have

$$\text{rank}\ \ \text{Skw}_{2m}(\mathbf{A},\mathbf{B})\geqslant\text{rank}\{\mathbf{n}\wedge\mathring{\mathbf{A}}\mathbf{n},\mathbf{n}\wedge\mathring{\mathbf{B}}\mathbf{n},\mathring{\mathbf{A}}\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}}\}=3,$$

$$\text{rank}\ \ \text{Sym}_{2m}(\mathbf{A},\mathbf{B})\geqslant\text{rank}\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\mathring{\mathbf{A}}\mathbf{n}\otimes\mathring{\mathbf{A}}\mathbf{n},\mathring{\mathbf{A}},\mathring{\mathbf{B}},\mathring{\mathbf{A}}\mathring{\mathbf{B}}+\mathring{\mathbf{B}}\mathring{\mathbf{A}}\}=6.$$

From the above results and Tables 2–3 in Sec. 2, we infer that the two sets $\mathrm{Skw}_{2m}(\mathbf{A},\mathbf{B})$ and $\mathrm{Sym}_{2m}(\mathbf{A},\mathbf{B})$ obey the criterion (2.3). Further, by considering the pair $\mathbf{A}_1 = \mathbf{D}_1$ and $\mathbf{B}_1 = \mathbf{n} \vee \mathbf{l}_1$ we infer that the generators $\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}$, $\mathring{\mathbf{B}}\;\mathbf{n}\wedge\; \mathring{\mathbf{A}}\mathring{\mathbf{B}}\;\mathbf{n}$ and $\mathring{\mathbf{A}}\mathring{\mathbf{B}} + \mathring{\mathbf{B}}\mathring{\mathbf{A}}$ are irreducible. Moreover, by considering the pair $\mathbf{A}_2 = \mathbf{n} \wedge \mathbf{l}_1$ and $\mathbf{B}_2 = \mathbf{D}_1$ we deduce that the generator $\mathring{\mathbf{A}}\;\mathbf{n}\wedge\; \mathring{\mathbf{B}}\mathring{\mathbf{A}}\;\mathbf{n}$ is also irreducible.

Next, we show that the presented set $I_{2m}(\mathbf{A},\mathbf{B})$ of invariants determines a functional basis of $(\mathbf{A},\mathbf{B})$ under the cylindrical group $D_{\infty h}(\mathbf{n})$. Indeed, the latter is obtainable from the four invariants $\mathrm{tr}\mathbf{C}$, $\mathbf{n}\cdot\mathbf{C}\mathbf{n}$, with $\mathbf{C} = \mathbf{A},\ \mathbf{B}$, as well as an isotropic functional basis of $(\mathring{\mathbf{A}}, \mathring{\mathbf{B}}, \mathbf{n}\otimes\mathbf{n})$ (see, e.g., BOEHLER [8]). By applying the related result for isotropic functions we know that the latter basis is given by

$$\mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}},\ \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,(\mathbf{n}\otimes\mathbf{n}),\ \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}},\ \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}^{\,2},\ \mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}}^{\,2},$$

as well as certain invariants of a single tensor $\mathbf{A}$ or $\mathbf{B}$. Each of the latter is covered or determined by the basis $I_{2m}(\mathbf{A})$ or $I_{2m}(\mathbf{B})$. The first four invariants above yield the last four invariants in the set $I_{2m}(\mathbf{A},\mathbf{B})$. Moreover, it is readily verified that the invariant $\mathrm{tr}\mathring{\mathbf{A}}^{\,2}\,\mathring{\mathbf{B}}^{\,2}$ is redundant for each of cases (c1)–(c3).

(viii) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{W})$ of a vector and a skewsymmetric tensor

$$V \quad V_{2m}(\mathbf{u}) \cup \{\mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathbf{W}\mathbf{n}) + (\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} \quad (\equiv V_{2m}(\mathbf{u},\mathbf{W})).$$

$$\mathrm{Skw} \quad \mathrm{Skw}_{2m}(\mathbf{u}) \cup \{\mathbf{W}, \mathbf{u}\wedge\mathbf{W}\mathbf{u}, \mathbf{u}\wedge\mathbf{W}^2\mathbf{u}\}.$$

$$\mathrm{Sym} \quad \mathrm{Sym}_{2m}(\mathbf{u}) \cup \mathrm{Sym}_{2m}(\mathbf{W}) \cup \{\mathbf{u}\vee\mathbf{W}\mathbf{u}, (\mathrm{tr}\mathbf{W}\mathbf{N})\,\mathring{\mathbf{u}}\vee(\mathbf{n}\times\mathring{\mathbf{u}})\}.$$

$$R \quad \mathbf{r}\cdot V_{2m}(\mathbf{u}); \mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{u}); \mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{u}), \mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{A});$$
$$\mathbf{r}\cdot\mathbf{W}\mathbf{u}, \mathbf{r}\cdot\mathbf{W}^2\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{r}}\cdot\eta_{2m-1}(\mathbf{W}\mathbf{n}) + (\mathbf{r}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{W}\mathbf{n});$$
$$\mathrm{tr}\mathbf{H}\mathbf{W}, \mathbf{u}\cdot\mathbf{H}\mathbf{W}\mathbf{u}, \mathbf{u}\cdot\mathbf{H}\mathbf{W}^2\mathbf{u}; \mathbf{u}\cdot\mathring{\mathbf{C}}\,\mathbf{W}\mathbf{u}, (\mathrm{tr}\mathbf{W}\mathbf{N})[\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{C}}\mathring{\mathbf{u}}];$$

$$I_{2m}(\mathbf{u}) \cup I_{2m}(\mathbf{W}) \cup \{(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \mathbf{u}\cdot\mathbf{W}^2\mathbf{u}\}.$$

Since $2r$th-order tensor-valued anisotropic functions of the variables $(\mathbf{u},\mathbf{W})$ under the group D_{2mh} are equivalent to those of the variables $(\mathbf{W}, \mathbf{u}\otimes\mathbf{u})$ under the same group, where $r \geqslant 0$, in the above table the results except those for the vector generators and their related invariants, can be derived by setting $\mathbf{A} = \mathbf{u}\otimes\mathbf{u}$ in the table for the variables $(\mathbf{W},\mathbf{A})$ in **(vi)**. In what follows we need only to show that the presented set $V_{2m}(\mathbf{u},\mathbf{W})$ supplies a desired vector generating set.

The D_{2mh}-irreducibility condition for $(\mathbf{u},\mathbf{W})$ is given by (see (3.1)):

$$\Gamma(\mathbf{u},\mathbf{W}) \cap D_{2mh} \neq \Gamma(\mathbf{z}) \cap D_{2mh}, \quad \mathbf{z} = \mathbf{u},\ \mathbf{W}.$$

It is evident that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{W} \neq \mathbf{O}$, and, moreover, $\Gamma(\mathbf{u}) \cap D_{2mh} \neq C_1$. The latter yields $(\mathbf{u} \cdot \mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0$. Three cases will be discussed.

First, let $\mathbf{u} \cdot \mathbf{n} = 0$ and $\beta_{2m}(\overset{\circ}{\mathbf{u}}) \neq 0$. Then $\mathbf{u} = \overset{\circ}{\mathbf{u}}$, and the two vectors $\overset{\circ}{\mathbf{u}}$ and $\eta_{2m-1}(\overset{\circ}{\mathbf{u}})$ are linearly independent. Hence, we have

$$\operatorname{rank} V_{2m}(\mathbf{u}, \mathbf{W}) \geqslant \operatorname{rank}\{\overset{\circ}{\mathbf{u}}, \eta_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{W}\mathbf{u}, (\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} = 3.$$

In the above, we have $\mathbf{W}\mathbf{n} \neq \mathbf{0}$. The case when $\mathbf{W}\mathbf{n} = \mathbf{0}$ violates the D_{2mh}-irreducibility condition for $(\mathbf{u}, \mathbf{W})$ (see (3.1)) and hence is excluded.

Second, let $\beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0$ and $(\mathbf{u} \cdot \mathbf{n})|\ \overset{\circ}{\mathbf{u}}\ | \neq 0$, i.e. $\mathbf{u} = a\mathbf{n} + b\mathbf{e}$ with $ab \neq 0$, and let

$$\mathbf{W} = x\mathbf{e} \wedge \mathbf{e}' + y\mathbf{e} \wedge \mathbf{n} + z\mathbf{e}' \wedge \mathbf{n},\ x^2 + z^2 \neq 0.$$

Then we have

$$\begin{aligned} \operatorname{rank} V_{2m}(\mathbf{u}, \mathbf{A}) \geqslant \{(\mathbf{u} \cdot \mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, (\mathbf{u} \cdot \mathbf{n})\eta_{2m-1}(\mathbf{W}\mathbf{n}) \\ +(\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} = \operatorname{rank}\{\mathbf{n}, \mathbf{e}, (az - bx)\mathbf{e}', y(bz + ax)\mathbf{e}', \\ \beta_{2m-1}(\mathbf{W}\mathbf{n})\mathbf{e}'\} = 3 \end{aligned}$$

for

$$az - bx \neq 0 \text{ or } y(bz + ax) \neq 0 \text{ or } \beta_{2m-1}(\mathbf{W}\mathbf{n}) \neq 0.$$

In the above, the case $az - bx = y(bz + ax) = \beta_{2m-1}(\mathbf{W}\mathbf{n}) = 0$ has been excluded, since this case yields $x = z = 0$ and $y \neq 0$, i.e. $\mathbf{W} = y\mathbf{e} \wedge \mathbf{n}$, which violates the D_{2mh}-irreducibility condition for $(\mathbf{u}, \mathbf{W})$.

Third, let $\overset{\circ}{\mathbf{u}} = \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{n} \neq \mathbf{0}$. According to Theorem 2 in XIAO [47], we know that an isotropic vector generating set of the extended variables $(\mathbf{u}, \mathbf{W}, \mathbf{\Phi}_{2m-2}(\mathbf{W}\mathbf{n}), \mathbf{n} \otimes \mathbf{n})$ offers a desired anisotropic vector generating set of $(\mathbf{u}, \mathbf{W})$. Applying this fact and the related result for isotropic functions we infer that the former is included in the set $V_{2m}(\mathbf{u}, \mathbf{W})$.

Finally, let $\mathbf{u} \cdot \mathbf{n} = \beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0$ and $\overset{\circ}{\mathbf{u}} \neq \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{e} \neq \mathbf{0}$, and let

$$\mathbf{W} = x\mathbf{e} \wedge \mathbf{e}' + y\mathbf{e} \wedge \mathbf{n} + z\mathbf{e}' \wedge \mathbf{n},\ x^2 + y^2 + z^2 \neq 0.$$

Then we have (note $x^2 + y^2 + z^2 \neq 0$)

$$\begin{aligned} \operatorname{rank} V_{2m}(\mathbf{u}) \geqslant \operatorname{rank}\{\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{u}, \mathbf{W}^2\mathbf{u}, (\mathbf{u} \cdot \mathbf{n})\eta_{2m-1}(\mathbf{W}\mathbf{n}) \\ + (\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{W}\mathbf{n}))\mathbf{n}\} = \operatorname{rank}\{\mathbf{e}, \alpha_{2m-1}(\mathbf{W}\mathbf{n})\mathbf{n}, x\mathbf{e}' + y\mathbf{n}, zy\mathbf{e}' - zx\mathbf{n}\} \\ = \begin{cases} 3 \text{ if } z(x^2 + y^2) \neq 0 \text{ or } z = 0,\ xy \neq 0, \\ 2 \text{ if } z = y = 0 \text{ or } z = x = 0, \\ 1 \text{ if } x = y = 0, \end{cases} \end{aligned}$$

and

$$\Gamma(\mathbf{u},\mathbf{W})\cap D_{2mh}=\begin{cases} C_{1h}(\mathbf{e}') \text{ if } z=x=0, & y\neq 0\,,\\ C_{1h}(\mathbf{n}) \text{ if } z=y=0, & x\neq 0\,,\\ C_2(\mathbf{e}) \text{ if } x=y=0; & z\neq 0\,. \end{cases}$$

From the above facts and Table 1 in Sec 2, we conclude that the set $V_{2m}(\mathbf{u},\mathbf{W})$ is a desired vector generating set. Further, by considering the two pairs: $\mathbf{u}_1=\mathbf{e}$ and $\mathbf{W}_1=\mathbf{N}+\mathbf{n}\wedge\mathbf{e}'$, $\mathbf{u}_2=\mathbf{n}+\mathbf{e}$ and $\mathbf{W}_2=\mathbf{E}(\mathbf{n}+\mathbf{e})$, we infer the last three generators in $V_{2m}(\mathbf{u},\mathbf{W})$ are irreducible.

(ix) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{A})$ of a vector and a symmetric tensor

$$V \quad V_{2m}(\mathbf{u})\cup\{\overset{\circ}{\mathbf{A}}\mathbf{u},\overset{\circ}{\mathbf{A}}{}^2\mathbf{u},(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}),(\overset{\circ}{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})+\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\,\overset{\circ}{\mathbf{A}})\mathbf{u}\}\ (\equiv V_{2m}(\mathbf{u},\mathbf{A})).$$

$$\mathrm{Skw} \quad \mathrm{Skw}_{2m}(\mathbf{u})\cup Skw_{2m}(\mathbf{A})\cup\{\mathbf{u}\wedge\overset{\circ}{\mathbf{A}}\mathbf{u},\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},(\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n})\,\overset{\circ}{\mathbf{u}}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}\}.$$

$$\mathrm{Sym} \quad \mathrm{Sym}_{2m}(\mathbf{u})\cup Sym_{2m}(\mathbf{A})\cup\{\mathbf{u}\vee\overset{\circ}{\mathbf{A}}\mathbf{u}\}.$$

$$R \quad \mathbf{r}\cdot V_{2m}(\mathbf{u});\mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{u}),\mathbf{H}:\mathrm{Skw}_{2m}(\mathbf{A});\mathbf{C}:\mathrm{Sym}_{2m}(\mathbf{u});$$
$$\mathbf{r}\cdot\overset{\circ}{\mathbf{A}}\mathbf{u},\mathbf{r}\cdot\overset{\circ}{\mathbf{A}}{}^2\mathbf{u},(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{r}}\cdot\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})+(\mathbf{r}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}),$$
$$\mathbf{r}\cdot(\overset{\circ}{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})+\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}})\,\overset{\circ}{\mathbf{A}})\mathbf{u};$$
$$\mathbf{u}\cdot\mathbf{H}\overset{\circ}{\mathbf{A}}\mathbf{u},\overset{\circ}{\mathbf{u}}\cdot\mathbf{H}\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},(\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\mathbf{H}\overset{\circ}{\mathbf{A}}\mathbf{n};\mathbf{u}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{A}}\mathbf{u};$$
$$I_{2m}(\mathbf{u})\cup I_{2m}(\mathbf{A})\cup\{\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}},(\mathbf{u}\cdot\mathbf{n})\,\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}\mathbf{n},\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{A}}{}^2\,\overset{\circ}{\mathbf{u}}\}.$$

Since $2r$th-order tensor-valued anisotropic functions of the variables $(\mathbf{u},\mathbf{A})$ under the group D_{2mh} are equivalent to those of the variables $(\mathbf{A},\mathbf{u}\otimes\mathbf{u})$ under the same group, where $r\geqslant 0$, in the above table the results except those for the vector generators and their related invariants are derived by setting $\mathbf{B}=\mathbf{u}\otimes\mathbf{u}$ in the table for the variables $(\mathbf{A},\mathbf{B})$ in **(vii)**. In deriving the functional basis in the above table, some redundant invariants have been removed.

In what follows we prove that the presented set $V_{2m}(\mathbf{u},\mathbf{A})$ supplies a desired vector generating set.

For $\overset{\circ}{\mathbf{u}}\neq\mathbf{0}$ we have

$$\Gamma(\mathbf{u})\cap D_{2mh}=\Gamma(\mathbf{u},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}))=\begin{cases} C_{1h}(\mathbf{n}) \text{ if } \mathbf{u}\cdot\mathbf{n}=0,\ \beta_{2m}(\overset{\circ}{\mathbf{u}})\neq 0\,,\\ C_{2v}(\mathbf{l},\mathbf{n},\mathbf{n}\times\mathbf{l}) \text{ if } \mathbf{u}=a\mathbf{l},\ a\neq 0\,,\\ C_{1h}(\mathbf{l}) \text{ if } \mathbf{u}=a\mathbf{n}+b\mathbf{n}\times\mathbf{l},\ ab\neq 0\,,\\ C_1 \text{ if } (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}})\neq 0\,. \end{cases}$$

Hence, for $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$ we infer

$$\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh} = (\Gamma(\mathbf{u})\cap D_{2mh})\cap \Gamma(\mathbf{A}) = \Gamma(\mathbf{u},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}}).$$

From the latter equality and the criterion (2.3) we know that, when $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$, isotropic generating sets for the variables $(\mathbf{u},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}),\overset{\circ}{\mathbf{A}})$ provide anisotropic generating sets for the variables $(\mathbf{u},\mathbf{A})$ under the group D_{2mh}. Moreover, when $\overset{\circ}{\mathbf{u}}= \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{n}$, isotropic generating sets for the variables $(\mathbf{u},\overset{\circ}{\mathbf{A}},\mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}),\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A})),\mathbf{n}\otimes\mathbf{n})$ supply anisotropic generating sets for the variables $(\mathbf{u},\mathbf{A})$ under D_{2mh} (see Theorem 2 in XIAO [47]). As a result, by applying the related result for isotropic functions, when $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$, we know that the desired vector generating set is formed by the generators in the set $V_{2m}(\mathbf{u},\mathbf{A})$ except $(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$. When $\overset{\circ}{\mathbf{u}}= \mathbf{0}$, i.e. $\mathbf{u} = a\mathbf{n}$, the desired vector generating set is formed by the five generators $(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, $\overset{\circ}{\mathbf{A}}\ \mathbf{u}$, $\overset{\circ}{\mathbf{A}}\ {}^2\mathbf{u}$, $(\mathbf{u}\cdot\mathbf{n})\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$ and $\boldsymbol{\phi}(\mathbf{u},\mathbf{A}) = (\mathbf{u}\cdot\mathbf{n})\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}))\ \overset{\circ}{\mathbf{A}}\ \mathbf{n}$. The first four generators are included in the presented set $V_{2m}(\mathbf{u},\mathbf{A})$.

We show that the generator $\boldsymbol{\phi}(\mathbf{u},\mathbf{A})$ is redundant. In fact, when $\overset{\circ}{\mathbf{A}}\ \mathbf{n}\neq\mathbf{0}$, the three vectors $(\mathbf{u}\cdot\mathbf{n})\mathbf{n}$, $\overset{\circ}{\mathbf{A}}\ \mathbf{u}$ and $\mathbf{r} = \mathbf{n}\times\ \overset{\circ}{\mathbf{A}}\ \mathbf{u}$ constitute an orthogonalized basis of V (note $\mathbf{u} = a\mathbf{n}\neq\mathbf{0}$). The components of the last three of the foregoing five generators with respect to $\hat{\mathbf{u}}$ are of the forms

$$\hat{\mathbf{u}}\cdot\ \overset{\circ}{\mathbf{A}}\ {}^2\mathbf{u} = \alpha x^2 y\sin(2\phi(\mathbf{A})-\psi(\mathbf{A})),\ \ \hat{\mathbf{u}}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n}) = \beta x^{2m}\sin 2m\phi(\mathbf{A}),$$

$$\hat{\mathbf{u}}\cdot\boldsymbol{\phi}(\mathbf{u},\mathbf{A}) = \gamma x^2 y^{m-1}\sin(2\phi(\mathbf{A})+(m-1)\psi(\mathbf{A})).$$

Here $x = |\ \overset{\circ}{\mathbf{A}}\ \mathbf{n}|$, $y = |\mathbf{q}(\mathbf{A})|$ and α, β and γ are nonvanishing. From these and the identity

$$\sin(2\phi(\mathbf{A})+(m-1)\psi(\mathbf{A})) = \sin 2m\phi(\mathbf{A})\cos(m-1)(2\phi(\mathbf{A})-\psi(\mathbf{A})) - \cos 2m\phi(\mathbf{A})\sin(m-1)(2\phi(\mathbf{A})-\psi(\mathbf{A})),$$

we deduce that the last one of the foregoing three components is determined by the other two. Hence, the generator $\boldsymbol{\phi}(\mathbf{u},\mathbf{A})$ is redundant.

Thus, we conclude that the presented set $V_{2m}(\mathbf{u},\mathbf{A})$ is a desired generating set. Moreover, from the property of this set concerning $\overset{\circ}{\mathbf{u}}\neq \mathbf{0}$ and $\overset{\circ}{\mathbf{u}}= \mathbf{0}$, indicated in the above proof, we know that the invariant $(\mathbf{u}\cdot\mathbf{n})\mathbf{r}\cdot\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{A}}\ \mathbf{n})$ given by the scalar product of the variable $\mathbf{r} \in V$ and the generator $(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})$, may be replaced by the third invariant at line 6 in the table given before.

Finally, from the criterion (2.3) and the facts

$$\mathbf{u}=\mathbf{n},\ \mathbf{A}=\mathbf{D}_1+\mathbf{D}_3:\ V_{2m}(\mathbf{u})=\{\mathbf{n}\},\ \overset{\circ}{\mathbf{u}}=\mathbf{0},$$

$$\overset{\circ}{\mathbf{A}}\,\mathbf{u}=(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{A}}\,\mathbf{n})=\mathbf{e};$$

$$\mathbf{u}=\mathbf{e},\ \mathbf{A}=\mathbf{D}_2+\mathbf{D}_3:\ (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{u}})=\mathbf{0},$$

$$\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}=2\mathbf{e},\ V_{2m}(\mathbf{u})=\{\mathbf{e}\};$$

$$\mathbf{u}=\mathbf{n},\ \mathbf{A}=\mathbf{n}\vee(\mathbf{e}+\mathbf{l}_1):\overset{\circ}{\mathbf{u}}=\mathbf{0},\ \overset{\circ}{\mathbf{A}}{}^2\mathbf{n}=\mathbf{n},\ V_{2m}(\mathbf{u})=\{\mathbf{n}\},$$

where $\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}=C_1$ for each pair $(\mathbf{u},\mathbf{A})$ given, we deduce that the generator $\overset{\circ}{\mathbf{A}}{}^2\mathbf{u}$, the generator $\overset{\circ}{\mathbf{A}}\,\mathbf{u}$, the last generator in the set $V_{2m}(\mathbf{u},\mathbf{A})$ and the generator $(\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{u}})$, are irreducible, respectively.

4.3. D_{2mh}-irreducible sets of three variables

(x) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{W})$ of two vectors and a skewsymmetric tensor

V $\{(\mathbf{u}\cdot\mathbf{n})\mathbf{n},\overset{\circ}{\mathbf{u}},(\mathbf{v}\cdot\mathbf{n})\mathbf{n},\overset{\circ}{\mathbf{v}},\mathbf{W}\mathbf{u},\mathbf{W}\mathbf{v}\}\ (\equiv V(\mathbf{u},\mathbf{v},\mathbf{W}))$.

Skw $\{\mathbf{u}\wedge\mathbf{v},\mathbf{W},\mathbf{u}\wedge\mathbf{W}\mathbf{v}+\mathbf{v}\wedge\mathbf{W}\mathbf{u}\}\ (\equiv \mathrm{Skw}(\mathbf{u},\mathbf{v},\mathbf{W}))$.

Sym $\{\mathbf{I},\mathbf{n}\otimes\mathbf{n},\overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}},\overset{\circ}{\mathbf{v}}\otimes\overset{\circ}{\mathbf{v}},\mathbf{u}\vee\mathbf{v},\mathbf{u}\vee\mathbf{W}\mathbf{u},\mathbf{v}\vee\mathbf{W}\mathbf{v},\mathbf{u}\vee\mathbf{W}\mathbf{v}+\mathbf{v}\vee\mathbf{W}\mathbf{u}\}$
$(\equiv \mathrm{Sym}(\mathbf{u},\mathbf{v},\mathbf{W}))$.

R $\mathbf{r}\cdot V(\mathbf{u},\mathbf{v},\mathbf{W});\mathbf{u}\cdot\mathbf{H}\mathbf{v},\mathrm{tr}\mathbf{H}\mathbf{W},\mathbf{u}\cdot(\mathbf{H}\mathbf{W}-\mathbf{W}\mathbf{H})\mathbf{v};$

$\mathrm{tr}\mathbf{C},\mathbf{n}\cdot\mathbf{C}\mathbf{n},\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{u}},\mathbf{u}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{u},\overset{\circ}{\mathbf{v}}\cdot\overset{\circ}{\mathbf{C}}\overset{\circ}{\mathbf{v}},\mathbf{v}\cdot\overset{\circ}{\mathbf{C}}\,\mathbf{W}\mathbf{v},\mathbf{u}\cdot(\overset{\circ}{\mathbf{C}}\,\mathbf{W}-\mathbf{W}\,\overset{\circ}{\mathbf{C}})\mathbf{v};$
$\{(\mathbf{u}\cdot\mathbf{n})^2,|\overset{\circ}{\mathbf{u}}|^2,(\mathbf{v}\cdot\mathbf{n})^2,|\overset{\circ}{\mathbf{v}}|^2\}\ (\equiv I(\mathbf{u},\mathbf{v},\mathbf{W}))$.

The proof for the above results is as follows. From the condition (3.2) with $\mathbf{x}=\mathbf{u}$, $\mathbf{y}=\mathbf{v}$ and $\mathbf{z}=\mathbf{A}$ and $g=D_{2mh}$, it is evident that the two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and $\mathbf{W}\neq\mathbf{O}$, and $\Gamma(\mathbf{u},\mathbf{v})\cap D_{2mh}\neq C_1$, $\Gamma(\mathbf{z},\mathbf{W})\cap D_{2mh}\neq C_1$, $\mathbf{z}=\mathbf{u},\ \mathbf{v}$. From the first expression above and $\Gamma(\mathbf{u},\mathbf{v})=C_{1h}(\mathbf{u}\times\mathbf{v})$, we infer that the vector $\mathbf{u}\times\mathbf{v}$ must be in the direction of one of the symmetry axis vectors $\mathbf{n}$ and $\mathbf{l}_1,\cdots,\mathbf{l}_{2m}$. Since the set $(\mathbf{u},\mathbf{v})$ should be a D_{2mh}-irreducible set, we further deduce that the two vectors $\mathbf{u}$ and $\mathbf{v}$ must be in directions of two of the symmetry axis vectors $\mathbf{n}$ and $\mathbf{l}_k$, $k=0,1,\cdots,2m-1$. In what follows we shall prove that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and either $\mathbf{W}\mathbf{u}=\mathbf{0}$ or $\mathbf{W}\mathbf{v}=\mathbf{0}$ holds.

In fact, the second expression for $(\mathbf{z}, \mathbf{W})$ given before implies that for each vector $\mathbf{z} \in \{\mathbf{u}, \mathbf{v}\}$, the axis vector $\mathbf{w} = \mathbf{E} : \mathbf{W}$ of $\mathbf{W}$ must be either normal to or in the direction of $\mathbf{z}$. This fact leads to the two possibilitie for $(\mathbf{u}, \mathbf{v}, \mathbf{W})$: (a) The vector $\mathbf{E} : \mathbf{W}$ is orthogonal to one of the vectors $\mathbf{u}$ and $\mathbf{v}$ and in the direction of the other, and (b) the vector $\mathbf{E} : \mathbf{W}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. The latter is excluded, since it results in $\Gamma(\mathbf{u}, \mathbf{v}, \mathbf{W}) = \Gamma(\mathbf{u}, \mathbf{v}) = C_{1h}(\mathbf{u} \times \mathbf{v})$, which violates the condition (3.2). Hence, we conclude that the fact stated before is true.

Thus, the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ is given by

$$\text{(c1) } \mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{e},\ \mathbf{W} = c\mathbf{N},\ abc \neq 0;$$

$$\text{(c2) } \mathbf{u} = a\mathbf{n},\ \mathbf{v} = b\mathbf{e},\ \mathbf{W} = c\mathbf{n} \wedge \mathbf{e}',\ abc \neq 0;$$

$$\text{(c3) } \mathbf{u} = a\mathbf{e},\ \mathbf{v} = b\mathbf{e}',\ \mathbf{W} = c\mathbf{n} \wedge \mathbf{e},\ abc \neq 0.$$

With the above cases one may readily verify that the four sets $V(\mathbf{u}, \mathbf{v}, \mathbf{W})$, $\mathrm{Skw}(\mathbf{u}, \mathbf{v}, \mathbf{W})$, $\mathrm{Sym}(\mathbf{u}, \mathbf{v}, \mathbf{W})$ and $I(\mathbf{u}, \mathbf{v}, \mathbf{W})$ providet desired generating sets and a functional basis for the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ under the group D_{2mh}. Further, by considering the set $(\mathbf{u}, \mathbf{v}, \mathbf{W})$ given by case (c1), we infer that the two tensor generators $\mathbf{u} \wedge \mathbf{W}\mathbf{v} + \mathbf{v} \wedge \mathbf{W}\mathbf{u}$ and $\mathbf{u} \vee \mathbf{W}\mathbf{v} + \mathbf{v} \vee \mathbf{W}\mathbf{u}$ are irreducible.

(xi) The D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ of two vectors and a symmetric tensor

$$V \quad \{(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \mathring{\mathbf{u}}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \mathring{\mathbf{v}}, \mathring{\mathbf{A}}\,\mathbf{u}, \mathring{\mathbf{A}}\,\mathbf{v}\}\ (\equiv V(\mathbf{u}, \mathbf{v}, \mathbf{A})).$$

$$\mathrm{Skw} \quad \{\mathbf{u} \wedge \mathbf{v}, \mathbf{u} \wedge \mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{v} \wedge \mathring{\mathbf{A}}\,\mathbf{v}, \mathbf{u} \wedge \mathring{\mathbf{A}}\,\mathbf{v} + \mathbf{v} \wedge \mathring{\mathbf{A}}\,\mathbf{u}\}\ (\equiv \mathrm{Skw}(\mathbf{u}, \mathbf{v}, \mathbf{A})).$$

$$\begin{aligned}\mathrm{Sym} \quad & \{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}}, \mathring{\mathbf{v}} \otimes \mathring{\mathbf{v}}, \mathbf{u} \vee \mathbf{v}, \mathbf{u} \vee \mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{v} \vee \mathring{\mathbf{A}}\,\mathbf{v}, \\ & \qquad \mathbf{u} \vee \mathring{\mathbf{A}}\,\mathbf{v} + \mathbf{v} \vee \mathring{\mathbf{A}}\,\mathbf{u}\}\ (\equiv \mathrm{Sym}(\mathbf{u}, \mathbf{v}, \mathbf{A})).\end{aligned}$$

$$\begin{aligned} R \quad & \mathbf{r} \cdot V(\mathbf{u}, \mathbf{v}, \mathbf{A}); \mathbf{u}\cdot\mathbf{H}\mathbf{v}, \mathbf{u}\cdot\mathbf{H}\mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{v}\cdot\mathbf{H}\mathbf{v}, \mathbf{v}\cdot\mathbf{H}\mathring{\mathbf{A}}\,\mathbf{v}, \\ & \qquad \mathbf{u}\cdot(\mathbf{H}\mathring{\mathbf{A}} - \mathring{\mathbf{A}}\mathbf{H})\mathbf{v}; \\ & \mathrm{tr}\mathbf{C}, \mathbf{n}\cdot\mathbf{C}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{u}}, \mathbf{u}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{u}, \mathring{\mathbf{v}}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{v}}, \mathbf{v}\cdot\mathring{\mathbf{C}}\mathring{\mathbf{A}}\,\mathbf{v}, \mathbf{u}\cdot(\mathring{\mathbf{C}}\mathring{\mathbf{A}} - \mathring{\mathbf{A}}\mathring{\mathbf{C}})\mathbf{v}; \\ & \{(\mathbf{u}\cdot\mathbf{n})^2, |\mathring{\mathbf{u}}|^2, (\mathbf{v}\cdot\mathbf{n})^2, |\mathring{\mathbf{v}}|^2, \mathbf{n}\cdot\mathbf{A}\mathbf{n}, \mathrm{tr}\mathbf{A}, |\mathring{\mathbf{A}}|^2, |\mathbf{q}(\mathbf{A})|^2, \\ & \qquad \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, \mathring{\mathbf{v}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{v}}\}\ (\equiv I(\mathbf{u}, \mathbf{v}, \mathbf{A})).\end{aligned}$$

To prove the above results, we work out the D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{v}, \mathbf{A})$ specified by the condition (3.2) with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{v}, \mathbf{A})$ and $g = D_{2mh}$. The latter yields

$$\Gamma(\mathbf{u},\mathbf{A})\cap D_{2mh}\neq C_1,\ \Gamma(\mathbf{v},\mathbf{A})\cap D_{2mh}\neq C_1\,. \tag{4.9}$$

According to the first half of the proof in (x), the vectors $\mathbf{u}$ and $\mathbf{v}$ are in directions of two of the symmetry axis vectors $\mathbf{n}$ and $\mathbf{l}_k$. Moreover, if one of the symmetric axis vectors is an eigenvector of the symmetric tensor $\mathbf{A}$, say $\mathbf{a}$, then $\mathbf{u}$ and $\mathbf{v}$ can not be normal to $\mathbf{a}$ simultaneously, since otherwise we would have $\Gamma(\mathbf{u},\mathbf{v},\mathbf{A})=\Gamma(\mathbf{u},\mathbf{v})\cap D_{2mh}=C_{1h}(\mathbf{a})$.

Consider the symmetric tensor $\mathbf{A}$. Since the centrosymmetric subgroups of the group $\Gamma(\mathbf{A})\cap D_{2mh}$ are given by S_2, $C_{2h}(\mathbf{a})$, $D_{2h}(\mathbf{n},\mathbf{a}_1,\mathbf{a}_2)$ and D_{2mh}, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are two mutually orthogonal two-fold axis vectors of the group D_{2mh}, and $\mathbf{a}$ is one of the symmetry axis vectors $\mathbf{l}_k$ and $\mathbf{n}$. It is evident that the following two cases for the symmetric tensor $\mathbf{A}$ can be excluded: $\Gamma(\mathbf{A})\cap D_{2mh}=S_2,\ D_{2mh}$. Hence, there are two cases for $\mathbf{A}$ left, which are discussed below.

Let $\Gamma(\mathbf{A})\cap D_{2mh}=C_{2h}(\mathbf{a})$, $\mathbf{a}\in\{\mathbf{n},\mathbf{l}_1,\cdots,\mathbf{l}_{2m}\}$. Then the two conditions given by (4.9) imply that either of the vectors $\mathbf{u}$ and $\mathbf{v}$ is normal to $\mathbf{a}$ or in the direction of $\mathbf{a}$, since the nontrivial proper subgroups of the group $C_{2h}(\mathbf{a})$ are merely $C_{1h}(\mathbf{a})$ and $C_2(\mathbf{a})$. From this fact and the foregoing fact concerning $\mathbf{u}$ and $\mathbf{v}$ we derive the three cases: (a) $\mathbf{a}=\mathbf{l}$, $\mathbf{u}=a\mathbf{l}$ and $\mathbf{v}=b\mathbf{n}$; (b) $\mathbf{a}=\mathbf{l}$, $\mathbf{u}=a\mathbf{l}$ and $\mathbf{v}=b\mathbf{n}\times\mathbf{l}$; and (c) $\mathbf{a}=\mathbf{n}$, $\mathbf{u}=a\mathbf{n}$ and $\mathbf{v}=b\mathbf{l}$. Here and henceforth, $ab\neq 0$ and $\mathbf{l}\in\{\mathbf{l}_1,\cdots,\mathbf{l}_{2m}\}$.

Let $\Gamma(\mathbf{A})\cap D_{2mh}=D_{2h}(\mathbf{n},\mathbf{a}_1,\mathbf{a}_2)$, $\mathbf{a}_1,\ \mathbf{a}_2\in\{\mathbf{l}_1,\cdots,\mathbf{l}_{2m}\}$. Then, from (2.5) and the two conditions given by (4.9) we deduce that either of the vectors $\mathbf{u}$ and $\mathbf{v}$ is normal to or in the direction of one of the vectors $\mathbf{n}$, $\mathbf{a}_1$ and $\mathbf{a}_2$. From this fact and the aforementioned fact concerning $\mathbf{u}$ and $\mathbf{v}$, we derive the only one case: (d) $\mathbf{u}=a\mathbf{n}$ and $\mathbf{v}=b\mathbf{l}$ with $\mathbf{l}\times\mathbf{a}_i\neq\mathbf{0}$, $i=1,\ 2$.

From the above analysis, we know that the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{A})$ is specified by the four cases (a)-(d) for $\mathbf{A}$ above. Without loss of generality, we set $\mathbf{l}=\mathbf{e}$ in these cases. Then we have

(c1) $\mathbf{u}=a\mathbf{e},\ \mathbf{v}=b\mathbf{n},\ \overset{\circ}{\mathbf{A}}=x\mathbf{D}_1+y\mathbf{D}_4,\ aby\neq 0;$

(c2) $\mathbf{u}=a\mathbf{e},\ \mathbf{v}=b\mathbf{e}',\ ab\neq 0,\ \overset{\circ}{\mathbf{A}}=x\mathbf{D}_1+y\mathbf{D}_4,\ aby\neq 0;$

(c3) $\mathbf{u}=a\mathbf{n},\ \mathbf{v}=b\mathbf{e},\ ab\neq 0,\ \overset{\circ}{\mathbf{A}}=x\mathbf{D}_1+y\mathbf{D}_2,\ aby\neq 0.$

In the above, we would mention that the cases (c) and (d) derived before have been combined into case (c3).

With cases (c1)–(c3), one may readily verify that the four sets $V(\mathbf{u},\mathbf{v},\mathbf{A})$, $\mathrm{Skw}(\mathbf{u},\mathbf{v},\mathbf{A})$, $\mathrm{Sym}(\mathbf{u},\mathbf{v},\mathbf{A})$ and $I(\mathbf{u},\mathbf{v},\mathbf{A})$ provide the desired generating sets and a functional basis for the D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{A})$. Further, by considering the set $(\mathbf{u},\mathbf{v},\mathbf{A})$ given by case (c1), we infer that the tensor generators $\mathbf{u}\wedge\overset{\circ}{\mathbf{A}}\mathbf{v}+\mathbf{v}\wedge\overset{\circ}{\mathbf{A}}\mathbf{u}$ and $\mathbf{u}\vee\overset{\circ}{\mathbf{A}}\mathbf{v}+\mathbf{v}\vee\overset{\circ}{\mathbf{A}}\mathbf{u}$ are irreducible.

(xii) The D_{2mh}-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{w})$ of three vectors

If the three vectors $(\mathbf{u},\mathbf{v},\mathbf{w})$ are linearly dependent, i.e. they lie on the same plane, then there are two of them, say $\mathbf{u}$ and $\mathbf{v}$, such that $\Gamma(\mathbf{u},\mathbf{v},\mathbf{w})=\Gamma(\mathbf{u},\mathbf{v})$, which violates the g-irreducibility condition (3.2) with $(\mathbf{x},\mathbf{y},\mathbf{z})=(\mathbf{u},\mathbf{v},\mathbf{w})$ and any given subgroup $g\subseteq$ Orth. Thus, we deduce that the three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent for each g-irreducible set $(\mathbf{u},\mathbf{v},\mathbf{w})$ with any given subgroup $g\subseteq$ Orth. Hence we construct the following table.

V $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ $(\equiv V(\mathbf{u},\mathbf{v},\mathbf{w}))$.

Skw $\{\mathbf{u}\wedge\mathbf{v},\mathbf{v}\wedge\mathbf{w},\mathbf{w}\wedge\mathbf{u}\}$ $(\equiv \mathrm{Skw}(\mathbf{u},\mathbf{v},\mathbf{w})$.

Sym $\{\mathbf{u}\otimes\mathbf{u},\mathbf{v}\otimes\mathbf{v},\mathbf{w}\otimes\mathbf{w},\mathbf{u}\vee\mathbf{v},\mathbf{v}\vee\mathbf{w},\mathbf{w}\vee\mathbf{u}\}$ $(\equiv \mathrm{Sym}(\mathbf{u},\mathbf{v},\mathbf{w}))$.

R $\mathbf{r}\cdot V(\mathbf{u},\mathbf{v},\mathbf{w});\mathbf{H}:\mathrm{Skw}(\mathbf{u},\mathbf{v},\mathbf{w});\mathbf{C}:\mathrm{Sym}(\mathbf{u},\mathbf{v},\mathbf{w});$
$\{|\mathbf{u}|^2,|\mathbf{v}|^2,|\mathbf{w}|^2,\mathbf{u}\cdot\mathbf{v},\mathbf{v}\cdot\mathbf{w},\mathbf{w}\cdot\mathbf{u}\}$ $(\equiv I(\mathbf{u},\mathbf{v},\mathbf{w}))$.

Evidently, each invariant and each generator in the four presented sets $M(\mathbf{u},\mathbf{v},\mathbf{w})$, where $M=I,V$, Skw, Sym, are isotropic and involve not more than two vector variables, and therefore they are determined by the functional basis and the generating sets for one and two vector variables under the group $g\subseteq$ Orth. As a result, for any given subgroup $g\subseteq$ Orth, the isotropic vector generating set $V(\mathbf{u},\mathbf{v},\mathbf{w})$ can be omitted. Further, if the set $(\mathbf{u},\mathbf{v})$ of two vector variables has been covered before, as is the case treated here, all the isotropic invariants and generators listed in the above table can be omitted.

(xiii) The D_{2mh}-irreducible sets $(\mathbf{u},\mathbf{W},\boldsymbol{\Omega})$ and $(\mathbf{u},\mathbf{W},\mathbf{A})$

According to Theorem 3.2 in XIAO [52], it suffices to supply generating sets for vector-valued functions for the above two sets of variables and the set of variables given later. The desired results are given as follows.

V $\{\mathbf{u},\mathbf{W}\mathbf{u},\boldsymbol{\Omega}\mathbf{u},\mathbf{W}^2\mathbf{u},\boldsymbol{\Omega}^2\mathbf{u},(\mathbf{W}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{W})\mathbf{u}\}$.

V $\{\mathbf{u},\mathbf{W}\mathbf{u},\overset{\circ}{\mathbf{A}}\mathbf{u},\mathbf{W}^2\mathbf{u},\overset{\circ}{\mathbf{A}}{}^2\mathbf{u},(\mathbf{W}\overset{\circ}{\mathbf{A}}-\overset{\circ}{\mathbf{A}}\mathbf{W})\mathbf{u}\}$.

The proof is as follows. Let X_0 be either of the two sets $(\mathbf{u},\mathbf{W},\boldsymbol{\Omega})$ and $(\mathbf{u},\mathbf{W},\overset{\circ}{\mathbf{A}})$. We first prove

$$\Gamma(X_0)\cap D_{2mh}=\Gamma(X_0)\,. \tag{4.10}$$

In fact, if $\Gamma(X_0)=C_1$, then it is evident that (4.10) holds. The other case, i.e. $\Gamma(X_0)\neq C_1$, implies that there is a unit vector $\mathbf{a}$ such that either $C_{1h}(\mathbf{a})\subseteq\Gamma(X_0)$ or $C_2(\mathbf{a})\subseteq\Gamma(X_0)$ holds. If $C_{1h}(\mathbf{a})\subseteq\Gamma(X_0)$, then from (2.5) – (2.6) we deduce that $\mathbf{u}$ is normal to $\mathbf{a}$ and $\mathbf{W}\mathbf{a}=\mathbf{0}$. Hence we have $\Gamma(X_0)=\Gamma(\mathbf{u},\mathbf{W})=C_{1h}(\mathbf{a})$, which violates the condition (3.2) and should be excluded. On the other hand, if $C_2(\mathbf{a}) \subseteq \Gamma(X_0)$, then again from (2.5) and (2.6) we deduce that $\mathbf{u} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{Wa} = \mathbf{0}$. Hence we have $\Gamma(X_0) \subseteq \Gamma(\mathbf{u}, \mathbf{W}) = C_\infty(\mathbf{a})$. Since the symmetry group of any vector or second-order tensor has nothing but 2-fold and ∞-fold symmetry axes, we infer that $\Gamma(X_0) = \Gamma(\mathbf{u}, \mathbf{W}) = C_\infty(\mathbf{a})$ or $\Gamma(X_0) = C_2(\mathbf{a})$. The former violates the condition (3.2) and is excluded. For the latter, we infer that (4.10) holds, if $\mathbf{a}$ is a symmetry axis vector of the group D_{2mh}. If the latter is not true, then we have $\Gamma(X_0) \cap D_{2mh} = \Gamma(\mathbf{u}, \mathbf{W}) \cap D_{2mh} = C_1$, which violates the condition (3.2) and is excluded.

Thus, we infer that (4.10) holds. Then, from (4.10) and criterion (2.3) it follows that a generating set for vector-valued isotropic functions of each D_{2mh}-irreducible set X_0 supplies a generating set for vector-valued form-invariant functions of X_0 under D_{2mh}. The former can be derived by applying the related results for isotropic functions, as given before. Moreover, the irreducibility of the generators $(\mathbf{W\Omega} - \mathbf{\Omega W})\mathbf{u}$ and $(\mathbf{W}\,\mathring{\mathbf{A}} - \mathring{\mathbf{A}}\,\mathbf{W})\mathbf{u}$ can be deduced by considering: $\mathbf{u}_0 = \mathbf{n},\ \mathbf{W}_0 = \mathbf{En},\ \mathbf{\Omega}_0 = \mathbf{n} \wedge \mathbf{e}$ and $\mathbf{u}_0 = \mathbf{e},\ \mathbf{W}_0 = \mathbf{En},\ \mathbf{A}_0 = \mathbf{n} \vee \mathbf{e}$.

(xiv) The D_{2mh}-irreducible set $(\mathbf{u}, \mathbf{A}, \mathbf{B})$ of a vector and two symmetric tensors

A desired generating set for vector-valued functions is given by

$$V \qquad V_{2m}(\mathbf{u}, \mathbf{A}) \cup V_{2m}(\mathbf{u}, \mathbf{B}) \cup \{(\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u}\}.$$

The proof is as follows. From the condition (3.2) with $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{u}, \mathbf{A}, \mathbf{B})$ and $g = D_{2mh}\ \Gamma(\mathbf{u}) \cap D_{2mh} \neq C_1$, it is evident that $\Gamma(\mathbf{u}) \cap D_{2mh} \neq C_1,\ D_{2mh}$ and

$$\Gamma(\mathbf{u}, \mathbf{C}) \cap D_{2mh} \neq C_1,\ \mathbf{C} = \mathbf{A},\ \mathbf{B}\,. \tag{4.11}$$

From the former and (2.5) we infer that $\mathbf{u}$ is normal to one of the symmetry axis vectors of the group D_{2mh}. Hence we derive the three cases for $\mathbf{u}$: (a) $\mathbf{u} = c\mathbf{a} \neq \mathbf{0}$ with $\mathbf{a} \in \{\mathbf{n}, \mathbf{l}_0, \cdots, \mathbf{l}_{2m-1}\}$, (b) $\mathbf{u} = a\mathbf{n} + b\mathbf{l}$ with $ab \neq 0$, and (c) $\mathbf{u} \cdot \mathbf{n} = 0$ with $\mathbf{u} \times \mathbf{l}_k \neq \mathbf{0}$, $k = 1, \cdots, 2m$. For the latter two cases we have $\Gamma(\mathbf{u}) \cap D_{2mh} = C_{1h}(\mathbf{a})$ with $\mathbf{a} = \mathbf{n} \times \mathbf{l}$ and $\mathbf{a} = \mathbf{n}$, respectively. Hence we deduce

$$\Gamma(\mathbf{u}, \mathbf{C}) \cap D_{2mh} = C_{1h}(\mathbf{a}) \cap \Gamma(\mathbf{A}) = \begin{cases} C_1 = \Gamma(\mathbf{u}, \mathbf{A}, \mathbf{B}) \cap D_{2mh} \\ \quad \text{if } \mathbf{a} \text{ is not an eigenvector of } \mathbf{A}\,, \\ C_{1h}(\mathbf{a}) = \Gamma(\mathbf{u}) \cap D_{2mh} \\ \qquad \text{if } \mathbf{a} \text{ is an eigenvector } \mathbf{A}\,. \end{cases}$$

From the above we know that the condition (3.2) is violated and hence the case at issue is excluded.

In what follows we are concerned with case (a) for $\mathbf{u}$ indicated before. From (4.11) and $\Gamma(\mathbf{u},\mathbf{C}) \cap D_{2mh} \subseteq \Gamma(\mathbf{u}) \cap D_{2mh}$ we derive

$$\Gamma(\mathbf{u},\mathbf{A}) \cap D_{2mh} = C_{1h}(\mathbf{a}'),\ C_2(\mathbf{a}) \subseteq \Gamma(\mathbf{u},\mathbf{A}) \cap D_{2mh},$$

$$\Gamma(\mathbf{u},\mathbf{B}) \cap D_{2mh} = C_{1h}(\mathbf{a}''),\ C_2(\mathbf{a}) \subseteq \Gamma(\mathbf{u},\mathbf{B}) \cap D_{2mh},$$

where $\mathbf{a}'$ and $\mathbf{a}''$ are symmetry axis vectors of D_{2mh} normal to $\mathbf{u} = c\mathbf{a}$. Evidently, $\mathbf{a}'$ and $\mathbf{a}''$ are not coincident, or else the condition (3.2) will be violated. Then, we derive the three cases:

(c1) $C_2(\mathbf{a}) \subseteq \Gamma(\mathbf{u},\mathbf{C}),\ \mathbf{C} = \mathbf{A},\ \mathbf{B};$

(c2) $\Gamma(\mathbf{u},\mathbf{A}) \cap D_{2mh} = C_{1h}(\mathbf{a}'),\ \Gamma(\mathbf{u},\mathbf{B}) \cap D_{2mh} = C_{1h}(\mathbf{a}''),\ \mathbf{a}' \times \mathbf{a}'' \neq \mathbf{0};$

(c3) $\Gamma(\mathbf{u},\mathbf{A}) \cap D_{2mh} = C_{1h}(\mathbf{a}'),\ C_2(\mathbf{a}) \subseteq \Gamma(\mathbf{u},\mathbf{B}) \cap D_{2mh}.$

For case (c1), we have $C_2(\mathbf{a}) \subseteq \Gamma(\mathbf{u},\mathbf{A},\mathbf{B}) \cap D_{2mh}$ and $\mathbf{u} = c\mathbf{a} \neq \mathbf{0}$, it is easy to show that the subset $V_{2m}(\mathbf{u})$ obeys the criterion (2.3).

For case (c2), using the formula (2.4) we have

$$\begin{aligned}\operatorname{rank}(V_{2m}(\mathbf{u},\mathbf{A})\ \cup\ & V_{2m}(\mathbf{u},\mathbf{B}))\\ &= \operatorname{rank}(V(\Gamma(\mathbf{u},\mathbf{A}) \cap D_{2mh}) \cup V(\Gamma(\mathbf{u},\mathbf{B}) \cap D_{2mh}))\\ &= \operatorname{rank}(V(C_{1h}(\mathbf{a}')) \cup V(C_{1h}(\mathbf{a}'')))\\ &= \operatorname{rank}\{\mathbf{a}, \mathbf{a}\times\mathbf{a}', \mathbf{a}\times\mathbf{a}''\} = 3\,.\end{aligned}$$

For case (c3), $\mathbf{u} = c\mathbf{a}$ is an eigenvector of $\mathbf{B}$ (see (2.8)). Moreover, $(\mathbf{a}, \mathbf{a}', \mathbf{r} \equiv \mathbf{a}\times\mathbf{a}')$ is an orthonormal basis of V. In terms of this basis, we have the expressions

$$\mathbf{A}\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{r},\ \mathbf{B} = x\mathbf{a}\otimes\mathbf{a} + y\mathbf{a}'\otimes\mathbf{a}' + z\mathbf{r}\otimes\mathbf{r} + w\mathbf{a}'\vee\mathbf{r},\ \beta z \neq 0.$$

For the former, the following facts are used: the vector $\mathbf{A}\mathbf{u} \in V(C_{1h}(\mathbf{a}))$ is normal to $\mathbf{a}$, and $\mathbf{a}$ is not an eigenvector of $\mathbf{A}$ (hence $z \neq 0$). Moreover, $w = 0$ has been excluded, since otherwise we have $C_{2h}(\mathbf{a}') \subseteq \Gamma(\mathbf{B})$ and hence

$$\Gamma(\mathbf{u},\mathbf{A},\mathbf{B}) \cap D_{2mh} = \Gamma(\mathbf{u},\mathbf{A}) \cap D_{2mh} = C_{1h}(\mathbf{a}'),$$

which violates the condition (3.2). Thus, for case (c3) we have

$$\operatorname{rank}(V_{2m}(\mathbf{u},\mathbf{A}) \cup \{(\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u}\}) = \operatorname{rank}\{\mathbf{a}, \mathbf{r}, \beta w\mathbf{a}'\} = 3.$$

From the above we conclude that the set of vector generators offers a desired generating set. The irreducibility of the generator $(\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{u}$ can be deduced by considering

$$\mathbf{u}_0 = \mathbf{n},\ \mathbf{A}_0 = \mathbf{n}\vee\mathbf{e},\ \mathbf{B}_0 = \mathbf{n}\vee\mathbf{e}'.$$

4.4 The general results

THEOREM 1. *The four sets given by*

$$I_{2m}(\mathbf{u}); I_{2m}(\mathbf{W}); I_{2m}(\mathbf{A}); (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{v}}), \mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{u}});$$

$$\mathrm{tr}\mathbf{W\Omega}, \eta_{2m-1}(\mathbf{Wn})\cdot\mathbf{\Omega n}, \eta_{2m-1}(\mathbf{\Omega n})\cdot\mathbf{Wn}, \mathrm{tr}\mathbf{W\Omega}^2\mathbf{\Phi}_{2m-2}(\mathbf{\Omega n}),$$
$$\mathrm{tr}\mathbf{\Omega W}^2\mathbf{\Phi}_{2m-2}(\mathbf{Wn});$$

$$(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathring{\mathbf{B}}\,\mathbf{n}, \mathrm{tr}\mathbf{A_e B_e}, \mathrm{tr}\mathring{\mathbf{A}}^2\mathring{\mathbf{B}}, \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}^2, \mathrm{tr}\mathbf{A_e}\mathbf{\Phi}_{m-1}(\mathbf{q}(\mathbf{B})),$$

$$\mathrm{tr}\mathbf{A_n B_n}\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{A}}\,\mathbf{n}), \mathrm{tr}\mathbf{A_e}\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{B}}\,\mathbf{n}), \mathrm{tr}\mathbf{B_e}\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathbf{Wn}, \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W}^2, \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathbf{Wn}), \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W\Phi}_{2m-2}(\mathbf{Wn}),$$
$$(\mathrm{tr}\mathbf{WN})\beta_m(\mathbf{q}(\mathbf{A})),$$

$$\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})\cdot\mathbf{Wn} - |\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-4}(\mathrm{tr}\mathbf{WN})J(\mathbf{A}),$$

$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-2}(\mathbf{Wn})\cdot\mathring{\mathbf{A}}^2\mathbf{n} - (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$(\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathbf{Wn}, \mathbf{u}\cdot\mathbf{W}^2\mathbf{u}, (\mathrm{tr}\mathbf{WN})\beta_{2m}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{Wn};$$

$$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^2\mathring{\mathbf{u}}, \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathring{\mathbf{A}}\,\mathbf{n};$$

$$\mathrm{tr}\mathbf{W\Omega H}; \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\mathring{\mathbf{C}}; \mathrm{tr}\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{W}, (\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}],$$
$$(\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{B}}\,\mathbf{n}, \mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n}];$$

$$\mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{W\Omega}, |\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})[\mathbf{n}, \mathbf{Wn}, \mathring{\mathbf{A}}\,\mathbf{Wn}] + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathbf{\Omega n}, \mathring{\mathbf{A}}\,\mathbf{\Omega n}];$$

$$\mathbf{u}\cdot\mathbf{Wv}, \mathbf{u}\cdot\mathbf{W}^2\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathbf{Wn}) + (\mathbf{v}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{Wn}),$$
$$|\mathbf{u}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{v}})\cdot\mathbf{Wu} + |\mathbf{v}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{Wv};$$

$$\mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{v}, \mathbf{u}\cdot\mathring{\mathbf{A}}^2\mathbf{v}, (\mathbf{u}\cdot\mathbf{n})\,\mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}) + (\mathbf{v}\cdot\mathbf{n})\,\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}),$$

$$\mathbf{u}\cdot(\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}) + \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}})\,\mathring{\mathbf{A}})\mathbf{v}, \mathbf{u}\cdot(\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{v}}) + \mathbf{\Phi}_{2m-2}(\mathring{\mathbf{v}})\,\mathring{\mathbf{A}})\mathbf{v},$$

$$|\mathbf{u}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{v}})\cdot\mathring{\mathbf{A}}\,\mathbf{u} + |\mathbf{v}|^{2m-2}\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathring{\mathbf{A}}\,\mathbf{v};$$

$$\mathbf{u}\cdot\mathbf{W\Omega u}, \mathbf{u}\cdot\mathbf{W\Omega}^2\mathbf{u}, \mathbf{u}\cdot\mathbf{W}^2\mathbf{\Omega u}; \mathbf{u}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{u}; \mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{Wu}, (\mathrm{tr}\mathbf{WN})[\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{A}}\mathring{\mathbf{u}}],$$

$$\overset{\circ}{\mathbf{u}} \cdot \mathbf{W} \overset{\circ}{\mathbf{A}} \overset{\circ}{\mathbf{u}}, (\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}} \mathbf{n}) \overset{\circ}{\mathbf{u}} \cdot \mathbf{W} \overset{\circ}{\mathbf{A}} \mathbf{n};$$

$$\mathbf{u} \cdot (\mathbf{W\Omega} - \mathbf{\Omega W})\mathbf{v}; \mathbf{u} \cdot (\mathbf{W} \overset{\circ}{\mathbf{A}} - \overset{\circ}{\mathbf{A}} \mathbf{W})\mathbf{v}; \mathbf{u} \cdot (\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}})\mathbf{v};$$

and

$$V_{2m}(\mathbf{u}); \mathbf{Wu}, \mathbf{W}^2\mathbf{u}, (\mathbf{u} \cdot \mathbf{n})\eta_{2m-1}(\mathbf{Wn}) + (\overset{\circ}{\mathbf{u}} \cdot \eta_{2m-1}(\mathbf{Wn}))\mathbf{n};$$

$$\overset{\circ}{\mathbf{A}} \mathbf{u}, \overset{\circ}{\mathbf{A}}{}^2\mathbf{u}, (\mathbf{u} \cdot \mathbf{n})\eta_{2m-1}(\overset{\circ}{\mathbf{A}} \mathbf{n}), (\overset{\circ}{\mathbf{A}} \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}) + \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}) \overset{\circ}{\mathbf{A}})\mathbf{u};$$

$$(\mathbf{W\Omega} - \mathbf{\Omega W})\mathbf{u}; (\mathbf{W} \overset{\circ}{\mathbf{A}} + \mathbf{W} \overset{\circ}{\mathbf{A}})\mathbf{u}; (\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}})\mathbf{u};$$

and

$$\mathrm{Skw}_{2m}(\mathbf{u}), \mathrm{Skw}_{2m}(\mathbf{W}), \mathrm{Skw}_{2m}(\mathbf{A});$$

$$\mathbf{u} \wedge \mathbf{v}, |\mathbf{u}|^{2m-2}\mathbf{u} \wedge \eta_{2m-1}(\overset{\circ}{\mathbf{v}}) + |\mathbf{v}|^{2m-2}\mathbf{v} \wedge \eta_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega} - \mathbf{\Omega W}; \overset{\circ}{\mathbf{A}} \mathbf{W} + \mathbf{W} \overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}} \mathbf{W}^2 - \mathbf{W}^2 \overset{\circ}{\mathbf{A}};$$

$$\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} - \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}}, \overset{\circ}{\mathbf{A}} \mathbf{n} \wedge \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}} \mathbf{n}, \overset{\circ}{\mathbf{B}} \mathbf{n} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} \mathbf{n};$$

$$\mathbf{u} \wedge \mathbf{Wu}, \mathbf{u} \wedge \mathbf{W}^2\mathbf{u}; \mathbf{u} \wedge \overset{\circ}{\mathbf{A}} \mathbf{u}, \overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{u}}, (\overset{\circ}{\mathbf{u}} \cdot \overset{\circ}{\mathbf{A}} \mathbf{n}) \overset{\circ}{\mathbf{u}} \wedge \overset{\circ}{\mathbf{A}} \mathbf{n};$$

$$\mathbf{u} \wedge \mathbf{Wv} + \mathbf{v} \wedge \mathbf{Wu}; \mathbf{u} \wedge \overset{\circ}{\mathbf{A}} \mathbf{v} + \mathbf{v} \wedge \overset{\circ}{\mathbf{A}} \mathbf{u};$$

and

$$\mathrm{Sym}_{2m}(\mathbf{u}), \mathrm{Sym}_{2m}(\mathbf{W}), \mathrm{Sym}_{2m}(\mathbf{A});$$

$$\mathbf{u} \vee \mathbf{v}, |\mathbf{u}|^{2m-2}\mathbf{u} \vee \eta_{2m-1}(\overset{\circ}{\mathbf{v}}) + |\mathbf{v}|^{2m-2}\mathbf{v} \vee \eta_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega} + \mathbf{\Omega W},$$

$$|\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn} \vee \mathbf{NWn} + |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n} \vee \mathbf{N\Omega n};$$

$$\overset{\circ}{\mathbf{A}} \mathbf{W} - \mathbf{W} \overset{\circ}{\mathbf{A}}, (\mathrm{tr}\mathbf{WN}) \overset{\circ}{\mathbf{A}} \mathbf{n} \vee \mathbf{N} \overset{\circ}{\mathbf{A}} \mathbf{n}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}} + \overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$\mathbf{u} \vee \mathbf{Wu}, (\mathrm{tr}\mathbf{WN}) \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \overset{\circ}{\mathbf{u}});$$

$$\mathbf{u} \vee \overset{\circ}{\mathbf{A}} \mathbf{u}; \mathbf{u} \vee \mathbf{Wv} + \mathbf{v} \vee \mathbf{Wu}; \mathbf{u} \vee \overset{\circ}{\mathbf{A}} \mathbf{v} + \mathbf{v} \vee \overset{\circ}{\mathbf{A}} \mathbf{u};$$

where $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_i, \mathbf{u}_j)$, $(\mathbf{W}, \mathbf{\Omega}, \mathbf{H}) = (\mathbf{W}_\sigma, \mathbf{W}_\tau, \mathbf{W}_\theta)$, $(\mathbf{A}, \mathbf{B}, \mathbf{C}) = (\mathbf{A}_L, \mathbf{A}_M, \mathbf{A}_N)$, $j > i = 1, \cdots, a$, $\theta > \tau > \sigma = 1, \cdots, b$, $N > M > L = 1, \cdots, c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1, \cdots, \mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1, \cdots, \mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1, \cdots, \mathbf{A}_c$ *under the group* D_{2mh} *for each* $m \geqslant 2$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* D_{2mh}.

In the above theorem, the nine invariants of two or three symmetric tensors are quoted from the established results (see Theorem 1 in XIAO, BRUHNS and MEYERS [53]).

5. Crystal and quasicrystal classes D_{2m} for $m \geqslant 2$

The classes at issue take forms

(5.1) $$D_{2m}(\mathbf{n},\mathbf{e}) = D_{2mh}(\mathbf{n},\mathbf{e}) \cap \mathrm{Orth}^+ = \{\mathbf{R}_{\mathbf{n}}^{k\pi/m}, \mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m}\mathbf{e},\ k = 1, \cdots, 2m\}.$$

They include the crystal classes D_4 and D_6 as particular cases when $m = 2, 3$.

Let $I^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$, $\mathrm{Skw}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ and $\mathrm{Sym}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ be, respectively, an irreducible functional basis and irreducible generating sets for scalar-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of $(a + b)$ skewsymmetric tensor variables and c symmetric tensor variables under a centrosymmetrical orthogonal subgroup g containing the central inversion $-\mathbf{I}$. Then, according to Theorems 2.1-2.2 in XIAO [43], the four sets

$$I^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

$$\mathbf{E} : \mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

$$\mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

$$\mathrm{Sym}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),$$

supply, respectively, an irreducible functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of a vector variables, b skewsymmetric tensor variables and c symmetric tensor variables under the rotation subgroup of g, i.e. $g \cap \mathrm{Orth}^+$. Here, the second set above is obtained by forming the double dot product between each skewsymmetric tensor generator and the third order Levi-Civita tensor $\mathbf{E}$.

From the above facts and Theorem 1, we obtain the following result.

THEOREM 2. *The four sets given by*

$$I_{2m}(\mathbf{u}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}}); I_{2m}(\mathbf{W}); I_{2m}(\mathbf{A}); I_{2m}(\mathbf{W}, \mathbf{\Omega}, \mathbf{H}, \mathbf{A}, \mathbf{B}, \mathbf{C});$$

$$\mathbf{u}\cdot\mathbf{v}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{v}}), \mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{u}}), [\mathbf{u},\mathbf{v},\eta_{2m-1}(\mathring{\mathbf{v}})], [\mathbf{v},\mathbf{u},\eta_{2m-1}(\mathring{\mathbf{u}})]; \quad [\mathbf{u},\mathbf{v},\mathbf{r}];$$

$$\mathrm{tr}\mathbf{W}(\mathbf{Eu}), [\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathbf{Wn})], [\mathbf{n},\mathbf{Wn},\eta_{2m-1}(\mathring{\mathbf{u}})],$$

$$\eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{Wu}, \mathrm{tr}(\mathbf{Eu})\mathbf{W}^2\mathbf{\Phi}_{2m-2}(\mathbf{Wn});$$

$$[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{A}}\,\mathbf{n}], \mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{u}, \mathrm{tr}\mathring{\mathbf{A}}\,\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), \mathrm{tr}\mathring{\mathbf{A}}\,(\mathbf{Eu})\mathbf{\Phi}_{2m-2}(\mathring{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A})),$$

$$[\mathbf{n},\mathring{\mathbf{u}},\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n})] - |\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-4}(\mathbf{u}\cdot\mathbf{n})J(\mathbf{A}),$$

$$|\mathring{\mathbf{A}}\,\mathbf{n}|^{2m-2}[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{A}}^{\,2}\mathbf{n}] - (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$\mathbf{u}\cdot\mathbf{Wv}; \mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{v}, |\mathbf{v}\cdot\mathbf{n}|(\mathbf{v}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{A}}\mathring{\mathbf{u}}] + |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{v}},\mathring{\mathbf{A}}\mathring{\mathbf{v}}];$$

$$\mathrm{tr}(\mathbf{Eu})\mathbf{W}\mathbf{\Omega}; \mathrm{tr}(\mathbf{Eu})\,\mathring{\mathbf{A}}\mathring{\mathbf{B}},$$

$$(\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{A}}\,\mathbf{n},\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}], (\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathring{\mathbf{B}}\,\mathbf{n},\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n}];$$

$$\mathrm{tr}(\mathbf{Eu})\mathbf{W}\,\mathring{\mathbf{A}}, |\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})[\mathbf{n},\mathbf{Wn},\mathring{\mathbf{A}}\,\mathbf{Wn}] - |\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})[\mathbf{n},\mathring{\mathbf{u}},\mathring{\mathbf{A}}\mathring{\mathbf{u}}];$$

and

$$\mathbf{u}, \eta_{2m-1}(\mathring{\mathbf{u}}), \mathbf{u}\times\eta_{2m-1}(\mathring{\mathbf{u}}); \mathbf{E}:\mathrm{Skw}_{2m}(\mathbf{W}); \mathbf{E}:Skw_{2m}(\mathbf{A});$$

$$\mathbf{u}\times\mathbf{v}; \mathbf{Wu}; \mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{u}\times\mathring{\mathbf{A}}\,\mathbf{u}; \mathbf{E}:(\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W});$$

$$\mathring{\mathbf{A}}\,(\mathbf{E}:\mathbf{W}), \mathbf{E}:(\mathring{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\mathring{\mathbf{A}});$$

$$\mathbf{E}:(\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}), \mathring{\mathbf{A}}\,\mathbf{n}\times\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{B}}\,\mathbf{n}\times\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n};$$

and

$$\mathbf{Eu}, \mathbf{E}\eta_{2m-1}(\mathring{\mathbf{u}}), \mathbf{u}\wedge\eta_{2m-1}(\mathring{\mathbf{u}}); \mathrm{Skw}_{2m}(\mathbf{W}); \mathrm{Skw}_{2m}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}; \mathbf{E}(\mathbf{Wu}); \mathbf{E}(\mathring{\mathbf{A}}\,\mathbf{u}), \mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{u};$$

$$\mathbf{W}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{W}; \mathring{\mathbf{A}}\,\mathbf{W} + \mathbf{W}\,\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}^2 - \mathbf{W}^2\,\mathring{\mathbf{A}};$$

$$\mathring{\mathbf{A}}\mathring{\mathbf{B}} - \mathring{\mathbf{B}}\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{B}}\,\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n};$$

and

$$\mathrm{Sym}_{2m}(\mathbf{Eu}); \mathrm{Sym}_{2m}(\mathbf{W}); \mathrm{Sym}_{2m}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v},|\mathbf{v}\cdot\mathbf{n}|(\mathbf{v}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}})+|\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{v}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$$

$$\mathbf{W\Omega}+\mathbf{\Omega W},|\mathrm{tr}\mathbf{\Omega N}|(\mathrm{tr}\mathbf{\Omega N})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\mathbf{\Omega n}\vee\mathbf{N\Omega n};$$

$$\overset{\circ}{\mathbf{A}}\mathbf{W}-\mathbf{W}\overset{\circ}{\mathbf{A}},(\mathrm{tr}\mathbf{WN})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\mathbf{N}\overset{\circ}{\mathbf{A}}\mathbf{n};\overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}+\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$(\mathbf{Eu})\mathbf{W}+\mathbf{W}(\mathbf{Eu}),|\mathbf{u}\cdot\mathbf{n}|(\mathbf{u}\cdot\mathbf{n})\mathbf{Wn}\vee\mathbf{NWn}$$

$$+|\mathrm{tr}\mathbf{WN}|(\mathrm{tr}\mathbf{WN})\overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$$

$$\overset{\circ}{\mathbf{A}}(\mathbf{Eu})-(\mathbf{Eu})\overset{\circ}{\mathbf{A}},(\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\mathbf{N}\overset{\circ}{\mathbf{A}}\mathbf{n};$$

where $(\mathbf{u},\mathbf{v},\mathbf{r})=(\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k)$, $(\mathbf{W},\mathbf{\Omega},\mathbf{H})=(\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C})=(\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $k>j>i=1,\cdots,a$, $\theta>\tau>\sigma=1,\cdots,b$, $N>M>L=1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* D_{2m} *for each* $m\geqslant 2$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* D_{2m}.

Here and henceforth, $I_{2m}(\mathbf{W},\mathbf{\Omega},\mathbf{H},\mathbf{A},\mathbf{B},\mathbf{C})$ is used to represent the invariants of two or three second order tensors given in THEOREM 1.

6. Crystal and quasicrystal classes C_{2mv} for $m\geqslant 2$

The classes at issue are of the form

$$C_{2mv}(\mathbf{n},\mathbf{e})=\{\mathbf{R}_{\mathbf{n}}^{k\pi/m},-\mathbf{R}_{\mathbf{l}_k}^{\pi}\mid\mathbf{l}_k=\mathbf{R}_{\mathbf{n}}^{k\pi/2m}\mathbf{e},\ k=1,\cdots,2m\}. \tag{6.1}$$

They include the crystal classes C_{4v} and C_{6v} as particular cases when $m=2$, 3.

For anisotropic functions under any subgroup $g\subseteq C_{\infty v}$, the general cases involving any number of vector variables and tensor variables may be reduced to the cases involving not more than two variables (see THEOREM 2.2 in XIAO [52]). As a result, the third step in the procedure outlined in Sec. 3 can be omitted. Further reduction is possible. Let X_0 represent any of the five sets of variables, $\mathbf{W}$, $\mathbf{A}$, $(\mathbf{W},\mathbf{\Omega})$, $(\mathbf{W},\mathbf{A})$ and $(\mathbf{A},\mathbf{B})$. Then each scalar-valued or tensor-valued anisotropic function of X_0 under the group $C_{2mv}(\mathbf{n},\mathbf{e})$ is a scalar-valued or tensor-valued anisotropic function of X_0 under the larger group $D_{2mh}(\mathbf{n},\mathbf{e})$ $(\supset C_{2mv}(\mathbf{n},\mathbf{e}))$. Thus, in the general results for the group C_{2mv} (THEOREM 3 below), we can directly cite the invariants and the tensor generators depending on skewsymmetric and/or symmetric tensor variables in THEOREM 1. Moreover, let Y_0 be any set of a single variable or two variables. Then each anisotropic function of Y_0 under the group C_{2mv} is an anisotropic function of $(Y_0, \mathbf{n})$ under the larger group D_{2mh} ($\supset C_{2mv}$). Thus, for all sets of variables, $(\mathbf{u})$, $(\mathbf{u}, \mathbf{W})$, $(\mathbf{u}, \mathbf{A})$ and $(\mathbf{u}, \mathbf{v})$, the desired results for the group C_{2mv} can be obtained by setting $\mathbf{v} = \mathbf{n}$ in the tables given in Sec. 4 (**iv**), (**x**), (**xi**) and setting $\mathbf{w} = \mathbf{n}$ in the table given in Sec. 4 (**xii**), respectively. In addition, for each of the sets $\mathbf{W}$, $\mathbf{A}$, $(\mathbf{W}, \mathbf{\Omega})$, $(\mathbf{W}, \mathbf{A})$ and $(\mathbf{A}, \mathbf{B})$, the desired vector generating set under the group C_{2mv} and the invariants from the scalar products related to this generating set can be derived by taking $\mathbf{u} = \mathbf{n}$ in the corresponding results in the tables given in Sec. 4 (**viii**), (**ix**), (**xiii**), (**xiv**), respectively. Combining these facts, we arrive at the general result for the group C_{2mv} as follows.

THEOREM 3. *The four sets given by*

$$\mathbf{u}\cdot\mathbf{n}, |\,\mathring{\mathbf{u}}\,|^2, \alpha_{2m}(\mathring{\mathbf{u}}); I_{2m}(\mathbf{W}); I_{2m}(\mathbf{A}); I_{2m}(\mathbf{W}, \mathbf{\Omega}, \mathbf{H}, \mathbf{A}, \mathbf{B}, \mathbf{C});$$

$$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{v}}), \mathring{\mathbf{v}}\cdot\eta_{2m-1}(\mathring{\mathbf{u}});$$

$$\mathring{\mathbf{u}}\cdot\mathbf{W}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathbf{W}^2\mathbf{n}, (\mathrm{tr}\mathbf{W}\mathbf{N})\beta_{2m}(\mathring{\mathbf{u}}), \eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathbf{W}\mathbf{n}, \mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathbf{W}\mathbf{n});$$

$$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}^{\,2}\mathbf{n}, \mathring{\mathbf{u}}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{u}}, \mathrm{tr}\mathring{\mathbf{A}}\,\Phi_{2m-2}(\mathring{\mathbf{u}}), \eta_{2m-1}(\mathring{\mathbf{u}})\cdot\mathring{\mathbf{A}}\,\mathbf{n},$$

$$\mathring{\mathbf{u}}\cdot\eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n});$$

$$\mathbf{u}\cdot\mathbf{W}\mathbf{v}; \mathbf{u}\cdot\mathring{\mathbf{A}}\,\mathbf{v}; \mathbf{u}\cdot(\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W})\mathbf{n}; \mathbf{u}\cdot(\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{n}, \mathbf{u}\cdot\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{u};$$

$$\mathbf{u}\cdot(\mathbf{W}\mathring{\mathbf{A}}-\mathring{\mathbf{A}}\mathbf{W})\mathbf{n}, \mathbf{u}\cdot\mathbf{W}\mathring{\mathbf{A}}\,\mathbf{u};$$

and

$$\mathbf{n}, \mathring{\mathbf{u}}, \eta_{2m-1}(\mathring{\mathbf{u}}); \mathbf{W}\mathbf{n}, \mathbf{W}^2\mathbf{n}, \eta_{2m-1}(\mathbf{W}\mathbf{n});$$

$$\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{A}}^{\,2}\mathbf{n}, \eta_{2m-1}(\mathring{\mathbf{A}}\,\mathbf{n}); \mathbf{W}\,\mathring{\mathbf{u}}; \mathring{\mathbf{A}}\mathring{\mathbf{u}};$$

$$(\mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W})\mathbf{n}; (\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}})\mathbf{n}; (\mathbf{W}\mathring{\mathbf{A}}-\mathring{\mathbf{A}}\mathbf{W})\mathbf{n};$$

and

$$\mathbf{n}\wedge\mathring{\mathbf{u}}, \mathbf{n}\wedge\eta_{2m-1}(\mathring{\mathbf{u}}), \beta_{2m}(\mathring{\mathbf{u}})\mathbf{N}; \mathrm{Skw}_{2m}(\mathbf{W}); \mathrm{Skw}_{2m}(\mathbf{A});$$

$$\mathbf{u}\wedge\mathbf{v}; \mathbf{W}\mathbf{\Omega}-\mathbf{\Omega}\mathbf{W}; \mathring{\mathbf{A}}\,\mathbf{W}+\mathbf{W}\,\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{W}^2-\mathbf{W}^2\,\mathring{\mathbf{A}};$$

$$\mathring{\mathbf{A}}\mathring{\mathbf{B}}-\mathring{\mathbf{B}}\mathring{\mathbf{A}}, \mathring{\mathbf{A}}\,\mathbf{n}\wedge\mathring{\mathbf{B}}\mathring{\mathbf{A}}\,\mathbf{n}, \mathring{\mathbf{B}}\,\mathbf{n}\wedge\mathring{\mathbf{A}}\mathring{\mathbf{B}}\,\mathbf{n};$$

$$\mathbf{n}\wedge\mathbf{W}\mathbf{u}+\mathbf{u}\wedge\mathbf{W}\mathbf{n}; \mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{u}, \mathbf{n}\wedge\mathring{\mathbf{A}}\,\mathbf{u}+\mathbf{u}\wedge\mathring{\mathbf{A}}\,\mathbf{n};$$

and

$$\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{\Phi}_{2m-2}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\overset{\circ}{\mathbf{u}});$$

$$\mathrm{Sym}_{2m}(\mathbf{W}); \mathrm{Sym}_{2m}(\mathbf{A});$$

$$\mathbf{u}\vee\mathbf{v}; \mathbf{W}\boldsymbol{\Omega}+\boldsymbol{\Omega}\mathbf{W}, |\mathrm{tr}\boldsymbol{\Omega}\mathbf{N}|(\mathrm{tr}\boldsymbol{\Omega}\mathbf{N})\mathbf{W}\mathbf{n}\vee\mathbf{N}\mathbf{W}\mathbf{n}$$

$$+|\mathrm{tr}\mathbf{W}\mathbf{N}|(\mathrm{tr}\mathbf{W}\mathbf{N})\boldsymbol{\Omega}\mathbf{n}\vee\mathbf{N}\boldsymbol{\Omega}\mathbf{n};$$

$$\overset{\circ}{\mathbf{A}}\mathbf{W}-\mathbf{W}\overset{\circ}{\mathbf{A}}, (\mathrm{tr}\mathbf{W}\mathbf{N})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee\mathbf{N}\overset{\circ}{\mathbf{A}}\mathbf{n}; \overset{\circ}{\mathbf{A}}\overset{\circ}{\mathbf{B}}+\overset{\circ}{\mathbf{B}}\overset{\circ}{\mathbf{A}};$$

$$\mathbf{u}\vee\mathbf{W}\mathbf{u}, \mathbf{n}\vee\mathbf{W}\mathbf{u}+\mathbf{u}\vee\mathbf{W}\mathbf{n}; \mathbf{u}\vee\overset{\circ}{\mathbf{A}}\mathbf{u}, \mathbf{n}\vee\overset{\circ}{\mathbf{A}}\mathbf{u}+\mathbf{u}\vee\overset{\circ}{\mathbf{A}}\mathbf{n};$$

where $(\mathbf{u},\mathbf{v})=(\mathbf{u}_i,\mathbf{u}_j)$, $(\mathbf{W},\boldsymbol{\Omega},\mathbf{H})=(\mathbf{W}_\sigma,\mathbf{W}_\tau,\mathbf{W}_\theta)$, $(\mathbf{A},\mathbf{B},\mathbf{C})=(\mathbf{A}_L,\mathbf{A}_M,\mathbf{A}_N)$, $j>i=1,\cdots,a$, $\theta>\tau>\sigma=1,\cdots,b$, $N>M>L=1,\cdots,c$, *supply a functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the a vectors* $\mathbf{u}_1,\cdots,\mathbf{u}_a$, *the b skewsymmetric tensors* $\mathbf{W}_1,\cdots,\mathbf{W}_b$ *and the c symmetric tensors* $\mathbf{A}_1,\cdots,\mathbf{A}_c$ *under the group* C_{2mv} *for each* $m\geqslant 2$. *In the presented result,* $\mathbf{n}$ *and* $\mathbf{e}$ *are two orthonormal vectors in the directions of the principal axis and a two-fold axis of the group* C_{2mv}.

Acknowledgement

This research was completed under the finiancial support from Deutsche Forschungsgemeinschaft (DFG) (Contract No.: Br 580/26-1) and Alexander von Humboldt-Stiftung. This support is gratefully acknowledged. Moreover, the authors are very grateful to the reviewer for the careful examination and the constructive and helpful comments on the early version of this work.

References

1. J.E. Adkins, *Dynamic properties of resilient materials: constitutive equations,* Phil. Trans. Roy. Soc., A**250**, 519–534, 1958.
2. J.E. Adkins, *Symmetry relations for orthotropic and transversely isotropic materials,* Arch. Rat. Mech. Anal., **4**, 193–213, 1960.
3. J. Betten, *Recent advances in applications of tensor functions in solid mechanics,* Advances in Mech., **14**(1), 79–109, 1991.
4. J. Betten, *Kontinuumsmechanik: Elasto-, Plasto- und Kriechmechanik,* Springer, Berlin 1993.
5. B. Bischhoff-Beiermann and O.T. Bruhns, *A physically motivated set of invariants and tensor generators in the case of transverse isotropy,* Int. J. Eng. Sci., **32**, 1531–1552, 1994.

6. J.P. BOEHLER, *On irreducible representations for isotropic scalar functions*, Zeits. Angew. Math. Mech., **57**, 323–327, 1977.

7. J.P. BOEHLER, *Lois de comportement anisotrope des milieux continus*, J. de Mechanique, **17**, 153–190, 1978.

8. J.P. BOEHLER, *A simple derivation of non-polynomial representations for constitutive equations in some cases of anisotropy*, Zeits. Angew. Math. Mech., **59**, 157–167, 1979.

9. J.P. BOEHLER [Ed.], *Applications of tensor functions in solid mechanics*, CISM Courses and Lectures no. 292, Springer-Verlag, New York, Wien, etc. 1987.

10. J.P. BOEHLER and J. RACLIN, *Representations irréducibles des functions tensorielles anisotropes non-polynomiales de deux tenseurs symétriques*, Arch. Mech., **29**, 263–274, 1977.

11. O.T. BRUHNS, H. XIAO and A. MEYERS, *On representations for yield functions of crystals, quasicrystals and transversely isotropic solids*, Eur. J. Mech. A/Solids, **18**, 47–67, 1999.

12. A.C. ERINGEN and G.A. MAUGIN, *Electrodynamics of continua I. Foundations and solid media*. Springer-Verlag, Berlin, New York 1989.

13. S. JEMIOŁA and J.J. TELEGA, *Non-polynomial representations of orthotropic tensor functions in the 3-dimensional case: an alternative approach*, Arch. Mech., **49**, 233–239, 1997.

14. E. KIRAL and A.C. ERINGEN, *Constitutive equations of electromagnetic-elastic crystals*, Springer-Verlag, Berlin, New York, etc. 1990.

15. E. KIRAL and G.F. SMITH, *On the constitutive relations for anisotropic materials: triclinic, monoclinic, rhombic, tetragonal and hexagonal crystal systems*, Int. J. Eng. Sci., **12**, 471–490, 1974.

16. I.S. LIU, *On representations of anisotropic invariants*, Int. J. Eng. Sci., **20**, 1099–1109, 1982.

17. V.V. LOKHIN and L.I. SEDOV, *Nonlinear tensor functions of several tensor arguments*, Prikl. Mat. Mekh., **29**, 393–417, 1963.

18. J.F. NYE, *Physical properties of crystals*, Clarendon, London, New York 1957.

19. S. PENNISI and M. TROVATO, *On irreducibility of Professor G.F. Smith's representations for isotropic functions*, Int. J. Eng. Sci., **25**, 1059–1065, 1987.

20. A.C. PIPKIN and R.S. RIVLIN, *The formulation of constitutive equations in continuum physics I*, Arch. Rat. Mech. Anal., **4**, 129–144, 1959.

21. A.C. PIPKIN and A.S. WINEMAN, *Material symmetry restrictions on nonpolynomial constitutive equations*, Arch. Rat. Mech. Anal., **12**, 420–426, 1963.

22. J. RYCHLEWSKI, *Symmetry of causes and effects*, SIAMM Research Report no. 8706, Shanghai Institute of Applied Mathematics and Mechanics, Shanghai 1987; also: Wydawnictwo Naukowe PWN, Warsaw 1991.

23. J. RYCHLEWSKI and J.M. ZHANG, *On representations of tensor functions: A review*, Advances in Mech., **14**(4), 75–94, 1990.

24. G.F. SMITH, *On isotropic functions of symmetric tensors, skewsymmetric tensors and vectors*, Int. J. Eng. Sci., **9**, 899–916, 1971.

25. G.F. SMITH, *On transversely isotropic functions of vectors, symmetric second order tensors and skewsymmetric second order tensors*, Q. Appl. Math., **39**, 509–516, 1982.

26. G.F. SMITH, *Constitutive equations for anisotropic and isotropic materials*, Elsevier, New York 1994.

27. G.F. SMITH and E. KIRAL, *Integrity bases for N symmetric second order tensors – the crystal classes*, Rend. Circ. Mat. Palermo. II, Ser. **18**, 5–22, 1969.

28. G.F. Smith and R.S. Rivlin, *The strain-energy functions for anisotropic elastic materials*, Trans. Am. Math. Soc., **88**, 175–193, 1958.

29. G.F. Smith and R.S. Rivlin, *Integrity bases for vectors – the crystal classes*, Arch. Rat. Mech. Anal., **15**, 169–221, 1964.

30. G.F. Smith, M.M. Smith and R.S. Rivlin, *Integrity bases for a symmetric tensor and a vector – the crystal classes*, Arch. Rat. Mech. Anal., **12**, 93–133, 1963.

31. A.J.M. Spencer, *The invariants of six symmetric* 3×3 *matrices*, Arch. Rat. Mech. Anal., **17**, 64–77, 1961.

32. A.J.M. Spencer, *Isotropic integrity bases for vectors and second order tensors, Part II*, Arch. Rat. Mech. Anal., **18**, 51–82, 1965.

33. A.J.M. Spencer, *Theory of invariants*, in *Continuum Physics*, Vol. I, A.C. Eringen [Ed.], Academic Press, 239–353, New York 1971.

34. A.J.M. Spencer and R.S. Rivlin, *The theory of matrix polynomials and its applications to the mechanics of isotropic continua*, Arch. Mech. Rat. Anal., **2**, 309–336, 1959.

35. A.J.M. Spencer and R.S. Rivlin, *Finite integrity bases for five or fewer symmetric* 3×3 *matrices*, Arch. Rat. Mech. Anal., **2**, 435–446, 1959.

36. A.J.M. Spencer and R.S. Rivlin, *Further results in the theory of matrix polynomials*, Arch. Rat. Mech. Anal., **4**, 214–230, 1960.

37. A.J.M. Spencer and R.S. Rivlin, *Isotropic integrity bases for vectors and second order tensors, Part I*, Arch. Rat. Mech. Anal., **9**, 45–63, 1962.

38. C. Truesdell and W. Noll, *The nonlinear field theories of mechanics*, [In] *Handbuch der Physik*, Vol. III/3, S. Flügge [Ed.], Springer-Verlag, Berlin, New York, etc. 1965.

39. B.L. Vainstein, *Modern crystallography 1: Fundamentals of crystals*, Springer-Verlag, Berlin, New York, etc. 1994.

40. C.C. Wang, *A new representation theorem for isotropic functions, Part I and II*, Arch. Rat. Mech. Anal., **36**, 166–223, 1970; *Corrigendum*, ibid, **43**, 392–395, 1971.

41. A.S. Wineman and A.C. Pipkin, *Material symmetry restrictions on constitutive equations*, Arch. Rat. Mech. Anal., **17**, 184–214, 1964.

42. H. Xiao, *General irreducible representations for constitutive equations of elastic crystals and transversely isotropic elastic solids*, J. Elasticity, **39**, 47–73, 1995.

43. H. Xiao, *On isotropic extension of anisotropic tensor functions*, Zeits. Angew. Math. Mech., **76**, 205–214, 1996.

44. H. Xiao, *Two general representation theorems for arbitrary-order tensor-valued isotropic and anisotropic tensor functions of vectors and second order tensors*, Zeits. Angew. Math. Mech., **76**, 151–162, 1996.

45. H. Xiao, *On anisotropic scalar functions of a single symmetric tensor*, Proc. Roy. Soc. London, A**452**, 1545–1561, 1996.

46. H. Xiao, *On minimal representations for constitutive equations of anisotropic elastic materials*, J. Elasticity, **45**, 13–32, 1996.

47. H. Xiao, *A unified theory of representations for scalar-, vector- and 2nd order tensor-valued anisotropic functions of vectors and 2nd order tensors*, Arch. Mech., **50**, 275–313, 1997.

48. H. Xiao, *On constitutive equations of Cauchy elastic solids: All kinds of crystals and quasicrystals*, J. Elasticity, **48**, 241–283, 1997.

49. H. XIAO, *On anisotropic invariants of a symmetric tensor: crystal classes, quasicrystal classes and others*, Proc. Roy. Soc. London, A**454**, 1217–1240, 1998.

50. H. XIAO, *On anisotropic functions of vectors and second order tensors: All subgroups of the transverse isotropy group* $C_{\infty h}$, Arch. Mech., **50**, 281–319, 1998.

51. H. XIAO, *On scalar-, vector- and 2nd order tensor-valued anisotropic functions of vectors and 2nd order tensors relative to all kinds of subgroups of the transverse isotropy group* $C_{\infty h}$, Phil. Trans. Roy. Soc. London, A**356**, 3087–3122, 1998.

52. H. XIAO, *Further results on general representation theorems for arbitrary-order tensor-valued isotropic and anisotropic tensor functions of vectors and second order tensors*, Zeits. Angew. Math. Mech., 1999 (to appear).

53. H. XIAO, O.T. BRUHNS amd A. MEYERS, *On anisotropic invariants of N symmetric 2nd order tensors: All crystal and quasicrystal classes as subgroups of the cylindrical group* $D_{\infty h}$, Proc. Roc. Soc. London, ser. A, **455**, 1993–2020, 1999.

54. J.M. ZHANG and J. RYCHLEWSKI, *On structural tensors for anisotropic solids*, Arch. Mech., **42**, 267–277, 1990.

55. Q.S. ZHENG, *On transversely isotropic, orthotropic and relatively isotropic functions of symmetric tensors, skewsymmetric tensors and vectors, Parts I-V*, Int. J. Engng. Sci., **31**, 1399–1453, 1993.

56. Q.S. ZHENG, *Two-dimensional tensor function representations for all kinds of material symmetry*, Proc. Roy. Soc. London, A**443**, 127–138, 1993.

57. Q.S. ZHENG, *Theory of representations for tensor functions: A unified invariant approach to constitutive equations*, Appl. Mech. Rev., **47**, 545–587, 1994.

58. Q.S. ZHENG and J.P. BOEHLER, *Tensor function representations as applied to constitutive laws for clinotropic materials,* Acta Mechanica Sinica, **10**, 336–348, 1994.

59. Q.S. ZHENG and A.J.M. SPENCER, *Tensors which characterize anisotropies*, Int. J. Eng. Sci., **31**, 679–693, 1993.

Received November 25, 1998; revised version August 5, 1999.