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# Multimode wave scattering problems in layered dissipative solids

G. CAVIGLIA (1) and A. MORRO (2)

(1) *Department of Mathematics*
*16146 Genova, Italy*

(2) *University of Genova, DIBE,*
*16145 Genova, Italy*
*e-mail: morro@dibe.unige.it*

A GENERAL SCHEME within the frequency domain is elaborated for the scattering of obliquely-incident waves at a stratified, anisotropic, viscoelastic solid. Existence and uniqueness of the asymptotic wave propagator is established. Possible nonexistence of the scattering matrix at specific values of the incident field is then shown. Conditions for nonuniqueness or incompatibility of the direct scattering problem are provided.

## 1. Introduction

SCATTERING BY STRATIFIED viscoelastic media is of interest, e.g., in seismics and nondestructive testing. The linear viscoelastic model allows for the introduction of dispersion and attenuation effects that are of importance in realistic calculations. In this paper, a frequency-domain approach is elaborated for the determination of the scattering matrix, with the specific aim at establishing properties of the transmitted and backscattered wave fields generated by a wave obliquely incident from infinity. The scattering of horizontally-polarized shear waves, in isotropic viscoelastic solids and within the time domain, is investigated in [1] for oblique incidence and in [2] for normal incidence.

The material functions or parameters of the continuously stratified viscoelastic medium vary in one space dimension, $z$ say. To fix ideas, we let $z$ be directed vertically upward. The usual assumption is made that dependence on time and transverse coordinates is through the complex factor $\exp[i(\mathbf{k}_\parallel \cdot \mathbf{x} - \omega t)]$, where $\mathbf{k}_\parallel$ is a (possibly complex-valued) constant horizontal vector, $\omega$ is the (real, constant) frequency, $\mathbf{x}$ and $t$ denote the position vector and time, respectively. The realness of $\mathbf{k}_\parallel$ amounts to considering the Fourier components of the unknown functions. It follows from these conditions that the dynamics of any stratified anisotropic solid may be modelled by a linear system of six first-order ordinary differential equations with complex-valued, $z$-dependent, coefficients in the unknown components of the displacement and traction vectors. Under the rather stringent assumption that the solid is isotropic, the equations of the system decouple and the usual formulations of scattering problems for the one-dimensional Helmholtz and Schrödinger equations is recovered [3 – 7].

We assume that the body is asymptotically homogeneous. Accordingly we express the unknown functions as superpositions, with $z$-dependent coefficients, of the inhomogeneous waves [8] associated with the eigenvectors and the eigenvalues of the governing system at $\pm\infty$. By looking at the differential equations for the pertinent coefficients, we can find a formal expression for the asymptotic wave propagator matrix $\mathbf{T}$ that provides the solution at $+\infty$ in terms of the solution at $-\infty$.

While the wave propagator matrix $\mathbf{T}$ is uniquely defined, existence and uniqueness or nonexistence and nonuniqueness of the solution to the direct scattering problem may occur, depending on the value of $\mathbf{k}_{\parallel}$. To our mind this feature has not been adequately investigated in the literature. Our approach in terms of the asymptotic eigenvectors allows a systematic treatment of the problem and a simple understanding of the conditions for existence and uniqueness of the solution.

Nonuniqueness seems to be related to the phenomenon of mode conversion and shows some analogy with the possible existence of interfacial waves at the common boundary between homogeneous solid half-spaces [9 – 11]. Yet nonuniqueness or nonexistence are not confined to the occurrence of interfacial waves. As an example, in the last section we consider horizontally-polarized waves in an inhomogeneous half-space, bonded to a homogeneous incidence half-space, and determine numerically the matrix $\mathbf{T}$. We then show that conditions on $\mathbf{T}$, such that the solution to the reflection-transmission problem does not exist, are realized by admissible values of the material parameters.

## 2. Time-harmonic waves in anisotropic media

Consider a body, in an unstressed configuration, occupying an unbounded region $\Omega$ which is described by the Cartesian coordinates $(x, y, z) =: \mathbf{x}$. Let $V$ be the translational space associated with the three-dimensional Euclidean point space. Also, denote by $\mathbf{e}_3$ the unit vector of the $z$-axis. Let $\mathbf{u}$ denote the displacement vector with values in $V$; $\mathbf{u}(\mathbf{x}, t)$ maps $\Omega \times \mathbb{R}$ onto $V$ and represents the displacement, at time $t$, of the point labelled by $\mathbf{x}$ [12].

Let $\mathcal{T}$ be the symmetric (Cauchy) stress tensor, $\mathcal{T} : V \to V$, and let the time-dependence be expressed through the common factor $\exp(-i\omega t)$. In the absence of body forces, the evolution of time-harmonic waves is governed by the equation

$$-\rho\omega^2 \mathbf{u} = \nabla \cdot \mathcal{T}. \tag{2.1}$$

The tensor $\mathcal{T}$ depends linearly on the spatial derivatives of $\mathbf{u}$, namely

$$\mathcal{T} = \mathbf{C}(\nabla \otimes \mathbf{u}),$$

where $\mathbf{C}$ is a complex-valued fourth-order tensor. In components,

$$\mathcal{T}_{jh} = C_{jhkl}\partial_k u_l.$$

The standard symmetry properties

$$C_{jhkl} = C_{hjkl} = C_{hjlk} = C_{lkhj}$$

are taken to hold. The tensor $\mathbf{C}$ is real-valued in (linear) elasticity and is strictly complex-valued and dependent on $\omega$ in common dissipative models. For later convenience, for any two vectors $\mathbf{a}$ and $\mathbf{b}$ we let $\mathbf{aCb}$ be a second-order tensor defined by

$$(\mathbf{aCb})_{jk} = a_h C_{hjkl} b_l.$$

We assume that the material properties $\rho$ and $\mathbf{C}$ depend only on the vertical coordinate $z$. Hence we look for solutions of the form

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(z) \exp[i(\mathbf{k}_\parallel \cdot \mathbf{x} - \omega t)],$$

where $\mathbf{k}_\parallel$ is a horizontal, complex-valued, wave vector. Correspondingly the gradient takes the form

$$\nabla = i\mathbf{k}_\parallel + \mathbf{e}_3 \frac{d}{dz}.$$

Let $\mathbf{t} = \mathcal{T}\mathbf{e}_3 = \hat{\mathbf{t}}(z) \exp[i(\mathbf{k}_\parallel \cdot \mathbf{x} - \omega t)]$ be the traction at horizontal planes. Evaluation of $\nabla \otimes \mathbf{u}$ and substitution into the expression for $\mathcal{T}$ shows that the definition of $\hat{\mathbf{t}}$ and the equation of motion (2.1) become

$$\begin{aligned} \hat{\mathbf{t}} &= i(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel)\hat{\mathbf{u}} + (\mathbf{e}_3\mathbf{C}\mathbf{e}_3)\hat{\mathbf{u}}', \\ \rho\omega^2\hat{\mathbf{u}} &= -(\mathbf{k}_\parallel\mathbf{C}\mathbf{k}_\parallel)\hat{\mathbf{u}} + i(\mathbf{k}_\parallel\mathbf{C}\mathbf{e}_3)\hat{\mathbf{u}}' + \hat{\mathbf{t}}', \end{aligned} \tag{2.2}$$

where a prime stands for $d/dz$. As shown in [11], thermodynamics implies that $\mathbf{e}_3\mathbf{C}\mathbf{e}_3$ is invertible. Hence, a comparison allows $\hat{\mathbf{u}}'$ and $\hat{\mathbf{t}}'$ to be expressed in terms of $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ as

$$\hat{\mathbf{u}}' = -i(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel)\hat{\mathbf{u}} + (\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}\hat{\mathbf{t}}, \tag{2.3}$$

$$\begin{aligned} \hat{\mathbf{t}}' = [-\rho\omega^2\mathbf{1} + (\mathbf{k}_\parallel\mathbf{C}\mathbf{k}_\parallel) - (\mathbf{k}_\parallel\mathbf{C}\mathbf{e}_3)(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel)]\hat{\mathbf{u}} \\ -i(\mathbf{k}_\parallel\mathbf{C}\mathbf{e}_3)(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}\hat{\mathbf{t}}. \end{aligned} \tag{2.4}$$

To get a more compact notation, we let $\mathbf{w}$ be the column of the ordered set of components of $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$, i.e. $\mathbf{w} = [\hat{\mathbf{u}}, \hat{\mathbf{t}}]^T$ where the superscript $T$ means transpose, whence

$$[\mathbf{u}, \mathbf{t}]^T = \mathbf{w}(z) \exp[i(\mathbf{k}_\parallel \cdot \mathbf{x} - \omega t)]. \tag{2.5}$$

In accordance with (2.2) and (2.3), let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix},$$

where the four $3 \times 3$ blocks $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are given by

$$\mathbf{A}_1 = -i(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel), \quad \mathbf{A}_2 = (\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1} = \mathbf{A}_2^T, \quad \mathbf{A}_4 = \mathbf{A}_1^T.$$

$$\mathbf{A}_3 = -\rho\omega^2\mathbf{1} + (\mathbf{k}_\parallel\mathbf{C}\mathbf{k}_\parallel) - (\mathbf{k}_\parallel\mathbf{C}\mathbf{e}_3)(\mathbf{e}_3\mathbf{C}\mathbf{e}_3)^{-1}(\mathbf{e}_3\mathbf{C}\mathbf{k}_\parallel) = \mathbf{A}_3^T.$$

The matrix $\mathbf{A}$ is neither symmetric nor Hermitian. Yet, by means of the $6 \times 6$ matrix

$$\mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{1}$ is the $3 \times 3$ identity matrix and $\mathbf{0}$ is the $3 \times 3$ zero matrix, we can write the symmetry condition

$$\mathbf{K}\mathbf{A} = (\mathbf{K}\mathbf{A})^T. \tag{2.6}$$

In terms of $\mathbf{A}$, Eqs. (2.2) and (2.3) can be given the form

$$\mathbf{w}' = \mathbf{A}\mathbf{w}, \tag{2.7}$$

where $\mathbf{w}$ and $\mathbf{A}$ depend only on the space variable $z$ and the vector parameter $\mathbf{k}_\parallel$. This is the sought Stroh-like form of differential equations governing the behaviour of the body [10]. The general solution to an equation of the form (2.6) is sometimes termed as a multimode wave [13].

In electromagnetism, Stroh-like forms are more familiar (cf. [14]) also because Maxwell's equations as such are first-order equations. This is a further motivation for the investigation of (2.6) as a model of wave propagation.

For later purposes, we consider the mechanical model of isotropic bodies where

$$C_{hjkl}(z) = \mu(z)[\delta_{hk}\delta_{jl} + \delta_{hl}\delta_{jk}] + \lambda(z)\delta_{hj}\delta_{kl}. \tag{2.8}$$

If the body is a viscoelastic solid then the tensor (2.8) is complex-valued and $\mu$ and $\lambda$ are parameterized by $\omega \in \mathbb{R}$ in the form

$$\mu = \mu_0 + \int_0^\infty \mu'(\eta)\exp(i\omega\eta)d\eta, \qquad \lambda = \lambda_0 + \int_0^\infty \lambda'(\eta)\exp(i\omega\eta)d\eta,$$

where $\mu_0, \lambda_0 \in \mathbb{R}$ are taken to be positive and $\mu', \lambda' \in L^1(\mathbb{R})$ are real-valued; the dependence of $\mu_0, \lambda_0$ and $\mu', \lambda'$ on $z$ is understood and not written. Further restrictions are due to thermodynamics. As shown in [15] (Eq. (3.2.11)), the dissipative character of the stress results in the inequalities

$$\mu_s'(\omega) < 0, \qquad 2\mu_s'(\omega) + \lambda_s'(\omega) < 0, \qquad \forall\omega > 0, \tag{2.9}$$

where the subscript $s$ denotes the half-range Fourier-sine transform. Since

$$\mathbf{e}_3\mathbf{C}\mathbf{e}_3 = \mathrm{diag}[\mu, \mu, 2\mu + \lambda],$$

the invertibility of $\mathbf{e}_3\mathbf{C}\mathbf{e}_3$ amounts to the requirement that

$$\mu \neq 0, \qquad 2\mu + \lambda \neq 0,$$

which is a direct consequence of (2.9).

For formal simplicity we let $\mathbf{k}_\parallel$ have a real direction and choose the $x$-axis to be in the direction of $\mathbf{k}_\parallel$ so that $k_y = 0$. Upon (2.8), the system (2.7) then decouples in two subsystems,

$$\mathbf{w}_v' = \mathbf{A}_v \mathbf{w}_v, \qquad \mathbf{w}_h' = \mathbf{A}_h \mathbf{w}_h. \tag{2.10}$$

The vectors $\mathbf{w}_v, \mathbf{w}_h$ and matrices $\mathbf{A}_v, \mathbf{A}_h$ are given by

$$\mathbf{w}_v = \begin{bmatrix} \hat{u}_x \\ \hat{u}_z \\ \hat{t}_x \\ \hat{t}_z \end{bmatrix}, \qquad \mathbf{A}_v = \begin{bmatrix} 0 & -ik_x & 1/\mu & 0 \\ -i\gamma k_x & 0 & 0 & \gamma/\lambda \\ \zeta k_x^2 - \rho\omega^2 & 0 & 0 & -i\gamma k_x \\ 0 & -\rho\omega^2 & -ik_x & 0 \end{bmatrix},$$

where $\gamma = \lambda/(2\mu + \lambda)$, $\zeta = 4\mu(\mu + \lambda)/(2\mu + \lambda)$, and

$$\mathbf{w}_h = \begin{bmatrix} \hat{u}_y \\ \hat{t}_y \end{bmatrix}, \qquad \mathbf{A}_h = \begin{bmatrix} 0 & 1/\mu \\ \mu k_x^2 - \rho\omega^2 & 0 \end{bmatrix}. \tag{2.11}$$

The systems (2.10) describe vertically-polarized (with polarization in the $xz$-plane) waves and horizontally-polarized ($y$-polarized) waves [16]. The decoupling implies that the horizontal and vertical polarizations are conserved through the body. The system for $\mathbf{w}_h$ in (2.10) is equivalent to the ordinary differential equation

$$(\mu \hat{u}_y')' + (\rho\omega^2 - \mu k_x^2)\hat{u}_y = 0 \tag{2.12}$$

which is typical of scattering problems [3 – 7]; here, though, $\mu$ and $k_x$ are complex-valued. Upon the transformation

$$\hat{u}_y = \sigma(z) \exp\Big[ - (1/2) \int_0^z (\mu'/\mu)(\zeta) d\zeta \Big],$$

Eq. (2.12) can be written in the normal form

$$\sigma''(z) + f(z, \omega, k_x)\sigma(z) = 0, \tag{2.13}$$

where

$$f(z, \omega, k_x) = \frac{\rho\omega^2}{\mu} - k_x^2 - \frac{1}{2}\frac{\mu''}{\mu} + \left(\frac{\mu'}{2\mu}\right)^2$$

is complex-valued and parameterized by $\omega$ and $k_x$.

## 3. Asymptotic wave propagator

Let $\mathbf{A}(z)$ be a matrix function on $\mathbb{R}$ with values in $M_6(\mathbb{C})$, namely the set of $6 \times 6$ matrices with complex entries. Let the matrix $\mathbf{A}$ be continuous on $\mathbb{R}$ except a point, say $z = 0$, where it may suffer a jump discontinuity, $\mathbf{0} \neq [\![\mathbf{A}]\!] := \mathbf{A}(0^+) - \mathbf{A}(0^-)$. Hence $\mathbf{w}(z) : \mathbb{R} \to \mathbb{C}^6$ satisfies the differential Eq. (2.7) as $z \in \mathbb{R} \setminus \{0\}$ and is continuous at $z = 0$, i.e.

$$\mathbf{w}(0^+) = \mathbf{w}(0^-).$$

This equality represents the welded contact condition such that $\mathbf{u}$ and $\mathbf{t}$ are continuous at any discontinuity surface for material parameters.

We now examine the asymptotic properties of the fields $\mathbf{u}$ and $\mathbf{t}$. It is convenient to introduce a representation in terms of a column vector $\mathbf{v}$ which is related to $\mathbf{w}$ through a wave-splitting technique (cf. [17]). To give evidence to the asymptotic properties, we apply the wave splitting in terms of the eigenvectors of $\mathbf{A}$ at $\pm\infty$.

Let $\mathbf{A}^-$ and $\mathbf{A}^+$ be the limit values of $\mathbf{A}$ as $z$ approaches $-\infty$ and $\infty$. The matrices $\mathbf{A}^{\pm}$ are taken to be simple and the eigenvalues to be nonzero. Simplicity is a generic property; examples of non-simple matrices $\mathbf{A}$ correspond to peculiar values of $k_x$ (cf. [16]). Let $m^+$ ($m^-$) be the maximal real part of the eigenvalues of $\mathbf{A}^+$ ($\mathbf{A}^-$). Also let $\|\cdot\|$ be a matrix norm in $M_6(\mathbb{C})$ like, e.g., $\|\mathbf{A}\| = \max|A_{ij}|, i,j = 1,...,6$. We assume that

$$\|\mathbf{A}(z) - \mathbf{A}^{\pm}\| = o([|z|\exp(2m^{\pm}z)]^{-1}), \quad \text{as } z \to \pm\infty. \tag{3.1}$$

For the sake of convenience we now restrict the analysis to the half-space $z > 0$ and hence denote by the superscript $+$ the pertinent asymptotic values. Strictly analogous relations hold for the half-space $z < 0$.

Denote by $i\sigma_\alpha^+$ and $\mathbf{p}_\alpha^+$, $\alpha = 1,...,6$, the eigenvalues and eigenvectors of $\mathbf{A}^+$. Let $\mathbf{P}^+$ be the matrix whose columns are the eigenvectors $\mathbf{p}_\alpha^+$, namely

$$\mathbf{P}^+ = [\mathbf{p}_1^+, ..., \mathbf{p}_6^+].$$

Since $\mathbf{A}^+$ is simple we have [18]

$$(\mathbf{P}^+)^{-1}\mathbf{A}^+\mathbf{P}^+ = \mathbf{S}^+, \tag{3.2}$$

where $\mathbf{S}^+$ is the diagonal matrix

$$\mathbf{S}^+ = \operatorname{diag}[i\sigma_1^+, ..., i\sigma_6^+].$$

It is convenient to consider the new variables $\mathbf{v}(z) : \mathbb{R} \to \mathbb{C}^6$ such that

$$\mathbf{w} = \mathbf{P}^+\mathbf{E}\mathbf{v},$$

where

$$\mathbf{E}(z) = \operatorname{diag}[\exp(i\sigma_1^+ z), ..., \exp(i\sigma_6^+ z)].$$

Incidentally, by definition $\mathbf{E}$ satisfies the differential equation

$$\mathbf{E}' = \mathbf{E}\mathbf{S}^{+} = \mathbf{S}^{+}\mathbf{E}.$$

Substitution of $\mathbf{w}$ in (2.7) provides the differential equation for $\mathbf{v}$ in the form

$$\mathbf{v}' = \mathbf{M}\mathbf{v}, \tag{3.3}$$

where

$$\mathbf{M}(z) = \mathbf{E}^{-1}(z)(\mathbf{P}^{+})^{-1}\mathbf{A}(z)\mathbf{P}^{+}\mathbf{E}(z) - \mathbf{S}^{+}.$$

Once the vector function $\mathbf{v}$ is determined we obtain the original unknown $\mathbf{w}(z) = [\mathbf{u}(z), \mathbf{t}(z)]^{T}$ in the form

$$\mathbf{w}(z) = \sum_{\alpha=1}^{6} v_{\alpha}(z)\mathbf{p}_{\alpha}^{+} \exp[i(\mathbf{k}_{\alpha}^{+} \cdot \mathbf{x} - \omega t)],$$

where $\mathbf{k}_{\alpha}^{+} = \mathbf{k}_{\parallel} + \sigma_{\alpha}^{+}\mathbf{e}_3$. The components $v_{\alpha}$ can be viewed as the amplitudes of inhomogeneous waves [8]. Indeed, since the functions $v_{\alpha}$ have a limit as $z \to \pm\infty$, asymptotically $\mathbf{w}$ is just a superposition of inhomogeneous waves.

We then investigate the existence and uniqueness of the solution $\mathbf{v}$ to (3.3) with a given initial value, e.g. $\mathbf{v}(0^{+})$. By replacing $\mathbf{S}^{+}$ with (3.2) we can write $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{E}^{-1}(\mathbf{P}^{+})^{-1}[\mathbf{A}(z) - \mathbf{A}^{+}]\mathbf{P}^{+}\mathbf{E}.$$

Hence, because $-\Im\sigma_{\alpha}^{+} \leqslant m^{+}$, $\alpha = 1, ..., 6$, we have the estimate

$$\|\mathbf{M}(z)\| \leqslant \|(\mathbf{P}^{+})^{-1}\| \, \|\mathbf{A}(z) - \mathbf{A}^{+}\| \, \|\mathbf{P}^{+}\| \exp(2m^{+}z).$$

Two consequences follow at once. First, if $\mathbf{A}$ is constant as $z \geqslant 0$ then $\mathbf{M}(z) = 0$, $\forall z \in (0, \infty)$. By (3.3) this implies that $\mathbf{v}$ is constant in homogeneous half-spaces. Second, the assumption (3.1) makes $\|\mathbf{M}\|$ to be integrable on $\mathbb{R}^{+}$ namely

$$\int_{0}^{\infty} \|\mathbf{M}(z)\| dz < \infty. \tag{3.4}$$

The integrability (3.4) and the continuity of $\mathbf{M} : \mathbb{R}^{+} \to M_6(\mathbb{C})$ allows us to argue, step by step, as in [19], and conclude that there exists a fundamental matrix $\mathbf{U}(z)$ such that

$$\mathbf{U}' = \mathbf{M}\mathbf{U}, \qquad \mathbf{U}(0) = \mathbf{1}, \qquad \mathbf{v}(z) = \mathbf{U}(z)\mathbf{v}(0^{+}), \tag{3.5}$$

in $(0, \infty)$ and $\mathbf{v}(z)$ has a limit as $z \to \infty$. Letting

$$\mathbf{v}^{+} = \lim_{z \to \infty} \mathbf{v}(z), \qquad \mathbf{U}^{+} = \lim_{z \to \infty} \mathbf{U}(z),$$

we can write

$$\mathbf{v}^{+} = \mathbf{U}^{+}\mathbf{v}(0^{+}). \tag{3.6}$$

Apart from purely formal changes, the same statements and results hold as $z \in (-\infty, 0)$. For instance,

$$\mathbf{E} = \mathrm{diag}[\exp(i\sigma_1^- z), ..., \exp(i\sigma_6^- z)],$$

and

$$\mathbf{v}^- = \mathbf{U}^- \mathbf{v}(0^-). \tag{3.7}$$

Owing to a possible discontinuity of $\mathbf{A}$, and hence of $\mathbf{v}$, at $z = 0$, we need a relation between $\mathbf{v}(0^-)$ and $\mathbf{v}(0^+)$. We know that since $\mathbf{w}$ is continuous, whence

$$\mathbf{w}(0^-) = \mathbf{w}(0^+),$$

that

$$\mathbf{E}(0^-) = \mathbf{E}(0^+) = \mathbf{1},$$

and that

$$\mathbf{w}(0^-) = \mathbf{P}^- \mathbf{v}(0^-), \qquad \mathbf{w}(0^+) = \mathbf{P}^+ \mathbf{v}(0^+).$$

As a consequence, $\mathbf{v}(0^-)$ and $\mathbf{v}(0^+)$ are connected by

$$\mathbf{v}(0^+) = (\mathbf{P}^+)^{-1} \mathbf{P}^- \mathbf{v}(0^-). \tag{3.8}$$

By means of (3.6), (3.7) and (3.8) we have

$$\mathbf{v}^+ = \mathbf{T} \mathbf{v}^-, \tag{3.9}$$

where

$$\mathbf{T} = \mathbf{U}^+ (\mathbf{P}^+)^{-1} \mathbf{P}^- (\mathbf{U}^-)^{-1} \tag{3.10}$$

may be viewed as the asymptotic wave-propagator matrix (cf. [20]). The matrix $\mathbf{T}$ is non-singular and is parameterized by $\mathbf{k}_\parallel$ through $\mathbf{P}^\pm$ and $\mathbf{U}^\pm$.

To sum up, the existence and uniqueness of the fundamental matrix $\mathbf{U}(z)$ implies the existence and uniqueness of the solution $\mathbf{v}(z)$ to the Cauchy problem for (3.3) in $(0, \infty)$ with initial value $\mathbf{v}(0^+)$, possibly through (3.8). In direct scattering problems, though, the values $\mathbf{v}(0^+)$ and $\mathbf{v}(0^-)$ are unknown. Rather, we consider an incident wave which comes, e.g., from $-\infty$; it is partially backscattered and partially transmitted by the stratified medium. This means that neither $\mathbf{v}^+$ nor $\mathbf{v}^-$ in (3.9) can be regarded as known. The associated reflection-transmission problem for $\mathbf{v}^+$ and $\mathbf{v}^-$ is not a Cauchy problem and hence existence and uniqueness of the solution are not guaranteed. This is examined in the next section.

## 4. Existence and uniqueness of reflected and transmitted waves

The asymptotic limits of the matrix $\mathbf{A}$ allows the identification of incident, reflected and transmitted waves. The real part of $\sigma_\alpha^+$ is the $z$-component of the phase speed of the corresponding asymptotic wave mode

$$\mathbf{w} = v_\alpha^+ \mathbf{p}_\alpha^+ \exp[i(\mathbf{k}_\alpha^+ \cdot \mathbf{x} - \omega t)].$$

Accordingly, we assume that the real part of $\sigma_\alpha$ is positive for three eigenvalues and negative for the three remaining ones. For definiteness we let $\Re\sigma_\alpha^+ > 0$ as $\alpha = 1, 2, 3$ and $\Re\sigma_\alpha^+ < 0$ as $\alpha = 4, 5, 6$. Hence we regard $v_1, v_2, v_3$ as being associated with forward-propagating waves, and $v_4, v_5, v_6$ with backward-propagating waves, in the $z$-direction. Alternative, non-equivalent partitions of the forward-backward propagating modes can be considered that are based on the direction of the energy flow or amplitude growth [21, 13]. They result in a different partition of the asymptotic wave modes. Yet the essence of the subsequent analysis holds, irrespective of the criterion adopted to select forward- and backward-propagating wave solutions.

For definiteness, at first let the incident wave come from $-\infty$. A reflected wave is going back to $-\infty$, a transmitted wave is going to $+\infty$ and no backward-propagating wave occurs at $+\infty$. Let $\mathbf{v}_i, \mathbf{v}_o$ be triples associated with input and output waves. The sixtuples $\mathbf{v}^-$ and $\mathbf{v}^+$ at $-\infty$ and $+\infty$ are given by

$$\mathbf{v}^- = \begin{bmatrix} \mathbf{v}_i^- \\ \mathbf{v}_o^- \end{bmatrix}, \qquad \mathbf{v}^+ = \begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ represent the reflected and transmitted waves. The matrix $\mathbf{T}$ can be viewed as given by $3 \times 3$ blocks. We then write (3.9) in the form

$$\begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_i^- \\ \mathbf{v}_o^- \end{bmatrix},$$

or

$$\mathbf{T}_1\mathbf{v}_i^- + \mathbf{T}_2\mathbf{v}_o^- = \mathbf{v}_o^+, \qquad \mathbf{T}_3\mathbf{v}_i^- + \mathbf{T}_4\mathbf{v}_o^- = \mathbf{0}. \tag{4.1}$$

If, instead, the incident wave is coming from $+\infty$ then the reflected triple $\mathbf{v}_o^+$ and the transmitted triple $\mathbf{v}_o^-$ are given by

$$\mathbf{v}_o^+ = \mathbf{T}_2\mathbf{v}_o^-, \qquad \mathbf{v}_i^+ = \mathbf{T}_4\mathbf{v}_o^-, \tag{4.2}$$

where $\mathbf{v}_i^+$ is the incident triple. Equations (4.1) and (4.2) are to be solved in the unknowns $\mathbf{v}_o^-, \mathbf{v}_o^+$.

If $\mathbf{T}_4$ is non-singular then $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ are determined at once. By (4.1) we have

$$\mathbf{v}_o^- = -\mathbf{T}_4^{-1}\mathbf{T}_3\mathbf{v}_i^-, \qquad \mathbf{v}_o^+ = (\mathbf{T}_1 - \mathbf{T}_2\mathbf{T}_4^{-1}\mathbf{T}_3)\mathbf{v}_i^-.$$

Similarly, by (4.2) we determine $\mathbf{v}_o^+$ and $\mathbf{v}_o^-$ in terms of the incident wave $\mathbf{v}_i^+$. For generality, both $\mathbf{v}_i^-$ and $\mathbf{v}_i^+$ are allowed to occur. The output waves $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ are given in terms of the input waves $\mathbf{v}_i^-$ and $\mathbf{v}_i^+$ as

$$\begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{v}_o^- \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_i^- \\ \mathbf{v}_i^+ \end{bmatrix},$$

where

$$\mathbf{S}_1 = \mathbf{T}_1 - \mathbf{T}_2\mathbf{T}_4^{-1}\mathbf{T}_3, \quad \mathbf{S}_2 = \mathbf{T}_2\mathbf{T}_4^{-1}, \quad \mathbf{S}_3 = -\mathbf{T}_4^{-1}\mathbf{T}_3, \quad \mathbf{S}_4 = \mathbf{T}_4^{-1}.$$

The matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix},$$

is called scattering matrix and is uniquely determined by $\mathbf{T}$. Of course it provides the reflected and transmitted waves in terms of the incident ones.

The matrix $\mathbf{T}_4$ depends on the complex vector parameter $\mathbf{k}_\parallel$ and hence may become singular by appropriate choices of $\mathbf{k}_\parallel$. Let $\det \mathbf{T}_4 = 0$. By (4.1) nonzero triples $\mathbf{v}_o^-$ in the nullspace of $\mathbf{T}_4$, $\mathcal{N}(\mathbf{T}_4)$, exist and make the case $\mathbf{v}_i^- = \mathbf{0}$ and $\mathbf{v}_o^- \neq \mathbf{0}$ to be possible while the remaining equation of (4.1) determines $\mathbf{v}_o^+$. Similar conclusions follow from (4.2). Hence scattering solutions $\mathbf{v}_s(z)$ can exist such that

$$\mathbf{v}_s^- = \begin{bmatrix} \mathbf{0} \\ \mathbf{v}_o^- \end{bmatrix}, \qquad \mathbf{v}_s^+ = \begin{bmatrix} \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix}. \tag{4.3}$$

In the picture of plane waves, a wave solution satisfying (4.3) is regarded as an interfacial wave [9].

Now let the incident wave $\tilde{\mathbf{v}}_i^-$ meet the condition $\mathbf{T}_3\tilde{\mathbf{v}}_i^- \in \mathcal{R}(\mathbf{T}_4)$ where $\mathcal{R}(\mathbf{T}_4)$ is the range of $\mathbf{T}_4$. By (4.1) at least one pair of output triples exists, say $\tilde{\mathbf{v}}_o^-, \tilde{\mathbf{v}}_o^+$. Hence a solution $\tilde{\mathbf{v}}$ occurs subject to

$$\tilde{\mathbf{v}}^- = \begin{bmatrix} \tilde{\mathbf{v}}_i^- \\ \tilde{\mathbf{v}}_o^- \end{bmatrix}, \qquad \tilde{\mathbf{v}}^+ = \begin{bmatrix} \tilde{\mathbf{v}}_o^+ \\ \mathbf{0} \end{bmatrix}.$$

Since the system (3.3) is linear, the field $\mathbf{v}_s(z) + \tilde{\mathbf{v}}(z)$ is a solution such that

$$\left[\mathbf{v}_s + \tilde{\mathbf{v}}\right]^- = \begin{bmatrix} \tilde{\mathbf{v}}_i^- \\ \tilde{\mathbf{v}}_o^- + \mathbf{v}_o^- \end{bmatrix}, \qquad \left[\mathbf{v}_s + \tilde{\mathbf{v}}\right]^+ = \begin{bmatrix} \tilde{\mathbf{v}}_o^+ + \mathbf{v}_o^+ \\ \mathbf{0} \end{bmatrix}.$$

Hence $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{v}} + \mathbf{v}_s$ are two fields associated with the same incident field, which shows the nonuniqueness of the direct scattering problem.

Something else can also occur. The range of $\mathbf{T}_4$, $\mathcal{R}(\mathbf{T}_4)$, is two-dimensional, at most. Hence there are vectors $\mathbf{v}_i^+$ such that (4.2) is not compatible. Similarly, if $\mathbf{T}_3$ is non-singular, there are vectors $\mathbf{v}_i^-$ such that

$$\mathbf{T}_3\mathbf{v}_i^- \notin \mathcal{R}(\mathbf{T}_4).$$

Hence we say that, for such vectors $\mathbf{v}_i^+$ and $\mathbf{v}_i^-$, no vector $\mathbf{v}_o^-$ and $\mathbf{v}_o^+$ satisfies the second relation in (4.1) or (4.2) and then there is no solution for the scattering problem. In conclusion we can write

$$\det \mathbf{T}_4 \neq 0 \quad \Longrightarrow \quad \text{existence and uniqueness of } \mathbf{v}_o^-, \mathbf{v}_o^+, \text{ and } \mathbf{S},$$

$$\det \mathbf{T}_4 = 0,\ \mathbf{v}_i = \mathbf{v}_i^-, \quad \Longrightarrow \begin{cases} \text{no solution } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{T}_3\mathbf{v}_i \notin \mathcal{R}(\mathbf{T}_4), \\ \text{nonuniqueness of } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{T}_3\mathbf{v}_i \in \mathcal{R}(\mathbf{T}_4), \end{cases}$$

$$\det \mathbf{T}_4 = 0, \mathbf{v}_i = \mathbf{v}_i^+, \quad \Longrightarrow \begin{cases} \text{no solution } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{v}_i \notin \mathcal{R}(\mathbf{T}_4), \\ \text{nonuniqueness of } \mathbf{v}_o^-, \mathbf{v}_o^+ \text{ if } \mathbf{v}_i \in \mathcal{R}(\mathbf{T}_4). \end{cases}$$

It is of interest to compare this conclusion with existence and uniqueness results appearing in the literature. For example, [5] proves that there is at most one pair of reflection and transmission coefficients consistent with the wave reflection problem. Here we show how a less restrictive hypothesis makes nonuniqueness to be possible. Roughly, the differential equation

$$y'' + N(z)y = 0, \quad z \in \mathbb{R}, \tag{4.4}$$

is considered where $N(z) \geqslant \delta > 0$, $N' \in L(\mathbb{R})$ and $N$ has finite limits as $z \to \pm\infty$. Also in view of (2.13), we observe that the qualitative difference with the approach in [5] is that we allow for a complex-valued coefficient $N$, which does not guarantee existence and uniqueness. A similar remark holds for [6]. Moreover, the nonexistence of $\mathbf{S}$, due to the vanishing of $\det \mathbf{T}_4$, is consistent with a remark made in [3] that $\mathbf{S}$ may not exist at some values of a suitable parameter. The equation examined in [3] is a particular case of (2.13).

As a comment on the condition $\det \mathbf{T}_4 = 0$, we observe that if the half-spaces are uniform then $\mathbf{U}^\pm$ is the identity. Hence $\mathbf{T} = (\mathbf{P}^+)^{-1}\mathbf{P}^-$. Now, by (3.10) we have

$$\mathbf{T} = \begin{bmatrix} (\mathbf{P}_3^+)^T\mathbf{P}_1^- + (\mathbf{P}_1^+)^T\mathbf{P}_3^- & (\mathbf{P}_3^+)^T\mathbf{P}_2^- + (\mathbf{P}_1^+)^T\mathbf{P}_4^- \\ (\mathbf{P}_4^+)^T\mathbf{P}_1^- + (\mathbf{P}_2^+)^T\mathbf{P}_3^- & (\mathbf{P}_4^+)^T\mathbf{P}_2^- + (\mathbf{P}_2^+)^T\mathbf{P}_4^- \end{bmatrix}.$$

Hence $\det \mathbf{T}_4 = 0$ takes the form

$$\det[(\mathbf{P}_4^+)^T\mathbf{P}_2^- + (\mathbf{P}_2^+)^T\mathbf{P}_4^-] = 0,$$

which corresponds to the condition for the occurrence of interfacial waves (see, e.g., the equivalent Eq. (2.11) of [9]). This is consistent with [11] where the condition for the existence of interfacial waves is related to possible nonexistence or nonuniqueness of the solution.

## 5. Application to horizontally-polarized waves

By (2.11), horizontally-polarized waves are described by

$$\mathbf{A} = \begin{bmatrix} 0 & 1/\mu \\ -\nu^2\mu & 0 \end{bmatrix},$$

where $\nu = \sqrt{\rho\omega^2/\mu - k_x^2}$, $\Re\nu > 0$. Accordingly we have

$$\mathbf{A}^+ = \begin{bmatrix} 0 & 1/\mu^+ \\ -(\nu^+)^2\mu^+ & 0 \end{bmatrix}, \qquad \mathbf{P}^+ = \begin{bmatrix} 1 & 1 \\ i\mu^+\nu^+ & -i\mu^+\nu^+ \end{bmatrix},$$

$$\mathbf{S} = \begin{bmatrix} i\nu^+ & 0 \\ 0 & -i\nu^+ \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} \exp(i\nu^+ z) & 0 \\ 0 & \exp(-i\nu^+ z) \end{bmatrix}, \qquad z > 0.$$

The matrix $\mathbf{M}$ is then given by

$$\mathbf{M}(z) = \begin{bmatrix} f(1+a) - i\nu^+ & -f(1-a)\exp(-2i\nu^+ z) \\ f(1-a)\exp(2i\nu^+ z) & -f(1+a) + i\nu^+ \end{bmatrix},$$

where

$$a(z) = \frac{\mu^2(z)\nu^2(z)}{(\mu^+\nu^+)^2}, \qquad f(z) = i\frac{\mu^+\nu^+}{2\mu(z)}.$$

In homogeneous media we have $\mu = \mu^+, \nu = \nu^+$ whence $a = 1$, $f = i\nu^+/2$ and $\mathbf{M}$ vanishes. This in turn implies that $v_1$ and $v_2$ are constant in homogeneous regions.

We now determine numerically a case where $\det \mathbf{T}_4 = 0$. First we look for the $2 \times 2$ fundamental matrix $\mathbf{U}^+$ such that

$$\begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} U_1^+ & U_2^+ \\ U_3^+ & U_4^+ \end{bmatrix} \begin{bmatrix} v_1(0^+) \\ v_2(0^+) \end{bmatrix}.$$

We regard $\mathbf{U}^+$ as given which means that (3.5) is taken to be solved.

We assume that the half-space $z < 0$ is homogeneous and hence

$$\begin{bmatrix} v_1^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} v_1(0^-) \\ v_2(0^-) \end{bmatrix}, \qquad \mathbf{U}^- = \mathbf{1}.$$

Since $\mathbf{T} = \mathbf{U}^+(\mathbf{P}^+)^{-1}\mathbf{P}^-$ and

$$\mathbf{P}^- = \begin{bmatrix} 1 & 1 \\ i\mu^-\nu^- & -i\mu^-\nu^- \end{bmatrix}$$

we obtain

$$\mathbf{T} = \frac{1}{2\mu^+\nu^+}\begin{bmatrix} \mu^+\nu^+(U_1^+ + U_2^+) + \mu^-\nu^-(U_1^+ - U_2^+) & \mu^+\nu^+(U_1^+ + U_2^+) - \mu^-\nu^-(U_1^+ - U_2^+) \\ \mu^+\nu^+(U_3^+ + U_4^+) + \mu^-\nu^-(U_3^+ - U_4^+) & \mu^+\nu^+(U_3^+ + U_4^+) - \mu^-\nu^-(U_3^+ - U_4^+) \end{bmatrix}.$$

Now we show that there is a value of $\mu^-$ such that

$$0 = T_4 = \mu^+\nu^+(U_3^+ + U_4^+) - \mu^-\nu^-(U_3^+ - U_4^+).$$

Such is the case if

$$\mu^-\nu^- = \frac{U_3^+ + U_4^+}{U_3^+ - U_4^+}\mu^+\nu^+ =: \alpha.$$

Upon substitution of the expression for $\nu$ we have

$$(k_x\mu^-)^2 - \rho^-\omega^2\mu^- + \alpha^2 = 0,$$

whence

$$\mu^- = \frac{\rho^-\omega^2}{2k_x^2}(1 \pm \sqrt{1 - (2\alpha k_x/\rho^-\omega^2)^2}).$$

By thermodynamics [15], only the solution such that $\Im\mu^- < 0$ is admissible.

As an example, let

$$\mu(z) = \begin{cases} \mu_0 \exp(z^2), & z \in (0,2), \\ \mu_0 \exp(4) = \mu^+, & z \in [2,\infty), \end{cases}$$

where $\mu_0 = (12.10 - 0.40i)10^{10}$ g/cm s$^2$ is the value of $\mu$ in the model of Berkeley crust while

$$\rho^+ = 2.1 \text{ g/cm}^3,\ z \in (0,\infty), \qquad \rho^- = 1.5 \text{ g/cm}^3,\ z \in (-\infty, 0)$$

and

$$\omega = 30 \text{ s}^{-1}, \qquad k_x = 0.1 + 0.5i \text{ cm}^{-1}.$$

We find that

$$\mu^- = (.3676 - 2.468i)10^{11}\text{g/cm}^3$$

is the admissible solution. Accordingly, if such is the value of $\mu$ in the incidence half-space ($\mu^- \neq \mu(0^+)$) then $T_4 = 0$ and hence, since $T_3 \neq 0$, the problem (4.1) is incompatible.

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